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*Research Article*

# Estimates for the Multiplicative Square Function of Solutions to Nondivergence Elliptic Equations

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We prove distributional inequalities that imply the comparability of the $L^p$ norms of the multiplicative square function of $u$ and the nontangential maximal function of $\log u$, where $u$ is a positive solution of a nondivergence elliptic equation. We also give criteria for singularity and mutual absolute continuity with respect to harmonic measure of any Borel measure defined on a Lipschitz domain based on these distributional inequalities. This extends recent work of M. González and A. Nicolau where the term multiplicative square functions is introduced and where the case when $u$ is a harmonic function is considered.



## 1. Preliminaries and notations

An open set $D \subset \mathbb{R}^n$ is a *star-like Lipschitz domain* centered at the origin with character $M$ if, letting $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, there is a function $\varphi : S^{n-1} \to \mathbb{R}$ with $|\varphi(t) - \varphi(s)| \leq M|t-s|$ and $\varphi(t) \geq \delta > 0$, and such that in polar coordinates $D = \{(\rho,s) : 0 \leq \rho \leq \varphi(s), s \in S^{n-1}\}$. The surface measure of $\partial D$ is denoted by $\sigma$. For $N > 0$ set $ND = \{(\rho,s) : 0 \leq \rho \leq N\varphi(s)\}$, for $Q \in \partial D$, $Q = \varphi(s_0)$, we let $NQ \in ND$ be the point $NQ = (N\varphi(s_0), s_0)$, and for $r > 0$, define $A_r(Q) = (\varphi(s_0) - r, s_0)$.

The surface cubes are defined by $\Delta_r(Q) = B_r(Q) \cap \partial D$, where the Euclidian balls in $\mathbb{R}^n$ are denoted by $B_r(Q) = \{X \in \mathbb{R}^n : |Q - X| < r\}$. The Carleson regions are defined as $\Psi_r(Q) = \{(\rho,s) \in D : s \in \Delta_r(Q), \varphi(s) - r < \rho < \varphi(s)\}$.

Let $L$ be the operator defined as

$$Lu = \sum_{i,j} a_{i,j}(X) \frac{\partial^2 u}{\partial x_i x_j} \tag{1.1}$$

in a Lipschitz domain $D \subset \mathbb{R}^n$. The matrix $A = (a_{i,j})$ of coefficients is assumed to satisfy the ellipticity condition

$$\frac{|\xi|^2}{\lambda} \leq \sum_{i,j} a_{i,j}(X)\xi_i\xi_j \leq \lambda|\xi|^2 \tag{1.2}$$

for every $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$ and every $X \in \mathbb{R}^n$ and the coefficients are assumed to be smooth, although the estimates depend at most on $\lambda$, $n$, the Lipschitz character and the diameter of $D$. It is also assumed that $A$ coincides with the identity matrix for $|X|$ sufficiently large. When all of these conditions hold we write $L \in \mathscr{E}$.

When $L$ has smooth coefficients, it is well known that for each continuous function $f : \partial D \to \mathbb{R}$ there exists a unique function $u_f$ smooth in $D$ and continuous in $\overline{D}$, and such that

$$\begin{gathered} Lu_f = 0 \quad \text{on } D, \\ u_f|_{\partial D} = f. \end{gathered} \tag{1.3}$$

This implies the existence of the *elliptic measure* associated to $L \in \mathscr{E}$. This is the unique probability Borel measure $\omega(X;\cdot)$ defined on $\partial D$ that represents the solution $u_f$ in (1.3) in the following sense:

$$u_f(X) = \int_{\partial D} f(Y)\,d\omega(X,Y), \tag{1.4}$$

by Riesz representation theorem and the maximum principle. We denote by $\omega(\cdot)$ the measure $\omega(\vec{0},\cdot)$, which by Harnack's inequality are mutually absolutely continuous.

Let $g(X,Y)$ denote the Green's function for $L$ on $D$ (see, e.g., [1]), and $G(Y) \equiv G(X_0,Y)$ the Green's function for $L$ on $20D$, with $X_0 \in \partial 10D$. Finally, let $\widetilde{g}(X,Y) = g(X,Y)/G(Y)$. To shorten the notation we often write $G(E) = \int_E G(Y)dY$ for any Borel set $E \subset \mathbb{R}^n$. For $X \in D$ we let $d(X) = \inf\{|X-Q| : Q \in \partial D\}$, and $B(X) = B_{d(X)}(X)$. Also $d(X,Y) = |X-Y|$ denotes the distance from $X$ to $Y$. The connection between elliptic measure and Green's function is given by the following identity:

$$v(X) = \int_{\partial D} v(Y)\,d\omega(X,Y) - \int_D g(X,Y)Lv(Y)\,dY, \tag{1.5}$$

which holds for every $X \in D$, and every $v$ sufficiently smooth in $D$.

Given two Borel measures $\mu_1$ and $\mu_2$ defined on $\partial D$, we say that $\mu_1$ belongs to the class $A_\infty(\mu_2)$ if there exist constants $C,\theta > 0$ such that for every $\Delta \subset \partial D$ and any Borel set $E \subseteq \Delta$ we have

$$\frac{\mu_1(E)}{\mu_1(\Delta)} \leq C\left(\frac{\mu_2(E)}{\mu_2(\Delta)}\right)^\theta. \tag{1.6}$$

Define for a positive solution to $Lu = 0$ the measure

$$d\mu_u(X) = \frac{|\nabla u(X)|^2}{|u(X)|^2}dX. \tag{1.7}$$

Using an idea from [2] combined with the definition in [3], we define for $P \in \partial D$ the *multiplicative square function of u* as

$$\mathcal{M}_\alpha u(P) = \left( \int_{\Gamma_\alpha(P)} \frac{d(X)^2}{G(B(X))} G(X) d\mu_u(X) \right)^{1/2}. \tag{1.8}$$

Similarly, the *nontangential maximal function* of $\log u$ is

$$N_\alpha \log u(P) = \sup_{X \in \Gamma_\alpha(P)} |\log u|. \tag{1.9}$$

The *nontangential approach region* $\Gamma_\alpha(P)$ is the cone with vertex at $P$, aperture $\alpha > 0$, with principal axis in the radial direction and truncated at the origin. We denote by $\Gamma_\alpha^r(P) = \Gamma_\alpha(P) \cap \{X \in D : |X - P| < r$ the *truncated cone* at height $r > 0\}$. The superscript will be added to either $\mathcal{M}$ or $N$ when we substitute cones by truncated cones.

The motivation for using the logarithm of $u$, as well as the explanation of the term *multiplicative square function*, can be found in [2], where the analogues of our main theorems are proved for harmonic functions. In turn, the definition in [2] follows the idea of the area function for subharmonic functions (see, e.g., [4] and references therein).

For many basic facts about solutions and adjoint solutions associated to $L$ we refer the reader to [5] and references therein. More recent works include [3, 6, 7] and we will use and quote results from those works.

For easy reference though, we quote a substitute of a well-known comparison theorem, to point out the inclusion of an adjoint solution (in this case $G(Y)$) as a weight that appears in this and related estimates.

Proposition 1.1 (comparison between Green's functions and the elliptic measures [3, Lemma 2]). *There exists $r_0$ depending on the Lipschitz character of $D$, such that for every $Q \in \partial D$, $r < r_0$ and $Y \in \partial B_r(Q) \cap \Gamma_1(Q)$ and $X \notin \Psi_{4r}(Q)$ we have*

$$\tilde{g}(X,Y) \frac{G(B(Y))}{d(Y)^2} \approx \omega(X, \Delta_r(Q)), \tag{1.10}$$

*whenever $\Delta_{2r}(Q) \subset \partial D$.*

In the next section we will focus on the results related to estimates for the multiplicative square function (Theorems 2.1, 2.4) that may have an independent interest. On the other hand, it is of special interest the problem of describing operators $L \in \mathscr{E}$ for which $\omega \in A_\infty(\sigma)$, and to our knowledge there are no complete characterizations. In Section 3, we will state and prove the results related to the singularity and mutual absolute continuity for harmonic measure and Borel measures defined in $\partial D$ (Theorems 3.1 and 3.2), which represent steps towards a better understanding of this problem, and which are applications of the results in Section 2.

## 2. Distributional inequalities

Our first distributional inequality is based on techniques of [4], as developed in [3]. We observe that the exponential decay in the right-hand side of (2.1) is in certain way sharp, as observed for instance in [8] for harmonic functions.

Theorem 2.1. *Suppose $\mu \in A_\infty(d\sigma)$. If $0 < \alpha < \beta < \infty$, then there exist constants $C_1, C_2 > 0$ such that for all $\lambda > 0$ and $\gamma \geq C_1$,*

$$\mu(\{Q \in \partial D : \mathcal{M}_\alpha u(Q) > \gamma\lambda,\ N_\beta \log u(Q) \leq \lambda\}) \leq e^{-c\gamma^2\lambda^2}\mu(\{Q \in \partial D : \mathcal{M}_{2\alpha} u > \lambda\}). \tag{2.1}$$

Fix $0 < \alpha < \beta < \infty$ and let $u$ be any strictly positive solution to $Lu = 0$. We describe the proof, where we assume $\lambda = 1$. Define

$$E_u = \{Q \in \partial D : N_\beta \log u(Q) \leq 1\}, \qquad \Gamma_u \equiv \Gamma_\alpha(E_u) = \bigcup_{Q \in E_u} \Gamma_\alpha(Q). \tag{2.2}$$

For a Borel set $F \subset D$ we define

$$\nu_u(F) = \int_{\Gamma_u \cap F} g(\vec{0}, Y) \frac{|\nabla u(Y)|^2}{|u(Y)|^2} dY. \tag{2.3}$$

Lemma 2.2. *The measure $\nu_u$ is a Carleson measure with respect to $\omega$, that is,*

$$\sup_{\substack{Q \in \partial D \\ r > 0}} \frac{\nu_u(\Psi_r(Q))}{\omega(\Delta_r(Q))} < \infty. \tag{2.4}$$

*Proof.* The proof of [3, Lemma 5] can be easily adapted. Accordingly, if we define $D_\varepsilon = \{X \in D : d(X) > \varepsilon\}$, it suffices to prove that $\nu_u(\Psi_r(Q) \cap D_\varepsilon) \lesssim \omega(\Delta_r(Q))$ independent of $\varepsilon$. One observes first that by Harnack's principle and Caccioppoli's inequality

$$\nu_u(\Psi_r(Q) \cap D_\varepsilon) \lesssim \int_{W_0} d^{-2}(X) g(\vec{0}, X) dX, \tag{2.5}$$

where $W_0 \subset D$ is exactly the same set of [3, page 282]. The proof in that paper can now be followed verbatim. □

Once we have proved this proposition, setting

$$\mathcal{K}(X,Q) = \varphi(X)\psi\left(\frac{|X-Q|}{d(X)}\right)\frac{d(X)^2}{G(B(X))}\frac{1}{\widetilde{g}(\vec{0},X)} \tag{2.6}$$

for $X \in D$ and $Q \in \partial D$, it is proved in [3, Lemma 7] that

$$\mathcal{K}\nu(Q) = \int_D \mathcal{K}(X,Q)\,d\nu(Q) \tag{2.7}$$

is in BMO($d\sigma$) with BMO norm controlled by the Carleson norm of $\nu$. To finish the proof we observe that for $\gamma \gtrsim 1$, by Harnack's inequality, Caccioppoli's inequality, and the argument in [3, page 285],

$$\{Q \in \partial D : \mathcal{M}_\alpha u > \gamma,\ N_\beta \log u \leq 1\} \subseteq \left\{Q \in \partial D : \widetilde{\mathcal{M}}_\alpha u > \frac{\gamma}{2},\ N_\beta \log u \leq 1\right\}, \tag{2.8}$$

where

$$\widetilde{\mathcal{M}}_\alpha u = \left( \int_{\Gamma_\alpha(P)} \frac{d(X)^2}{G(B(X))} G(X)\varphi(X)\,d\mu_u(X) \right)^{1/2}, \tag{2.9}$$

and where $\varphi(X) \in C_0^\infty$ with $\varphi \equiv 1$ on $D \setminus D_{r_0/10}$ and $\varphi \equiv 0$ on $D_{r_0/5}$. This already suffices to conclude the proof of Theorem 2.1 (see details in [3, page 285]).

The second main result is again a distributional inequality. As stated below the decay of the constant in the right-hand side of (2.15) is far from being sharp. However, in the next section we will describe how one can obtain a better decay as in (2.1).

The proof follows the lines of well-known techniques (see, e.g., [9]), and that is the reason why in the statement we stated only local estimates on balls arising from certain Whitney decompositions, and use local operators. For its proof we also have a use for the following Poincaré-type inequality for $\log u$.

Proposition 2.3. *Let $u$ be a positive solution of $Lu = 0$ on $B_{2r} \equiv B_{2r}(X_0) \subset D$. Then*

$$\sup_{X \in B_r(X_0)} |\log u(X) - \log u(X_0)| \lesssim \frac{r^2}{G(B_{2r}(X_0))} \int_{B_{2r}(X_0)} \frac{|\nabla u(X)|^2}{|u(X)|^2} G(X)\,dX. \tag{2.10}$$

*Proof.* By (1.5) applied to $v(X) = \log u(X) - \log u(X_0)$ with $X = X_0$ we have

$$\int_{\partial B_{2r}} [\log u(Q) - \log u(X_0)]\,d\omega_{2r}(X_0, Q) = \int_{B_{2r}} g_{2r}(X_0, Y) L(\log u(Y))\,dY, \tag{2.11}$$

where $g_{2r}$ and $\omega_{2r}$ denote the Green's function and the elliptic measure for $L$ in $B_{2r}$. On the other hand, if $X \in B_r(X_0)$ and again by (1.5),

$$\begin{aligned} \log u(X) - \log u(X_0) = &\int_{\partial B_{2r}} [\log u(Q) - \log u(X_0)]\,d\omega_{2r}(X, Q) \\ &- \int_{B_{2r}} g_{2r}(X_0, Y) L(\log u(Y))\,dY. \end{aligned} \tag{2.12}$$

Therefore, since $|L(\log u(Y))| \lesssim |\nabla u(Y)|^2/|u(Y)|^2$ and $d\omega_{2r}(X_0, \cdot)/d\omega_{2r}(X, \cdot)$ is essentially bounded by 1,

$$|\log u(X) - \log u(X_0)| \le \int_{B_{2r}} g_{2r}(X_0, Y) \frac{|\nabla u(Y)|^2}{|u(Y)|^2}\,dY. \tag{2.13}$$

Applying Harnack's inequality to $u$, and using an integral estimate in [3, page 286] we obtain

$$\begin{aligned} |\log u(X) - \log u(X_0)| &\le \frac{1}{[\inf_{B_{2r}} u(Y)]^2} \int_{B_{2r}} g_{2r}(X_0, Y) |\nabla u(Y)|^2\,dY \\ &\lesssim \frac{1}{[\sup_{B_{2r}} u(Y)]^2} \frac{r^2}{G(B_{2r})} \int_{B_{2r}} |\nabla u(Y)|^2 G(Y)\,dY \\ &\lesssim \frac{r^2}{G(B_{2r}(X_0))} \int_{B_{2r}(X_0)} \frac{|\nabla u(Y)|^2}{|u(Y)|^2} G(Y)\,dY. \end{aligned} \tag{2.14}$$

□

Theorem 2.4. *Suppose $u$ is a positive solution to $Lu=0$ in $D$, $\Delta \equiv \Delta_r(P_0)$ is a surface ball in $\partial D$, $0<\alpha<\beta$, $\mu \in A_\infty(\omega)$. Assume that for some $\lambda>0$, $N_\alpha \log u(P_1)<\lambda$ for some $P_1 \in S$ with $d(P_1;\Delta)\approx r$. Then given $\gamma>1$ there exist $\epsilon>0$ and $0<\delta<1/2$ (depending only on the $A_\infty$ property of $\mu$, the Lipschitz character of $D$, the ellipticity of $L$, $\alpha$, $\beta$, and $n$) such that*

$$\mu\{P\in\Delta : N_\alpha^r \log u(P)>\gamma\lambda,\ [\mathcal{M}_\beta^r u(P)]^2 \le \epsilon\lambda,\ M_\mu(\chi_{G_{\epsilon\lambda}})\le\delta\} \le C\gamma^{-\theta}\mu(\Delta), \tag{2.15}$$

*where $G_\lambda=\{P\in\Delta : [\mathcal{M}_\beta^r u(P)]^2>\lambda\}$, and where $C$ is a constant depending on $\gamma$ and the $A_\infty$ property of $\mu$.*

*Proof of Theorem 2.4.* Let $E$ be the set in the left-hand side of (2.15) and let $W=\Gamma_{(\alpha+\beta)/2}(E)\cap D$, denote its sawtooth region. It is well known that there is a point $X_0\in W$ with the property that $d(X_0,\partial W)\approx r$. Define for, $Q\in E$,

$$\widetilde{\mathcal{N}}\log u(Q)=\sup_{X\in\Gamma_\alpha^r(Q)\cap W}|\log u(X)-\log u(X_0)|. \tag{2.16}$$

Observe that $\widetilde{\mathcal{N}}\log u \le CN_\alpha \log u$ on $\partial D$. In particular, $\widetilde{\mathcal{N}}$ satisfies the well-known weak-$(1,1)$ boundedness property, by Hardy-Littlewood's maximal theorem.

The following lemma follows from the doubling property of $\omega$, as in the "Main Lemma" of [9] (see also [3, page 282]). □

Lemma 2.5. *Let $\nu$ be the harmonic measure of $W$ with pole at $X_0$, and let $F\subset 2\Delta\equiv\Delta_{2r}(P_0)$ a Borel set. Define*

$$\widetilde{\nu}(F)=\nu(E\cap F)+\sum_j \frac{\omega(F\cap I_j)}{\omega(I_j\cap S)}\nu(\widetilde{I}_j\cap(\partial W\cap D)), \tag{2.17}$$

*where $\{I_j\}$ is a Whitney decomposition of $2\Delta\setminus E$, and $\widetilde{I}_j$ is the projection of $I_j$ on $W$. Then there exists $\theta, C>0$ such that*

$$\frac{\omega(F)}{\omega(\Delta')}\le C\left(\frac{\widetilde{\nu}(F)}{\widetilde{\nu}(\Delta')}\right)^\theta, \tag{2.18}$$

*where $\Delta'\subset\Delta$ any surface ball.*

The projection used above is a function mapping any point $P\in D$ to the point $\widetilde{P}\in W$ in the radial direction of $P$. Observe that in particular one has $d(I_j;\widetilde{I}_j)\approx d(I_j;E)\approx \operatorname{diam} I_j\equiv r_j$. The "Main Lemma" mentioned above states that, with the notation of the previous lemma, we actually have

$$\left(\frac{\omega(F)}{\omega(\Delta)}\right)^\theta \le C\nu(F). \tag{2.19}$$

On the other hand, using that $[\mathcal{M}_\beta^r \log u(P_1)]^2<\epsilon\lambda$ and (2.10) one may proceed as in [9, page 104], to obtain $E\subset\{P : \widetilde{\mathcal{N}}\log u(P)\ge\gamma\lambda\}$.

To prove (2.15), since $\mu \in A_\infty(\omega)$, it suffices to prove that if $\xi = \gamma\lambda$, then, for some $\theta > 0$,

$$\omega(\{Q \in \Delta : \widetilde{\mathcal{N}} \log u(Q) > \xi\}) \leq C\xi^{-\theta}\omega(\Delta). \tag{2.20}$$

*Proof of (2.20).* Let $H_\xi = \{Q \in \Delta : \widetilde{\mathcal{N}} \log u(Q) > \xi\}$. By (2.19), it will be enough to prove

$$\widetilde{\nu}(H_\xi) \leq C\xi^{-\theta} \tag{2.21}$$

for $\xi \gg 1$. We give the proof (2.21) in three steps. Observe first that by Chebyshev's inequality

$$\widetilde{\nu}(H_\xi) \leq \frac{1}{\xi}\int_E (\widetilde{\mathcal{N}} \log u)\,d\nu + \sum_j \frac{\omega(H_\xi \cap I_j)}{\omega(I_j \cap S)}\nu(\widetilde{I}_j \cap (\partial W \cap D)). \tag{2.22}$$

*Step 1.* We prove first that $\omega(H_\xi \cap I_j) \approx \omega(I_j \cap S)$.

For $Q \in H_\xi \cap I_j$ one has $d(Q,E) \approx \operatorname{diam} I_j$; also, if $P \in E$ satisfies $d(Q;P) \approx \operatorname{diam} I_j$, then there exists $\mathbf{X} \in \Gamma_\alpha(Q) \cap \Gamma_{(\alpha+\beta)/2}(P)$ with $d(\mathbf{X};\partial D) \approx \operatorname{diam} I_j$, and such that $|\log u(\mathbf{X}) - \log u(X_0)| > \xi$. So one may choose $\beta$ as a large multiple of $\alpha$, and for a constant $\rho$ one has $\widetilde{\mathcal{N}} \log u > \xi$ on the set $\Delta_{\rho \operatorname{diam} I_j}(Q)$. The doubling property of $\omega$ implies the first claim.

*Step 2.* Next we prove that

$$\widetilde{\nu}(H_\xi)\xi \leq C\int_W g_0(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}\,dY, \tag{2.23}$$

where $g_0$ is the Green's function of $L$ in $W$.

For $Z \in \widetilde{I}_j \cap (\partial W \cap D)$, by (2.10) and the choice of $\mathbf{X}$,

$$|\log u(Z) - \log u(\mathbf{X})| \leq (\mathcal{M}_\alpha u(P))^2 < \epsilon\lambda. \tag{2.24}$$

Hence, for $\epsilon > 0$ small, $|\log u(Z) - \log u(X_0)| > C$. Therefore,

$$\widetilde{\nu}(H_\xi)\xi \leq C\left[\int_E (\widetilde{\mathcal{N}} \log u)\,d\nu + \sum_j \int_{\widetilde{I}_j} |\log u(Q) - \log u(X_0)|\,d\nu\right]. \tag{2.25}$$

Since there is only a finite overlapping, we may use weak (1,1) estimates theorem to obtain with a different constant $C$,

$$\widetilde{\nu}(H_\xi)\xi \leq C\int_{\partial W} |\log u(Q) - \log u(X_0)|\,d\nu. \tag{2.26}$$

By (1.5) applied to $\log u(X) - \log u(X_0)$ with respect to $\nu$,

$$\widetilde{\nu}(H_\xi)\xi \leq \int_W g_0(X_0,Y)L\log u(Y)\,dY \lesssim \int_W g_0(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}\,dY. \tag{2.27}$$

*Step 3.* We finally prove that

$$\int_W g_0(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}dY \le C. \tag{2.28}$$

Set $W^r = \{X \in W : d(X;\partial D) \le \tau r\}$ so that we can split the integral over $W$ in the part away from the boundary and the one close to the boundary. Observe that by Harnack's inequality

$$\int_{W\setminus W^r} g_0(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}dY \le \frac{1}{\inf_{W\setminus W^r} u^2}\int_{W\setminus W^r} g_0(X_0,Y)|\nabla u(Y)|^2 dY. \tag{2.29}$$

Observe now that, applying (1.5),

$$\begin{aligned}\int_{W\setminus W^r} g_0(X_0,Y)|\nabla u|^2 dY &\lesssim \int_{W\setminus W^r} g_0(X_0,Y)\langle A\nabla u,\nabla u\rangle dY\\ &\lesssim \int_{W\setminus W^r} g_0(X_0,Y)Lu^2(Y)dY \lesssim \sup_{X\in W\setminus W^{r/2}} |u(X)|^2.\end{aligned} \tag{2.30}$$

By Harnack's inequality again

$$\int_{W\setminus W^r} g_0(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}dY \le C. \tag{2.31}$$

To handle the part close to the boundary observe that

$$C \ge \int_{\partial D\setminus G_{\varepsilon\lambda}} [\mathcal{M}_\beta^r u(Q)]^2 d\omega(Q) \ge \int_W \frac{|\nabla u(Y)|^2}{|u(Y)|^2}G(Y)\frac{d(Y)^2}{G(B(Y))}\psi(Y)dY, \tag{2.32}$$

where $\psi(Y) = \omega(X_0;\{P \in \partial D \setminus G_{\varepsilon\lambda} : Y \in \Gamma_\beta^r(P)\})$, and $G_{\varepsilon\lambda}$ is as in the statement of the theorem.

Observe also that for $Y \in W^r$ there exists $\tilde{Y} \in E \subset \partial D \setminus G_{\varepsilon\lambda}$ such that $\|Y-\tilde{Y}\| \approx d(Y)$ and $Y \in \Gamma_\beta(\tilde{Y})$. Hence, setting $\tilde{\Delta} \equiv \Delta_{\gamma d(Y)}(\tilde{Y})$ we have $\tilde{\Delta}\cap\partial D\setminus G_{\varepsilon\lambda} \subset \{P \in \partial D\setminus G_{\varepsilon\lambda} : Y \in \Gamma_\beta^r(P)\}$ and consequently, by (1.10) and Harnack's inequality,

$$\begin{aligned}\psi(Y) \ge \omega(X_0;\tilde{\Delta}\cap\partial D\setminus G_{\varepsilon\lambda}) &= \frac{\omega(X_0;\tilde{\Delta}\cap\partial D\setminus G_{\varepsilon\lambda})}{\omega(X_0;\tilde{\Delta})}\omega(X_0;\tilde{\Delta})\\ \approx \frac{\omega(\tilde{\Delta}\cap\partial D\setminus G_{\varepsilon\lambda})}{\omega(\tilde{\Delta})}\omega(X_0;\tilde{\Delta}) &\approx \left(\frac{\omega(\tilde{\Delta}\cap\partial D\setminus G_{\varepsilon\lambda})}{\omega(\tilde{\Delta})}\right)\left(\frac{\tilde{g}(X_0,Y)G(B(Y))}{d^2(Y)}\right).\end{aligned} \tag{2.33}$$

By the definition of $E$, since $\tilde{Y} \in E$, then $\mu(\tilde{\Delta}\cap\partial D\setminus G_{\varepsilon\lambda})/\mu(\tilde{\Delta}) > \varepsilon\lambda$, and by the $A_\infty$ property of $\mu$,

$$\psi(Y) \ge \frac{\tilde{g}(X_0,Y)G(B(Y))}{d^2(Y)}. \tag{2.34}$$

Thus by (2.32)

$$\int_{W^r} g_0(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}dY \le \int_W g(X_0,Y)\frac{|\nabla u(Y)|^2}{|u(Y)|^2}dY \le C, \tag{2.35}$$

which completes the proof of (2.21). □

## 3. Applications to harmonic measure

The operators $L \in \mathscr{E}$ for which their harmonic measures are in $A_\infty(d\sigma)$ are not well characterized. The preservation of the $A_\infty$ property under small perturbations of the main coefficients of $L$ was proved in [10], and more recently in [11] a class of operators for which the harmonic measure is in $A_\infty(d\sigma)$ is described. To have some criteria to determine absolute continuity or singularity with respect to harmonic measure may therefore be of interest, and the results in this section go in this direction.

Given a Borel measure $\nu$ defined on $\partial D$, we define

$$u(X) = \int_{\partial D} K(X,Q)d\nu(Q), \tag{3.1}$$

where $K(X,Q) = (d\omega^X/d\omega)(Q)$ is the *kernel function* associated to $L$ (see [12]).

Theorem 3.1. *Fix $\alpha > 0$ and for a positive Borel measure $\nu$ let $u$ be the solution given by (3.1). Then $\nu$ is singular with respect to $\omega$ if and only if $\mathcal{M}_\alpha u(Q) = \infty$ for $\omega$-almost every $Q$.*

The proof is a direct application of Theorems 2.1 and 2.4, by proving that the sets

$$A = \left\{P \in \partial D : \lim_{\substack{X\to P\\ X\in\Gamma(P)}} u(X) > 0\right\}, \qquad B = \{P \in \partial D : \mathcal{M}_\alpha u(P) < \infty\} \tag{3.2}$$

only differ in a set of null $\omega$ measure. However, there is an alternative proof based on sawtooth region techniques, independent from the arguments to prove Theorems 2.1 and 2.4. This was observed originally in [2] for the case of harmonic functions, and the proof that we include for its simplicity and for completeness is based in that original argument.

*Proof.* We divide the proof in two claims.

*Claim 1.* $\omega$-almost every point of $A$ is in $B$.

Divide $\partial D$ into surface balls of finite overlapping $\Delta_i \equiv \Delta_{r_i}(P_i)$ with $r_i = r_0/2$, where $r_0$ is the constant of (1.10). Let $E \subset \Delta_i \cap A$ be a closed set and $\varepsilon > 0$ such that

$$\frac{1}{\varepsilon} > \lim_{X\to P,\, X\in\Gamma(P)} u(X) > \varepsilon \tag{3.3}$$

for every $P \in E$. Let $\Gamma(E) = \bigcup_{P\in E}\Gamma_\alpha(P)$ and recall that there exists $X_0 \in \Gamma(E)$ whose distance to $\partial\Gamma(E)$ is proportional to $r_i$. We denote by $\omega_\Gamma$ the harmonic measure of $\Gamma(E)$ with pole at $X_0$.

Using (2.19) we can conclude that $\omega_\Gamma(F) = 0$ implies $\omega(F) = 0$ whenever $F \subset \Delta_i$, and so we will prove that $\omega_\Gamma$-almost every element in $A$ is also in $B$. By Harnack's inequality, renormalizing $u$, we may assume that for every $P \in E$,

$$\inf_{X \in \widetilde{\Gamma}(P)} u(X) > \varepsilon, \qquad \sup_{X \in \widetilde{\Gamma}(P)} u(X) < \frac{1}{\varepsilon}, \tag{3.4}$$

where $\widetilde{\Gamma}(P)$ has a slightly bigger aperture than $\Gamma(P)$. Thus we have $\varepsilon < u(X) < 1/\varepsilon$ for $X \in \Gamma(E)$.

By Fubini's theorem

$$\begin{aligned} \int_E \mathcal{M}_\alpha u(P) d\omega_\Gamma(P) &\le \int_E \int_{\Gamma(P)} \frac{|\nabla u(X)|^2}{|u(X)|^2} \frac{d^2(X)}{G(B(X))} G(X) dX \, d\omega_\Gamma(P) \\ &\le \int_{\Gamma(E)} \Psi(X) \frac{|\nabla u(X)|^2}{|u(X)|^2} \frac{d^2(X)}{G(B(X))} G(X) dX, \end{aligned} \tag{3.5}$$

where $\Psi(X) = \omega_\Gamma(\Delta_{\alpha d(X)}(\widetilde{X}))$ and $\widetilde{X}$ is the radial projection of $X$ onto $\partial\Gamma(E)$. By (1.10) the last quantity is controlled by

$$\int_{\Gamma(E)} \frac{|\nabla u(X)|^2}{|u(X)|^2} g_\Gamma(X_0, X) dX < \frac{1}{\varepsilon} \int_{\Gamma(E)} |\nabla u(X)|^2 g_\Gamma(X_0, X) dX, \tag{3.6}$$

where $g_\Gamma$ denotes the Green's function for $L$ on $\Gamma(E)$. Since for any constant $k$ one has $L[(u-k)^2] = 2\langle A\nabla u, \nabla u\rangle$, we conclude by Green's identity (1.5) and Harnack's inequality that

$$\int_E \mathcal{M}_\alpha u(P) d\omega_\Gamma(P) \lesssim \frac{1}{\varepsilon} \left( \sup_{X \in \partial\Gamma(E) \setminus E} |u(X) - u(X_0)|^2 \right) < \frac{1/\varepsilon^2}{\varepsilon}. \tag{3.7}$$

This implies $\mathcal{M}_\alpha u(P) < \infty$ for $\omega_\Gamma$-almost every $P \in E$, which as observed above yields the claim.

*Claim 2.* $\omega$-almost every point of $B$ is in $A$

Once again divide $\partial D$ into the surface balls $\Delta_i \equiv \Delta_{r_i}(P_i)$ as above, and let $E \subset \Delta_i$ be a closed set of $B$, where $u$ is nontangentially bounded and where $\mathcal{M}_\alpha u(P) \le 1$ and $\lim_{X \to P} u(X) = 0$ nontangentially. We define $\Gamma_\alpha(E) = \bigcup_{P \in E} \Gamma_\alpha(P)$ and use the same notation used in the previous claim.

The proof is by contradiction, and so we assume that $\omega_\Gamma(E) > 0$. Applying once again (1.5)

$$\log u(\vec{0}) + \int_{\Gamma(E)} g_\Gamma(X_0, Y) L \log u(Y) dY = \int_{\partial\Gamma(E)} \log u(P) d\omega_\Gamma(P). \tag{3.8}$$

The proof will finish if we prove that the second term in the left is finite, since the right-hand side is unbounded, by the assumption $\omega_\Gamma(E) > 0$. The contradiction will prove the claim.

Note that $|L\log u| \lesssim |\nabla u|^2/u^2$ and so we estimate that term as follows:

$$\begin{aligned}\int_{\Gamma(E)} g_\Gamma(X_0,Y)L\log u(Y)\,dY &\lesssim \int_{\Gamma_\alpha(E)} \frac{|\nabla u(X)|^2}{u^2(X)} g_\Gamma(X_0,X)\,dX \\ &\lesssim \int_{\Gamma_\alpha(E)} \frac{|\nabla u(X)|^2}{u^2(X)} \omega_\Gamma(\Delta_{\alpha d(X)}(\widetilde{X}))\frac{d^2(X)}{G(B(X))}G(X)\,dX \\ &\lesssim \int_E \mathcal{M}_\alpha u(P)\,d\omega_\Gamma(P) < \infty,\end{aligned} \tag{3.9}$$

where in the second to last estimate we used (1.10), and in the last one Fubini's theorem is applied. □

The proof of the next theorem is also based on the argument given originally for harmonic functions in [2, page 700]. We first record a consequence of the proof of Theorem 2.4 that we will explicitly use in the proof of the theorem, and that it was actually observed in [3] for solutions to $Lu = 0$. Notice that (2.15) implies $\|N_\alpha \log u\|_{L^q(d\omega)} \lesssim 1$ for some $q > 0$ (see [3, page 291]). This, along with the argument in [3, page 288], implies $N_\alpha \log u \in \mathrm{BMO}$ with BMO norm $\lesssim 1$, which suffices to prove the following improvement of the decay in the right-hand side of (2.15): with the notations of Theorem 2.4, there are constants $c_1$, $c_2$ such that for $\gamma > c_1$ and $\lambda > 0$ one has

$$\begin{aligned}&\mu\{P \in \partial D : N_\alpha \log u(P) > \gamma\lambda, [\mathcal{M}_\beta u(P)]^2 \le \lambda\} \\ &\quad \le c_1 \exp(-c_2\gamma\lambda)\mu\{P \in \partial D : N_\alpha \log u(P) > \lambda\}.\end{aligned} \tag{3.10}$$

Theorem 3.2. *With the notation introduced above, there exists a constant $C = C(n,\lambda) > 0$ such that $\exp(C\mathcal{M}_\alpha^2 u) \in L^1(\partial D, d\omega)$ implies that $\omega$ and $\nu$ are mutually absolutely continuous.*

*Proof.* Define

$$\mathfrak{M}(P) = \sup\left\{\frac{\nu(\Delta)}{\omega(\Delta)}, \left(\frac{\nu(\Delta)}{\omega(\Delta)}\right)^{-1}\right\}, \qquad \mathfrak{N}(P) = \sup\left|\log\frac{\nu(\Delta)}{\omega(\Delta)}\right|, \tag{3.11}$$

where in both cases the supremum is taken over dyadic surface balls $\Delta$ containing $P \in \partial D$. It suffices then to prove that $\mathfrak{M} \in L^1(\partial D, d\omega)$, and since $e^{\mathfrak{N}} = \mathfrak{M}$, we can just prove that $e^{\mathfrak{N}} \in L^1(\partial D, d\omega)$. Now observe that by [12, Theorem I.2.5], if $P \in \partial D$ and $\Delta$ is a surface ball containing $P$, then

$$\frac{\nu(\Delta)}{\omega(\Delta)} \approx u(P_\Delta) \le Nu(P), \tag{3.12}$$

where $u(X)$ is as in (3.1), and $P_\Delta$ is a point in $D$ whose distance to $P$ and to $\partial D$ are both proportional to the radius of $\Delta$; in fact $P_\Delta$ can be chosen so that $P_\Delta \in \Gamma_\alpha(P)$. This implies

$N_\alpha u(P) \geq \mathfrak{N}(P)$ and so we can conclude

$$\begin{aligned}\int_{\partial D} (e^{\mathfrak{N}} - 1)\, d\omega(P) &= \gamma \int_0^\infty e^{\gamma\lambda} \omega(\{P \in \partial D : \mathfrak{N}(P) > \gamma\lambda\})\, d\lambda \\ &\leq \gamma \int_0^\infty e^{\gamma\lambda} \omega(\{P \in \partial D : N_\alpha(P) > \gamma\lambda\})\, d\lambda.\end{aligned} \tag{3.13}$$

Then by (3.10), and with $\epsilon > 0$ to be chosen,

$$\begin{aligned}\int_{\partial D} (e^{\mathfrak{N}} - 1)\, d\omega(P) \leq{}& \gamma \int_0^\infty e^{\gamma\lambda} \omega(\{P \in \partial D : N_\alpha \log u(P) > \gamma\lambda, [\mathcal{M}_\alpha u(X)]^2 \leq \epsilon\lambda\})\, d\lambda \\ &+ \gamma \int_0^\infty e^{\gamma\lambda} \omega(\{P \in \partial D : [\mathcal{M}_\alpha u(P)]^2 > \epsilon\lambda\})\, d\lambda \\ \leq{}& \gamma c_1 \int_0^\infty e^{\gamma\lambda} e^{-c_2\gamma\lambda/\epsilon} \omega(\{P \in \partial D : N_\alpha \log u(P) > \epsilon\lambda\})\, d\lambda \\ &+ \int_{\partial D} \left[\exp\left(\frac{\gamma \mathcal{M}^2(P)}{\epsilon}\right) - 1\right] d\omega(P) \\ \leq{}& \frac{\gamma c_1}{\epsilon} \int_{\partial D} N_\alpha \log u(P) d\omega(P) + \int_{\partial D} \left[\exp\left(\frac{\gamma \mathcal{M}^2(P)}{\epsilon}\right) - 1\right] d\omega(P),\end{aligned} \tag{3.14}$$

where $\epsilon$ has been chosen sufficiently small so that $1 - c_2/\epsilon \leq 0$. Now we bound the first term in the right-hand side by $\int_{\partial D} \mathcal{M}_\alpha u(P) d\omega(P)$ and in conclusion,

$$\int_{\partial D} e^{\mathfrak{N}} d\omega(P) \lesssim \int_{\partial D} \exp\left(\frac{\gamma \mathcal{M}^2(P)}{\epsilon}\right) d\omega(P) < \infty \tag{3.15}$$

and thus choosing $C = \gamma/\epsilon$ will prove the theorem. □

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