

## Research Article

# Series Solutions of Time-Fractional PDEs by Homotopy Analysis Method

**O. Abdulaziz, I. Hashim, and A. Saif**

*Centre for Modelling and Data Analysis, School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 UKM Bangi Selangor, Malaysia*

Correspondence should be addressed to I. Hashim, [ishak.h@ukm.my](mailto:ishak.h@ukm.my)

Received 12 August 2008; Accepted 30 October 2008

Recommended by Shijun Liao

The homotopy analysis method (HAM) is applied to solve linear and nonlinear fractional partial differential equations (fPDEs). The fractional derivatives are described by Caputo's sense. Series solutions of the fPDEs are obtained. A convergence theorem for the series solution is also given. The test examples, which include a variable coefficient, inhomogeneous and hyperbolic-type equations, demonstrate the capability of HAM for nonlinear fPDEs.

Copyright © 2008 O. Abdulaziz et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## 1. Introduction

Fractional calculus has been given considerable popularity and importance during the past three decades, due mainly to its applications in numerous fields of science and engineering. For example, phenomena in the areas of fluid flow, rheology, electrical networks, probability and statistics, control theory of dynamical systems, electrochemistry of corrosion, chemical physics, optics and signal processing, and so on can be successfully modelled by linear or nonlinear fractional differential equations (fDEs) [1–4].

Finding accurate methods for solving nonlinear differential equations has become important. Some of the analytical methods for nonlinear differential equations are the Adomian decomposition method (ADM) [5–14], the homotopy-perturbation method (HPM) [15–19], variational iteration method (VIM) [12, 20–24], and the EXP-function method [25]. Another analytical approach that can be applied to solve nonlinear differential equations is to employ the homotopy analysis method (HAM) [26–29]. Some of the recent applications of HAM can be found in [30–41]. An account of the recent developments of HAM was given by Liao [42]. HAM has been successfully applied into engineering fields. The method has been applied to give an explicit solution for the Riemann problem of the nonlinear shallow-water equations [43]. The obtained Riemann solver has been implemented into a numerical model to simulate long waves, such as storm surge or tsunami, propagation and run-up.

Very recently, Song and Zhang [44] applied HAM to solve fractional KdV-Burgers-Kuramoto equation. Cang et al. [45] solved nonlinear Riccati differential equations of fractional order using HAM. Hashim et al. [46] employed HAM to solve fractional initial value problems (fIVPs) for ordinary differential equations. In [47], the applicability of the HAM was extended to construct numerical solution for the fractional BBM-Burgers equation. The HAM solutions for systems of nonlinear fractional differential equations were presented by Bataineh et al. [48].

A specific linear, nonhomogeneous time fractional partial differential equation (fPDE) with variable coefficients was first transformed to two fractional ordinary differential equations which were then solved by HAM in [49]. Recently, Xu et al. [50] applied the HAM to linear, homogeneous one- and two-dimensional fractional heat-like PDEs subject to the Neumann boundary conditions. Jafari and Seifi [51] applied HAM to linear and nonlinear homogeneous fractional diffusion-wave equations. Very recently, the HAM was shown to be capable of solving linear and nonlinear systems of fPDEs [52].

In this paper, we shall consider linear and nonlinear fPDEs of the form

$$D_t^\alpha u(x, t) = f(u, u_x, u_{xx}), \quad n-1 < \alpha \leq n, \quad t > 0, \quad (1.1)$$

subject to the initial conditions

$$u^{(k)}(x, 0) = g_k(x), \quad k = 0, 1, 2, \dots, n-1, \quad (1.2)$$

where  $n$  is an integer,  $f$  is a linear/nonlinear function, and  $D_t^\alpha(\cdot) = \partial^\alpha(\cdot)/\partial t^\alpha$  is a fractional differential operator. We shall demonstrate the applicability of HAM to fPDEs through several linear and nonlinear test examples.

## 2. Preliminaries

The fractional derivative is defined in the Caputo sense as in [53],

$$D^\alpha w(t) = J^{n-\alpha} D^n w(t). \quad (2.1)$$

Here  $D^n$  is the usual integer differential operator of order  $n$  and  $J^\beta$  is the Riemann-Liouville fractional integral operator of order  $\beta > 0$ , defined by

$$J^\beta w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} w(\tau) d\tau, \quad (t > 0), \quad (2.2)$$

$$J^0 w(t) = w(t).$$

Some of the properties of the operator  $J^\beta$ , which we will need in our work, are as follows [2, 3]:

- (1)  $J^\beta J^\mu w(t) = J^{\beta+\mu} w(t)$ ,
- (2)  $J^\beta J^\mu w(t) = J^\mu J^\beta w(t)$ ,
- (3)  $J^\beta t^\gamma = (\Gamma(\gamma+1)/\Gamma(\beta+\gamma+1)) t^{\beta+\gamma}$ .

Caputo's fractional derivative has a useful property [54]

$$(J^\mu D^\mu)w(t) = w(t) - \sum_{k=0}^{n-1} w^{(k)}(0^+) \frac{t^k}{k!}, \quad (n-1 < \mu \leq n). \quad (2.3)$$

The operator form of the nonlinear fPDEs (1.1) can be written as follows:

$$D_t^\alpha(u(x, t)) = A(u, u_x, u_{xx}) + B(u, u_x, u_{xx}) + C(x, t), \quad n-1 < \alpha \leq n, \quad t > 0, \quad (2.4)$$

subject to the initial conditions

$$u^{(k)}(x, 0) = g_k(x), \quad k = 0, 1, 2, \dots, n-1, \quad (2.5)$$

where  $A$  is a linear operator which might include other fractional derivatives of order less than  $\alpha$ ,  $B$  is a nonlinear operator which also might include other fractional derivatives of order less than  $\alpha$  and  $C$  is a known analytic function.

Applying the operator  $J^\alpha$ , the inverse operator of  $D^\alpha$ , to both sides of (2.4) with considering the initial conditions (2.5) according to (2.3), we obtain

$$u(x, t) = \sum_{k=0}^{n-1} g_k(x) \frac{t^k}{k!} + J^\alpha C(x, t) + J^\alpha A(u, u_x, u_{xx}) + J^\alpha B(u, u_x, u_{xx}), \quad n-1 < \alpha \leq n, \quad t > 0. \quad (2.6)$$

### 3. Homotopy analysis method (HAM)

#### 3.1. The zeroth-order deformation equation

Let  $\mathcal{L}$  denote an auxiliary linear operator,  $u_0(x, t)$  is an initial approximation of  $u(x, t)$  which satisfies the initial conditions (2.5). Note that, in this paper, the auxiliary linear operator  $\mathcal{L}$  is not the same linear operator  $A$  of (2.4).

Note that the original equation (1.1) contains the linear operator  $D_t^\alpha$ . So, it is straightforward for us to choose the auxiliary linear operator

$$\mathcal{L}(\phi) = D_t^\alpha(\phi). \quad (3.1)$$

According to (2.6), we can choose the initial approximation to be

$$u_0(x, t) = \sum_{k=0}^{m-1} g_k(x) \frac{t^k}{k!} + J^\alpha C(x, t). \quad (3.2)$$

For simplicity, let us define, according to (2.4), the nonlinear operator

$$\mathcal{N}(\phi) = D_t^\alpha(\phi) - A(\phi, \phi_x, \phi_{xx}) - B(\phi, \phi_x, \phi_{xx}) - C(x, t). \quad (3.3)$$

Hence, in the frame of HAM [29], we can construct the so-called zeroth-order deformation

$$(1 - q)\mathcal{L}(U(x, t; q) - u_0(x, t)) = q\hbar\mathcal{N}(U(x, t; q)), \quad (3.4)$$

subject to the following initial conditions:

$$U^{(k)}(x, 0; q) = g_k(x), \quad k = 0, 1, 2, \dots, m - 1, \quad (3.5)$$

where  $q \in [0, 1]$  is the embedding parameter,  $\hbar \neq 0$  is an auxiliary parameter, and  $U(x, t; q)$  is an unknown function on the independent variables  $x$ ,  $t$ , and  $q$ .

When  $q = 0$ , since  $u_0(x, t)$  satisfies all the initial conditions (2.5), and  $\phi = 0$  is a solution of  $\mathcal{L}\phi = 0$ , we have obviously

$$U(x, t; 0) = u_0(x, t), \quad (3.6)$$

and when  $q = 1$ , the zeroth-order deformation equations (3.4) and (3.5) are equivalent to the original equations (2.4) and (2.5), provided

$$U(x, t; 1) = u(x, t). \quad (3.7)$$

Using the parameter  $q$ , we expand  $U(x, t; q)$  in Taylor series as follows:

$$U(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \quad (3.8)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m U(x, t; q)}{\partial q^m} \right|_{q=0}. \quad (3.9)$$

Assume that the auxiliary linear operator  $\mathcal{L}$ , the initial guess  $u_0$  and the auxiliary parameter  $\hbar$  are properly chosen such that the series (3.8) is convergent at  $q = 1$ . Thus, due to (3.7) we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t). \quad (3.10)$$

### 3.2. The $m$ th-order deformation equation

Let us define the vector

$$\vec{u}_n = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}. \quad (3.11)$$

Following Liao [26–29], differentiating (3.4)  $m$  times with respect to the embedding parameter  $q$ , then setting  $q = 0$ , and finally dividing them by  $m!$ , we have the so-called  $m$ th-order deformation equation

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathcal{R}_m(\vec{u}_{m-1}), \quad (3.12)$$

subject to the initial conditions

$$u_m^{(k)}(x, 0) = 0, \quad k = 0, 1, 2, \dots, m-1, \quad (3.13)$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}(U(x, t; q))}{\partial^{m-1} q} \Big|_{q=0}, \quad (3.14)$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (3.15)$$

Substituting (3.3) into (3.14), and since  $A$  is a linear operator,  $\mathcal{R}_m(\vec{u}_{m-1})$  can be given by

$$\begin{aligned} \mathcal{R}_m(\vec{u}_{m-1}) &= D_t^\alpha u_{m-1} - A(u_{(m-1)}, u_{(m-1)x}, u_{(m-1)xx}) \\ &\quad - \frac{1}{(m-1)!} \frac{\partial^{m-1} B(U, U_x, U_{xx})}{\partial^{m-1} q} \Big|_{q=0} - (1 - \chi_m) C(x, t). \end{aligned} \quad (3.16)$$

According to (3.1), we can apply the operator  $J^\alpha$  to both sides of (3.12) to obtain

$$J^\alpha D^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar J^\alpha [\mathcal{R}_m(\vec{u}_{m-1})]. \quad (3.17)$$

Using the property (2.3) and the initial conditions (1.1), we have

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar J^\alpha [\mathcal{R}_m(\vec{u}_{m-1})]. \quad (3.18)$$

Finally, for the purpose of computation, we will approximate the HAM solution (3.10) by the following truncated series:

$$\phi_m(x, t) = \sum_{k=0}^{m-1} u_k(x, t). \quad (3.19)$$

### 3.3. Convergence theorem

**Theorem 3.1.** *As long as the series  $u(x, t) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t)$  converges, where  $u_m(x, t)$  is governed by (3.12) under the definitions (3.14) and (3.15), it must be a solution of (2.4).*

*Proof.* If the series  $\sum_{m=0}^{+\infty} u_m(x, t)$  is convergent, we can write

$$S(x, t) = \sum_{m=0}^{+\infty} u_m(x, t), \quad (3.20)$$

and it holds

$$\lim_{n \rightarrow +\infty} u_n(x, t) = 0. \quad (3.21)$$

From (3.12) and by using (3.15), it yields

$$\begin{aligned} \hbar \sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) &= \sum_{m=1}^{+\infty} \mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] \\ &= \lim_{n \rightarrow +\infty} \sum_{m=1}^n \mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] \\ &= \mathcal{L} \left[ \lim_{n \rightarrow +\infty} \sum_{m=1}^n (u_m(x, t) - \chi_m u_{m-1}(x, t)) \right] \\ &= \mathcal{L} \left[ \lim_{n \rightarrow +\infty} u_n(x, t) \right] = 0. \end{aligned} \quad (3.22)$$

Since  $\hbar \neq 0$ , then

$$\sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) = 0. \quad (3.23)$$

Substituting (3.16) into the above equation and simplifying it, due to the convergence of the series  $u(x, t) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t)$  and since  $A$  is a linear operator, yield

$$\begin{aligned} \sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) &= \sum_{m=1}^{+\infty} [D_t^\alpha u_{m-1} - A(u_{(m-1)}, u_{(m-1)x}, u_{(m-1)xx})] \\ &\quad - \sum_{m=1}^{+\infty} \left[ \frac{1}{(m-1)!} \frac{\partial^{m-1} B(U, U_x, U_{xx})}{\partial^{m-1} q} \Big|_{q=0} - (1 - \chi_m) C(x, t) \right] \\ &= D_t^\alpha \left( \sum_{m=0}^{+\infty} u_m \right) - A \left( \sum_{m=0}^{+\infty} u_m, \sum_{m=0}^{+\infty} u_{mx}, \sum_{m=0}^{+\infty} u_{mxx} \right) \\ &\quad - \sum_{m=0}^{+\infty} \left[ \frac{1}{m!} \frac{\partial^m B(U, U_x, U_{xx})}{\partial^m q} \Big|_{q=0} \right] - C(x, t). \end{aligned} \quad (3.24)$$

Now, expanding the nonlinear term  $B(U(x, t; q), U_x(x, t; q), U_{xx}(x, t; q))$  by using the general Taylor theorem at  $q = 0$  yields

$$B(U(x, t; q), U_x(x, t; q), U_{xx}(x, t; q)) = \sum_{m=0}^{+\infty} \left[ \frac{1}{m!} \frac{\partial^m B(U, U_x, U_{xx})}{\partial^m q} \Big|_{q=0} q^m \right]. \quad (3.25)$$

Setting  $q = 1$  in the above equation and using (3.8), we obtain

$$B\left(\sum_{m=0}^{+\infty} u_m, \sum_{m=0}^{+\infty} u_{mx}, \sum_{m=0}^{+\infty} u_{mxx}\right) = \sum_{m=0}^{+\infty} \left[ \frac{1}{m!} \frac{\partial^m B(U, U_x, U_{xx})}{\partial^m q} \Big|_{q=0} \right]. \quad (3.26)$$

Then

$$\begin{aligned} & \sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) \\ &= D_t^\alpha \left( \sum_{m=0}^{+\infty} u_m \right) - A \left( \sum_{m=0}^{+\infty} u_m, \sum_{m=0}^{+\infty} u_{mx}, \sum_{m=0}^{+\infty} u_{mxx} \right) - B \left( \sum_{m=0}^{+\infty} u_m, \sum_{m=0}^{+\infty} u_{mx}, \sum_{m=0}^{+\infty} u_{mxx} \right) - C(x, t) \\ &= D_t^\alpha (S(x, t)) - A(S(x, t), S_x(x, t), S_{xx}(x, t)) - B(S(x, t), S_x(x, t), S_{xx}(x, t)) - C(x, t). \end{aligned} \quad (3.27)$$

From the initial conditions (3.5) and (3.13), it holds that

$$S^{(k)}(x, 0) = \sum_{m=0}^{+\infty} u_m^{(k)}(x, 0) = u_0^{(k)}(x, 0) + \sum_{m=1}^{+\infty} u_m^{(k)}(x, 0) = g_k(x). \quad (3.28)$$

Thus,  $S(x, t)$  is satisfied and also must be the exact solution for (2.4).  $\square$

#### 4. Test examples

In this section, we shall illustrate the applicability of HAM to several linear and nonlinear fpDEs.

##### 4.1. Problem 1

Let us consider the following linear time-fractional wave-like equations:

$$D_t^\alpha u(x, t) = \frac{1}{2} x^2 u_{xx}(x, t), \quad t > 0, \quad x \in R, \quad 1 < \alpha \leq 2, \quad (4.1)$$

$$u(x, 0) = x, \quad u_t(x, 0) = x^2. \quad (4.2)$$

We note that the heat-like counterpart of (4.1) was solved by HAM in [50] without direct comparison with the result by the ADM. According to (3.2), we can choose the initial guess to be

$$u_0(x, t) = x + x^2 t. \quad (4.3)$$

From (3.18), we have

$$u_m = (\chi_m + \hbar)u_{m-1} - \frac{\hbar}{2}x^2 J^\alpha [(u_{m-1})_{xx}]. \quad (4.4)$$

Consequently, the first few terms of HAM series solutions are as follows:

$$\begin{aligned} u_1(x, t) &= -\hbar \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2, \\ u_2(x, t) &= -\hbar(\hbar+1) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2 + \hbar^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2, \\ u_3(x, t) &= -\hbar(\hbar+1)^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2 + 2\hbar^2(\hbar+1) \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2 - \hbar^3 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} x^2, \end{aligned} \quad (4.5)$$

and so on. Hence, the HAM series solution is

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\ &= x + x^2 t - \hbar [1 + (\hbar+1) + (\hbar+1)^2 + \dots] \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2 \\ &\quad + \hbar^2 [1 + 2(\hbar+1) + \dots] \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2 \\ &\quad - \hbar^3 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} x^2 + \dots. \end{aligned} \quad (4.6)$$

Since we choose the initial guess  $u_0(x, t)$  to be the same initial guess used by ADM [12], we can notice that when  $\hbar = -1$ , the above expression gives the same solution given by ADM. Table 1 shows the HAM approximation solutions for (4.1)-(4.2) when  $\alpha = 1.5, 1.75$ , and 2 with  $\hbar = -1$  and  $-1.0453$ . It is to be noted that the first four terms of the HAM series were used to evaluate the approximate solutions in Table 1.

#### 4.2. Problem 2

In this example, we consider the following one-dimensional linear inhomogeneous time-fractional equation:

$$D^\alpha u(x, t) + x u_x(x, t) + u_{xx}(x, t) = 2t + 2x^2 + 2, \quad t > 0, x \in R, 0 < \alpha \leq 1, \quad (4.7)$$

**Table 1:** Approximate solution of (4.1) for some values of  $\alpha$  using the 4-term HAM approximation,  $\phi_4$ , with  $\hbar = -1$  and  $\tilde{\hbar} = -1.0453$ .

$t$	$x$	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		Exact
		$\hbar = -1$	$\tilde{\hbar} = -1.045$	$\hbar = -1$	$\tilde{\hbar} = -1.045$	$\hbar = -1$	$\tilde{\hbar} = -1.045$	
0.2	0.25	0.26284062	0.26284062	0.26266989	0.26266991	0.26258350	0.26258350	0.26258350
	0.50	0.55136246	0.55136250	0.55067959	0.55067964	0.55033400	0.55033402	0.55033400
	0.75	0.86556554	0.86556562	0.86402909	0.86402919	0.86325150	0.86325156	0.86325150
	1.0	1.20544985	1.20544999	1.20271839	1.20271856	1.20133600	1.20133610	1.20133600
0.4	0.25	0.27697114	0.27697109	0.27615668	0.27615669	0.27567202	0.27567205	0.27567202
	0.50	0.60788455	0.60788437	0.60462675	0.60462679	0.60268808	0.60268820	0.60268808
	0.75	0.99274025	0.99273983	0.98541019	0.98541027	0.98104818	0.98104846	0.98104818
	1.0	1.43153821	1.43153748	1.41850702	1.41850715	1.41075232	1.41075282	1.41075232
0.6	0.25	0.29309481	0.29309501	0.29109009	0.29108997	0.28979084	0.28979084	0.28979084
	0.50	0.67237924	0.67238006	0.66436039	0.66435990	0.65916338	0.65916339	0.65916339
	0.75	1.13785328	1.13785513	1.11981088	1.11980978	1.10811762	1.10811764	1.10811764
	1.0	1.68951694	1.68952024	1.65744156	1.65743961	1.63665355	1.63665358	1.63665358

subject to the initial condition

$$u(x, 0) = x^2. \quad (4.8)$$

In Section 3, we chose the initial guess to contain the initial conditions and the source term  $C(x, t)$ . In this example, due to the appearance of noise terms and also to get the exact solution, we will modify the way we choose the initial guess. The initial guess is set to contain only the initial condition (4.8), and the source term,  $C(x, t) = 2t + 2x^2 + 2$ , will be added to  $u_1(x, t)$ . The other terms are obtained the same as described in Section 3.

Hence, the initial guess is given by

$$u_0(x, t) = x^2, \quad (4.9)$$

and according to (3.18), we have

$$\begin{aligned} u_1 &= J^\alpha [2t + 2x^2 + 2] + \hbar J^\alpha [x(u_0)_x + (u_0)_{xx}], \\ u_m &= (\chi_m + \hbar)u_{m-1} + \hbar J^\alpha [x(u_{m-1})_{xx} + (u_{m-1})_{xx}], \quad m \geq 2. \end{aligned} \quad (4.10)$$

The terms of the HAM solution series can be given by

$$\begin{aligned} u_1(x, t) &= \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + 2(\hbar+1)(x^2+1)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2(x, t) &= 2(\hbar+1)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2(\hbar+1)(x^2+1)\left[\frac{(\hbar+1)t^\alpha}{\Gamma(\alpha+1)} + \frac{2\hbar t^{2\alpha}}{\Gamma(2\alpha+1)}\right], \end{aligned} \quad (4.11)$$

and so on. Hence, the HAM series solution is

$$\begin{aligned}
u(x, t) &= u_0 + u_1 + u_2 + \dots \\
&= x^2 + 2[1 + (\hbar + 1) + \dots] \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \\
&\quad + 2(\hbar + 1)(x^2 + 1)[1 + (\hbar + 1) + \dots] \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
&\quad + 4\hbar(\hbar + 1)(x^2 + 1) \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 1)} x^2 + \dots .
\end{aligned} \tag{4.12}$$

Taking  $\hbar = -1$  in (4.12), we obtain the exact solution,

$$u(x, t) = x^2 + \frac{2t^\alpha}{\Gamma(\alpha + 2)}. \tag{4.13}$$

### 4.3. Problem 3

Consider the following nonlinear time-fractional hyperbolic equation:

$$D_t^\alpha u(x, t) = \frac{\partial}{\partial x} \left( u(x, t) \frac{\partial u(x, t)}{\partial x} \right), \quad t > 0, \quad x \in R, \quad 1 < \alpha \leq 2, \tag{4.14}$$

subject to the initial conditions

$$u(x, 0) = x^2, \quad u_t(x, 0) = -2x^2. \tag{4.15}$$

Equation (4.14) can be rewritten as follows:

$$D_t^\alpha u(x, t) = \left( \frac{\partial u(x, t)}{\partial x} \right)^2 + u(x, t) \frac{\partial^2 u(x, t)}{\partial x^2}. \tag{4.16}$$

From (3.4), construct the following zeroth-order deformation:

$$(1 - q)\mathcal{L}(U(x, t; q) - u_0(x, t)) = q\hbar\mathcal{N}(U(x, t; q)), \tag{4.17}$$

subject to the following initial conditions:

$$U(x, 0) = x^2, \quad U_t(x, 0) = -2x^2, \tag{4.18}$$

where

$$\mathcal{N}(\theta) = D_t^\alpha \theta - \theta_x^2 - \theta\theta_{xx}. \tag{4.19}$$

The auxiliary linear operator can be chosen as follows:

$$\mathcal{L}(\theta) = D_t^\alpha(\theta), \quad (4.20)$$

with the property

$$\mathcal{L}(\theta) = 0 \quad \text{when } \theta = 0, \quad (4.21)$$

while, the initial guess is

$$u_0(x, t) = x^2 - 2x^2t. \quad (4.22)$$

Again from (3.12), the high-order deformation equation can be given by

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathcal{R}_m(\vec{u}_{m-1}), \quad (4.23)$$

subject to the initial conditions

$$u_m^{(k)}(x, 0) = 0, \quad k = 0, 1, \quad (4.24)$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}(U(x, t; q))}{\partial q^{m-1}} \Big|_{q=0}. \quad (4.25)$$

Then,  $\mathcal{R}_m(\vec{u}_{m-1})$  can be given by

$$\mathcal{R}_m(\vec{u}_{m-1}) = D_t^\alpha u_{m-1} - \sum_{i=0}^{m-1} u_{ix} u_{(m-1-i)x} - \sum_{i=0}^{m-1} u_i u_{(m-1-i)xx}. \quad (4.26)$$

Accordingly, the governing equation is as follows:

$$u_m = \chi_m u_{m-1} + \hbar J^\alpha \left[ D_t^\alpha u_{m-1} - \sum_{i=0}^{m-1} u_{ix} u_{(m-1-i)x} \right] - \hbar J^\alpha \left[ \sum_{i=0}^{m-1} u_i u_{(m-1-i)xx} \right], \quad m \geq 1. \quad (4.27)$$

Consequently, the first few terms of HAM series solutions are given by

$$\begin{aligned}
u_1(x, t) &= -6\hbar x^2 \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \right], \\
u_2(x, t) &= -6\hbar x^2 \left[ (\hbar+1)\frac{t^\alpha}{\Gamma(\alpha+1)} - 4(\hbar+1)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8(\hbar+1)\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \right] \\
&\quad + 72\hbar^2 x^2 \left[ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2\left(\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} + 2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \right] \\
&\quad + 576\hbar^2 x^2 \left[ \left(\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+2)} + 1\right)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + 2\Gamma(\alpha+4)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \right],
\end{aligned} \tag{4.28}$$

and so on. Hence, the HAM series solution is

$$\begin{aligned}
u(x, t) &= u_0 + u_1 + u_2 + u_3 + \dots \\
&= x^2 - 2x^2 t - 6\hbar x^2 [1 + (\hbar+1) + \dots] \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\quad + 24\hbar x^2 [1 + (\hbar+1) + \dots] \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\
&\quad - 48\hbar x^2 [1 + (\hbar+1) + \dots] \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\
&\quad + 72\hbar^2 x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots.
\end{aligned} \tag{4.29}$$

The four-term HAM approximate solutions for (4.14)-(4.15), when  $\alpha = 1.5, 1.75,$  and  $2$  with  $\hbar = -1$  and  $-1.0966$ , are shown in Table 2. Notice that the HAM approximate solution when  $\alpha = 2$  with  $\hbar = -1.0966$  is in good agreement with the exact solution,  $u(x, t) = (x/(t+1))^2$ .

#### 4.4. Problem 4

Consider the following nonlinear time-fractional Fisher's equation:

$$D^\alpha u(x, t) = u_{xx}(x, t) + 6u(x, t)(1 - u(x, t)), \tag{4.30}$$

for  $t > 0$ ,  $x \in R$ ,  $0 < \alpha \leq 1$ , subject to the initial condition

$$u(x, 0) = \frac{1}{(1 + e^x)^2}. \tag{4.31}$$

According to (3.2), we can choose the initial guess to be

$$u_0(x, t) = \frac{1}{(1 + e^x)^2}, \tag{4.32}$$

**Table 2:** Approximate solution of (4.14) for some values of  $\alpha$  using the 4-term HAM approximation,  $\phi_4$ , with  $\hbar = -1$  and  $\hbar = -1.0966$ .

$t$	$x$	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		Exact
		$\hbar = -1$	$\hbar = -1.0966$	$\hbar = -1$	$\hbar = -1.0966$	$\hbar = -1$	$\hbar = -1.0966$	
0.2	0.25	0.060049	0.060199	0.048780	0.048788	0.043403	0.043404	0.043403
	0.50	0.240195	0.240795	0.195119	0.195152	0.173610	0.173617	0.173611
	0.75	0.540439	0.541789	0.439018	0.439092	0.390623	0.390638	0.390625
	1.0	0.960781	0.963180	0.780477	0.780607	0.694441	0.694468	0.694444
0.4	0.25	0.074536	0.076976	0.045434	0.045821	0.031853	0.031904	0.031888
	0.50	0.298142	0.307903	0.181736	0.183285	0.127412	0.127616	0.127551
	0.75	0.670821	0.692783	0.408907	0.412392	0.286677	0.287137	0.286990
	1.0	1.192570	1.231614	0.726946	0.733141	0.509649	0.510465	0.510204
0.6	0.25	0.091469	0.099717	0.045907	0.048048	0.023920	0.024414	0.024414
	0.50	0.365878	0.398867	0.183626	0.192191	0.095680	0.097656	0.097656
	0.75	0.823225	0.897450	0.413159	0.432430	0.215280	0.219727	0.219727
	1.0	1.463511	1.595467	0.734504	0.768765	0.382720	0.390626	0.390625

**Table 3:** Approximate solution of (4.30) for some values of  $\alpha$  using the 3-term HAM approximation,  $\phi_3$ , with  $\hbar = -1$  and  $-1.05133$ .

$t$	$x$	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		Exact
		$\hbar = -1$	$\hbar = -1.0513$	$\hbar = -1$	$\hbar = -1.0513$	$\hbar = -1$	$\hbar = -1.0513$	
0.1	0.25	0.946129	0.984066	0.488195	0.496911	0.317948	0.319612	0.316042
	0.50	0.843908	0.883590	0.405740	0.414926	0.250500	0.252292	0.250000
	0.75	0.715013	0.752540	0.324453	0.333180	0.190964	0.192689	0.191689
	1.0	0.576466	0.609185	0.249683	0.257315	0.140979	0.142501	0.142537
0.2	0.25	1.475318	1.551785	0.791250	0.816533	0.481199	0.488421	0.461284
	0.50	1.359827	1.439679	0.690146	0.716646	0.396941	0.404576	0.387456
	0.75	1.180981	1.256421	0.574405	0.599498	0.315266	0.322535	0.316042
	1.0	0.970078	1.035807	0.456647	0.478543	0.241175	0.247540	0.250000
0.3	0.25	1.967448	2.082543	1.124230	1.171123	0.681440	0.698116	0.604195
	0.50	1.845234	1.965336	1.009482	1.058531	0.581860	0.599391	0.534447
	0.75	1.622908	1.736324	0.859509	0.905896	0.475833	0.492464	0.461284
	1.0	1.345505	1.444292	0.695479	0.735922	0.372917	0.387446	0.387456

and according to (3.18), we have

$$u_m = (\chi_m + \hbar)u_{m-1} - \hbar J^\alpha \left[ (u_{m-1})_{xx} + 6u_{m-1} - 6 \sum_{i=0}^{m-1} u_i u_{m-1-i} \right]. \tag{4.33}$$

Consequently, the first few terms of HAM series solutions are as follows:

$$u_1(x, t) = -\frac{10\hbar e^x}{(1 + e^x)^3} \frac{t^\alpha}{\Gamma(\alpha + 1)}, \tag{4.34}$$

$$u_2(x, t) = -\frac{10\hbar e^x}{(1 + e^x)^3} \left[ (\hbar + 1) \frac{t^\alpha}{\Gamma(\alpha + 1)} - 5\hbar \frac{(2e^x - 1)}{(1 + e^x)} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \right],$$

and so on. Hence, the HAM series solution is

$$\begin{aligned}
 u(x, t) &= u_0 + u_1 + u_2 + \dots \\
 &= \frac{1}{(1 + e^x)^2} - \frac{10\hbar e^x}{(1 + e^x)^3} (1 + (\hbar + 1) + \dots) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 &\quad + 50\hbar^2 \frac{e^x(2e^x - 1)}{(1 + e^x)^4} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \dots
 \end{aligned} \tag{4.35}$$

Table 3 shows the 3-term HAM approximate solutions for (4.30)-(4.31),  $\phi_3$ , when  $\alpha = 0.5, 0.75$ , and 1 with  $\hbar = -1$  and  $-1.05133$ . We notice that the HAM approximate solution when  $\alpha = 2$  with  $\hbar = -1.05133$  is in good agreement with the exact solution,  $u(x, t) = 1/(1 + \exp(x - 5t))^2$ .

## 5. Conclusions

In this work, the homotopy analysis method (HAM) was implemented to derive exact and approximate analytical solutions for both linear and nonlinear partial differential equations of fractional order. The convergence region of the series solution obtained by HAM can be controlled and adjusted by the auxiliary parameter  $\hbar$ . We give some examples to show the efficiency and accuracy of the suggested method. It was also demonstrated that the Adomian decomposition method (ADM) is a special case of HAM for the first and second test examples.

## Acknowledgment

The financial support received from the Academy of Sciences Malaysia under tSAGA Grant no. P24c (STGL-011-2006) is gratefully acknowledged.

## References

- [1] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, vol. 111 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1974.
- [2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.
- [3] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.
- [4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2006.
- [5] N. T. Shawagfeh, "Analytical approximate solutions for nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 131, no. 2-3, pp. 517-529, 2002.
- [6] S. S. Ray and R. K. Bera, "Solution of an extraordinary differential equation by Adomian decomposition method," *Journal of Applied Mathematics*, vol. 2004, no. 4, pp. 331-338, 2004.
- [7] S. S. Ray and R. K. Bera, "An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method," *Applied Mathematics and Computation*, vol. 167, no. 1, pp. 561-571, 2005.
- [8] O. Abdulaziz, I. Hashim, M. S. H. Chowdhury, and A. K. Zulkifl, "Assessment of decomposition method for linear and nonlinear fractional differential equations," *Far East Journal of Applied Mathematics*, vol. 28, no. 1, pp. 95-112, 2007.

- [9] O. Abdulaziz, I. Hashim, and E. S. Ismail, "Approximate analytical solutions to fractional modified KdV equations by decomposition method," *Far East Journal of Applied Mathematics*, vol. 29, no. 3, pp. 455–468, 2007.
- [10] I. Hashim and M. S. Selamat, "Simulation of time-dependent enzyme kinetics by an explicit numeric-analytic technique," *Far East Journal of Applied Mathematics*, vol. 30, no. 1, pp. 115–124, 2008.
- [11] S. Momani, "Non-perturbative analytical solutions of the space- and time-fractional Burgers equations," *Chaos, Solitons & Fractals*, vol. 28, no. 4, pp. 930–937, 2006.
- [12] S. Momani and Z. Odibat, "Analytical approach to linear fractional partial differential equations arising in fluid mechanics," *Physics Letters A*, vol. 355, no. 4-5, pp. 271–279, 2006.
- [13] Q. Wang, "Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1048–1055, 2006.
- [14] Q. M. Al-Mdallal, "An efficient method for solving fractional Sturm-Liouville problems," *Chaos, Solitons & Fractals*. In press.
- [15] J.-H. He, "Application of homotopy perturbation method to nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 695–700, 2005.
- [16] Z. Odibat and S. Momani, "Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order," *Chaos, Solitons & Fractals*, vol. 36, no. 1, pp. 167–174, 2008.
- [17] S. Momani and Z. Odibat, "Homotopy perturbation method for nonlinear partial differential equations of fractional order," *Physics Letters A*, vol. 365, no. 5-6, pp. 345–350, 2007.
- [18] Q. Wang, "Homotopy perturbation method for fractional KdV-Burgers equation," *Chaos, Solitons & Fractals*, vol. 35, no. 5, pp. 843–850, 2008.
- [19] Z. Odibat, "Exact solitary solutions for variants of the KdV equations with fractional time derivatives," *Chaos, Solitons & Fractals*. In press.
- [20] Z. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *International Journal of Nonlinear Science and Numerical Simulation*, vol. 7, no. 1, pp. 27–34, 2006.
- [21] S. Momani and Z. Odibat, "Numerical comparison of methods for solving linear differential equations of fractional order," *Chaos, Solitons & Fractals*, vol. 31, no. 5, pp. 1248–1255, 2007.
- [22] J.-H. He and X.-H. Wu, "Construction of solitary solution and compacton-like solution by variational iteration method," *Chaos, Solitons & Fractals*, vol. 29, no. 1, pp. 108–113, 2006.
- [23] Y. Molliq R, M. S. M. Noorani, and I. Hashim, "Variational iteration method for fractional heat- and wave-like equations," *Nonlinear Analysis: Real World Applications*. In press.
- [24] S. Abbasbandy, "An approximation solution of a nonlinear equation with Riemann-Liouville's fractional derivatives by He's variational iteration method," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 53–58, 2007.
- [25] X.-H. Wu and J.-H. He, "EXP-function method and its application to nonlinear equations," *Chaos, Solitons & Fractals*, vol. 38, no. 3, pp. 903–910, 2008.
- [26] S.-J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.
- [27] S.-J. Liao, "An approximate solution technique not depending on small parameters: a special example," *International Journal of Non-Linear Mechanics*, vol. 30, no. 3, pp. 371–380, 1995.
- [28] S.-J. Liao, "A kind of approximate solution technique which does not depend upon small parameters—II. An application in fluid mechanics," *International Journal of Non-Linear Mechanics*, vol. 32, no. 5, pp. 815–822, 1997.
- [29] S.-J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, vol. 2 of *CRC Series: Modern Mechanics and Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.
- [30] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, "Solving systems of ODEs by homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 10, pp. 2060–2070, 2008.
- [31] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, "Approximate analytical solutions of systems of PDEs by homotopy analysis method," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2913–2923, 2008.
- [32] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, "Approximate solutions of singular two-point BVPs by modified homotopy analysis method," *Physics Letters A*, vol. 372, no. 22, pp. 4062–4066, 2008.
- [33] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, "Modified homotopy analysis method for solving systems of second-order BVPs," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 2, pp. 430–442, 2009.

- [34] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, "The homotopy analysis method for Cauchy reaction-diffusion problems," *Physics Letters A*, vol. 372, no. 5, pp. 613–618, 2008.
- [35] S. Abbasbandy, M. Yürütsoy, and M. Pakdemirli, "The analysis approach of boundary layer equations of power-law fluids of second grade," *Zeitschrift für Naturforschung*, vol. 63a, pp. 564–570, 2008.
- [36] S. Abbasbandy, "Soliton solutions for the Fitzhugh-Nagumo equation with the homotopy analysis method," *Applied Mathematical Modelling*, vol. 32, no. 12, pp. 2706–2714, 2008.
- [37] T. Hayat and Z. Abbas, "Channel flow of a Maxwell fluid with chemical reaction," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 59, no. 1, pp. 124–144, 2008.
- [38] T. Hayat, Z. Abbas, and M. Sajid, "Heat and mass transfer analysis on the flow of a second grade fluid in the presence of chemical reaction," *Physics Letters A*, vol. 372, no. 14, pp. 2400–2408, 2008.
- [39] Z. Abbas and T. Hayat, "Radiation effects on MHD flow in a porous space," *International Journal of Heat and Mass Transfer*, vol. 51, no. 5-6, pp. 1024–1033, 2008.
- [40] M. M. Rashidi, G. Domairry, and S. Dinarvand, "Approximate solutions for the Burger and regularized long wave equations by means of the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 3, pp. 708–717, 2009.
- [41] G. Domairry and M. Fazeli, "Homotopy analysis method to determine the fin efficiency of convective straight fins with temperature-dependent thermal conductivity," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 2, pp. 489–499, 2009.
- [42] S.-J. Liao, "Notes on the homotopy analysis method: some definitions and theorems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 983–997, 2009.
- [43] Y. Wu and K. F. Cheung, "Explicit solution to the exact Riemann problem and application in nonlinear shallow-water equations," *International Journal for Numerical Methods in Fluids*, vol. 57, no. 11, pp. 1649–1668, 2008.
- [44] L. Song and H. Zhang, "Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation," *Physics Letters A*, vol. 367, no. 1-2, pp. 88–94, 2007.
- [45] J. Cang, Y. Tan, H. Xu, and S.-J. Liao, "Series solutions of non-linear Riccati differential equations with fractional order," *Chaos, Solitons & Fractals*. In press.
- [46] I. Hashim, O. Abdulaziz, and S. Momani, "Homotopy analysis method for fractional IVPs," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 3, pp. 674–684, 2009.
- [47] L. Song and H. Zhang, "Solving the fractional BBM-Burgers equation using the homotopy analysis method," *Chaos, Solitons & Fractals*. In press.
- [48] A. S. Bataineh, A. K. Alomari, M. S. M. Noorani, I. Hashim, and R. Nazar, "Series solutions of systems of nonlinear fractional differential equations," *Acta Applicandae Mathematicae*, vol. 105, no. 2, pp. 189–198, 2009.
- [49] H. Xu and J. Cang, "Analysis of a time fractional wave-like equation with the homotopy analysis method," *Physics Letters A*, vol. 372, no. 8, pp. 1250–1255, 2008.
- [50] H. Xu, S.-J. Liao, and X.-C. You, "Analysis of nonlinear fractional partial differential equations with the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1152–1156, 2009.
- [51] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, 2009.
- [52] H. Jafari and S. Seifi, "Solving a system of nonlinear fractional partial differential equations using homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 1962–1969, 2009.
- [53] M. Caputo, "Linear models of dissipation whose  $Q$  is almost frequency independent-II," *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, 1967.
- [54] R. Gorenflo and F. Mainardi, "Fractional calculus: integral and differential equations of fractional order," in *Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996)*, vol. 378 of *CISM Courses and Lectures*, pp. 223–276, Springer, Vienna, Austria, 1997.