Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2007, Article ID 56360, 12 pages
doi:10.1155/2007/56360

Research Article

Finite Element Formulation of Forced Vibration Problem of a Prestretched Plate Resting on a Rigid Foundation

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Received 27 April 2007; Revised 21 June 2007; Accepted 3 September 2007

Recommended by Heinz H. Bauschke

The three-dimensional linearized theory of elastodynamics mathematical formulation of the forced vibration of a prestretched plate resting on a rigid half-plane is given. The variational formulation of corresponding boundary-value problem is constructed. The first variational of the functional in the variational statement is equated to zero. In the framework of the virtual work principle, it is proved that appropriate equations and boundary conditions are derived. Using these conditions, finite element formulation of the prestretched plate is done. The numerical results obtained coincide with the ones given by Ufly and in 1963 for the static loading case.

1. Introduction

Elastodynamics problems of prestretched media arise in many areas of applied mathematics, engineering, and natural sciences. Classical linear theory of elastic waves is not sufficient for solving elastodynamics problems involving initially stressed bodies. That is why a general nonlinear theory of elastic waves has been developed since the second half of the 20th century. An analysis of the studies up to 1986 was made in [1, 2]. Later researches are given in [3]. Recent researches involving dynamic stress field in multilayered media with initial stress are given in [4–7].

In the present paper, a boundary-value problem of elastodynamics involving initially stressed bodies which has no analytical solution considered and finite element method is utilized to solve the problem numerically. In a study by Akbarov [7], the layers of the slab have infinite length in the radial direction. In a recent paper by Akbarov and Guler [8], the stress field in a half-plane covered by the prestretched layer under the action of arbitrarily linearly located time-harmonic forces is investigated. In [8], the layer considered

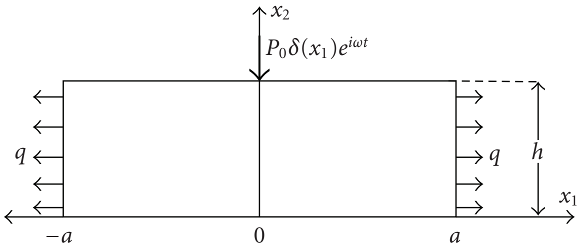

FIGURE 2.1. The geometry of the strip plate resting on a rigid foundation.

is extending to infinity in the x_1-axis direction. In this study, a prestretched strip plate which has a finite domain both for the x_1-axis and x_2-axis is considered. So the method of solution used in [7, 8] is not suitable for the problem at hand.

2. Formulation of the problem

The problem of forced vibration of a prestretched strip plate resting on a rigid foundation is considered. Strip plate and rigid half-plane occupy the regions

$$B = \{(x_1,x_2) : -a \le x_1 \le a,\ 0 \le x_2 \le h\} \tag{2.1}$$

and $\{(x_1,x_2) : -\infty < x_1 < \infty, -\infty < x_2 \le 0\}$, respectively, in Cartesian coordinate system Ox_1x_2 (see Figure 2.1).

We assume that strip plate is made of linearly elastic material, homogeneous and isotropic. We also assume that, before contact, the plate is stressed from both sides by normal forces having amplitude q. A time-harmonic point-located normal load, $P_0\delta(x_1)e^{i\omega t}$, is applied to the upper surface of the plate, where $\delta(x_1)$ stands for the Dirac delta function. Following on from the above, it can be assumed that the plane deformation state prevails.

According to Guz [1], for the case considered, the equations of motion are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + q\frac{\partial^2 u_i}{\partial x_1^2} = \rho_0 \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1,2,\ j = 1,2. \tag{2.2}$$

In (2.2), ρ_0 denotes the density of the material in the natural state, $u_1(x_1,x_2,t)$ and $u_2(x_1,x_2,t)$ denote the displacement in the axis x_1 and x_2, respectively. For an isotropic compressible material, we can write the following mechanical relations:

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \theta = \varepsilon_{11} + \varepsilon_{22}, \tag{2.3}$$

where λ and μ are Láme constants and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.4}$$

Here

$$\varepsilon = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\}^T \tag{2.5}$$

denotes the deformation tensor and

$$\sigma = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T \tag{2.6}$$

denotes the stress tensor. In (2.3), δ_{ij} denotes the Kronecker delta

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{2.7}$$

In the case considered the relations

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} = G \tag{2.8}$$

are valid, where E denotes the elasticity moduli and ν denotes the Possion ratio. We assume that the following boundary conditions exist:

$$\begin{aligned} u_1 \big|_{x_2=0} &= 0, & u_2 \big|_{x_2=0} &= 0, \\ \left(q\frac{\partial u_1}{\partial x_1} + \sigma_{11}\right)\bigg|_{x_1=\pm a} &= 0, & \left(q\frac{\partial u_2}{\partial x_1} + \sigma_{12}\right)\bigg|_{x_1=\pm a} &= 0, \\ \sigma_{21} \big|_{x_2=h} &= 0, & \sigma_{22} \big|_{x_2=h} &= P_0 \delta(x_1) e^{i\omega t}. \end{aligned} \tag{2.9}$$

Since the applied point-located load is time-harmonic, all the dependent variables are also harmonic and can be represented as

$$\{u_i, \sigma_{ij}, \varepsilon_{ij}\} = \{\hat{u}_i, \hat{\sigma}_{ij}, \hat{\varepsilon}_{ij}\} e^{i\omega t}, \tag{2.10}$$

where the superposed caret denotes the amplitude of the corresponding quantity. Hereafter the carets will be omitted. Using (2.3) and (2.4) in (2.2), we have the linearized equations of motion in terms of displacement as follows:

$$\begin{aligned} (\lambda + 2\mu + q)\frac{\partial^2 u_1}{\partial x_1^2} + \mu \frac{\partial^2 u_1}{\partial x_2^2} + (\lambda + \mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} &= -\rho_0\, \omega^2\, u_1, \\ (\mu + q)\frac{\partial^2 u_2}{\partial x_1^2} + (\lambda + 2\mu)\frac{\partial^2 u_2}{\partial x_2^2} + (\lambda + \mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} &= -\rho_0\, \omega^2\, u_2. \end{aligned} \tag{2.11}$$

3. Finite element formulation

We have the dimensionless coordinate system by the following transformation:

$$\hat{x}_1 = \frac{x_1}{h}, \qquad \hat{x}_2 = \frac{x_2}{h}. \tag{3.1}$$

By multiplying both sides of the equations with h^2 after substituting (3.1) in (2.2), we have

$$
\begin{aligned}
h\frac{\partial\sigma_{11}}{\partial\hat{x}_1}+h\frac{\partial\sigma_{12}}{\partial\hat{x}_2}+q\frac{\partial^2 u_1}{\partial\hat{x}_1^2}&=-\rho_0\,\omega^2\,h^2\,u_1,\\
h\frac{\partial\sigma_{21}}{\partial\hat{x}_1}+h\frac{\partial\sigma_{22}}{\partial\hat{x}_2}+q\frac{\partial^2 u_2}{\partial\hat{x}_1^2}&=-\rho_0\,\omega^2\,h^2\,u_2.
\end{aligned}
\tag{3.2}
$$

Under the coordinate transformations (3.1), boundary conditions (2.9) will be as follows:

$$
\begin{aligned}
u_1\big|_{\hat{x}_2=0}&=0, & u_2\big|_{\hat{x}_2=0}&=0,\\
\left(q\frac{\partial u_1}{\partial\hat{x}_1}+\sigma_{11}\right)\bigg|_{\hat{x}_1=\pm a/h}&=0, & \left(q\frac{\partial u_2}{\partial\hat{x}_1}+\sigma_{12}\right)\bigg|_{\hat{x}_1=\pm a/h}&=0,\\
\sigma_{21}\big|_{\hat{x}_2=1}&=0, & \sigma_{22}\big|_{\hat{x}_2=1}&=P_0\delta(h\hat{x}_1)e^{i\omega t}.
\end{aligned}
\tag{3.3}
$$

We first multiply (3.2) by the test functions $v_1=v_1(\hat{x}_1,\hat{x}_2)$ and $v_2=v_2(\hat{x}_1,\hat{x}_2)$, respectively, and then add the resultant equations side by side. After integrating the equation over the domain

$$
\hat{B}=\{(\hat{x}_1,\hat{x}_2):-a/h\le\hat{x}_1\le a/h,\ 0\le\hat{x}_2\le 1\},
\tag{3.4}
$$

we get

$$
\begin{aligned}
&\int_0^1\int_{-a/h}^{a/h}\left[h\frac{\partial\sigma_{11}}{\partial\hat{x}_1}v_1+h\frac{\partial\sigma_{21}}{\partial\hat{x}_1}v_2+h\frac{\partial\sigma_{12}}{\partial\hat{x}_2}v_1+h\frac{\partial\sigma_{22}}{\partial\hat{x}_2}v_2+q\frac{\partial^2 u_1}{\partial\hat{x}_1^2}v_1+q\frac{\partial^2 u_2}{\partial\hat{x}_1^2}v_2\right]d\hat{x}_1\,d\hat{x}_2\\
&\quad=-\int_0^1\int_{-a/h}^{a/h}\rho_0\omega^2h^2(u_1v_1+u_2v_2)\,d\hat{x}_1d\hat{x}_2.
\end{aligned}
\tag{3.5}
$$

Applying integration by parts to (3.5) we get

$$
\begin{aligned}
&\int_{\partial\hat{B}}\Big[h\sigma_{11}v_1\cos(\vec{n},\hat{x}_1)+h\sigma_{21}v_2\cos(\vec{n},\hat{x}_1)+h\sigma_{12}v_1\cos(\vec{n},\hat{x}_2)+h\sigma_{22}v_2\cos(\vec{n},\hat{x}_2)\\
&\qquad+q\frac{\partial u_1}{\partial\hat{x}_1}v_1\cos(\vec{n},\hat{x}_1)+q\frac{\partial u_2}{\partial\hat{x}_1}v_2\cos(\vec{n},\hat{x}_1)\Big]ds\\
&\quad-\int_0^1\int_{-a/h}^{a/h}\left[h\sigma_{11}\frac{\partial v_1}{\partial\hat{x}_1}+h\sigma_{21}\frac{\partial v_2}{\partial\hat{x}_1}+h\sigma_{12}\frac{\partial v_1}{\partial\hat{x}_2}+h\sigma_{22}\frac{\partial v_2}{\partial\hat{x}_2}+q\frac{\partial u_1}{\partial\hat{x}_1}\frac{\partial v_1}{\partial\hat{x}_1}+q\frac{\partial u_2}{\partial\hat{x}_1}\frac{\partial v_2}{\partial\hat{x}_1}\right]d\hat{x}_1d\hat{x}_2\\
&\quad=-\int_0^1\int_{-a/h}^{a/h}\rho_0\omega^2h^2(u_1v_1+u_2v_2)\,d\hat{x}_1d\hat{x}_2,
\end{aligned}
\tag{3.6}
$$

where $\partial\hat{B}$ denotes the boundary of the domain $\hat{B}$. If we collect the domain integrals in (3.6) together and define

$$
T_{ij}=\sigma_{ij}+\sigma_{\text{in}}^0\frac{\partial u_j}{\partial\hat{x}_n},\quad \sigma_{11}^0=q\,,\ \sigma_{\text{in}}^0=0\ \text{for in}\ne 11,
\tag{3.7}
$$

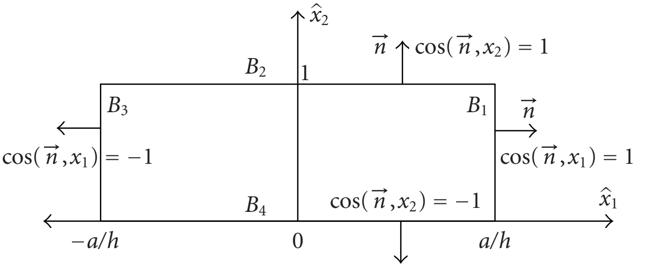

Figure 3.1. The form of the boundary $\partial\hat{B}$.

we get

$$\begin{aligned}
&\int_0^1 \int_{-a/h}^{a/h} \left[hT_{ij} \frac{\partial v_j}{\partial \hat{x}_i} - \rho_0 \omega^2 h^2 u_i v_i \right] d\hat{x}_1 d\hat{x}_2 \\
&\quad = \int_{\partial\hat{B}} \Big[h\sigma_{11} v_1 \cos(\vec{n}, \hat{x}_1) + h\sigma_{21} v_2 \cos(\vec{n}, \hat{x}_1) + h\sigma_{12} v_1 \cos(\vec{n}, \hat{x}_2) \\
&\qquad + h\sigma_{22} v_2 \cos(\vec{n}, \hat{x}_2) + q \frac{\partial u_1}{\partial \hat{x}_1} v_1 \cos(\vec{n}, \hat{x}_1) + q \frac{\partial u_2}{\partial \hat{x}_1} v_2 \cos(\vec{n}, \hat{x}_1) \Big] ds.
\end{aligned} \tag{3.8}$$

The integral over the boundary $\partial\hat{B}$ in (3.8) can be calculated as follows: let the boundary $\partial\hat{B}$ be in the form given in Figure 3.1.

According to Figure 3.1 boundary $\partial\hat{B}$ can be written as $B_1 \cup B_2 \cup B_3 \cup B_4$. The right-hand side of (3.8) can be written as follows:

$$\int_{\partial\hat{B}} \left\{ \cos(\vec{n}, \hat{x}_1) \left[h\sigma_{11} v_1 + h\sigma_{21} v_2 + q \frac{\partial u_1}{\partial \hat{x}_1} v_1 + q \frac{\partial u_2}{\partial \hat{x}_1} v_2 \right] + \cos(\vec{n}, \hat{x}_2) \left[h\sigma_{12} v_1 + h\sigma_{22} v_2 \right] \right\} ds. \tag{3.9}$$

We have the integrals

$$\begin{aligned}
&\int_0^1 1 \cdot \left[h\sigma_{11} v_1 + h\sigma_{21} v_2 + q \frac{\partial u_1}{\partial \hat{x}_1} v_1 + q \frac{\partial u_2}{\partial \hat{x}_1} v_2 \right] d\hat{x}_2, \\
&\int_{-a/h}^{a/h} 1 \cdot \left[h\sigma_{12} v_1 + h\sigma_{22} v_2 \right] d\hat{x}_1, \\
&\int_0^1 (-1) \left[h\sigma_{11} v_1 + h\sigma_{21} v_2 + q \frac{\partial u_1}{\partial \hat{x}_1} v_1 + q \frac{\partial u_2}{\partial \hat{x}_1} v_2 \right] d\hat{x}_2, \\
&\int_{-a/h}^{a/h} (-1) \left[h\sigma_{12} v_1 + h\sigma_{22} v_2 \right] d\hat{x}_1,
\end{aligned} \tag{3.10}$$

for the boundaries

$$B_1 = \left\{(\hat{x}_1,\hat{x}_2) : \hat{x}_1 = \frac{a}{h}, 0 \le \hat{x}_2 \le 1\right\}, \qquad B_2 = \left\{(\hat{x}_1,\hat{x}_2) : \frac{-a}{h} \le \hat{x}_1 \le \frac{a}{h}, \hat{x}_2 = 1\right\},$$
$$B_3 = \left\{(\hat{x}_1,\hat{x}_2) : \hat{x}_1 = \frac{-a}{h}, 0 \le \hat{x}_2 \le 1\right\}, \qquad B_4 = \left\{(\hat{x}_1,\hat{x}_2) : \frac{-a}{h} \le \hat{x}_2 \le \frac{a}{h}, \hat{x}_2 = 0\right\}, \tag{3.11}$$

respectively. Using the conditions (2.9) in (3.10) we get the integral term

$$\int_{-a/h}^{a/h} h\sigma_{22}\big|_{\hat{x}_2=1} v_2\big|_{\hat{x}_2=1} d\hat{x}_1. \tag{3.12}$$

Consequently, (3.8) can be written as

$$\int_0^1\int_{-a/h}^{a/h}\left[hT_{ij}\frac{\partial v_j}{\partial \hat{x}_i} - \rho_0\omega^2h^2u_iv_i\right]d\hat{x}_1 d\hat{x}_2 = \int_{-a/h}^{a/h} h\sigma_{22}\big|_{\hat{x}_2=1} v_2\big|_{\hat{x}_2=1} d\hat{x}_1 \tag{3.13}$$

or

$$\begin{aligned}\int_0^1\int_{-a/h}^{a/h}\bigg[&h\sigma_{11}\frac{\partial v_1}{\partial \hat{x}_1} + h\sigma_{21}\frac{\partial v_2}{\partial \hat{x}_1} + h\sigma_{12}\frac{\partial v_1}{\partial \hat{x}_2} + h\sigma_{22}\frac{\partial v_2}{\partial \hat{x}_2} + q\frac{\partial u_1}{\partial \hat{x}_1}\frac{\partial v_1}{\partial \hat{x}_1} + q\frac{\partial u_2}{\partial \hat{x}_1}\frac{\partial v_2}{\partial \hat{x}_1}\\ &- \rho_0\omega^2h^2(u_1v_1+u_2v_2)\bigg]d\hat{x}_1 d\hat{x}_2 = \int_{-a/h}^{a/h} h\sigma_{22}\Big|_{\hat{x}_2=1} v_2\Big|_{\hat{x}_2=1} d\hat{x}_1.\end{aligned} \tag{3.14}$$

The mechanical relations (2.3) and (2.4) under the transformation (3.1) can be written explicity as follows:

$$\begin{aligned}\sigma_{11} &= (\lambda+2\mu)\frac{1}{h}\frac{\partial u_1}{\partial \hat{x}_1} + \lambda\frac{1}{h}\frac{\partial u_2}{\partial \hat{x}_2},\\ \sigma_{22} &= \lambda\frac{1}{h}\frac{\partial u_1}{\partial \hat{x}_1} + (\lambda+2\mu)\frac{1}{h}\frac{\partial u_2}{\partial \hat{x}_2},\\ \sigma_{12} &= \mu\frac{1}{h}\left(\frac{\partial u_1}{\partial \hat{x}_2} + \frac{\partial u_2}{\partial \hat{x}_1}\right).\end{aligned} \tag{3.15}$$

After using boundary condition

$$\sigma_{22}\big|_{\hat{x}_2=1} = P_0\delta(h\hat{x}_1)e^{i\omega t} \tag{3.16}$$

and the property

$$\delta(a(x)) = \frac{1}{a'(x)}\delta(x), \tag{3.17}$$

the right-hand side of the (3.13) can be written as

$$\int_{-a/h}^{a/h} P_0\delta(\hat{x}_1)v_2\big|_{\hat{x}_2=1} d\hat{x}_1. \tag{3.18}$$

After substituting relations (3.15) in (3.13) and using (3.18) as right-hand side we get

$$\begin{aligned}&\int_0^1\int_{-a/h}^{a/h}\bigg[\left\{(\lambda+2\mu+q)\frac{\partial u_1}{\partial\hat{x}_1}+\lambda\frac{\partial u_2}{\partial\hat{x}_2}\right\}\frac{\partial v_1}{\partial\hat{x}_1}+\left\{\mu\frac{\partial u_1}{\partial\hat{x}_2}+(\mu+q)\frac{\partial u_2}{\partial\hat{x}_1}\right\}\frac{\partial v_2}{\partial\hat{x}_1}+\mu\left\{\frac{\partial u_1}{\partial\hat{x}_2}+\frac{\partial u_2}{\partial\hat{x}_1}\right\}\frac{\partial v_1}{\partial\hat{x}_2}\\&\qquad+\left\{\lambda\frac{\partial u_1}{\partial\hat{x}_1}+(\lambda+2\mu)\frac{\partial u_2}{\partial\hat{x}_2}\right\}\frac{\partial v_2}{\partial\hat{x}_2}-\rho_0\omega^2h^2(u_1v_1+u_2v_2)\bigg]d\hat{x}_1d\hat{x}_2\\&\quad=\int_{-a/h}^{a/h}P_0\delta(\hat{x}_1)v_2\big|_{\hat{x}_2=1}d\hat{x}_1.\end{aligned}\tag{3.19}$$

By dividing both sides of (3.19) to Láme constant μ, we get

$$\begin{aligned}&\int_0^1\int_{-a/h}^{a/h}\bigg[\left\{\left(\frac{q}{\mu}+\frac{\lambda}{\mu}+2\right)\frac{\partial u_1}{\partial\hat{x}_1}+\frac{\lambda}{\mu}\frac{\partial u_2}{\partial\hat{x}_2}\right\}\frac{\partial v_1}{\partial\hat{x}_1}+\left\{\frac{\partial u_1}{\partial\hat{x}_2}+\left(\frac{q}{\mu}+1\right)\frac{\partial u_2}{\partial\hat{x}_1}\right\}\frac{\partial v_2}{\partial\hat{x}_1}+\left\{\frac{\partial u_1}{\partial\hat{x}_2}+\frac{\partial u_2}{\partial\hat{x}_1}\right\}\frac{\partial v_1}{\partial\hat{x}_2}\\&\qquad+\left\{\frac{\lambda}{\mu}\frac{\partial u_1}{\partial\hat{x}_1}+\left(\frac{\lambda}{\mu}+2\right)\frac{\partial u_2}{\partial\hat{x}_2}\right\}\frac{\partial v_2}{\partial\hat{x}_2}-\frac{\rho_0\omega^2h^2}{\mu}(u_1v_1+u_2v_2)\bigg]d\hat{x}_1d\hat{x}_2\\&\quad=\int_{-a/h}^{a/h}\frac{P_0}{\mu}\delta(\hat{x}_1)v_2\big|_{\hat{x}_2=1}d\hat{x}_1.\end{aligned}\tag{3.20}$$

By (3.20), we get bilinear form $a(u,v)$ and linear form $l(v)$. Introducing the distortion wave velocity

$$c_2=\sqrt{\frac{\mu}{\rho_0}}\tag{3.21}$$

and the dimensionless frequency

$$\Omega=\frac{\omega h}{c_2},\tag{3.22}$$

we obtain $J(u)=(1/2)a(u,u)-l(u)$, the total energy functional, where $u=u(u_1,u_2)$ as follows:

$$\begin{aligned}&J(u_1,u_2)\\&=\frac{1}{2}\int_0^1\int_{-a/h}^{a/h}\bigg[\frac{q}{\mu}\bigg[\left(\frac{\partial u_1}{\partial\hat{x}_1}\right)^2+\left(\frac{\partial u_2}{\partial\hat{x}_1}\right)^2\bigg]+\left\{\frac{\partial u_1}{\partial\hat{x}_2}+\frac{\partial u_2}{\partial\hat{x}_1}\right\}^2+2\frac{\lambda}{\mu}\cdot\frac{\partial u_1}{\partial\hat{x}_1}\cdot\frac{\partial u_2}{\partial\hat{x}_2}\\&\quad+\left(\frac{\lambda}{\mu}+2\right)\bigg[\left(\frac{\partial u_1}{\partial\hat{x}_1}\right)^2+\left(\frac{\partial u_2}{\partial\hat{x}_2}\right)^2\bigg]-\Omega^2(u_1^2+u_2^2)\bigg]d\hat{x}_1d\hat{x}_2-\int_{-a/h}^{a/h}\frac{P_0}{\mu}\delta(\hat{x}_1)u_2\big|_{\hat{x}_2=1}d\hat{x}_1.\end{aligned}\tag{3.23}$$

As known from calculus of variation [9], by equating the first variational of the total energy functional, $J(u)$ given in (3.23), to zero one must derive (2.11) and boundary conditions (3.3). In order to achieve this result we use

$$\delta J(u_1,u_2)=0.\tag{3.24}$$

In (3.24) we equate the integrands involving δu_1 and δu_2 to zero. Consequently, we obtain

$$-u_1\rho_0\omega^2 = \frac{q}{\mu}\cdot\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\lambda}{\mu}\frac{\partial^2 u_2}{\partial x_1\partial x_2} + \left(\frac{\lambda}{\mu}+2\right)\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1\partial x_2},$$
$$-u_2\rho_0\omega^2 = \frac{q}{\mu}\cdot\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1\partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\lambda}{\mu}\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \left(\frac{\lambda}{\mu}+2\right)\frac{\partial^2 u_2}{\partial x_2^2}, \tag{3.25}$$

$$\left(q\frac{\partial u_1}{\partial x_1}+\sigma_{11}\right)\Bigg|_{\hat{x}_1=\pm a/h} = 0, \qquad \sigma_{12}|_{\hat{x}_2=1} = 0,$$
$$\left(q\frac{\partial u_2}{\partial x_1}+\sigma_{12}\right)\Bigg|_{\hat{x}_1=\pm a/h} = 0, \qquad \sigma_{22}|_{\hat{x}_2=1} = P_0\delta(\hat{x}_1), \tag{3.26}$$

as boundary conditions. It is worthy of noting that we get (3.25) by using functional (3.23) which is obtained by (3.20). Therefore, if we get the total energy functional using (3.19), we obtain (2.2) the linearized equations of motion.

The total energy functional, $J(u)$, will be minimized using Rayleigh-Ritz method. Firstly, we divide the domain $\widehat{B}$ into finitely many B_i subdomains. We utilize displacement-based finite element method, so the functions to be sought in each finite element will be displacements. Thus, in the eth finite element, we get

$$u_1^{(e)} = \sum_{k=1}^{M} a_k N_k(r,s),$$
$$u_2^{(e)} = \sum_{k=1}^{M} b_k N_k(r,s). \tag{3.27}$$

In (3.27), M denotes the nodes in eth finite element. The shape functions $N_k(r,s)$, defined on the unit square $[-1,1]\times[-1,1]$, are

$$N_1(r,s) = \frac{1}{4}(r^2-r)(s^2-s) \quad N_2(r,s) = \frac{1}{4}(r^2+r)(s^2-s) \quad N_3(r,s) = \frac{1}{4}(r^2+r)(s^2+s),$$
$$N_4(r,s) = \frac{1}{4}(r^2-r)(s^2+s) \quad N_5(r,s) = -\frac{1}{2}(r^2-1)(s^2-s) \quad N_6(r,s) = -\frac{1}{2}(r^2+r)(s^2-1),$$
$$N_7(r,s) = -\frac{1}{2}(r^2-1)(s^2+s) \quad N_8(r,s) = -\frac{1}{2}(r^2-r)(s^2-1) \quad N_9(r,s) = (r^2-1)(s^2-1) \tag{3.28}$$

(see Figure 3.2). Substituting (3.27) in (3.23) we get the linear algebraic system

$$Au = f, \tag{3.29}$$

according to Rayleigh-Ritz method. In (3.29), matrix A denotes the stiffness matrix of the system and is given by

$$A^{(e)} = \begin{bmatrix} [A_{11}^{(e)}] & [A_{12}^{(e)}] \\ [A_{21}^{(e)}] & [A_{22}^{(e)}] \end{bmatrix} \tag{3.30}$$

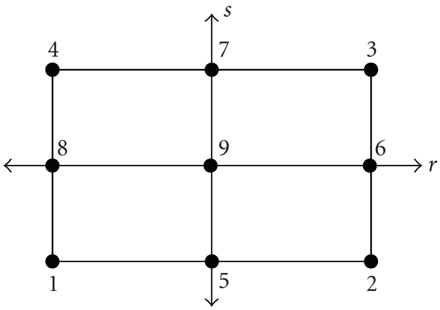

FIGURE 3.2. The order of the nodes of a finite element defined on $[-1,1]\times[-1,1]$.

on the eth finite element. In (3.30) we have

$$\begin{aligned}
[A_{11}^{(e)}] &= \int_0^1\int_{-a/h}^{a/h}\left[\frac{\partial N_{ij}}{\partial x_2}\cdot\frac{\partial N_{kl}}{\partial x_2}+\left(\frac{\lambda}{\mu}+2\right)\frac{\partial N_{ij}}{\partial x_1}\cdot\frac{\partial N_{kl}}{\partial x_1}-\Omega^2 N_{ij}N_{kl}\right]d\hat{x}_1 d\hat{x}_2,\\
[A_{12}^{(e)}] &= \int_0^1\int_{-a/h}^{a/h}\left[\frac{\partial N_{ij}}{\partial x_1}\cdot\frac{\partial N_{kl}}{\partial x_2}+\frac{\lambda}{\mu}\cdot\frac{\partial N_{ij}}{\partial x_2}\cdot\frac{\partial N_{kl}}{\partial x_1}\right]d\hat{x}_1 d\hat{x}_2,\\
[A_{21}^{(e)}] &= \int_0^1\int_{-a/h}^{a/h}\left[\frac{\partial N_{ij}}{\partial x_2}\cdot\frac{\partial N_{kl}}{\partial x_1}+\frac{\lambda}{\mu}\cdot\frac{\partial N_{ij}}{\partial x_1}\cdot\frac{\partial N_{kl}}{\partial x_2}\right]d\hat{x}_1 d\hat{x}_2,\\
[A_{22}^{(e)}] &= \int_0^1\int_{-a/h}^{a/h}\left[\frac{\partial N_{ij}}{\partial x_1}\cdot\frac{\partial N_{kl}}{\partial x_1}+\left(\frac{\lambda}{\mu}+2\right)\frac{\partial N_{ij}}{\partial x_2}\cdot\frac{\partial N_{kl}}{\partial x_2}-\Omega^2 N_{ij}N_{kl}\right]d\hat{x}_1 d\hat{x}_2.
\end{aligned} \tag{3.31}$$

In (3.31), the subscripts i, j, k, and l should take the values $i, j, k, l = 1,\dots,M$. In (3.29) the vector u is as follows:

$$u = \{[a_k][b_k]\}^T, \tag{3.32}$$

and the components of the vector u gives values of the displacements at the nodes in the directions x_1 and x_2. The vector f in (3.29) is given by

$$f = \left\{\frac{P_0}{\mu}N_{ij}\Big|_{\substack{\hat{x}_1=0\\ \hat{x}_2=1}}\right\}^T. \tag{3.33}$$

By using the displacements obtained from solving (3.29) in

$$\sigma = DBu, \tag{3.34}$$

we obtain the stresses. In (3.34), matrix D is given by

$$D = \frac{2}{1-2\nu}\begin{bmatrix} 1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1-2\nu\end{bmatrix}, \tag{3.35}$$

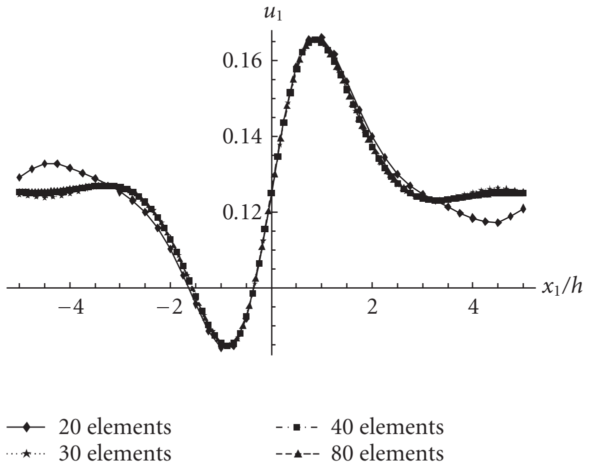

Figure 4.1

and matrix B is given by

$$B = \begin{bmatrix} \dfrac{\partial N_1}{\partial r} & \cdots & \dfrac{\partial N_9}{\partial r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial s} & \cdots & \dfrac{\partial N_9}{\partial s} \\ \dfrac{\partial N_1}{\partial s} & \cdots & \dfrac{\partial N_9}{\partial s} & \dfrac{\partial N_1}{\partial r} & \cdots & \dfrac{\partial N_9}{\partial r} \end{bmatrix}_{3\times 18}. \tag{3.36}$$

4. Numerical results

In order to see the validity of the algorithm and programmes, the case where $\Omega = 0$ and $q = 0$ is selected and the number of finite elements in the direction of x_1-axis is increased. The results obtained in the case considered approach the corresponding ones in the static loading case which are given in Uflyand [10]. Let η denote the dimensionless parameter characterizing the initial stress in the strip plate and is given by $\eta = q/\mu$. Consider the distribution of stresses and displacements in the interface plane (where $x_2/h = 0$) when $\eta = 0$. It is seen that the problem is an axisymmetric problem with respect to $x_1 = 0$ plane. The numerical results in Figures 4.1, 4.2, and 4.3 are obtained under the following assumptions: $\Omega = 0$, $q = 0$, $\nu = 0.33$, and $h/2a = 0.2$.

In Figures 4.1 and 4.2, the distribution of the displacements u_1 and u_2 in the direction of x_1-axis is given. Increasing the number of finite elements in the direction of x_1-axis, it is seen that the displacements are approaching to each other asymptotically.

In Figure 4.3 the stress distribution in the direction of x_1-axis is given. The results obtained here coincide with the ones given in [10]. It is seen that the values of σ_{22} decrease towards the sides of the strip plate where $x_1 = \pm a$.

In Figure 4.4 the stress distribution in the direction of x_1-axis for various Ω values is given. It is seen that increasing the dimensonless frequency Ω increases the stress σ_{22}.

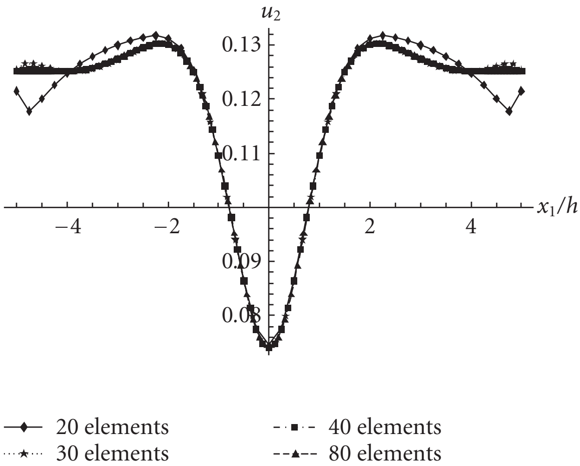

Figure 4.2

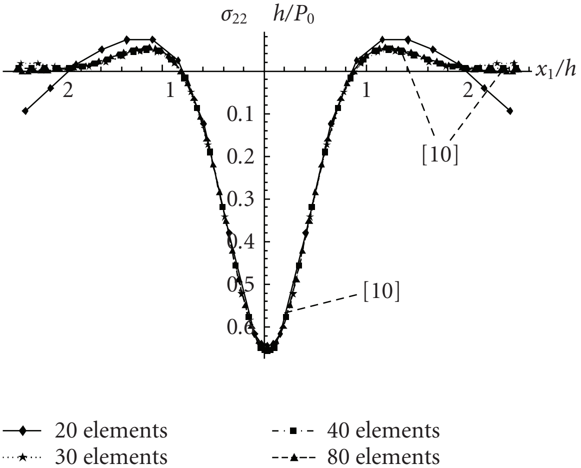

Figure 4.3

However, there are such points on the interface plane along the x_1-axis at which the dimensonless frequency Ω does not affect the values of the stress σ_{22}.

5. Conclusions

In this paper, mathematical formulation of forced vibration of a prestretched strip plate resting on a rigid foundation is given in the framework of the three-dimensional linearized theory of elastodynamics. A numerical algorithm is developed for both the static and dynamic loading cases. The numerical results are presented for the distribution of displacements and stresses. These numerical results indicate the validity of the formulation. It is seen that there are some points on the interface plane at which the dimensionless frequency does not affect the stress σ_{22}.

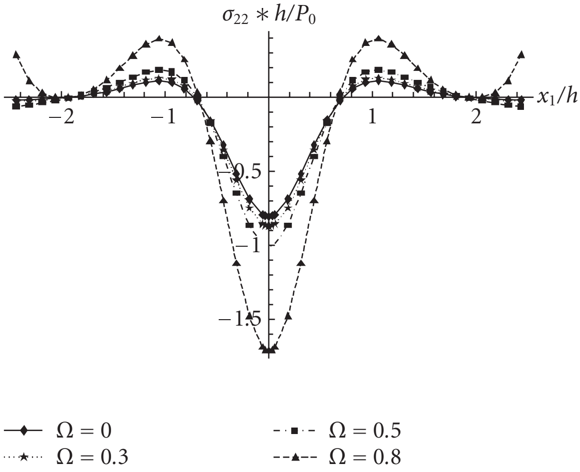

Figure 4.4

References

[1] A. N. Guz, *Elastic Waves in a Body with Initial Stresses, I.General Theory*, Naukova Dumka, Kiev, Russia, 1986.

[2] A. N. Guz, *Elastic Waves in a Body with Initial Stresses, II. Propagation Laws*, Naukova Dumka, Kiev, Russia, 1986.

[3] A. N. Guz, "Elastic waves in bodies with initial (Residual) stresses," *International Applied Mechanics*, vol. 38, no. 1, pp. 35–78, 2002.

[4] V. B. Rudnitsky and N. N. Dikbtaryaruk, "Inintially stressed elastic strip strengthenedby elastic stringers," *International Applied Mechanics*, vol. 38, no. 11, pp. 81–87, 2002.

[5] S. V. Matniak, "On Rolling rigid cylinder along a band with initial stresses," *International Applied Mechanics*, vol. 39, no. 3, pp. 81–87, 2003.

[6] S. D. Akbarov, A. D. Zamanov, and T. R. Suleimanov, "Forced vibration of a prestretched two-layer slab on a rigid foundation," *Mechanics of Composite Materials*, vol. 41, no. 3, pp. 229–240, 2005.

[7] S. D. Akbarov, "On the dynamical axisymmetric stress field in a finite pre-stretched bilayered slab resting on a rigid foundation," *Journal of Sound and Vibration*, vol. 294, no. 1-2, pp. 221–237, 2006.

[8] S. D. Akbarov and C. Guler, "On the stress field in a half-plane covered by the pre-stretched layer under the action of arbitrary linearly located time-harmonic forces," *Applied Mathematical Modelling*, vol. 31, no. 11, pp. 2375–2390, 2007.

[9] A. Cherkaev and E. Cherkaev, "Calculus of Variations and Applications," Lecture Notes, 2003.

[10] S. I. Uflyand, *Integral Transformations in the Theory of Elastcity*, Nauka, Moscow, Russia, 1963.

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