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Research Article

A Perron-Frobenius Theorem for Positive Quasipolynomial Matrices Associated with Homogeneous Difference Equations

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We extend the classical Perron-Frobenius theorem for positive quasipolynomial matrices associated with homogeneous difference equations. Finally, the result obtained is applied to derive necessary and sufficient conditions for the stability of positive system.

1. Introduction

Differential equations with dependence on the past state have various applications in physics, biology, economics, and so forth, and have also attracted many researchers as in the introduction and reference of [1–4]. The neutral differential difference equation is one of general classes. And one of important neutral differential difference equations is

$$\frac{d}{dt}[D(r,A)y_t] = f(t,y_t), \tag{1.1}$$

where $D(r,A): C([-h;0],\mathbb{R}^n)\to\mathbb{R}^n$ is linear continuous defined by

$$D(r,A)\phi = \phi(0) - \sum_{i=1}^{N} A_i\phi(-r_i), \quad \phi \in C([-h;0],\mathbb{R}^n); \tag{1.2}$$

here each A_i is an $n\times n$ matrix, each r_i is a constant satisfying $r_i > 0$ and $h = \max\{r_i : i \in \overline{N}\}$, $\overline{N} = \{1,2,\dots,N\}$, and $y_t : [-h;0]\to\mathbb{R}^n$ is defined by $y_t(s) = y(t+s)$, $s\in[-h;0]$. To get some properties of solution of (3.3), the stability of system (1.3) is required for the homogeneous difference equation of the form $D(r,A)y_t = 0$, $t \geq 0$; see [2]. This equation can be rewritten as follows:

$$y(t) - \sum_{i=1}^{N} A_i y(t - r_i) = 0. \tag{1.3}$$

We recall the definition in [2]: "*the operator* $D(r, A)$ *or the system (1.3) is called stable if the zero solution of (1.3) with* $y_0 \in C_D(r, A) = \{\phi \in C([-h, 0], \mathbb{R}^n) : D(r, A)\phi = 0\}$ *is uniformly asymptotically stable.*"

Associated with the system (1.3), we define the quasipolynomial

$$H(s) = I - \sum_{k=1}^{N} e^{-sr_k} A_k. \tag{1.4}$$

For $s \in \mathbb{C}$, if $\det H(s) = 0$, then s is called a characteristic root of the quasipolynomial matrix (1.4). Then, a nonzero vector $x \in \mathbb{C}^n$ satisfying $H(s)x = 0$ is called an eigenvector of $H(\cdot)$ corresponding to the characteristic root s. We set $\sigma(H(\cdot)) = \{\lambda \in \mathbb{C} : \det H(\lambda) = 0\}$, the spectral set of (1.4), and $a_H = \sup\{\Re\lambda : \lambda \in \sigma(H(\cdot))\}$, the spectral abscissa of (1.4) (noticing that we cannot replace sup by max). The following theorem is a well-known result in [2].

Theorem 1.1. *The system (1.3) is stable if and only if* $a_H < 0$.

Thus, studying the stability of the system (1.3) turns out considering the characteristics of quasipolynomial (1.4). It is well known that the principal tool for the analysis of the stability and robust stability of a positive system is the Perron-Frobenius theorem; see [5–8]. In this work, we give an extension of the classical Perron-Frobenius theorem to positive quasipolynomial matrices of the form (1.4). Then the result obtained is applied to derive necessary and sufficient conditions for stability of positive systems of the form (1.3). An outline of this paper is as follows. In the next section, we summarize some notations and recall the classical Perron-Frobenius theorem which will be used in the remainder. The main results will be addressed in Section 3, where we extend the classical Perron-Frobenius theorem to positive quasipolynomial matrices. Finally, we apply the obtained results to give necessary and sufficient conditions for the stability of the positive systems of the form (1.3).

2. Preliminaries

We first introduce some notations. Let n, l, q be positive integers, a matrix $P = [p_{ij}] \in \mathbb{R}^{l\times q}$ is said to be nonnegative ($P \geq 0$) if all its entries p_{ij} are nonnegative. It is said to be positive ($P > 0$) if all its entries p_{ij} are positive. For $P, Q \in \mathbb{R}^{l\times q}$, $P > Q$ means that $P - Q > 0$. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}^{l\times q}$. A similar notation will be used for vectors. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, then for any $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$, we defined $|x| \in \mathbb{R}^n_+$ and $|P| \in \mathbb{R}^{l\times q}_+$ by $|x| = (|x_i|)$, $|P| = [|p_{ij}|]$. For any matrix $A \in \mathbb{K}^{n\times n}$ the spectral radius and the spectral abscissa of A are defined by $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ and $\mu(A) = \max\{\Re\lambda : \lambda \in \sigma(A)\}$, respectively, where $\sigma(A)$ is the spectrum of A. We recall some useful results; see [9].

Theorem 2.1 (Perron-Frobenius). *Suppose that* $A \in \mathbb{R}_+^{n\times n}$. *Then,*

(i) $r(A)$ *is an eigenvalue of* A *and there is a nonnegative eigenvector* $x \geq 0$, $x \neq 0$ *such that* $Ax = r(A)x$;

(ii) *if* $\lambda \in \sigma(A)$ *and* $|\lambda| = r(A)$, *then the algebraic multiplicity of* λ *is not greater than the algebraic multiplicity of the eigenvalue* $r(A)$;

(iii) *given* $\alpha > 0$, *there exists a nonzero vector* $x \geq 0$ *such that* $Ax \geq \alpha x$ *if and only if* $r(A) \geq \alpha$;

(iv) $(tI - A)^{-1}$ *exists and is nonnegative if and only if* $t > r(A)$.

Theorem 2.2. *Let* $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{R}_+^{n\times n}$. *If* $|A| \leq B$, *then*

$$r(A) \leq r(|A|) \leq r(B). \tag{2.1}$$

3. Main results

In this section, we will extend to Perron-Frobenius for positive quasipolynomial matrices of the form (1.4).

Theorem 3.1. *if the quasipolynomial (1.4) is positive, that is,* A_i *is nonnegative matrix for all* $i \in \overline{N} = \{1,2,\dots,N\}$, *then* $a_H \in \sigma(H(\cdot))$.

Proof. Assume that $(\lambda_m)_{m\in\mathbb{N}}$ is a sequence in $\sigma(H(\cdot))$ such that $\lim \Re\lambda_m = a_H$. Then, for every $m \in \mathbb{N}$, there exists $x_m \in \mathbb{C}^n$ such that

$$x_m = \sum_{k=1}^{N} e^{-\lambda_m r_k} A_k x_m. \tag{3.1}$$

It follows that

$$|x_m| \leq \left(\sum_{k=1}^{N} e^{-\Re\lambda_m r_k} A_k\right) |x_m|. \tag{3.2}$$

By Theorem 2.1,

$$r\left(\sum_{k=1}^{N} e^{-\Re\lambda_m r_k} A_k\right) \geq 1. \tag{3.3}$$

Now let us define the function $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(t) = r\left(\sum_{k=1}^{N} e^{-t r_k} A_k\right), \quad t \in \mathbb{R}. \tag{3.4}$$

It is clear that $f(\cdot)$ is continuous and strictly decreasing on $\mathbb{R}$ and $\lim_{t\to+\infty} r(t) = 0$. Moreover, from (3.3) we have $f(\Re\lambda_m) \geq 1$. Thus, from the continuity of $f(\cdot)$, there exists $\alpha_m \geq \Re\lambda_m$ satisfying $r(\sum_{k=1}^{N} e^{-\alpha_m r_k} A_k) = 1$ which implies $\alpha_m \in \sigma(H(\cdot))$.

Briefly, we have constructed a real sequence (α_m) that has the following properties: $\alpha_m \in \sigma(H(\cdot))$ and $\alpha_m \geq \Re\lambda_m$, for all $m \in \mathbb{N}$. This follows from $\lim \alpha_m = a_H$. Furthermore, $\sigma(H(\cdot))$ is closed. Thus, $a_H \in \sigma(H(\cdot))$. The proof is complete. $\square$

Theorem 3.2. *Let quasipolynomial (1.4) be positive. Then,*

(i) a_H *is a characteristic root of* $H(a_H)$ *and there is a nonnegative eigenvector* $x \geq 0$, $x \neq 0$ *such that* $H(a_H)x = 0$;

(ii) *given* $\alpha > 0$, *there exists a nonzero vector* $x \geq 0$ *such that* $(\sum_{k=1}^{N} e^{-\alpha r_k} A_k)x \geq x$ *if and only if* $a_H \geq \alpha$;

(iii) *for* $t \in \mathbb{R}$, *then one has*

$$H(t)^{-1} \geq 0 \Longleftrightarrow t > a_H. \tag{3.5}$$

Proof. (i) First, we recall that in the proof of Theorem 3.1, we have constructed a real sequence (α_n) that has the following properties: $\alpha_n \in \sigma(H(\cdot))$, for all $n \in \mathbb{N}$, $\lim \alpha_n = a_H$, and $r(\sum_{k=1}^{N} e^{-\alpha_n r_k} A_k) = 1$. Thus, this implies $r(\sum_{k=1}^{N} e^{-a_H r_k} A_k) = 1$. By Theorem 2.1, there exists a nonnegative eigenvector $x \geq 0$, $x \neq 0$ such that $(\sum_{k=1}^{N} e^{-a_H r_k} A_k)x = x$, or $[I - (\sum_{k=1}^{N} e^{-a_H r_k} A_k)]x = 0$, then we get (i).

(ii) Assume that there exists a nonzero vector $x \geq 0$ such that $(\sum_{k=1}^{N} e^{-\alpha r_k} A_k)x \geq x$. It follows that $f(\alpha) = r(\sum_{k=1}^{N} e^{-\alpha r_k} A_k) \geq 1$. By similar argument in the proof of Theorem 3.1, there exists $\overline{\alpha} \geq \alpha$ such that $f(\overline{\alpha}) = r(\sum_{k=1}^{N} e^{-\overline{\alpha} r_k} A_k) = 1$. Thus, by definition of a_H, $a_H \geq \overline{\alpha} \geq \alpha$. Conversely, if $a_H \geq \alpha$, then $r(\sum_{k=1}^{N} e^{-\alpha r_k} A_k) = f(\alpha) =\geq f(a_H) = 1$. Therefore, applying Theorem 2.1 for positive matrix $\sum_{k=1}^{N} e^{-\alpha r_k} A_k$, we get (ii).

(iii) For $t > a_H$, and the decrease of $f(\cdot)$, $f(t) < f(a_H) = 1$. Again applying Theorem 2.1(iii), we obtain $H(t)^{-1} \geq 0$. Conversely, assume that $t \in \mathbb{R}$ and $H(t)^{-1} \geq 0$. By Theorem 2.1(iii), $f(a_H) = 1 > r(\sum_{k=1}^{N} e^{-t r_k} A_k) = f(t)$. Thus, $t > a_H$. The proof is complete. □

Remark 3.3. From Theorem 3.2, it is easy to see that under the positivity assumption, the spectral abscissa, a_H, is continuous with respect to the delay parameters. This is not the case if the positivity assumption is dropped; see [1, 3].

4. Application to positive homogeneous difference equation

We now apply the results obtained in the previous section to derive some necessary and sufficient conditions for the stability of the positive homogeneous difference equation of the form (1.3). The following results are obtained by applying directly Theorems 3.1 and 3.2.

Theorem 4.1. *Let quasipolynomial (1.4) be positive. The homogeneous difference equation (1.3) is stable if and only if all characteristic roots of* $H(\cdot)$ *lie inside a half plane* $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$.

Theorem 4.2. *Let quasipolynomial (1.4) be positive. The homogeneous difference equation (1.3) is stable if and only if* $r(\sum_{i=1}^{N} A_i) < 1$.

Proof. Assume that (1.3) is stable and positive. By (iii) in Theorem 3.2, from $a_H < 0$, we obtain $H^{-1}(0) = (I - \sum_{i=1}^{N} A_i) \geq 0$. Applying Theorem 2.1, $r(\sum_{i=1}^{N} A_i) < 1$.

Inversely, if $r(\sum_{i=1}^{N} A_i) < 1$, $H^{-1}(0) = (I - \sum_{i=1}^{N} A_i) \geq 0$. By (iii) in Theorem 3.2, $a_H < 0$. This completes the proof. □

Remark 4.3. (i) The stability of positive systems do not depend on delay parameters.

(ii) It is clear that in the positive system, applying Theorem 4.2 to check the existing stability for delay-difference equations is much more simpler and easier than the general results in [2–4].

Next, we address some remarks about the relationship between independent-delay stability and stability of the system (1.3).

Definition 4.4. The system (1.3) is stable globally in the delays if it is stable for each $(r_i)_{i\in\overline{N}} \in \mathbb{R}_+^N$.

The concept of global stability, which is sometimes called independent-delay stability, has interested many researchers as in [2, 4, 10] and references therein. In fact, if the system is globally stable, then it is stable, but the conversion is not true. However, in the case of positive system, both concepts are the same. This is stated in implications following corollary of Theorem 4.2.

Corollary 4.5. *Let A_i be all nonnegative matrices for all $i \in \overline{N}$. Then, the following statements are equivalent:*

(i) *the system (1.3) is stable;*
(ii) *equation (1.3) is independent-delay stable;*
(iii) $r(A_1 + \cdots + A_N) < 1$.

Now we consider some simple examples to illustrate the results obtained.

Example 4.6. Consider the system

$$y(t) = A_1 y(t-r) + A_2 y(t-s), \tag{4.1}$$

where

$$A_1 = \begin{pmatrix} 0.121 & 0.231 \\ 0.431 & 0.386 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.236 & 0.521 \\ 0.267 & 0.431 \end{pmatrix}. \tag{4.2}$$

Since $r(A_1 + A_2) = 1.347128936 > 1$, by Corollary 4.5, the system (4.1) is neither stable nor independent-delay stable.

Example 4.7. Consider the following system:

$$y(t) = A_1 y(t-r) + A_2 y(t-s) + A_3 y(t-h), \tag{4.3}$$

where

$$A_1 = \begin{pmatrix} 0.121 & 0.231 \\ 0.131 & 0.116 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.231 & 0.221 \\ 0.127 & 0.331 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0.131 & 0.126 \\ 0.112 & 0.132 \end{pmatrix}. \tag{4.4}$$

Since $r(A_1 + A_2) = .9959344040 < 1$, by Corollary 4.5, the system (4.3) is both stable and independent-delay stable.

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