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Research Article

On Local α-Times Integrated C-Semigroups

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This paper presents several characterizations of a local α-times integrated C-semigroup $\{T(t);\ 0 \leq t < \tau\}$ by means of functional equation, subgenerator, and well-posedness of an associated abstract Cauchy problem. We also discuss properties concerning the nondegeneracy of $T(\cdot)$, the injectivity of C, the closability of subgenerators, the commutativity of $T(\cdot)$, and extension of solutions of the associated abstract Cauchy problem.

1. Introduction

Let X be a complex Banach space and let $B(X)$ be the Banach algebra of all bounded (linear) operators on X. Let $j_{-1} := \delta_0$, the Dirac measure at 0, and for $r > -1$, let $j_r : [0,\infty) \to \mathbb{R}$ be defined as $j_r(t) := t^r/(\Gamma(r+1))$, $t \geq 0$, where $\Gamma(\cdot)$ is the Gamma function.

Let $C \in B(X)$ and $\tau \in (0,\infty]$. A strongly continuous family $\{T(t);\ 0 \leq t < \tau\} \subset B(X)$ is called a *local α-times* ($\alpha \geq 0$) *integrated C-semigroup on X* if it satisfies $T(t)C = CT(t)$ for $0 \leq t < \tau$, $T(0) = 0$, and

$$\begin{aligned} T(s)T(t)x &= \left(\int_0^{s+t} - \int_0^{s} - \int_0^{t}\right) j_{\alpha-1}(s+t-r)CT(r)x\,dr \\ &= \int_0^s \left[j_{\alpha-1}(r)CT(s+t-r) - j_{\alpha-1}(s+t-r)CT(r)\right]x\,dr \\ &= \int_0^t \left[j_{\alpha-1}(r)CT(s+t-r) - j_{\alpha-1}(s+t-r)CT(r)\right]x\,dr \end{aligned} \tag{1.1}$$

for $x \in X, 0 \leq s,t \leq s+t < \tau$. In case $\tau = \infty$, a local α-times integrated C-semigroup is named an α-times integrated C-semigroup (see [1] for general $\alpha \in [0,\infty)$, and [2] for the case $\alpha \in \mathbb{N}$). When $C = I$, the identity operator, $T(\cdot)$ is called an α-times integrated semigroup (cf. [3, 4]).

We say that $\{T(t);\ 0 \le t < \tau\}$ is a *local (0-times integrated) C-semigroup* (cf. [5–11]) if $T(0) = C$ and

$$T(t)T(s) = T(s+t)C \quad \forall 0 \le t,\ s \le s+t < \tau. \tag{1.2}$$

In case $\tau = \infty$, a local C-semigroup is called a *C-semigroup* (cf. [12–15]).

Local α-times integrated C-semigroups were first studied in [16] for the case $\alpha = n \in \mathbb{N}$ and under the assumption that C is injective and $T(\cdot)$ satisfies the condition

$$T(t)x = 0 \quad \forall 0 < t < \tau \text{ implies } x = 0. \tag{1.3}$$

Clearly, (1.3) is implied by the following condition:

$$T(t)x = 0 \quad \forall 0 < t < \frac{\tau}{2} \text{ implies } x = 0. \tag{1.4}$$

For the case $\tau = \infty$, both conditions (1.3) and (1.4) become the ordinary definition of nondegeneracy, that is,

$$T(t)x = 0 \quad \forall t > 0 \text{ implies } x = 0. \tag{1.5}$$

When $\tau < \infty$ and $\alpha = 0$, (1.4) is strictly stronger than (1.3) and is equivalent to that C is injective (cf. [6]). It will be seen that in the case $\alpha > 0$, (1.4) still implies (1.3) and the injectivity of C (Lemma 4.1). These facts suggest that a proper definition of *nondegeneracy* for a local α-times integrated C-semigroup seems to be (1.4). In the present paper, we use this definition.

The aim of this paper is to analyze in detail several characterizations for degenerate and nondegenerate local α-times integrated C-semigroups, by means of functional equation, subgenerator, and well-posedness of an associated abstract Cauchy problem.

In Section 2, we give the following general characterization of local α-times integrated C-semigroups in terms of functional equations:

$$\begin{gathered} T(0) = \delta_{0,\alpha}C, \qquad T(t)C = CT(t), \\ S(s)\left[T(t) - j_\alpha(t)C\right] = \left[T(s) - j_\alpha(s)C\right]S(t) \quad \forall 0 \le s,\ t \le s+t < \tau, \end{gathered} \tag{1.6}$$

where $\delta_{a,b}$ is the Kronecker delta and $S(t) := \int_0^t T(s)ds$, $0 \le t < \tau$ (see Theorem 2.3).

In Sections 3 and 4, we will define subgenerator and generator of a nondegenerate local α-times integrated C-semigroup $T(\cdot)$. Then, we discuss some properties concerning the nodegeneracy of $T(\cdot)$, the injectivity of C, the closability of subgenerators, and the commutativity of the family $\{T(t);\ 0 \le t < \tau\}$. For instance, we will see that nondegeneracy is equivalent to the injectivity of C when $T(\cdot)$ has a subgenerator G (Lemma 4.1), and nondegeneracy implies that $T(\cdot)$ has the generator and $\{T(t);\ 0 \le t < \tau\}$ is a commutative family (Theorem 3.5 and Proposition 4.6). Notice that (1.1) implies that $T(t)T(s) = T(s)T(t)$ holds for any pair of $s, t \ge 0$ which satisfies $s+t < \tau$, but, when $T(\cdot)$ is degenerate, in general, the commutativity does not hold for $\tau < s+t < 2\tau$ (see [6] for an example).

We also prove a characterization (Theorem 4.15) for nondegenerate local α-times integrated C-semigroups, which states that $\{T(t);\ 0 \le t < \tau\}$ is a nondegenerate local α-times integrated C-semigroup if and only if C is injective and there is a closed operator G satisfying

$$T(t)x - j_\alpha(t)Cx = \begin{cases} S(t)Gx, & x \in D(G); \\ GS(t)x, & x \in X \end{cases} \tag{1.7}$$

for all $0 \le t < \tau$. In this case, $C^{-1}GC$ is the generator of $T(\cdot)$.

In Section 5, we discuss the relation between a local α-times integrated C-semigroup with generator A and the associated abstract Cauchy problem:

$$\begin{aligned} &u'(t) = Au(t) + Cf(t), \quad 0 < t < \tau; \\ &u(0) = 0. \end{aligned} \tag{ACP($A;Cf,0$)}$$

Let $C \in B(X)$ be injective and $\alpha \ge 0$, and let A be a closed linear operator such that $CA \subset AC$. It will be shown (see Theorem 5.1) that the abstract Cauchy problem ACP$(A; j_\alpha Cx, 0)$ has a unique solution u_x for every $x \in X$ if and only if A is a subgenerator of a local α-times integrated C-semigroup $T(\cdot)$. Moreover, the solution is given by $u_x(t) = \int_0^t T(s)x\,ds$.

In Section 6, we apply Theorem 4.15 to show that the generator A of a local α-times integrated C-semigroup on $[0,\tau)$ also generates a local $(\alpha+n)$-times integrated C^2-semigroup on $[0,2\tau)$ for any integer n which is not less than α (see Theorem 6.1). This is a generalization to α-times integrated C-semigroups of a result in [17] on n-times integrated semigroups. This generalization (for the case $\alpha = n$) has been proved in [16] by different approach, and the case $n = 0$ was treated in [10].

As is well known, there is the Hille-Yosida generation theorem for a (C_0)-semigroup in terms of the resolvent of the generator (or equivalently, the Lapalace transform of the (C_0)-semigroup). For an exponentially bounded nondegenerate α-times integrated C-semigroup, we also have a Hille-Yosida type generation theorem in terms of the *C-resolvent* of the generator (or equivalently, the Lapalace transform of the C-semigroup) (cf. [1, 2]). For nonexponentially bounded C-semigroups and local C-semigroups, the Lapalace transform does not exist. In this case, there is a Hille-Yosida type generation theorem in terms of the *asymptotic C-resolvent* of the generator (cf. [9, 7]). See also [18] for a similar Hille-Yosida type generation theorem for nondegenerate local C-cosine functions. Finally, we remark that it is also possible to establish a similar Hille-Yosida type generation theorem for a nondegenerate local α-times integrated C-semigroup with $\alpha > 0$.

2. Degenerate local α-times integrated C-semigroups

Let $h : [0,b] \to \mathbb{C}$ be integrable and let $f : [0,b] \to X$ be Bochner integrable, where $b > 0$. The convolution of h and f is the function $h * f$ defined by $(h * f)(t) := \int_0^t h(t-s)f(s)ds$, $0 \le t \le b$ whenever the integral is well-defined at every point $t \in [0,b]$. When $h = j_{-1}$, the Dirac measure, we define $(j_{-1} * f)(t) := f(t)$ for $t \in [0,b]$. We will need the following lemma: (a) can be verified by using the Laplace transform and (b) is a modification of Titchmarsh's theorem (cf. [19, Corollary 2.2.5]).

Lemma 2.1. *The following hold for* $r,s \geq -1$.

(a) $j_r * j_s = j_{r+s+1}$.

(b) *Let* $f : [0,b] \to X$ *be Bochner integrable. If* $j_r * f \equiv 0$ *on* $[0,b]$, *then* $f = 0$ *almost everywhere.*

We will also need the following lemma whose proof we omit.

Lemma 2.2. *Let* $\alpha \geq 0$ *and let* $T(\cdot) : [0,\tau) \to B(X)$ *be a strongly continuous function satisfying* $T(0) = \delta_{0,\alpha}C$. *Let* $S(t)x := \int_0^t T(s)x\,ds$ *for all* $x \in X$ *and* $0 \leq t < \tau$. *Then,* $S(\cdot)$ *is a local* $(\alpha+1)$*-times integrated* C*-semigroup if and only if* $T(\cdot)$ *is a local* α*-times integrated* C*-semigroup.*

Theorem 2.3. *Let* $\alpha \geq 0$ *and let* $T(\cdot) : [0,\tau) \to B(X)$ *be a strongly continuous function satisfying* $T(0) = \delta_{0,\alpha}C$. *Let* $S(t)x := \int_0^t T(s)x\,ds$ *for all* $x \in X$ *and* $0 \leq t < \tau$. *Then,* $T(\cdot)$ *is a local* α*-times integrated* C*-semigroup on* X *if and only if* $T(t)C = CT(t)$ *for all* $0 \leq t < \tau$ *and*

$$S(s)[T(t) - j_\alpha(t)C] = [T(s) - j_\alpha(s)C]S(t) \quad \forall 0 \leq s,\ t \leq s+t < \tau. \tag{2.1}$$

Proof. Suppose $T(\cdot)$ is an α-times integrated C-semigroup on X. Integrating (1.1) with respect to t, and using integration by parts, we obtain the following equation:

$$\begin{aligned} T(s)S(t)x &= \int_0^s j_{\alpha-1}(r)C[S(s+t-r) - j_\alpha(s+t-r)CT(r)]x\,dr \\ &= \left(\int_t^{s+t} - \int_0^s\right) j_{\alpha-1}(s+t-r)CS(r)x\,dr - j_\alpha(t)CS(s)x. \end{aligned} \tag{2.2}$$

Integrating (1.1) with respect to s, we also have

$$\begin{aligned} S(s)T(t)x &= \int_0^t [j_{\alpha-1}(r)CS(s+t-r) - j_\alpha(s+t-r)CT(r)]x\,dr \\ &= \left(\int_s^{s+t} - \int_0^t\right) j_{\alpha-1}(s+t-r)CS(r)x\,dr - j_\alpha(s)CS(t)x \end{aligned} \tag{2.3}$$

for $x \in X$ and $0 \leq s,t \leq s+t < \tau$. Comparing (2.2) and (2.3), we obtain

$$\begin{aligned} T(s)S(t)x + j_\alpha(t)CS(s)x &= \left(\int_0^{s+t} - \int_0^t - \int_0^s\right) j_{\alpha-1}(s+t-r)CS(r)x\,dr \\ &= S(s)T(t)x + j_\alpha(s)CS(t)x. \end{aligned} \tag{2.4}$$

Since $T(\cdot)$ commutes with C, so does $S(\cdot)$. Therefore, (2.1) holds.

Conversely, we suppose that $T(\cdot)$ satisfies (2.1). By Lemma 2.2, it suffices to show that $S(\cdot)$ is an $(\alpha+1)$-times integrated C-semigroup. First, we replace s by $s+t-r$ and t by r in (2.1). Then, we have for $x \in X$

$$S(s+t-r)T(r)x - T(s+t-r)S(r)x = S(s+t-r)j_\alpha(r)Cx - j_\alpha(s+t-r)CS(r)x. \tag{2.5}$$

By integrating the right-hand side with respect to r from 0 to t, we obtain from $CT(\cdot) = T(\cdot)C$ that

$$\begin{aligned}\int_0^t & S(s+t-r)j_\alpha(r)Cx\,dr - \int_0^t j_\alpha(s+t-r)CS(r)x\,dr \\ &= \int_s^{s+t} S(r)j_\alpha(s+t-r)Cx\,dr - \int_0^t j_\alpha(s+t-r)CS(r)x\,dr \\ &= \left(\int_0^{s+t} - \int_0^s - \int_0^t\right) j_\alpha(s+t-r)CS(r)x\,dr.\end{aligned} \tag{2.6}$$

On the other hand, from the left-hand side, we have

$$\begin{aligned}\int_0^t & S(s+t-r)T(r)x\,dr - \int_0^t T(s+t-r)S(r)x\,dr \\ &= S(s+t-r)S(r)x|_0^t + \int_0^t T(s+t-r)S(r)x\,dr - \int_0^t T(s+t-r)S(r)x\,dr \\ &= S(s)S(t) - S(s+t)S(0) = S(s)S(t)\end{aligned} \tag{2.7}$$

for $0 \le t, s < s+t < \tau$. Therefore, $S(\cdot)$ is an $(\alpha+1)$-times integrated C-semigroup. The result follows from Lemma 2.2. □

Corollary 2.4. *Let $\alpha > 0$, $\beta \ge -1$. If $T(\cdot)$ is a local α-times integrated C-semigroup, then $j_\beta * T(\cdot)$ is an $(\alpha+\beta+1)$-times integrated C-semigroup.*

Proof. Let $U(t) := j_\beta * T(t)$ for all $0 \le t < \tau$. Using Lemma 2.1(a) and Theorem 2.3, we have for every $0 \le s, t \le s+t < \tau$ and $x \in X$,

$$\begin{aligned}[U(s) & - j_{\alpha+\beta+1}(s)C]\int_0^t U(r)x\,dr \\ &= \int_0^s j_\beta(s-u)[T(u) - j_\alpha(u)C]j_0 * j_\beta * T(t)x\,du \\ &= \int_0^s j_\beta(s-u)[T(u) - j_\alpha(u)C]\int_0^t j_\beta(t-v)(j_0 * T)(v)x\,dv\,du \\ &= \int_0^s\int_0^t j_\beta(s-u)j_\beta(t-v)[T(u) - j_\alpha(u)C](j_0 * T)(v)x\,dv\,du \\ &= \int_0^s\int_0^t j_\beta(s-u)j_\beta(t-v)(j_0 * T)(u)[T(v) - j_\alpha(v)C]x\,dv\,du \\ &= \int_0^s j_\beta(s-u)(j_0 * T)(u)du\int_0^t j_\beta(t-v)[T(v) - j_\alpha(v)C]x\,dv \\ &= j_\beta * (j_0 * T)(s)[j_\beta * T(t) - j_\beta * j_\alpha(t)C]x \\ &= \int_0^s U(r)dr[U(t) - j_{\alpha+\beta+1}(t)C]x.\end{aligned} \tag{2.8}$$

Therefore, $U = j_\beta * T$ is an $(\alpha+\beta+1)$-times integrated C-semigroup by Theorem 2.3 again. □

3. (C,α)-subgenerators

Let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function. We consider properties of those linear operators G which satisfy $R(S(t)) \subset D(G)$ and $S(t)G \subset GS(t) = T(t)x - j_\alpha(t)C$, that is, the following two conditions hold:

$$T(t)x - j_\alpha(t)Cx = S(t)Gx \quad \text{for } x \in D(G),\ 0 \le t < \tau, \tag{3.1}$$

$$R(S(t)) \subset D(G), \qquad T(t)x - j_\alpha(t)Cx = GS(t)x \quad \text{for } x \in X,\ 0 \le t < \tau. \tag{3.2}$$

Such an operator G will be called a (C,α)-*subgenerator* of $T(\cdot)$. There may or may not exist (C,α)-subgenerators for a given local α-times integrated C-semigroup and there may be many ones. If there is a (C,α)-subgenerator which contains all (C,α)-subgenerators of $T(\cdot)$, then we call this maximal (C,α)-subgenerator the (C,α)-*generator* of $T(\cdot)$.

It will be seen in Theorem 3.5(c) that if C is injective and if there is a closed (C,α)-subgenerator G of $T(\cdot)$, then $T(\cdot)$ is a local α-times integrated C-semigroup and $A := C^{-1}GC$ is its (C,α)-generator. (C,α)-subgenerators and (C,α)-generator of a local α-times integrated C-semigroup will be called simply subgenerators and generator, respectively.

Lemma 3.1. *Let $C \in B(X)$ be injective and let $T(\cdot) : [0,\tau) \to B(X)$ be strongly continuous. If an operator G satisfies condition (3.1), then it satisfies the following condition:*

$$u \equiv 0 \text{ is the only solution of the equation } u(t) = G(1 * u)(t), \quad 0 \le t < \tau. \tag{3.3}$$

In particular, (3.3) holds for any (C,α)-subgenerator G of $T(\cdot)$.

Proof. Let u be a solution of $u(t) = G\int_0^t u(s)ds$. By (3.1), we have

$$\begin{aligned} S * u = S * G(1 * u) &= \left[T - j_\alpha C\right] * (1 * u) \\ &= \left[S - j_{\alpha+1}C\right] * u = S * u - j_{\alpha+1}C * u. \end{aligned} \tag{3.4}$$

This proves $j_{\alpha+1}C * u \equiv 0$. It follows from Lemma 2.1(b) and the continuity of u that $Cu \equiv 0$ and hence $u \equiv 0$. □

Remark 3.2. Whenever C is injective, Lemma 3.1 implies that an operator G can be a (C,α)-subgenerator of at most one strongly continuous local α-times integrated C-semigroup $T(\cdot)$.

Lemma 3.3. *Let $T(\cdot) : [0,\tau) \to B(X)$ be strongly continuous. If $CT(t) = T(t)C$ for $0 \le t < \tau$, and if $T(\cdot)$ has a (C,α)-subgenerator G, then $T(\cdot)$ is a local α-times integrated C-semigroup with G a subgenerator.*

Proof. By (3.1) and (3.2), we have for every $0 \le s, t < \tau$ and $x \in X$

$$[T(t) - j_\alpha(t)C]S(s)x = S(t)GS(s)x = S(t)[T(s) - j_\alpha(s)C]x. \tag{3.5}$$

Hence it follows from Theorem 2.3 that $T(\cdot)$ is an α-times integrated C-semigroup. □

Proposition 3.4. *Let $C \in B(X)$ be an injection. Let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function and G be a closed operator satisfying (3.2) and (3.3). Suppose that B is a closed operator such that $BG \subset GB$, that is, $D(BG) \subset D(GB)$ and $BG = GB$ on $D(BG)$, and such that $S(t)D(B) \subset D(B)$ for all $0 \le t < \tau$, and $BS(\cdot)x \in C([0,\tau),X)$ for all $x \in D(B)$. Then the following two conditions are equivalent:*

(a) *$CB \subset BC$;*

(b) *$S(t)B \subset BS(t)$ and $G(1 * S)(t)D(B) \subset D(B)$ for all $0 \le t < \tau$.*

Proof. (a)⇒(b). Integrating (3.2), we have from the closedness of G that

$$S(t)x - j_{\alpha+1}(t)Cx = (1 * GS)(t)x = G(1 * S)(t)x \quad \text{for } x \in X. \tag{3.6}$$

Let $x \in D(B)$. By assumption, $S(t)x \in D(B)$. Also, by (a) we have $j_{\alpha+1}(t)Cx \in D(B)$ and $Bj_{\alpha+1}(t)Cx = j_{\alpha+1}(t)CBx$ for $0 \le t < \tau$. Hence it follows from (3.6) that $G(1 * S)(t)x \in D(B)$ for all $0 \le t < \tau$. Then, by the closedness of B and the assumption on B we obtain that

$$BG(1 * S)(t)x = GB(1 * S)(t)x = G(1 * BS)(t)x \quad \forall 0 \le t < \tau. \tag{3.7}$$

Therefore, using (3.6) and (3.7), we have for $x \in D(B)$ and $0 \le t < \tau$,

$$\begin{aligned} S(t)Bx - G(1 * S)(t)Bx &= j_{\alpha+1}(t)CBx = Bj_{\alpha+1}(t)Cx \\ &= B[S(t)x - G(1 * S)(t)x] \\ &= BS(t)x - G(1 * BS)(t)x. \end{aligned} \tag{3.8}$$

This implies $S(t)Bx - BS(t)x = G1 * [S(\cdot)B - BS(\cdot)](t)x$ for all $0 \le t < \tau$. Since $u = S(\cdot)Bx - BS(\cdot)x$ is a strongly continuous solution of $u = G1 * u$, it follows from (3.3) that $S(\cdot)Bx - BS(\cdot)x \equiv 0$ for all $x \in D(B)$. Therefore, (b) holds.

(b)⇒(a). Let $x \in D(B)$. By (b) and (3.6), we have

$$j_{\alpha+1}(t)Cx = S(t)x - G(1 * S)(t)x \in D(B) \quad \forall 0 \le t < \tau. \tag{3.9}$$

So, $Cx \in D(B)$. By the closedness of B and the assumption on B, this implies that $BG(1 * S)(t)x = BS(t)x - Bj_{\alpha+1}(t)Cx = S(t)Bx - j_{\alpha+1}(t)BCx$ is strongly continuous on $0 \le t < \tau$. It follows from the assumption on B, the closedness of B, and condition (b) that

$$BG(1 * S)(t)x = GB(1 * S)(t)x = G(1 * BS)(t)x = G(1 * S)(t)Bx \tag{3.10}$$

for all $0 \leq t < \tau$. Therefore, by (3.6) and (b) again, we obtain that

$$\begin{aligned} Bj_{\alpha+1}(t)Cx &= BS(t)x - BG(1 * S)(t)x \\ &= S(t)Bx - G(1 * S)(t)Bx = j_{\alpha+1}(t)CBx \quad \forall 0 \leq t < \tau. \end{aligned} \tag{3.11}$$

This proves (a). □

Note that if $B \in B(X)$, the assumption that $S(t)D(B) \subset D(B)$ for all $0 \leq t < \tau$ and $BS(\cdot)x \in C([0,\tau),X)$ for $x \in D(B)$ always holds.

Theorem 3.5. *Let $C \in B(X)$ be injective, and let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function with a closed (C,α)-subgenerator G. Then, the following hold:*

(a) *$CT(t) = T(t)C$ for all $0 \leq t < \tau$ (or equivalently, $CS(t) = S(t)C$ for all $0 \leq t < \tau$), so that $T(\cdot)$ is a local α-times integrated C-semigroup.*
(b) *$T(t)T(s) = T(s)T(t)$ for all $0 \leq s,\ t < \tau$.*
(c) *$CG \subset GC$, and $C^{-1}GC$ is the generator of $T(\cdot)$.*

Proof. By the definition of (C,α)-subgenerator, we have $R(S(s)) \subset D(G)$ and $S(s)G \subset GS(s)$ for all $s \in [0,\tau)$. Also, by Lemma 3.1, (3.3) holds. Hence the hypothesis and Proposition 3.4 (b) hold with B replaced by G, so that Proposition 3.4 (a) also holds with B replaced by G, that is, the first part of the above condition (c) is true. Then, the hypothesis and Proposition 3.4 (a) hold with B replaced by C, and consequently Proposition 3.4 (b) also holds with B replaced by C, that is, $S(t)C = CS(t)$ for all $0 \leq t < \tau$. Then Lemma 3.3 implies that $T(\cdot)$ is a local α-times integrated C-semigroup. Finally, applying (a) and Proposition 3.4 with B replaced by $S(s)$ for any $(0 \leq s < \tau)$ yields that Proposition 3.4 (b) also holds with B replaced by $S(s)$, that is, $S(t)S(s) = S(s)S(t)$ for all $0 \leq t < \tau$. Then, by differentiation with respect to s and t, we obtain the above condition (b).

To show the second part of (c), we first show that $C^{-1}GC$ is a subgenerator of $T(\cdot)$. Since G is a closed (C,α)-subgenerator of $T(\cdot)$ and $G \subset C^{-1}GC$, we have $T(t) - j_\alpha(t)C = GS(t) = C^{-1}GCS(t)$ for all $0 \leq t < \tau$. Moreover, if $x \in D(C^{-1}GC)$, then $Cx \in D(G)$ and $GCx \in R(C)$, so that, by (a),

$$\begin{aligned} C[T(t)x - j_\alpha(t)Cx] &= [T(t) - j_\alpha(t)C]Cx = S(t)GCx \\ &= S(t)CC^{-1}GCx = CS(t)C^{-1}GCx. \end{aligned} \tag{3.12}$$

It follows from the injectivity of C that $T(t)x - j_\alpha(t)Cx = S(t)C^{-1}GCx$ for all $0 \leq t < \tau$. Therefore, $C^{-1}GC$ is a subgenerator of $T(\cdot)$.

Let B be any subgenerator of $T(\cdot)$. It follows from (3.1) and (3.2) that for every $x \in D(B)$, $j_{\alpha+1}(t)Cx = S(t)x - (1 * S)(t)Bx \in D(G)$. This together with (3.2) and the closedness of G implies

$$\begin{aligned} GS(t)x - Gj_{\alpha+1}(t)Cx &= G(1 * S)(t)Bx = (1 * [T - j_\alpha C])(t)Bx \\ &= S(t)Bx - j_{\alpha+1}(t)CBx = BS(t)x - j_{\alpha+1}(t)CBx. \end{aligned} \tag{3.13}$$

Since $GS(t) = T(t) - j_\alpha(t)C = BS(t)$ by (3.2), we have $Gj_{\alpha+1}(t)Cx = j_{\alpha+1}(t)CBx$ for all $0 \leq t < \tau$. Since C is injective, this implies $Bx = C^{-1}GCx$, that is, $B \subset C^{-1}GC$. Hence $C^{-1}GC$ is the generator of $T(\cdot)$. □

The next corollary is about the converse of (c) of Theorem 3.5.

Corollary 3.6. *Let $C \in B(X)$ be injective, let G be a closed operator satisfying $G \subset C^{-1}GC$, and let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function. If $C^{-1}GC$ is a (C,α)-subgenerator of $T(\cdot)$, and if for every $0 \leq t < \tau$, there is a dense subspace D_t of X such that $S(t)D_t \subset D(G)$, then G is also a (C,α)-subgenerator of $T(\cdot)$. In particular, the conclusion holds when C has dense range.*

Proof. $C^{-1}GC$ and $T(\cdot)$ satisfy

$$T(t)x - j_\alpha(t)Cx = S(t)C^{-1}GCx \quad \text{for } x \in D(C^{-1}GC); \tag{3.14}$$

$$T(t)x - j_\alpha(t)Cx = C^{-1}GCS(t)x \quad \text{for } x \in X \tag{3.15}$$

for $0 \leq t < \tau$. Since $G \subset C^{-1}GC$, (3.14) implies that G satisfies (3.1). Equation (3.15) and the assumption $CG \subset GC$ imply that for every $x \in D_t$,

$$C[T(t) - j_\alpha(t)C]x = GCS(t)x = CGS(t)x. \tag{3.16}$$

Since C is injective, this implies $T(t)x - j_\alpha(t)Cx = GS(t)x$ for $x \in D_t$. It follows from $\overline{D_t} = X$ and the closedness of G that, for every $x \in X$, $S(t)x \in D(G)$, and $T(t)x - j_\alpha(t)Cx = GS(t)x$ for all $x \in X$, that is, G satisfies (3.2). Therefore G is a closed (C,α)-subgenerator of $T(\cdot)$.

Since (3.15) shows that $S(t)Cx = CS(t)x \in D(G)$ for all $x \in X$ and $0 \leq t < \tau$, we can take $D_t = R(C)$ if C has dense range. □

Corollary 3.7. *Let $C \in B(X)$ be injective and let $T, H : [0,\tau) \to B(X)$ be strongly continuous functions with closed (C,α)-subgenerators G and K, respectively. Suppose $KG \subset GK$ and $(1 * T)(t)D(K) \subset D(K)$ for all $0 \leq t < \tau$ and $K(1 * T)(\cdot)x \in C([0,\tau),X)$ for all $x \in D(K)$. Then $T(t)H(s) = H(s)T(t)$ for all $0 \leq s,t < \tau$.*

Proof. By Theorem 3.5, we have $CK \subset KC$, $CG \subset GC$, $CS(t) = S(t)C$, and $CH(t) = H(t)C$. Using these facts together with $KG \subset GK$, we obtain from Proposition 3.4 (by taking $B = K$) that $S(t)K \subset KS(t)$ for all $0 \leq t < \tau$. Fix a $t \geq 0$. Since $S(t)K \subset KS(t)$ and $CS(t) = S(t)C$, taking $B = S(t)$ in Proposition 3.4 we deduce that $H(s)S(t) = S(t)H(s)$ for all $0 \leq s < \tau$. This completes the proof. □

4. Generators of nondegenerate local α-times integrated C-semigroups

The results discussed so far are formulated under the assumption of existence of a (C,α)-subgenerator of a strongly continuous local α-times integrated C-semigroup $T(\cdot)$. In this section, we will see that subgenerators and generator do exist if $T(\cdot)$ is a nondegenerate local α-times integrated C-semigroup.

Lemma 4.1. *Let $T(\cdot)$ be a local α-times integrated C-semigroup on $[0,\tau)$. The following conditions have the implication relations (c)⇒(a)⇒(b):*

(a) $T(\cdot)$ *is nondegenerate;*
(b) C *is injective;*
(c) $u \in C([0,\tau/2),X)$ *and* $T * u \equiv 0$ *imply* $u \equiv 0$.

Moreover, when $T(\cdot)$ *has a subgenerator, these three conditions are equivalent.*

Proof. (a)⇒(b). If $Cx = 0$, then from (1.1) we see that $T(s)T(t)x = 0$ for all $0 < s,t < \tau/2$, which implies $x = 0$ by our definition of nondegeneracy. Hence C is injective.

(c)⇒(a). If $x \in X$ is such that $T(t)x = 0$ for all $0 < t < \tau/2$, then for $u \equiv x$ we have $(T * u)(t) = (1 * T)(t)x = 0$ for all $0 < t < \tau/2$. Thus, (a) follows from (c).

Next, suppose there is a subgenerator. We show "(b)⇒(c)." If $u \in C([0,\tau/2),X)$ satisfies $T * u \equiv 0$, then $S * u \equiv 1 * (T * u) \equiv 0$. It follows from (3.2) that

$$0 \equiv GS * u = T * u - j_\alpha C * u = -j_\alpha * Cu. \tag{4.1}$$

By Lemma 2.1(b), we have $Cu \equiv 0$. Since C is injective, this proves $u \equiv 0$. Therefore, (b) implies (c) when $T(\cdot)$ has a subgenerator. □

Lemma 4.2. *Let* $C \in B(X)$ *be injective and* $\{T(t);\ 0 \le t < \tau\}$ *be a local* α*-times integrated* C*-semigroup. If* $x \in X$ *is such that* $T(r)x = 0$ *for all* $0 < r \le s$ *for some number* $s \in (0,\tau)$*, then* $T(r)x = 0$ *for all* $0 < r < \tau$*. In particular, if* $T(\cdot)$ *is nondegenerate, then* $T(r)x = 0$ *for all* $0 < r \le s$ *with some number* $0 < s < \tau$ *implies* $x = 0$.

Proof. For an arbitrary $0 \le t < \tau$, choose an $s_0 \in (0,\min\{s,\tau - t\})$. The assumption implies $T(s_0)x = 0$ and $(1 * T)(s_0)x = 0$. Then, it follows from Theorem 2.3 that

$$\begin{aligned} -j_\alpha(s_0)C(1 * T)(t)x &= (1 * T)(t)[T(s_0) - j_\alpha(s_0)C]x \\ &= [T(t) - j_\alpha(t)C](1 * T)(s_0)x = 0. \end{aligned} \tag{4.2}$$

Since C is injective, this implies that $(1 * T)(t)x = 0$ for all $0 \le t < \tau$, and hence $T(t)x = 0$ for all $0 \le t < \tau$. □

We are ready to show the existence of subgenerators and generator for a nondegenerate local α-times integrated C-semigroup.

Definition 4.3. Let $C \in B(X)$ and let $T(\cdot)$ be a nondegenerate local α-times integrated C-semigroup. We define for every $0 < t < \tau$ a linear operator $G_t : D(G_t) \to X$ by

$$\begin{aligned} D(G_t) &:= \left\{ \sum_{k=1}^{n} S(t_k)x_k;\ 0 \le t_k < t,\ x_k \in X,\ k = 1,2,\ldots,\ n = 1,2,\ldots \right\}, \\ G_t y &:= \sum_{k=1}^{n} [T(t_k) - j_\alpha(t_k)C]x_k \quad \text{for } y = \sum_{k=1}^{n} S(t_k)x_k \in D(G_t). \end{aligned} \tag{4.3}$$

Fix a $0 < t < \tau$. We see that G_t is well-defined. Indeed, if $\sum_{k=1}^{n} S(t_k)x_k = 0$, then, by Theorem 2.3, for every $0 \le r < \tau - t$

$$S(r)\sum_{k=1}^{n} [T(t_k) - j_\alpha(t_k)C]x_k = \sum_{k=1}^{n} [T(r) - j_\alpha(r)C]S(t_k)x_k = 0. \tag{4.4}$$

Since $T(\cdot)$ is nondegenerate, it follows from Lemma 4.2 that $\sum_{k=1}^{n}[T(t_k)-j_\alpha(t_k)C]x_k = 0$. This proves that G_t is well-defined. These operators G_t form an increasing net. Let us define $G_\tau : D(G_\tau) \to X$ by

$$D(G_\tau) := \bigcup_{0<t<\tau} D(G_t),$$
$$G_\tau x := G_t x \quad \text{if } x \in D(G_t) \text{ for some } 0<t<\tau. \tag{4.5}$$

Proposition 4.4. *Let $T(\cdot)$ be a nondegenerate local α-times integrated C-semigroup on X, and let operators G_t, G_τ be defined as above.*

(i) *For $0 \le s < t < \tau$, we have*

$$S(s)X \subset D(G_t), \qquad S(s)G_t \subset G_t S(s) = T(s) - j_\alpha(s)C. \tag{4.6}$$

(ii) *G_τ is a subgenerator of $T(\cdot)$, that is,*

$$S(s)X \subset D(G_\tau), \quad S(s)G_\tau \subset G_\tau S(s) = T(s) - j_\alpha(s)C \quad \forall 0 \le s < \tau. \tag{4.7}$$

Proof. (i) Since $s<t$, by the definition of G_t, we have $S(s)x \in D(G_t)$ and $G_t S(s)x = [T(s) - j_\alpha(s)C]x$ for all $x \in X$. To show $S(s)G_t \subset G_t S(s) = T(s) - j_\alpha(s)C$, let $0 \le r < \tau - t$. Then, (1.1) implies that $S(r)$ commutes with $T(u)$ and $S(u)$ for $0 \le u \le t$. If $y \in D(G_t)$, then $y = \sum_{k=1}^{n} S(t_k)x_k$ for some $t_k \in [0,t)$, $x_k \in X$, $k = 1,\dots,n$. By Theorem 2.3, we have

$$\begin{aligned} S(r)S(s)G_t y &= S(s)S(r)\sum_{k=1}^{n}[T(t_k) - j_\alpha(t_k)C]x_k \\ &= S(s)[T(r) - j_\alpha(r)C]\sum_{k=1}^{n} S(t_k)x_k = S(s)[T(r) - j_\alpha r)C]y \\ &= [T(s) - j_\alpha(s)C]S(r)y = S(r)[T(s) - j_\alpha(s)C]y. \end{aligned} \tag{4.8}$$

This being true for all $r \in [0, \tau - t)$, it follows from Lemma 4.2 that $S(s)G_t y = [T(s) - j_\alpha(s)C]y$.

(ii) follows easily from (i) and the definition of G_τ. □

Lemma 4.5. *Suppose G and B are subgenerators of $T(\cdot)$. Define a linear operator $K : D(G) + D(B) \to X$ by $Ky := Gx_1 + Bx_2$ whenever $y = x_1 + x_2$ for some $x_1 \in D(G)$ and $x_2 \in D(B)$. Then, K is well-defined and it is also a subgenerator of $T(\cdot)$.*

Proof. Suppose G and B are two subgenerators of $T(\cdot)$. If $y = x_1 + x_2 = z_1 + z_2$ for some $x_1, z_1 \in D(G)$ and $x_2, z_2 \in D(B)$, then (3.1) implies

$$\begin{aligned} S(t)(Gx_1 + Bx_2) &= [T(t) - j_\alpha(t)C](x_1 + x_2) \\ &= [T(t) - j_\alpha(t)C](z_1 + z_2) = S(t)(Gz_1 + Bz_2) \end{aligned} \tag{4.9}$$

and hence $T(t)(Gx_1 + Bx_2) = T(t)(Gz_1 + Bz_2)$ for every $0 \le t < \tau$. Since $T(\cdot)$ is nondegenerate, $Gx_1 + Bx_2 = Gz_1 + Bz_2$. Therefore, K is a well-defined linear operator which satisfies (3.1). Clearly, K contains both G and B. Hence

$$T(t) - j_\alpha(t)C = GS(t) = KS(t) \quad \text{for } 0 \le t < \tau, \tag{4.10}$$

that is, (3.2) holds for K. □

Proposition 4.6. *Let $T(\cdot)$ be a local α-times integrated C-semigroup.*

(i) *If $T(\cdot)$ has a subgenerator, then $T(\cdot)$ has a maximal subgenerator which contains all subgenerators of $T(\cdot)$; it is called the generator of $T(\cdot)$.*

(ii) *If $T(\cdot)$ is nondegenerate, then $T(\cdot)$ has a generator.*

(iii) *Suppose $T(\cdot)$ is nondegenerate. Any subgenerator G is closable and its closure $\overline{G}$ is also a subgenerator of $T(\cdot)$, and $A := C^{-1}\overline{G}C$ is the generator of $T(\cdot)$. In particular, the operator G_τ is closable and $A := C^{-1}\overline{G_\tau}C$ is the generator of $T(\cdot)$.*

Proof. (i) Suppose B is a subgenerator of $T(\cdot)$. Let $\mathcal{S}$ be the set of all subgenerators of $T(\cdot)$. Then, $B \in \mathcal{S}$. If $G \in \mathcal{S}$, the definition of subgenerator implies $S(t)X \subset D(G)$.

Let $\{G_i\}_{i\in I}$ be an arbitrary chain in $(\mathcal{S}, \subset)$. Define $G : \bigcup_{i\in I} D(G_i) \to X$ by $Gx := G_i x$ for $x \in G_i$ for some $i \in I$. It is clear that G is well-defined and $D(G) = \bigcup_{i\in I} G_i$. If $x \in D(G)$, say $x \in D(G_i)$ for an $i \in I$, then

$$S(t)Gx = S(t)G_i x = T(t)x - j_\alpha(t)Cx = G_i S(t)x = GS(t)x \quad \forall t \ge 0. \tag{4.11}$$

Therefore, G is a subgenerator of $T(\cdot)$ and so is an upper bound of the chain $\{G_i\}_{i\in I}$. By the Zorn's lemma, $\mathcal{S}$ has a maximal subgenerator, say G.

We claim that G contains all subgenerators. Suppose there were $B \in \mathcal{S}$ such that $D(B) \not\subset D(G)$. Then, the operator K as defined in Lemma 4.5 is a subgenerator which is a proper extension of G. This contradicts the maximality of G and so we must have $D(B) \subset D(G)$ for any subgenerator B of $T(\cdot)$.

(ii) follows from (i) and Proposition 4.4(ii).

(iii) Let $\{x_n\}$ be a sequence in $D(G)$ such that $x_n \to 0$ and $Gx_n \to y$ as $n \to \infty$ for some $y \in X$. It follows from (3.1) that for every $0 \le t < \tau$

$$S(t)y = \lim_{n\to\infty} S(t)Gx_n = \lim_{n\to\infty} \left[T(t) - j_\alpha(t)C\right]x_n = 0. \tag{4.12}$$

Since $T(\cdot)$ is nondegenerate, this implies $y = 0$. Therefore, G is closable. Finally, let $y \in D(\overline{G})$ and $0 \le t < \tau$. Then, there is a sequence $\{y_n\}$ in $D(G)$ such that $(y_n, Gy_n) \to (y, \overline{G}y)$ as $n \to \infty$. By (3.1), we have

$$S(t)\overline{G}y = \lim_{n\to\infty} S(t)Gy_n = \lim_{n\to\infty} \left[T(t) - j_\alpha(t)C\right]y_n = \left[T(t) - j_\alpha(t)C\right]y. \tag{4.13}$$

Since $\overline{G}$ is an extension of G, we also have that $\overline{G}S(t) = GS(t) = T(t) - j_\alpha(t)C$, that is, $\overline{G}$ is also a subgenerator of $T(\cdot)$. That $C^{-1}\overline{G}C$ is the generator follows from Theorem 3.5(c). □

Remark 4.7. It is seen from Proposition 4.6 (ii) and Theorem 3.5(c) that any nondegenerate local α-times integrated C-semigroup has a unique generator A, which is closed and satisfies $C^{-1}AC = A$, and that the generator A is precisely the operator defined by

$$x \in D(A), \quad Ax = y \Longleftrightarrow S(t)y = T(t)x - j_\alpha(t)Cx \quad \forall 0 \le t < \tau. \tag{4.14}$$

Example 4.8. If G is a (C,α)-subgenerator of a strongly continuous function $T(\cdot)$ and $C_1 \in B(X)$ is such that $CC_1 = C_1C$ and $C_1G \subset GC_1$, then G is (CC_1,α)-subgenerator of $C_1T(\cdot)$.

Example 4.9. Let $T_0 : C_b[0,\infty) \to C_b[0,\infty)$ be the translation semigroup. Then, $T_0(\cdot)$ is not a (C_0)-semigroup but $\{(j_\alpha * T_0)(t)\}_{t\ge 0}$ is an α-times integrated semigroup on $[0,\infty)$ for all $\alpha > 0$.

Example 4.10. Let $C \in B(X)$. $T(t) := j_\alpha(t)C$, $t \ge 0$, is an α-times integrated C-semigroup. It is easily seen from (3.1) and (3.2) that an operator $G \in B(X)$ is a subgenerator of $T(\cdot)$ if and only if $CG = GC = 0$. For example, for any 2×2 matrix H the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & H \end{smallmatrix}\right)$ is a maximal subgenerator of the α-times integrated $\left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$-semigroup $T(t) := \left(\begin{smallmatrix} 2j_\alpha(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

Example 4.11. More generally, let $T(\cdot)$ be a nondegenerate local α-times integrated C_X-semigroup on a Banach space X with generator G. If $Y \ne \{0\}$ is another Banach space and $C_Y \in B(Y)$, then

$$\widetilde{T}(\cdot) := \begin{pmatrix} T(\cdot) & 0 \\ 0 & j_\alpha(\cdot)C_Y \end{pmatrix} \tag{4.15}$$

is a local α-times integrated $\left(\begin{smallmatrix} C_X & 0 \\ 0 & C_Y \end{smallmatrix}\right)$-semigroup on $X \oplus Y$. $\widetilde{T}(\cdot)$ is nondegenerate if and only if C_Y is injective. If C_Y is not injective, then for any $H \in B(Y)$ which satisfies $C_YH = HC_Y = 0$, the operator $\left(\begin{smallmatrix} G & 0 \\ 0 & H \end{smallmatrix}\right)$ is a maximal subgenerator of $\widetilde{T}(\cdot)$. If C_Y is injective, then $\left(\begin{smallmatrix} G & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is the generator of $\widetilde{T}(\cdot)$.

Thus a degenerate local α-times integrated C-semigroup may have more than one maximal subgenerator, and hence has no generator. This is in contrast to the nondegenerate case (Proposition 4.6(ii)).

Example 4.12. Let $T(\cdot)$ be the family of operators on c_0 (or ℓ^1) defined by $T(t)x := ((n-k)e^{-n}\int_0^t j_{\alpha-1}(t-s)e^{ns}ds x_n)$, for $x = (x_n) \in c_0$ (or ℓ^1) and for $t \in [0,1]$. Let C denote the operator defined by $Cx := ((n-k)e^{-n}x_n)$. $T(\cdot)$ is a local α-times integrated C-semigroup which cannot be extended beyond 1. If $k = 0$, then C is injective and the generator of $T(\cdot)$ is the operator $G : (x_n) \to (nx_n)$. If $k = 1$, $T(\cdot)$ is a degenerate local α-times integrated C-semigroup and for each $a \in \mathbb{C}$ the operator G_a defined by $G_a(x) := (ax_1, 2x_2, 3x_3, \ldots)$ is a maximal subgenerator of $T(\cdot)$.

From Lemma 4.1, Proposition 4.4, and Theorem 3.5, we deduce the next corollary.

Corollary 4.13. *If $T(\cdot)$ is a nondegenerate local α-times integrated C-semigroup, then $T(s)T(t) = T(t)T(s)$ for all $0 \le s,t < \tau$.*

Remark 4.14. In the proof of Proposition 4.4 (i), we have used the commutativity: $T(s)T(t) = T(t)T(s)$ only for $0 \le s,t < \tau$ with $s+t < \tau$, as given by (1.1). Now, Corollary 4.13 shows that the restriction $s+t < \tau$ can be removed, and consequently, one can show that the relation in Proposition 4.4 (i) actually holds for all $s,t \in [0,\tau)$.

We can deduce the following characterization theorem for nondegenerate local α-times integrated C-semigroups.

Theorem 4.15. *Let $C \in B(X)$ and let $T(\cdot) : [0,\tau) \to B(X)$ be a strongly continuous function. Then, $T(\cdot)$ is a nondegenerate local α-times integrated C-semigroup if and only if C is injective and there is a closed (C,α)-subgenerator G (i.e., satisfying (3.1) and (3.2)) of $T(\cdot)$. In this case, G is a closed subgenerator and $A := C^{-1}GC$ is the generator of $T(\cdot)$.*

Proof. The necessity follows from Lemma 4.1 and Proposition 4.4; the sufficiency follows from Theorem 3.5(a) and Lemma 4.1. □

5. Relation with abstract Cauchy problems

Theorem 5.1. *Let $C \in B(X)$ be injective and $\alpha \geq 0$, and let A be a closed linear operator on X. Then, the following statements are equivalent*

(i) *A is a subgenerator of a local α-times integrated C-semigroup $T(\cdot)$.*

(ii) *$CA \subset AC$ (i.e., $Cx \in D(A)$ and $CAx = ACx$ for $x \in D(A)$) and the equation: $v(t) = A(1 * v)(t) + j_\alpha(t)Cx$, $0 \leq t < \tau$, has a unique solution v_x for every $x \in X$.*

(ii′) *$CA \subset AC$ and the equation: $u'(t) = Au(t) + j_\alpha(t)Cx$, $0 \leq t < \tau$; $u(0) = 0$, has a unique solution u_x for every $x \in X$.*

Moreover, the solutions are given by $v_x = T(\cdot)x$ and $u_x(t) = \int_0^t T(s)x\,ds$, $t \geq 0$.

Proof. (i)⇒(ii). Since $T(\cdot)$ is an α-times integrated C-semigroup with A as a subgenerator and C is injective, (3.1)–(3.3) hold. Thus (ii) can be deduced from (3.2), Lemmas 3.1 and 4.1, and Theorem 3.5(c).

(ii)⇒(i). We define the operator $T(t)$ by $T(t)x := v_x(t)$ for $x \in X$. Then, $T(\cdot)x$ is strongly continuous on $[0,\tau)$ for every $x \in X$. Since both A and C are linear, the uniqueness of solution implies that $T(t)$ is a linear operator on X for all $0 \leq t < \tau$.

Next, we show that $T(t)$ is a bounded operator for each $0 \leq t < \tau$. Let $C([0,\tau),X)$ be the Frechét space with the quasinorm $|\|v\|| := \sum_{k=1}^{\infty} \|v\|_k/(2^k(1+\|v\|_k))$ for $v \in C([0,\tau),X)$, where $\|v\|_k := \max_{t\in[0,p_k]} \|v(t)\|$, $k = 1,2,\ldots$, and $0 < p_k \nearrow \tau$. Consider the map $\eta : X \to C([0,\tau),X)$ defined by $\eta(x) := T(\cdot)x = v_x$. We show that η is a closed linear operator. Let $\{x_n\}$ be a sequence in X such that $(x_n,\eta(x_n)) \to (x,v(\cdot))$ strongly as $n \to \infty$ for some $x \in X$ and $v \in C([0,\tau),X)$. Since A is closed and $v_{x_n} = A(1 * v_{x_n}) + j_\alpha Cx_n$, we obtain that $v = A(1 * v) + j_\alpha Cx$. It follows from the uniqueness of solutions that $v = v_x = T(\cdot)x = \eta(x)$. Hence η is closed. It follows from the closed graph theorem that η is continuous. This shows that $T(\cdot)$ is a strongly continuous function of bounded linear operators on X and it satisfies (3.2).

If A is shown to be a (C,α)-subgenerator of $T(\cdot)$, then by Theorem 3.5(c) we conclude that $T(\cdot)$ is a local α-times integrated C-semigroup with subgenerator A. This will be done if we can show $S(t)Ax = AS(t)x$ for all $x \in D(A)$ and $0 \leq t < \tau$. Since A is closed, we obtain from (3.2) that $AS(\cdot)x \in C([0,\tau),X)$ for all $x \in X$. Since (ii) implies that condition (3.3) holds for $G = A$ and Proposition 3.4 (a) holds for $B = A$, applying Proposition 3.4 we obtain $S(t)A \subset AS(t)$ $(0 \leq t < \tau)$ as desired. Thus, A is a subgenerator of $T(\cdot)$.

Clearly, (ii) and (ii′) are equivalent. This completes the proof. □

Lemma 5.2. *Let $C \in B(X)$ be injective and $\alpha \geq 0$, and let A be a closed subgenerator of a local α-times integrated C-semigroup $S(\cdot)$ on X, and let $1 \leq k \leq [\alpha]+1$. Then, for every $x \in D(A^k)$, the problem ACP($A; j_{\alpha-k}Cx, \delta_{\alpha,[\alpha]}Cx$) has a unique solution, which is given by*

$$u_k(t) := S^{(k-1)}(t)x = S(t)A^{k-1}x + \sum_{j=0}^{k-2} j_{\alpha-1-j}(t)CA^{k-2-j}x, \quad 0 \leq t < \tau. \tag{5.1}$$

Proof. Let $X_k = D(A^k)$ be equipped with the norm $\|x\|_k$ by $\|x\|_k = \sum_{i=0}^k \|A^i x\|_k$ for $x \in X_k$, $k = 1,2,\ldots$. If $y \in D(A)$, then (3.1) and (3.2) imply that $S(\cdot)y \in C^1((0,\infty),X) \cap C([0,\infty),X_1)$ and

$$S'(t)y = S(t)Ay + j_{\alpha-1}(t)Cy, \quad 0 \leq t < \tau. \tag{5.2}$$

If $x \in D(A^k)$, then $x, Ax, A^2x,\ldots,A^{k-1}x \in D(A)$, so that by applying (5.2) repeatedly, we obtain that $S(\cdot)x \in C^k((0,\tau),X) \cap C([0,\tau),X_k)$ (where $X_k = D(A^k)$ with $\|x\|_k = \sum_{i=0}^k \|A^i x\|$ for $x \in X_k$) and

$$S^{(k)}(t)x = S(t)A^k x + \sum_{j=0}^{k-1} j_{\alpha-1-j}(t)CA^{k-1-j}x, \quad 0 \leq t < \tau. \tag{5.3}$$

Let $u_k(t)$ be defined as in (5.1). Then, $u_k(0) = \delta_{\alpha,k-1}Cx$ and

$$\begin{aligned} u_k'(t) = S^{(k)}(t)x &= A\left(S(t)A^{k-1}x + \sum_{j=0}^{k-2} j_{\alpha-1-j}(t)CA^{k-2-j}x\right) + j_{\alpha-k}(t)Cx \\ &= Au_k(t) + j_{\alpha-k}(t)Cx. \end{aligned} \tag{5.4}$$

This shows that u_k is a solution of ACP($A; j_{\alpha-k}Cx, \delta_{\alpha,[\alpha]}Cx$), or equivalently, $v_k = u_k'$ is a solution of $v = A(1 * v) + j_{\alpha-k}Cx$. The uniqueness of solution follows from Lemma 3.1. $\square$

6. Extension of local α-times integrated C-semigroups

Let $T(\cdot)$ be a local α-times integrated C-semigroup on $[0,\tau)$ with generator A, and let n be an integer greater than or equal to α. We will show that A also generates a local $(\alpha+n)$-times integrated C^2-semigroup on $[0,2\tau)$. Let $H(t) := (j_{n-\alpha-1} * T)(t)$, $\tau > t \geq 0$. Then, $H(\cdot)$ is an n-times integrated C-semigroup. Fix any $\tau_0 \in (0,\tau)$. Define an operator-valued function $S_{\tau_0} : [0,2\tau_0) \to B(X)$ by

$$S_{\tau_0}(t) := \begin{cases} (j_{n-1} * T)(t)C & \text{for } 0 \leq t \leq \tau_0, \\ T(\tau_0)H(t-\tau_0) + \sum j_{\alpha-k-1}(\tau_0)(j_k * H)(t-\tau_0)C & \\ \quad + \sum_{k=0}^{n-1} j_{n-k-1}(t-\tau_0)(j_k * T)(\tau_0)C & \text{for } \tau_0 \leq t < 2\tau_0, \end{cases} \tag{6.1}$$

where the k in the first summation runs over those nonnegative integers such that $k-\alpha$ is not a nonnegative integer, that is, k runs from 0 to $\alpha-1$ when α is an integer and runs over all nonnegative integers when α is not an integer.

Clearly, $S_{\tau_0}(\cdot)$ is a local $(\alpha+n)$-times integrated C^2-semigroup on $[0,\tau_0]$ with generator A. It is easy to see for every $x\in X$ that

$$\lim_{t\to\tau_0^+} S_{\tau_0}(t)x = (j_{n-1} * T)(\tau_0)Cx = S_{\tau_0}(\tau_0)x. \tag{6.2}$$

Therefore, $S_{\tau_0}(\cdot)$ is strongly continuous on $[0,2\tau_0)$. Since A is the generator of $T(\cdot)$, we see that A and $S_{\tau_0}(\cdot)$ commute.

Theorem 6.1. *Let $T(\cdot)$ be a local α-times integrated C-semigroup on $[0,\tau)$ with generator A. For any $\tau_0\in(0,\tau)$, the function $S_{\tau_0}(\cdot)$, defined in (6.1), is a local $\alpha+n$-times integrated C^2-semigroup on $[0,2\tau_0)$ with generator A. Thus the function $S(\cdot):[0,2\tau)\to B(X)$, defined by $S(t):=S_{\tau_0}(t)$ for $0\le t<2\tau_0<2\tau$, is a local $(\alpha+n)$-times integrated C^2-semigroup on $[0,2\tau)$ with generator A.*

Proof. Since $S_{\tau_0}(\cdot)$ is a local $(\alpha+n)$-times integrated C^2-semigroup on $[0,\tau_0]$ with generator A, by Theorem 4.15 we need only to show that A and $S_{\tau_0}(\cdot)$ satisfy

$$R((1 * S_{\tau_0})(t)) \subset D(A), \qquad A(1 * S_{\tau_0})(t) = S_{\tau_0}(t)x - j_{\alpha+n}(t)Cx \tag{6.3}$$

for $x\in X$ and $\tau_0\le t<2\tau_0$.

We need the following equations which follow from (4.14):

$$\begin{gathered} A(j_{k+1} * H)(t) = [(j_k * H)(t) - j_{n+k+1}(t)C], \\ A(j_k * T)(t) = (j_{k-1} * T)(t) - j_{k+\alpha}(t)C \quad \text{for } k=-1,0,1,2,\dots. \end{gathered} \tag{6.4}$$

From the Taylor expansion, we have the next identity:

$$\begin{aligned} j_{\alpha+n}(t+\tau) &= \frac{\tau^{\alpha+n}}{\Gamma(\alpha+n+1)}\sum_{k=0}^{\infty}\binom{\alpha+n}{k}\left(\frac{t}{\tau}\right)^k = \sum_{k=0}^{\infty} j_k(t)j_{\alpha+n-k}(\tau) \\ &= j_{\alpha+n}(\tau) + \left(\sum_{k=n+1}^{\infty} + \sum_{k=1}^{n}\right) j_k(t)j_{\alpha+n-k}(\tau) \\ &= j_{\alpha+n}(\tau) + \sum_{k=0}^{\infty} j_{\alpha-k-1}(\tau)j_{n+k+1}(t) + \sum_{k=0}^{n-1} j_{n-k}(t)j_{\alpha+k}(\tau) \end{aligned} \tag{6.5}$$

for $0\le t<\tau$. Note that when α is an integer, all those terms with $k>\alpha-1$ in the first summation vanish.

It is easy to see that $(1 * S_{\tau_0})(t) = (j_n * T)(t)C$ for $0 \le t \le \tau_0$, and

$$\begin{aligned}(1 * S_{\tau_0})(t) &= (1 * S_{\tau_0})(\tau_0) + \int_0^{t-\tau_0} S_{\tau_0}(r+\tau_0)\,dr\\ &= (j_n * T)(\tau_0)C + T(\tau_0)(1 * H)(t-\tau_0)\\ &\quad + \sum j_{\alpha-k-1}(\tau_0)(j_{k+1} * H)(t-\tau_0)C\\ &\quad + \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0)(j_k * T)(\tau_0)C\end{aligned} \tag{6.6}$$

for $\tau_0 \le t < 2\tau_0$. Then, using (6.4)-(6.5), we have for every $\tau_0 \le t < 2\tau_0$,

$$\begin{aligned}A(1 * S_{\tau_0})(t) &= A(j_n * T)(\tau_0)C + T(\tau_0)A(1 * H)(t-\tau_0)\\ &\quad + \sum j_{\alpha-k-1}(\tau_0)A(j_{k+1} * H)(t-\tau_0)C + \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0)A(j_k * T)(\tau_0)C\\ &= (j_{n-1} * T)(\tau_0)C - j_{\alpha+n}(\tau_0)C^2 + T(\tau_0)[H(t-\tau_0) - j_n(t-\tau_0)C]\\ &\quad + \sum j_{\alpha-k-1}(\tau_0)[(j_k * H)(t-\tau_0)C - j_{n+k+1}(t-\tau_0)C^2]\\ &\quad + \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0)[(j_{k-1} * T)(\tau_0)C - j_{\alpha+k}(\tau_0)C^2]\\ &= T(\tau_0)H(t-\tau_0) + \sum j_{\alpha-k-1}(\tau_0)(j_k * H)(t-\tau_0)C\\ &\quad + \sum_{k=0}^{n-1} j_{n-k-1}(t-\tau_0)(j_k * T)(\tau_0)C\\ &\quad - \left[j_{\alpha+n}(\tau_0) + \sum j_{\alpha-k-1}(\tau_0)j_{n+k+1}(t-\tau_0) + \sum_{k=0}^{n-1} j_{n-k}(t-\tau_0)j_{\alpha+k}(\tau_0)\right]C^2\\ &= S_{\tau_0}(t) - j_{\alpha+n}(t)C^2.\end{aligned} \tag{6.7}$$

Since $S_{\tau_0}(\cdot)$ is a local $\alpha+n$-times integrated C^2-semigroup on $[0,\tau_0]$ generated by $C^2AC^{-2} = A$, (6.7) implies that $S_{\tau_0}(\cdot)$ is a local $(\alpha+n)$-times integrated C^2-semigroup on $[0,2\tau_0)$ with generator A, by Theorem 4.15. □

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