Ark. Mat., **54** (2016), 277–297
DOI: 10.1007/s11512-016-0236-4

Euler sequence and Koszul complex of a module

Björn Andreas, Darío Sánchez Gómez and Fernando Sancho de Salas

Abstract. We construct relative and global Euler sequences of a module. We apply it to prove some acyclicity results of the Koszul complex of a module and to compute the cohomology of the sheaves of (relative and absolute) differential p-forms of a projective bundle. In particular we generalize Bott's formula for the projective space to a projective bundle over a scheme of characteristic zero.

Introduction

This paper deals with two related questions: the acyclicity of the Koszul complex of a module and the cohomology of the sheaves of (relative and absolute) differential p-forms of a projective bundle over a scheme.

Let M be a module over a commutative ring A. One has the Koszul complex $\operatorname{Kos}(M)=\Lambda^{\cdot} M\otimes_A S^{\cdot} M$, where $\Lambda^{\cdot} M$ and $S^{\cdot} M$ stand for the exterior and symmetric algebras of M. It is a graded complex $\operatorname{Kos}(M)=\bigoplus_{n\geq 0}\operatorname{Kos}(M)_n$, whose n-th graded component $\operatorname{Kos}(M)_n$ is the complex:

$$0\longrightarrow \Lambda^n M\longrightarrow \Lambda^{n-1}M\otimes M\longrightarrow \Lambda^{n-2}M\otimes S^2M\longrightarrow \dots\longrightarrow S^nM\longrightarrow 0$$

It has been known for many years that $\operatorname{Kos}(M)_n$ is acyclic for $n>0$, provided that M is a flat A-module or n is invertible in A (see [3] or [10]). It was conjectured in [11] that $\operatorname{Kos}(M)$ is always acyclic. A counterexample in characteristic 2 was given in [5], but it is also proved there that $H_\mu(\operatorname{Kos}(M)_\mu)=0$ for any M, where μ is the minimal number of generators of M. Leaving aside the case of characteristic 2 (whose pathology is clear for the exterior algebra), we prove two new evidences for the validity of the conjecture (for A Noetherian): firstly, we prove (Theorem 1.6) that, for any finitely generated M, $\operatorname{Kos}(M)_n$ is acyclic for $n\gg 0$; secondly, we prove

This work was supported by the SFB 647 'Space-Time-Matter:Arithmetic and Geometric Structures' of the DFG (German Research Foundation) and by the Spanish grants MTM2013-45935-P (MINECO) and FS/12-2014 (Samuel Solórzano Barruso Foundation).

(Theorem 1.7) that if I is an ideal locally generated by a regular sequence, then $\mathrm{Kos}(I)_n$ is acyclic for any $n>0$. These two results are a consequence of relating the Koszul complex $\mathrm{Kos}(M)$ with the geometry of the space $\mathbb{P}=\mathrm{Proj}\, S^{\cdot}M$, as follows.

First of all, we shall reformulate the Koszul complex in terms of differential forms of $S^{\cdot}M$ over A: the canonical isomorphism $\Omega_{S^{\cdot}M/A}=M\otimes_A S^{\cdot}M$ allows us to interpret the Koszul complex $\mathrm{Kos}(M)$ as the complex of differential forms $\Omega^{\cdot}_{S^{\cdot}M/A}$ whose differential, $i_D\colon \Omega^p_{S^{\cdot}M/A}\to\Omega^{p-1}_{S^{\cdot}M/A}$, is the inner product with the A-derivation $D\colon S^{\cdot}M\to S^{\cdot}M$ consisting in multiplication by n on S^nM. By homogeneous localization, one obtains a complex of $\mathcal{O}_{\mathbb{P}}$-modules $\widetilde{\mathrm{Kos}}(M)$ on $\mathbb{P}$. Our first result (Theorem 1.4) is that the complex $\widetilde{\mathrm{Kos}}(M)$ is acyclic with factors (cycles or boundaries) the sheaves $\Omega^p_{\mathbb{P}/A}$. Moreover, one has a natural morphism

$$\mathrm{Kos}(M)_n\longrightarrow\pi_*[\widetilde{\mathrm{Kos}}(M)\otimes\mathcal{O}_{\mathbb{P}}(n)]$$

with $\pi\colon\mathbb{P}\to\mathrm{Spec}\, A$ the canonical morphism. In Theorem 1.5 we give (cohomological) sufficient conditions for the acyclicity of the complexes $\pi_*[\widetilde{\mathrm{Kos}}(M)\otimes\mathcal{O}_{\mathbb{P}}(n)]$ and $\mathrm{Kos}(M)_n$. These conditions, under Noetherian hypothesis, are satisfied for $n\gg0$, thus obtaining Theorem 1.6. The acyclicity of the Koszul complex of a locally regular ideal follows then from Theorem 1.5 and the theorem of formal functions.

The advantage of expressing the Koszul complex $\mathrm{Kos}(M)$ as $(\Omega^{\cdot}_{S^{\cdot}M/A},i_D)$ is two-fold. Firstly, it makes clear its relationship with the De Rham complex $(\Omega^{\cdot}_{S^{\cdot}M/A},\mathrm{d})$: The Koszul and De Rham differentials are related by Cartan's formula: $i_D\circ\mathrm{d}+\mathrm{d}\circ i_D=$ multiplication by n on $\mathrm{Kos}(M)_n$. This yields a splitting result (Proposition 1.10 or Corollary 1.11) which will be essential for some cohomological results in Section 3 as we shall explain later on. Secondly, it allows a natural generalization (which is the subject of Section 2): If A is a k-algebra, we define the complex $\mathrm{Kos}(M/k)$ as the complex of differential forms (over k), $\Omega^{\cdot}_{S^{\cdot}M/k}$ whose differential is the inner product with the same D as before. Again, one has that $\mathrm{Kos}(M/k)=\bigoplus_{n\geq0}\mathrm{Kos}(M/k)_n$ and it induces, by homogeneous localization, a complex $\widetilde{\mathrm{Kos}}(M/k)$ of modules on $\mathbb{P}$ which is also acyclic and whose factors are the sheaves $\Omega^p_{\mathbb{P}/k}$ (Theorem 2.1). We can reproduce the aforementioned results about the complexes $\mathrm{Kos}(M)_n$, $\widetilde{\mathrm{Kos}}(M)$, for the complexes $\mathrm{Kos}(M/k)_n$, $\widetilde{\mathrm{Kos}}(M/k)$.

Section 3 deals with the second subject of the paper: let $\mathcal{E}$ be a locally free module of rank $r+1$ on a k-scheme X and let $\pi\colon\mathbb{P}\to X$ be the associated projective bundle, i.e., $\mathbb{P}=\mathrm{Proj}\, S^{\cdot}\mathcal{E}$. There are well known results about the (global and relative) cohomology of the sheaves $\Omega^p_{\mathbb{P}/X}(n)$ and $\Omega^p_{\mathbb{P}/k}(n)$ (we are using the standard abbreviated notation $\mathcal{N}(n)=\mathcal{N}\otimes\mathcal{O}_{\mathbb{P}}(n)$) due to Deligne, Verdier and Berthelot-Illusie ([4], [12], [1]) and about the cohomology of the sheaves $\Omega^p_{\mathbb{P}_r}(n)$ of the ordinary projective space due to Bott (the so called Bott's formula, [2]). We shall not use their results; instead, we reprove them and we obtain some new results, overall when X is a $\mathbb{Q}$-scheme. Let us be more precise:

In Theorem 3.4 we compute the relative cohomology sheaves $R^i\pi_*\Omega^p_{\mathbb{P}/X}(n)$, obtaining Deligne's result (see [4] and also [12]) and a new (splitting) result, in the case of a $\mathbb{Q}$-scheme, concerning the sheaves $\pi_*\Omega^p_{\mathbb{P}/X}(n)$ and $R^r\pi_*\Omega^p_{\mathbb{P}/X}(-n)$ for $n>0$. We obtain Bott formula for the projective space as a consequence. In Theorem 3.11 we compute the relative cohomology sheaves $R^i\pi_*\Omega^p_{\mathbb{P}/k}(n)$, obtaining Verdier's results (see [12]) and improving them in two ways: first, we give a more explicit description of $\pi_*\Omega^p_{\mathbb{P}/k}(n)$ and of $R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)$ for $n>0$; secondly, we obtain a splitting result for these sheaves when X is a $\mathbb{Q}$-scheme (as in the relative case).

Regarding Bott's formula, we are able to generalize it for a projective bundle, computing the dimension of the cohomology vector spaces $H^q(\mathbb{P},\Omega^p_{\mathbb{P}/X}(n))$ and $H^q(\mathbb{P},\Omega^p_{\mathbb{P}/k}(n))$ when X is a proper k-scheme of characteristic zero (Corollaries 3.7 and 3.14).

It should be mentioned that these results make use of the complexes $\widetilde{\mathrm{Kos}}(\mathcal{E})$ (as Deligne and Verdier) and $\widetilde{\mathrm{Kos}}(\mathcal{E}/k)$. The complex $\widetilde{\mathrm{Kos}}(\mathcal{E})$ is essentially equivalent to the exact sequence

$$0\longrightarrow\Omega_{\mathbb{P}/X}\longrightarrow(\pi^*\mathcal{E})\otimes\mathcal{O}_{\mathbb{P}}(-1)\longrightarrow\mathcal{O}_{\mathbb{P}}\longrightarrow 0$$

which is usually called Euler sequence. The complex $\widetilde{\mathrm{Kos}}(\mathcal{E}/k)$ is equivalent to the exact sequence

$$0\longrightarrow\Omega_{\mathbb{P}/k}\longrightarrow\widetilde{\Omega}_{B/k}\longrightarrow\mathcal{O}_{\mathbb{P}}\longrightarrow 0$$

with $B=S^{\cdot}\mathcal{E}$, which we have called global Euler sequence. These sequences still hold for any A-module M (which we have called relative and global Euler sequences of M). The aforementioned results about the acyclicity of the Koszul complexes of a module obtained in Sections 1 and 2 are a consequence of this fact.

1. Relative Euler sequence of a module and Koszul complexes

Let $(X,\mathcal{O})$ be a scheme and let $\mathcal{M}$ be quasi-coherent $\mathcal{O}$-module. Let $\mathcal{B}=S^{\cdot}\mathcal{M}$ be the symmetric algebra of $\mathcal{M}$ (over $\mathcal{O}$), which is a graded $\mathcal{O}$-algebra: the homogeneous component of degree n is $\mathcal{B}_n=S^n\mathcal{M}$. The module $\Omega_{\mathcal{B}/\mathcal{O}}$ of Kähler differentials is a graded $\mathcal{B}$-module in a natural way: $\mathcal{B}\otimes_{\mathcal{O}}\mathcal{B}$ is a graded $\mathcal{O}$-algebra, with $(\mathcal{B}\otimes_{\mathcal{O}}\mathcal{B})_n=\underset{p+q=n}{\oplus}\mathcal{B}_p\otimes_{\mathcal{O}}\mathcal{B}_q$ and the natural morphism $\mathcal{B}\otimes_{\mathcal{O}}\mathcal{B}\to\mathcal{B}$ is a degree 0 homogeneous morphism of graded algebras. Hence, the kernel Δ is a homogeneous ideal and $\Delta/\Delta^2=\Omega_{\mathcal{B}/\mathcal{O}}$ is a graded $\mathcal{B}$-module. If $b_p,b_q\in\mathcal{B}$ are homogeneous of degree p,q, then $b_p\,\mathrm{d}\,b_q$ is an element of $\Omega_{\mathcal{B}/\mathcal{O}}$ of degree $p+q$. We shall denote by $\Omega^p_{\mathcal{B}/\mathcal{O}}$ the p-th exterior power of $\Omega_{\mathcal{B}/\mathcal{O}}$, that is $\Lambda^p_{\mathcal{B}}\Omega_{\mathcal{B}/\mathcal{O}}$, which is also a graded

$\mathcal{B}$-module in a natural way. For each $\mathcal{O}$-module $\mathcal{N}$, $\mathcal{N}\otimes_{\mathcal{O}}\mathcal{B}$ is a graded $\mathcal{B}$-module with gradation: $(\mathcal{N}\otimes_{\mathcal{O}}\mathcal{B})_n=\mathcal{N}\otimes_{\mathcal{O}}\mathcal{B}_n$. Then one has the following basic result:

Theorem 1.1. *The natural morphism of graded $\mathcal{B}$-modules*

$$\begin{aligned}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-1] &\longrightarrow \Omega_{\mathcal{B}/\mathcal{O}}\\ m\otimes b &\longmapsto b\,\mathrm{d}\,m\end{aligned}$$

is an isomorphism. Hence $\Omega^p_{\mathcal{B}/\mathcal{O}}\simeq\Lambda^p\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-p]$, where $\Lambda^i\mathcal{M}=\Lambda^i_{\mathcal{O}}\mathcal{M}$.

The natural morphism $\mathcal{M}\otimes_{\mathcal{O}}S^i\mathcal{M}\to S^{i+1}\mathcal{M}$ defines a degree zero homogeneous morphism of $\mathcal{B}$-modules $\Omega_{\mathcal{B}/\mathcal{O}}=\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-1]\to\mathcal{B}$ which induces a (degree zero) $\mathcal{O}$-derivation $D\colon\mathcal{B}\to\mathcal{B}$, such that $\Omega_{\mathcal{B}/\mathcal{O}}\to\mathcal{B}$ is the inner product with D. This derivation consists in multiplication by n in degree n. It induces homogeneous morphisms of degree zero:

$$i_D\colon \Omega^p_{\mathcal{B}/\mathcal{O}}\longrightarrow\Omega^{p-1}_{\mathcal{B}/\mathcal{O}}$$

and we obtain:

Definition 1.2. The Koszul complex, denoted by $\operatorname{Kos}(\mathcal{M})$, is the complex:

$$\text{(1)}\qquad \dots\longrightarrow\Omega^p_{\mathcal{B}/\mathcal{O}}\xrightarrow{i_D}\Omega^{p-1}_{\mathcal{B}/\mathcal{O}}\xrightarrow{i_D}\dots\xrightarrow{i_D}\Omega_{\mathcal{B}/\mathcal{O}}\xrightarrow{i_D}\mathcal{B}\longrightarrow 0$$

Via Theorem 1.1, this complex is

$$\dots\xrightarrow{i_D}\Lambda^p\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-p]\xrightarrow{i_D}\dots\xrightarrow{i_D}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-1]\xrightarrow{i_D}\mathcal{B}\longrightarrow 0$$

Taking the homogeneous components of degree $n\geq 0$, we obtain a complex of $\mathcal{O}$-modules, which we denote by $\operatorname{Kos}(\mathcal{M})_n$:

$$0\longrightarrow\Lambda^n\mathcal{M}\longrightarrow\Lambda^{n-1}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}\longrightarrow\dots\longrightarrow\mathcal{M}\otimes_{\mathcal{O}}S^{n-1}\mathcal{M}\longrightarrow S^n\mathcal{M}\longrightarrow 0$$

such that $\operatorname{Kos}(\mathcal{M})=\bigoplus_{n\geq 0}\operatorname{Kos}(\mathcal{M})_n$.

Now let $\mathbb{P}=\operatorname{Proj}\mathcal{B}$ and $\pi\colon\mathbb{P}\to X$ the natural morphism. We shall use the following standard notations: for each $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{N}$, we shall denote by $\mathcal{N}(n)$ the twisted sheaf $\mathcal{N}\otimes_{\mathcal{O}_{\mathbb{P}}}\mathcal{O}_{\mathbb{P}}(n)$ and for each graded $\mathcal{B}$-module N we shall denote by $\widetilde{N}$ the sheaf of $\mathcal{O}_{\mathbb{P}}$-modules obtained by homogeneous localization. We shall use without mention the following facts: homogeneous localization commutes with exterior powers and for any quasi-coherent module $\mathcal{L}$ on X one has $(\widetilde{\mathcal{L}\otimes_{\mathcal{O}}\mathcal{B}[r]})=(\pi^*\mathcal{L})(r)$.

Definition 1.3. Taking homogeneous localization on the Koszul complex (1), we obtain a complex of $\mathcal{O}_{\mathbb{P}}$-modules, which we denote by $\widetilde{\mathrm{Kos}}(\mathcal{M})$:

$$\ldots \longrightarrow \widetilde{\Omega}^{d}_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \widetilde{\Omega}^{d-1}_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \ldots \xrightarrow{i_D} \widetilde{\Omega}_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \mathcal{O}_{\mathbb{P}} \longrightarrow 0 \tag{2}$$

By Theorem 1.1, $\widetilde{\Omega}^d_{\mathcal{B}/\mathcal{O}}=(\pi^*\Lambda^d\mathcal{M})(-d)$, hence $\widetilde{\mathrm{Kos}}(\mathcal{M})$ can be written as

$$\ldots \xrightarrow{i_D} (\pi^*\Lambda^d\mathcal{M})(-d) \xrightarrow{i_D} \ldots \longrightarrow (\pi^*\mathcal{M})(-1) \xrightarrow{i_D} \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

Theorem 1.4. *The complex* $\widetilde{\mathrm{Kos}}(\mathcal{M})$ *is acyclic* (*that is, an exact sequence*). *Moreover,*

$$\Omega^p_{\mathbb{P}/X} = \mathrm{Ker}\big(\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \widetilde{\Omega}^{p-1}_{\mathcal{B}/\mathcal{O}}\big)$$

Hence one has exact sequences

$$0 \longrightarrow \Omega^p_{\mathbb{P}/X} \longrightarrow \widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}} \longrightarrow \Omega^{p-1}_{\mathbb{P}/X} \longrightarrow 0$$

and right and left resolutions of $\Omega^p_{\mathbb{P}/X}$:

$$0 \longrightarrow \Omega^p_{\mathbb{P}/X} \longrightarrow \widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}} \longrightarrow \widetilde{\Omega}^{p-1}_{\mathcal{B}/\mathcal{O}} \longrightarrow \ldots \longrightarrow \widetilde{\Omega}_{\mathcal{B}/\mathcal{O}} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$
$$\ldots \longrightarrow \widetilde{\Omega}^{r+1}_{\mathcal{B}/\mathcal{O}} \longrightarrow \widetilde{\Omega}^{r}_{\mathcal{B}/\mathcal{O}} \longrightarrow \ldots \longrightarrow \widetilde{\Omega}^{p+1}_{\mathcal{B}/\mathcal{O}} \longrightarrow \Omega^p_{\mathbb{P}/X} \longrightarrow 0$$

In particular, for $p=1$ *the exact sequence*

$$0 \longrightarrow \Omega_{\mathbb{P}/X} \longrightarrow \widetilde{\Omega}_{\mathcal{B}/\mathcal{O}} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0 \tag{3}$$

is called the (*relative*) *Euler sequence.*

Proof. The morphism $\widetilde{\Omega}_{\mathcal{B}/\mathcal{O}}\to\mathcal{O}_{\mathbb{P}}$ is surjective, since $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-1]\to\mathcal{B}$ is surjective in positive degree. Let K be the kernel. We obtain an exact sequence

$$0 \longrightarrow K \longrightarrow \widetilde{\Omega}_{\mathcal{B}/\mathcal{O}} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

Since $\mathcal{O}_{\mathbb{P}}$ is free, this sequence splits locally; then, it induces exact sequences

$$0 \longrightarrow \Lambda^p K \longrightarrow \widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}} \longrightarrow \Lambda^{p-1} K \longrightarrow 0$$

Joining these exact sequences one obtains the Koszul complex $\widetilde{\mathrm{Kos}}(\mathcal{M})$. This proves the acyclicity of $\widetilde{\mathrm{Kos}}(\mathcal{M})$. To conclude, it suffices to prove that $K=\Omega_{\mathbb{P}/X}$.

Let us first define a morphism $\Omega_{\mathbb{P}/X}\to\widetilde{\Omega}_{\mathcal{B}/\mathcal{O}}$. Assume for simplicity that $X=\operatorname{Spec} A$. For each $b\in\mathcal{B}$ of degree 1, let U_b the standard affine open subset of $\mathbb{P}$ defined by $U_b=\mathrm{Spec}(\mathcal{B}_{(b)})$, with $\mathcal{B}_{(b)}$ the 0-degree component of $\mathcal{B}_b$. The natural inclusion $\mathcal{B}_{(b)}\to\mathcal{B}_b$ induces a morphism $\Omega_{\mathcal{B}_{(b)}/A}\to\Omega_{\mathcal{B}_b/A}=(\Omega_{\mathcal{B}/A})_b$ which takes values in the 0-degree component, $(\Omega_{\mathcal{B}/A})_{(b)}$. Thus one has a morphism $\Omega_{\mathcal{B}_{(b)}/A}\to(\Omega_{\mathcal{B}/A})_{(b)}$, i.e. a morphism $\Gamma(U_b,\Omega_{\mathbb{P}/X})\to\Gamma(U_b,\widetilde{\Omega}_{\mathcal{B}/A})$. One checks that these morphisms glue to a morphism $f\colon\Omega_{\mathbb{P}/X}\to\widetilde{\Omega}_{\mathcal{B}/A}$. This morphism is injective, because the inclusion $\mathcal{B}_{(b)}\to\mathcal{B}_b$ has a retract, $c_n/b^k\mapsto c_n/b^n$, which induces a retract in the differentials. The composition $\Omega_{\mathbb{P}/X}\to\widetilde{\Omega}_{\mathcal{B}/A}\to\mathcal{O}_{\mathbb{P}}$ is null, as one checks in each U_b:

$$(i_D\circ f)(\mathrm{d}(\frac{c_k}{b^k}))=i_D\left(\frac{b^k\,\mathrm{d}\,c_k-c_k\,\mathrm{d}\,b^k}{b^{2k}}\right)=\frac{b^k i_D\,\mathrm{d}\,c_k-c_k i_D\,\mathrm{d}\,b^k}{b^{2k}}=0$$

because $i_D\,\mathrm{d}\,c_r=rc_r$ for any element c_r of degree r. Thus, we have that $\Omega_{\mathbb{P}/X}$ is contained in the kernel of $\widetilde{\Omega}_{\mathcal{B}/A}\to\mathcal{O}_{\mathbb{P}}$. To conclude, it is enough to see that the image of $\widetilde{\Omega}^2_{\mathcal{B}/A}\xrightarrow{i_D}\widetilde{\Omega}_{\mathcal{B}/A}$ is contained in $\Omega_{\mathbb{P}/X}$. Again, this is a computation in each U_b; one checks the equality

$$i_D\left(\frac{\mathrm{d}\,c_p\wedge\mathrm{d}\,c_q}{b^{p+q}}\right)=p\frac{c_p}{b^p}\,\mathrm{d}\left(\frac{c_q}{b^q}\right)-q\frac{c_q}{b^q}\,\mathrm{d}\left(\frac{c_p}{b^p}\right)$$

and the right member belongs to $\Omega_{\mathcal{B}_{(b)}/A}$. $\square$

For each $n\in\mathbb{Z}$, we shall denote by $\widetilde{\mathrm{Kos}}(\mathcal{M})(n)$ the complex $\widetilde{\mathrm{Kos}}(\mathcal{M})$ twisted by $\mathcal{O}_{\mathbb{P}}(n)$ (notice that the differential of the Koszul complex is $\mathcal{O}_{\mathbb{P}}$-linear). The differential of the complex $\widetilde{\mathrm{Kos}}(\mathcal{M})(n)$ is still denoted by i_D.

1.1. Acyclicity of the Koszul complex of a module

Let $\widetilde{\mathrm{Kos}}(\mathcal{M})_n:=\pi_*(\widetilde{\mathrm{Kos}}(\mathcal{M})(n))$. The natural morphisms

$$[\Omega^p_{\mathcal{B}/\mathcal{O}}]_n\longrightarrow\pi_*[\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)]$$

give a morphism of complexes

$$\mathrm{Kos}(\mathcal{M})_n\longrightarrow\widetilde{\mathrm{Kos}}(\mathcal{M})_n$$

and one has:

Theorem 1.5. *Let $\mathcal{M}$ be a finitely generated quasi-coherent module on a scheme $(X,\mathcal{O})$, $\mathbb{P}=\mathrm{Proj}\,S^{\cdot}\mathcal{M}$ and $\pi\colon\mathbb{P}\to X$ the natural morphism. Let d be the minimal number of generators of $\mathcal{M}$ (i.e., it is the greatest integer such that $\Lambda^d\mathcal{M}\neq0$) and $n>0$. Then:*

1. *If $R^j\pi_*[\widetilde{\Omega}^i_{\mathcal{B}/\mathcal{O}}(n)]=0$ for any $j>0$ and any $0\le i\le d$, then $\widetilde{\mathrm{Kos}}(\mathcal{M})_n$ is acyclic.*

2. *If* (1) *holds and the natural morphism* $[\Omega^i_{\mathcal{B}/\mathcal{O}}]_n \to \pi_*[\widetilde{\Omega}^i_{\mathcal{B}/\mathcal{O}}(n)]$ *is an isomorphism for any* $0 \le i \le d$, *then* $\mathrm{Kos}(\mathcal{M})_n$ *is also acyclic.*

Proof. (1) By Theorem 1.4, the complex $\widetilde{\mathrm{Kos}}(\mathcal{M})(n)$ is acyclic. Since the (nonzero) terms of this complex are $\widetilde{\Omega}^i_{\mathcal{B}/\mathcal{O}}(n)$, the hypothesis tells us that $\pi_*(\widetilde{\mathrm{Kos}}(\mathcal{M}) \otimes \mathcal{O}_{\mathbb{P}}(n))$ is acyclic, that is, $\widetilde{\mathrm{Kos}}(\mathcal{M})_n$ is acyclic.

(2) By hypothesis, $\mathrm{Kos}(\mathcal{M})_n \to \widetilde{\mathrm{Kos}}(\mathcal{M})_n$ is an isomorphism and then $\mathrm{Kos}(\mathcal{M})_n$ is also acyclic. □

Theorem 1.6. *Let* X *be a Noetherian scheme and* $\mathcal{M}$ *a coherent module on* X. *The Koszul complexes* $\mathrm{Kos}(\mathcal{M})_n$ *and* $\widetilde{\mathrm{Kos}}(\mathcal{M})_n$ *are acyclic for* $n \gg 0$.

Proof. Indeed, the hypothesis (1) and (2) of Theorem 1.5 hold for $n \gg 0$ (see [8, Theorem 2.2.1] and [7, Section 3.3 and Section 3.4]). □

Theorem 1.7. *Let* I *be an ideal of a Noetherian ring* A. *If* I *is locally generated by a regular sequence, then* $\mathrm{Kos}(I)_n$ *and* $\widetilde{\mathrm{Kos}}(I)_n$ *are acyclic for any* $n > 0$.

Proof. In this case $\pi\colon \mathbb{P} \to X = \mathrm{Spec}\, A$ is the blow-up with respect to I, because $S^n I = I^n$, since I is locally a regular ideal ([9]). Let d be the minimum number of generators of I. By Theorem 1.5, it suffices to see that for any A-module M and any $0 \le i \le d$ one has:

$$H^j(\mathbb{P}, (\pi^* M)(n-i)) = \begin{cases} 0 & \text{if } j > 0 \\ M \otimes_A I^{n-i} & \text{if } j = 0 \end{cases}$$

This is a consequence of the Theorem of formal functions (see [8, Corollary 4.1.7]). Indeed, let $Y_r = \mathrm{Spec}\, A/I^r$, $E_r = \pi^{-1}(Y_r)$ and $\pi_r\colon E_r \to Y_r$. One has that $E_r = \mathrm{Proj}\, S^{\cdot}_{A/I^r}(I/I^{r+1})$ is a projective bundle over Y_r, because I/I^{r+1} is a locally free A/I^r-module of rank d, since I is locally regular. Hence, for any module N on Y_r and any $m > -d$ one has

$$H^j(E_r, (\pi_r^* N)(m)) = \begin{cases} 0 & \text{if } j > 0 \\ N \otimes_{A/I^r} I^m/I^{m+r} & \text{if } j = 0 \end{cases}$$

Now, by the theorem of formal functions (let $m = n - i$)

$$H^j(\mathbb{P}, (\pi^* M)(m))^{\wedge} = \varprojlim_r H^j(E_r, \pi_r^*(M/I^r M)(m)) = 0, \text{ for } j > 0.$$

For $j=0$, the natural morphism $M\otimes_A I^m \to H^0(\mathbb{P},(\pi^*M)(m))$ is an isomorphism because it is an isomorphism after completion by I:

$$\begin{aligned} H^0(\mathbb{P},(\pi^*M)(m))^\wedge &= \lim_{\substack{\leftarrow \\ r}} H^0(E_r, \pi_r^*(M/I^rM)(m)) \\ &= \lim_{\substack{\leftarrow \\ r}} (M/I^rM)\otimes_{A/I^r} I^m/I^{m+r} \\ &= \lim_{\substack{\leftarrow \\ r}} (M\otimes_A S^m I)\otimes_A A/I^r = (M\otimes_A I^m)^\wedge. \qquad \square \end{aligned}$$

Remark 1.8. Let d be the minimum number of generators of $\mathcal{M}$. Since $\widetilde{\mathrm{Kos}}(\mathcal{M})$ is acyclic and π_* is left exact, one has that $H_d(\widetilde{\mathrm{Kos}}(\mathcal{M})_n)=0$ for any n. One the other hand, it is proved in [5] that $H_d(\mathrm{Kos}(\mathcal{M})_d)=0$. One cannot expect $\mathrm{Kos}(\mathcal{M})_n \to \widetilde{\mathrm{Kos}}(\mathcal{M})_n$ to be an isomorphism in general. For instance, consider $X=\mathrm{Spec}\, A$ with $A=k[u,v,s_1,s_2,t_1,t_2]/I$ where k is a field and $I=(-us_1+vt_1+ut_2, vs_1+us_2-vt_2, vs_2, ut_1)$. Let $M=(Ax\oplus Ay)/A(\bar{u}x+\bar{v}y)$, where $\bar{u}$ (resp. $\bar{v}$) is the class of u (resp. v) in A. Then one can prove that the map $M\to\pi_*\mathcal{O}_{\mathbb{P}}(1)$ is not injective (for details we refer to section 26.21 of The Stacks project). So that the question which arises here is whether $\mathrm{Kos}(\mathcal{M})_n \to \widetilde{\mathrm{Kos}}(\mathcal{M})_n$ is a quasi-isomorphism. We do not know the answer, besides the acyclicity theorems for both complexes mentioned above.

1.2. Koszul versus De Rham

The exterior differential defines morphisms

$$\mathrm{d}\colon \Omega^p_{\mathcal{B}/\mathcal{O}} \longrightarrow \Omega^{p+1}_{\mathcal{B}/\mathcal{O}}$$

which are $\mathcal{O}$-linear, but not $\mathcal{B}$-linear. One has then the De Rham complex:

$$\mathrm{DeRham}(\mathcal{M}) \equiv 0 \longrightarrow \mathcal{B} \xrightarrow{\mathrm{d}} \Omega_{\mathcal{B}/\mathcal{O}} \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^p_{\mathcal{B}/\mathcal{O}} \xrightarrow{\mathrm{d}} \Omega^{p+1}_{\mathcal{B}/\mathcal{O}} \longrightarrow \dots$$

which can be reformulated as

$$0 \longrightarrow \mathcal{B} \xrightarrow{\mathrm{d}} \mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-1] \longrightarrow \dots\Lambda^p\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-p] \longrightarrow \Lambda^{p+1}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{B}[-p-1] \longrightarrow \dots$$

Taking into account that d is homogeneous of degree 0, one has for each $n\geq 0$ a complex of $\mathcal{O}$-modules

$$0 \longrightarrow S^n\mathcal{M} \longrightarrow \mathcal{M}\otimes_{\mathcal{O}} S^{n-1} \longrightarrow \dots \longrightarrow \Lambda^{n-1}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M} \longrightarrow \Lambda^n\mathcal{M} \longrightarrow 0$$

which we denote by $\mathrm{DeRham}(\mathcal{M})_n$.

The differentials of the Koszul and De Rham complexes are related by Cartan's formula: $i_D\circ \mathrm{d}+\mathrm{d}\circ i_D=$ multiplication by n on $\Lambda^p\mathcal{M}\otimes_{\mathcal{O}}S^{n-p}\mathcal{M}$. This immediately implies the following result:

Proposition 1.9. *If X is a scheme over $\mathbb{Q}$, then $\mathrm{Kos}(\mathcal{M})_n$ and $\mathrm{DeRham}(\mathcal{M})_n$ are homotopically trivial for any $n>0$. In particular, they are acyclic.*

Now we pass to homogeneous localizations. The differential $\mathrm{d}\colon \Omega^p_{\mathcal{B}/\mathcal{O}}\to\Omega^{p+1}_{\mathcal{B}/\mathcal{O}}$ is compatible with homogeneous localization, since for any $\omega_{k+n}\in\Omega^p_{\mathcal{B}/\mathcal{O}}$ of degree $k+n$ and any $b\in\mathcal{B}$ of degree 1, one has:

$$\mathrm{d}\left(\frac{\omega_{k+n}}{b^n}\right)=\frac{b^n\,\mathrm{d}\,\omega_{k+n}-(\mathrm{d}\,b^n)\wedge\omega_{k+n}}{b^{2n}}$$

Thus, for any $n\in\mathbb{Z}$, one has ($\mathcal{O}$-linear) morphisms of sheaves

$$\mathrm{d}\colon \widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)\longrightarrow\widetilde{\Omega}^{p+1}_{\mathcal{B}/\mathcal{O}}(n)$$

and we obtain, for each n, a complex of sheaves on $\mathbb{P}$:

$$\widetilde{\mathrm{DeRham}}(\mathcal{M},n)=0\longrightarrow\mathcal{O}_{\mathbb{P}}(n)\xrightarrow{\mathrm{d}}\widetilde{\Omega}_{\mathcal{B}/\mathcal{O}}(n)\xrightarrow{\mathrm{d}}\dots\xrightarrow{\mathrm{d}}\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)\longrightarrow\dots$$

which can be reformulated as

$$0\longrightarrow\mathcal{O}_{\mathbb{P}}(n)\xrightarrow{\mathrm{d}}(\pi^*\mathcal{M})(n-1)\longrightarrow\dots\longrightarrow(\pi^*\Lambda^p\mathcal{M})(n-p)\longrightarrow\dots$$

It should be noticed that $\widetilde{\mathrm{DeRham}}(\mathcal{M},n)$ is not the complex obtained for $n=0$ twisted by $\mathcal{O}_{\mathbb{P}}(n)$, because the differential is not $\mathcal{O}_{\mathbb{P}}$-linear.

Again, one has that $i_D\circ\mathrm{d}+\mathrm{d}\circ i_D=$ multiplication by n, on $\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)$. Hence, one has:

Proposition 1.10. *If X is a scheme over $\mathbb{Q}$, then the complexes $\widetilde{\mathrm{Kos}}(\mathcal{M})(n)$ and $\widetilde{\mathrm{DeRham}}(\mathcal{M},n)$ are homotopically trivial for any $n\neq 0$.*

Corollary 1.11. *Let X be a scheme over $\mathbb{Q}$. For any $n\neq 0$, the exact sequences*

$$0\longrightarrow\Omega^p_{\mathbb{P}/X}(n)\longrightarrow\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)\longrightarrow\Omega^{p-1}_{\mathbb{P}/X}(n)\longrightarrow 0$$

split as sheaves of $\mathcal{O}$-modules (but not as $\mathcal{O}_{\mathbb{P}}$-modules).

2. Global Euler sequence of a module and Koszul complexes

Assume that $(X,\mathcal{O})$ is a k-scheme, where k is a ring (just for simplicity, one could assume that k is another scheme). Let $\mathcal{M}$ be an $\mathcal{O}$-module and $\mathcal{B}=S^{\cdot}\mathcal{M}$ the symmetric algebra over $\mathcal{O}$. Instead of considering the module of Kähler differentials of $\mathcal{B}$ over $\mathcal{O}$, we shall now consider the module of Kähler differentials over k, that is, $\Omega_{\mathcal{B}/k}$. As it happened with $\Omega_{\mathcal{B}/\mathcal{O}}$ (Section 1), the module $\Omega_{\mathcal{B}/k}$ is a graded $\mathcal{B}$-module in a natural way. The $\mathcal{O}$-derivation $D\colon \mathcal{B}\to\mathcal{B}$ is in particular a k-derivation, hence it defines a morphism $i_D\colon \Omega_{\mathcal{B}/k}\to\mathcal{B}$, which is nothing but the composition of the natural morphism $\Omega_{\mathcal{B}/k}\to\Omega_{\mathcal{B}/\mathcal{O}}$ with the inner product $i_D\colon \Omega_{\mathcal{B}/\mathcal{O}}\to\mathcal{B}$ defined in Section 1. Again we obtain a complex of $\mathcal{B}$-modules $(\Omega^{\cdot}_{\mathcal{B}/k}, i_D)$ which we denote by $\operatorname{Kos}(\mathcal{M}/k)$:

$$\text{(4)} \qquad \dots \longrightarrow \Omega^p_{\mathcal{B}/k} \xrightarrow{i_D} \Omega^{p-1}_{\mathcal{B}/k} \xrightarrow{i_D} \dots \xrightarrow{i_D} \Omega_{\mathcal{B}/k} \xrightarrow{i_D} \mathcal{B} \longrightarrow 0$$

and for each $n\geq 0$ a complex of $\mathcal{O}$-modules

$$\operatorname{Kos}(\mathcal{M}/k)_n = \dots \longrightarrow [\Omega^p_{\mathcal{B}/k}]_n \xrightarrow{i_D} \dots \longrightarrow [\Omega_{\mathcal{B}/k}]_n \xrightarrow{i_D} S^n\mathcal{M} \longrightarrow 0$$

By homogeneous localization one has a complex of $\mathcal{O}_{\mathbb{P}}$-modules, denoted by $\widetilde{\operatorname{Kos}}(\mathcal{M}/k)$:

$$\dots \longrightarrow \widetilde{\Omega}^p_{\mathcal{B}/k} \xrightarrow{i_D} \widetilde{\Omega}^{p-1}_{\mathcal{B}/k} \xrightarrow{i_D} \dots \longrightarrow \widetilde{\Omega}_{\mathcal{B}/k} \xrightarrow{i_D} \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

Theorem 2.1. *The complex $\widetilde{\operatorname{Kos}}(\mathcal{M}/k)$ is acyclic (that is, an exact sequence). Moreover,*

$$\Omega^p_{\mathbb{P}/k} = \operatorname{Ker}\left(\widetilde{\Omega}^p_{\mathcal{B}/k} \xrightarrow{i_D} \widetilde{\Omega}^{p-1}_{\mathcal{B}/k}\right)$$

Hence one has exact sequences

$$0 \longrightarrow \Omega^p_{\mathbb{P}/k} \longrightarrow \widetilde{\Omega}^p_{\mathcal{B}/k} \longrightarrow \Omega^{p-1}_{\mathbb{P}/k} \longrightarrow 0$$

and right and left resolutions of $\Omega^p_{\mathbb{P}/k}$:

$$0 \longrightarrow \Omega^p_{\mathbb{P}/k} \longrightarrow \widetilde{\Omega}^p_{\mathcal{B}/k} \longrightarrow \widetilde{\Omega}^{p-1}_{\mathcal{B}/k} \longrightarrow \dots \longrightarrow \widetilde{\Omega}_{\mathcal{B}/k} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

$$\dots \longrightarrow \widetilde{\Omega}^e_{\mathcal{B}/k} \longrightarrow \widetilde{\Omega}^{e-1}_{\mathcal{B}/k} \longrightarrow \dots \longrightarrow \widetilde{\Omega}^{p+1}_{\mathcal{B}/k} \longrightarrow \Omega^p_{\mathbb{P}/k} \longrightarrow 0$$

In particular, for $p=1$ the exact sequence

$$\text{(5)} \qquad 0 \longrightarrow \Omega_{\mathbb{P}/k} \longrightarrow \widetilde{\Omega}_{\mathcal{B}/k} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

is called the (global) Euler sequence.

Proof. It is completely analogous to the proof of Theorem 1.4. □

Let $\widetilde{\mathrm{Kos}}(\mathcal{M}/k)_n := \pi_*(\widetilde{\mathrm{Kos}}(\mathcal{M}/k)(n))$. The natural morphisms

$$[\Omega^p_{\mathcal{B}/k}]_n \longrightarrow \pi_*\big(\widetilde{\Omega}^p_{\mathcal{B}/k}(n)\big)$$

give a morphism of complexes

$$\mathrm{Kos}(\mathcal{M}/k)_n \longrightarrow \widetilde{\mathrm{Kos}}(\mathcal{M}/k)_n$$

In complete analogy to the relative setting we have the following:

Theorem 2.2. *Let $\mathcal{M}$ be a finitely generated quasi-coherent module on a scheme $(X,\mathcal{O})$, $\mathcal{B}{=}S^{\cdot}\mathcal{M}$, $\mathbb{P}{=}\mathrm{Proj}\,\mathcal{B}$ and $\pi\colon\mathbb{P}\to X$ the natural morphism. Let d' be the greatest integer such that $\Omega^{d'}_{\mathcal{B}/k}\neq 0$ and $n>0$. Then:*

1. *If $R^j\pi_*(\widetilde{\Omega}^i_{\mathcal{B}/k}(n))=0$ for any $j>0$ and any $0\le i\le d'$, then $\widetilde{\mathrm{Kos}}(\mathcal{M}/k)_n$ is acyclic.*

2. *If* (1) *holds and the natural morphism $[\Omega^i_{\mathcal{B}/k}]_n\to\pi_*(\widetilde{\Omega}^i_{\mathcal{B}/k}(n))$ is an isomorphism for any $0\le i\le d'$, then $\mathrm{Kos}(\mathcal{M}/k)_n$ is also acyclic.*

Theorem 2.3. *Let X be a Noetherian scheme and $\mathcal{M}$ a coherent module on X. The Koszul complexes $\mathrm{Kos}(\mathcal{M}/k)_n$ and $\widetilde{\mathrm{Kos}}(\mathcal{M}/k)_n$ are acyclic for $n\gg 0$.*

2.1. Koszul versus De Rham (Global case)

Now we pass to the De Rham complex (over k). The k-linear differentials

$$\mathrm{d}\colon \Omega^p_{\mathcal{B}/k}\longrightarrow\Omega^{p+1}_{\mathcal{B}/k}$$

give a (global) De Rham complex

$$\mathrm{DeRham}(\mathcal{M}/k)\equiv 0\longrightarrow\mathcal{B}\xrightarrow{\mathrm{d}}\Omega_{\mathcal{B}/k}\xrightarrow{\mathrm{d}}\dots\xrightarrow{\mathrm{d}}\Omega^{p-1}_{\mathcal{B}/k}\xrightarrow{\mathrm{d}}\Omega^p_{\mathcal{B}/k}\longrightarrow\dots$$

which is bounded if X is of finite type over k. Since d is homogeneous of degree 0, one has for each $n\ge 0$ a complex of $\mathcal{O}$-modules (with k-linear differential)

$$\mathrm{DeRham}(\mathcal{M}/k)_n\equiv 0\longrightarrow S^n\mathcal{M}\xrightarrow{\mathrm{d}}[\Omega_{\mathcal{B}/k}]_n\xrightarrow{\mathrm{d}}\dots\xrightarrow{\mathrm{d}}[\Omega^p_{\mathcal{B}/k}]_n\longrightarrow\dots$$

One has again Cartan's formula: $i_D\circ\mathrm{d}+\mathrm{d}\circ i_D=$ multiplication by n, on $[\Omega^p_{\mathcal{B}/k}]_n$ and then:

Proposition 2.4. *If X is a scheme over $\mathbb{Q}$, then the complexes $\mathrm{Kos}(\mathcal{M}/k)_n$ and $\mathrm{DeRham}(\mathcal{M}/k)_n$ are homotopically trivial (in particular, acyclic) for any $n>0$.*

As in Section 1.2, we can take homogeneous localizations: for each $n\in\mathbb{Z}$, the differentials $\Omega^p_{\mathcal{B}/k}\to\Omega^{p+1}_{\mathcal{B}/k}$ induce k-linear morphisms

$$\mathrm{d}\colon \widetilde{\Omega}^p_{\mathcal{B}/k}(n)\longrightarrow\widetilde{\Omega}^{p+1}_{\mathcal{B}/k}(n)$$

and one obtains a complex of $\mathcal{O}_{\mathbb{P}}$-modules (with k-linear differential)

$$\widetilde{\mathrm{DeRham}}(\mathcal{M}/k,n)=0\longrightarrow\mathcal{O}_{\mathbb{P}}(n)\xrightarrow{\mathrm{d}}\widetilde{\Omega}_{\mathcal{B}/k}(n)\xrightarrow{\mathrm{d}}\dots\xrightarrow{\mathrm{d}}\widetilde{\Omega}^p_{\mathcal{B}/k}(n)\longrightarrow\dots$$

Again, the differentials of Koszul and De Rham complexes are related by Cartan's formula: $i_D\circ\mathrm{d}+\mathrm{d}\circ i_D=$ multiplication by n, on $\widetilde{\Omega}^p_{\mathcal{B}/k}(n)$, so one has:

Proposition 2.5. *Let X be a scheme over $\mathbb{Q}$. The complexes $\widetilde{\mathrm{Kos}}(\mathcal{M}/k)(n)$ and $\widetilde{\mathrm{DeRham}}(\mathcal{M}/k,n)$ are homotopically trivial (in particular, acyclic) for any $n\neq0$.*

Corollary 2.6. *If X is a scheme over $\mathbb{Q}$, then for any $n\neq0$, the exact sequences*

$$0\longrightarrow\Omega^p_{\mathbb{P}/k}(n)\longrightarrow\widetilde{\Omega}^p_{\mathcal{B}/k}(n)\longrightarrow\Omega^{p-1}_{\mathbb{P}/k}(n)\longrightarrow0$$

split as sheaves of k-modules (but not as $\mathcal{O}_{\mathbb{P}}$-modules).

3. Cohomology of projective bundles

In this section we assume that $\mathcal{E}$ is a locally free sheaf of rank $r+1$ on a k-scheme $(X,\mathcal{O})$. Let $\mathcal{B}=S^{\cdot}\mathcal{E}$ be its symmetric algebra over $\mathcal{O}$ and $\mathbb{P}=\operatorname{Proj}\mathcal{B}\xrightarrow{\pi}X$ the corresponding projective bundle. Our aim is to determine the cohomology of the sheaves $\Omega^p_{\mathbb{P}/X}(n)$ and $\Omega^p_{\mathbb{P}/k}(n)$.

3.1. Cohomology of $\Omega^p_{\mathbb{P}/X}(n)$

Notations: In order to simplify some statements, we shall use the following conventions:

1. $S^p\mathcal{E}=0$ whenever $p<0$, and analogously for exterior powers.
2. For any integer p, let $\bar{p}=r+1-p$.
3. For any $\mathcal{O}$-module $\mathcal{M}$, we shall denote by $\mathcal{M}^*$ its dual $\mathcal{H}om\,(\mathcal{M},\mathcal{O})$.

We shall use the following well known result on the cohomology of a projective bundle:

Proposition 3.1. *Let n be a non negative integer. Then*

$$R^i\pi_*\mathcal{O}_{\mathbb{P}}(n)=\begin{cases}0 & \text{for } i\neq 0\\ S^n\mathcal{E} & \text{for } i=0\end{cases}$$

If n is a positive integer, then

$$R^i\pi_*\mathcal{O}_{\mathbb{P}}(-n)=\begin{cases}0 & \text{for } i\neq r\\ S^{n-r-1}\mathcal{E}^*\otimes\Lambda^{r+1}\mathcal{E}^* & \text{for } i=r\end{cases}$$

We shall also use without further explanation a particular case of projection formula: for any quasi-coherent module $\mathcal{N}$ on X and any locally free module $\mathcal{L}$ on $\mathbb{P}$ such that $R^j\pi_*\mathcal{L}$ is locally free (for any j), one has

$$R^i\pi_*(\pi^*\mathcal{N}\otimes\mathcal{L})=\mathcal{N}\otimes R^i\pi_*\mathcal{L}$$

Proposition 3.2. *Let n be a non negative integer. Then*

$$R^i\pi_*\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)=\begin{cases}0 & \text{for } i\neq 0\\ \Lambda^p\mathcal{E}\otimes S^{n-p}\mathcal{E} & \text{for } i=0\end{cases}$$

For any positive integer n, one has

$$R^i\pi_*\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(-n)=\begin{cases}0 & \text{for } i\neq r\\ \Lambda^{\bar{p}}\mathcal{E}^*\otimes S^{n-\bar{p}}\mathcal{E}^* & \text{for } i=r \text{ with } \bar{p}=r+1-p\end{cases}$$

Proof. Since $\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}=(\pi^*\Lambda^p\mathcal{E})(-p)$, the results follows from Proposition 3.1. For the second formula we have also used the natural isomorphism $\Lambda^{\bar{p}}\mathcal{E}=\Lambda^p\mathcal{E}^*\otimes\Lambda^{r+1}\mathcal{E}$. □

Remark 3.3. Notice that $\Lambda^p\mathcal{E}\otimes S^{n-p}\mathcal{E}=[\Omega^p_{\mathcal{B}/\mathcal{O}}]_n$. Thus, Proposition 3.2 and Theorem 1.5 tell us that $\mathrm{Kos}(\mathcal{E})_n\to\widetilde{\mathrm{Kos}}(\mathcal{E})_n$ is an isomorphism for any $n\geq 0$ and the Koszul complexes $\widetilde{\mathrm{Kos}}(\mathcal{E})_n$ and $\mathrm{Kos}(\mathcal{E})_n$ are acyclic for any $n>0$ (thus we obtain the well known fact of the acyclicity of the Koszul complex of a locally free module).

Let us denote by $\mathcal{K}_{p,n}$ the kernels of the morphisms i_D in $\mathrm{Kos}(\mathcal{E})_n$, that is,

$$\mathcal{K}_{p,n}:=\mathrm{Ker}\left(\Lambda^p\mathcal{E}\otimes S^{n-p}\mathcal{E}\longrightarrow\Lambda^{p-1}\mathcal{E}\otimes S^{n-p+1}\mathcal{E}\right)$$

One has the following result (see [12] or [4, Exposé XI] for different approaches).

Theorem 3.4. *Let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on a k-scheme $(X,\mathcal{O})$ and $\mathbb{P}=\mathrm{Proj}\, S^{\cdot}\mathcal{E}\xrightarrow{\pi}X$ the corresponding projective bundle.*

Let n be a positive integer number.

1.
$$R^i\pi_*\Omega^p_{\mathbb{P}/X} = \begin{cases} \mathcal{O} & \textit{if } 0\le i=p\le r \\ 0 & \textit{otherwise} \end{cases}$$

2.
$$R^i\pi_*\Omega^p_{\mathbb{P}/X}(n) = \begin{cases} 0 & \textit{if } i\neq 0 \\ \mathcal{K}_{p,n} & \textit{if } i=0 \end{cases}$$

and, if X is a $\mathbb{Q}$-scheme, then

$$\mathcal{K}_{p,n}\oplus\mathcal{K}_{p-1,n} = \Lambda^p\mathcal{E}\otimes S^{n-p}\mathcal{E}.$$

3.
$$R^i\pi_*\Omega^p_{\mathbb{P}/X}(-n) = \begin{cases} 0 & \textit{if } i\neq r \\ \mathcal{K}^*_{r-p,n} & \textit{if } i=r \end{cases}$$

and, if X is a $\mathbb{Q}$-scheme, then

$$\mathcal{K}^*_{r-p,n}\oplus\mathcal{K}^*_{r-p+1,n} = \Lambda^{\bar p}\mathcal{E}^*\otimes S^{n-\bar p}\mathcal{E}^*$$

Proof. Let $n\geq 0$. By Theorem 1.4

$$0\longrightarrow\Omega^p_{\mathbb{P}/X}(n)\longrightarrow\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n)\longrightarrow\ldots\longrightarrow\widetilde{\Omega}_{\mathcal{B}/\mathcal{O}}(n)\longrightarrow\mathcal{O}_{\mathbb{P}}(n)\longrightarrow 0$$

is a resolution of $\Omega^p_{\mathbb{P}/X}(n)$ by π_*-acyclic sheaves (by Proposition 3.2). One concludes then by Proposition 3.2 and Remark 3.3.

(3) follows from (2) and (relative) Grothendieck duality: one has an isomorphism $\Omega^p_{\mathbb{P}/X}=\mathcal{H}om\,(\Omega^{r-p}_{\mathbb{P}/X},\Omega^r_{\mathbb{P}/X})$ and then

$$\mathbb{R}\pi_*\Omega^p_{\mathbb{P}/X}(-n)\simeq\mathbb{R}\pi_*\mathcal{H}om\,(\Omega^{r-p}_{\mathbb{P}/X}(n),\Omega^r_{\mathbb{P}/X})\simeq\mathbb{R}\mathcal{H}om\,(\mathbb{R}\pi_*\Omega^{r-p}_{\mathbb{P}/X}(n)[r],\mathcal{O})$$

and one concludes by (2).

Finally, the statements of (2) and (3) regarding the case that X is a $\mathbb{Q}$-scheme follow from Corollary 1.11. □

Corollary 3.5. (Bott's formula) *Let $\mathbb{P}_r$ be the projective space of dimension r over a field k. Let n be a positive integer number.*

1.
$$\dim_k H^q(\mathbb{P}_r,\Omega^p_{\mathbb{P}_r}) = \begin{cases} 1 & \textit{if } 0\le q=p\le r \\ 0 & \textit{otherwise} \end{cases}$$

2.
$$\dim_k H^q(\mathbb{P}_r,\Omega^p_{\mathbb{P}_r}(n)) = \begin{cases} 0 & \textit{if } q\neq 0 \\ \binom{n+r-p}{n}\binom{n-1}{p} & \textit{if } q=0 \end{cases}$$

3.
$$\dim_k H^q(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(-n)) = \begin{cases} 0 & \textit{if } q \neq r \\ \binom{n+p}{n}\binom{n-1}{r-p} & \textit{if } q = r \end{cases}$$

Proof. By Theorem 3.4, it is enough to prove that $\dim_k \mathcal{K}_{p,n} = \binom{n+r-p}{n}\binom{n-1}{p}$. From the exact sequence

$$0 \longrightarrow \mathcal{K}_{p,n} \longrightarrow \Lambda^p \mathcal{E} \otimes S^{n-p}\mathcal{E} \longrightarrow \mathcal{K}_{p-1,n} \longrightarrow 0$$

it follows that $\dim_k \mathcal{K}_{p,n} + \dim_k \mathcal{K}_{p-1,n} = \binom{r+1}{p}\binom{n-p+r}{r}$; hence it suffices to prove that

$$\binom{n+r-p}{n}\binom{n-1}{p} + \binom{n+r-p+1}{n}\binom{n-1}{p-1} = \binom{r+1}{p}\binom{n-p+r}{r}$$

which is an easy computation if one writes $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. $\square$

Remark 3.6. (1) We can give an interpretation of $H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n))$ in terms of differentials forms of the polynomial ring $k[x_0, ..., x_r]$; one has the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \longrightarrow [\Omega^p_{k[x_0,\dots,x_r]/k}]_n \xrightarrow{i_D} [\Omega^{p-1}_{k[x_0,\dots,x_r]/k}]_n$$

that is, $H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n))$ are those p-forms $\omega_p \in \Omega^p_{k[x_0,\dots,x_r]/k}$ which are homogeneous of degree n and such that $i_D \omega_p = 0$, where $D = \sum_{i=0}^r x_i \partial/\partial x_i$.

(2) From the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \longrightarrow \Lambda^p\mathcal{E} \otimes S^{n-p}\mathcal{E} \longrightarrow \dots \longrightarrow \mathcal{E} \otimes S^{n-1}\mathcal{E} \longrightarrow S^n\mathcal{E} \longrightarrow 0$$

we can give a different combinatorial expression of $\dim_k H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n))$ (as Verdier does):

$$\dim_k H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) = \sum_{i=0}^p (-1)^i \binom{r+1}{p-i}\binom{n+r-p+i}{r}.$$

It follows from Theorem 3.4 that $H^q(\mathbb{P}, \Omega^p_{\mathbb{P}/X}) = H^{q-p}(X, \mathcal{O})$. For the twisted case we have the following:

Corollary 3.7. *Let X be a proper scheme over a field k of characteristic zero. Let $\mathcal{E}$ be a locally free module on X of rank $r+1$ and $\mathbb{P} = \operatorname{Proj} S^{\cdot}\mathcal{E}$ the associated projective bundle. Then, for any positive integer n, one has:*

1. $\dim_k H^q(\mathbb{P}, \Omega^p_{\mathbb{P}/X}(n)) = \sum_{i=0}^p (-1)^i \dim H^q(X, \Lambda^{p-i}\mathcal{E} \otimes S^{n-p+i}\mathcal{E})$.
2. $\dim_k H^q(\mathbb{P}, \Omega^p_{\mathbb{P}/X}(-n)) = \sum_{i=0}^p (-1)^i \dim H^{q-r}(X, \Lambda^{\bar{p}+i}\mathcal{E}^* \otimes S^{n-\bar{p}-i}\mathcal{E}^*)$ *with* $\bar{p} = r+1-p$.

Proof. (1) By Corollary 1.11, one has

$$H^q(\mathbb{P},\Omega^p_{\mathbb{P}/X}(n))\oplus H^q(\mathbb{P},\Omega^{p-1}_{\mathbb{P}/X}(n))=H^q(\mathbb{P},\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n))$$

and $H^q(\mathbb{P},\widetilde{\Omega}^p_{\mathcal{B}/\mathcal{O}}(n))=H^q(X,\Lambda^p\mathcal{E}\otimes S^{n-p}\mathcal{E})$ by Proposition 3.2. Conclusion follows.
(2) is completely analogous. □

3.2. Cohomology of $\Omega^p_{\mathbb{P}/k}(n)$

Let us consider the exact sequence of differentials

$$0\longrightarrow\Omega_{X/k}\otimes_{\mathcal{O}}\mathcal{B}\longrightarrow\Omega_{\mathcal{B}/k}\longrightarrow\Omega_{\mathcal{B}/\mathcal{O}}\longrightarrow 0$$

This sequence locally splits: indeed, if $\mathcal{E}$ is trivial, then $\mathcal{E}=E\otimes_k\mathcal{O}$ and $\mathcal{B}=B\otimes_k\mathcal{O}$, with $B=S^{\cdot}E$; hence, $\Omega_{\mathcal{B}/\mathcal{O}}=\Omega_{B/k}\otimes_k\mathcal{O}$ and there is a natural morphism $\Omega_{B/k}\otimes_k\mathcal{O}\to\Omega_{\mathcal{B}/k}$ which is a section of $\Omega_{\mathcal{B}/k}\to\Omega_{\mathcal{B}/\mathcal{O}}$.

Remark 3.8. The exact sequence is a sequence of graded $\mathcal{B}$-modules, hence it gives an exact sequence of $\mathcal{O}$-modules in each degree. In particular, in degree 0 one obtains an isomorphism $\Omega_{X/k}=[\Omega_{\mathcal{B}/k}]_0$, and an exact sequence in degree 1:

$$0\longrightarrow\Omega_{X/k}\otimes_{\mathcal{O}}\mathcal{E}\longrightarrow[\Omega_{\mathcal{B}/k}]_1\longrightarrow\mathcal{E}\longrightarrow 0$$

which is nothing but the Atiyah extension.

Taking homogeneous localizations we obtain an exact sequence of $\mathcal{O}_{\mathbb{P}}$-modules

$$0\longrightarrow\pi^*\Omega_{X/k}\longrightarrow\widetilde{\Omega}_{\mathcal{B}/k}\longrightarrow\widetilde{\Omega}_{\mathcal{B}/\mathcal{O}}\longrightarrow 0$$

which splits locally (on X).

Proposition 3.9. *Let n be a positive integer. Then*:

1.
$$R^i\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}=\begin{cases}0 & \textit{for } i\neq 0,r\\ \Omega^p_{X/k} & \textit{for } i=0\\ \Omega^{p-r-1}_{X/k} & \textit{for } i=r\end{cases}$$

2.
$$R^i\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(n)=\begin{cases}0 & \textit{for } i\neq 0\\ [\Omega^p_{\mathcal{B}/k}]_n & \textit{for } i=0\end{cases}$$

3. *$R^i\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(-n)=0$ for $i\neq r$ and $R^r\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(-n)$ is locally isomorphic to*

$$\bigoplus_{q=0}^{p}(\Omega^{p-q}_{X/k}\otimes\Lambda^{\bar q}\mathcal{E}^*\otimes S^{n-\bar q}\mathcal{E}^*)$$

with $\bar q=r+1-q$.

4. *Furthermore, if X is a smooth k-scheme (of relative dimension d), then*

$$R^r\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(-n)=\left[\Omega^{d+\bar p}_{\mathcal{B}/k}\right]^*_n\otimes\Omega^d_{X/k}$$

Proof. If $\mathcal{E}$ is trivial, then $\widetilde{\Omega}_{\mathcal{B}/k}=\pi^*\Omega_{X/k}\oplus\widetilde{\Omega}_{\mathcal{B}/\mathcal{O}}$, so $\widetilde{\Omega}^p_{\mathcal{B}/k}=\bigoplus_{q=0}^p\pi^*\Omega^{p-q}_{X/k}\otimes\widetilde{\Omega}^q_{\mathcal{B}/\mathcal{O}}$ and (1)–(3) follow from Proposition 3.2 in this case. Since $\mathcal{E}$ is locally trivial, we obtain the vanishing statements of (1)–(3).

(1) The natural morphism $\Omega^p_{X/k}\to\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}$ is an isomorphism because it is locally so. The natural morphism $\widetilde{\Omega}^{r+1}_{\mathcal{B}/k}\to\Omega^{r+1}_{\mathcal{B}/\mathcal{O}}$ gives a morphism $R^r\pi_*\widetilde{\Omega}^{r+1}_{\mathcal{B}/k}\to R^r\pi_*\widetilde{\Omega}^{r+1}_{\mathcal{B}/\mathcal{O}}=\mathcal{O}$, which is an isomorphism because it is locally so. Finally, for any $p\geq 0$, the natural morphism $\widetilde{\Omega}^p_{\mathcal{B}/k}\otimes\widetilde{\Omega}^{r+1}_{\mathcal{B}/k}\to\widetilde{\Omega}^{p+r+1}_{\mathcal{B}/k}$ induces a morphism $\pi_*(\widetilde{\Omega}^p_{\mathcal{B}/k})\otimes R^r\pi_*(\widetilde{\Omega}^{r+1}_{\mathcal{B}/k})\to R^r\pi_*\widetilde{\Omega}^{p+r+1}_{\mathcal{B}/k}$, i.e. a morphism $\Omega^p_{X/k}\to R^r\pi_*\widetilde{\Omega}^{p+r+1}_{\mathcal{B}/k}$, which is an isomorphism because it is locally so.

(2) The natural morphism $[\Omega^p_{\mathcal{B}/k}]_n\to\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(n)$ is an isomorphism because it is locally so.

It only remains to prove (4), which is a consequence of (relative) Grothendieck duality. Indeed, notice that, under the smoothness hypothesis, $R^r\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(-n)$ is locally free, by (3). Hence, if suffices to compute its dual. This is given by duality: the relative dualizing sheaf is $\Omega^r_{\mathbb{P}/X}=\widetilde{\Omega}^{r+1}_{\mathcal{B}/\mathcal{O}}$ and one has isomorphisms $\widetilde{\Omega}^{d+r+1}_{\mathcal{B}/k}=\widetilde{\Omega}^{r+1}_{\mathcal{B}/\mathcal{O}}\otimes\pi^*\Omega^d_{X/k}$ and $\mathcal{H}om\,(\widetilde{\Omega}^p_{\mathcal{B}/k},\widetilde{\Omega}^{d+r+1}_{\mathcal{B}/k})=\widetilde{\Omega}^{d+\bar p}_{\mathcal{B}/k}$; then:

$$\begin{aligned}\left[R^r\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(-n)\right]^*&=\pi_*\mathcal{H}om_{\,\mathbb{P}}(\widetilde{\Omega}^p_{\mathcal{B}/k}(-n),\widetilde{\Omega}^{r+1}_{\mathcal{B}/\mathcal{O}})\\&=\pi_*[\mathcal{H}om_{\,\mathbb{P}}(\widetilde{\Omega}^p_{\mathcal{B}/k}(-n),\widetilde{\Omega}^{d+r+1}_{\mathcal{B}/k})\otimes\pi^*(\Omega^d_{X/k})^*]\\&=(\pi_*\widetilde{\Omega}^{d+\bar p}_{\mathcal{B}/k}(n))\otimes(\Omega^d_{X/k})^*\overset{(2)}{=}[\Omega^{d+\bar p}_{\mathcal{B}/k}]_n\otimes(\Omega^d_{X/k})^*.\end{aligned}$$ □

Corollary 3.10. *The Koszul complexes* $\mathrm{Kos}(\mathcal{E}/k)_n$ *and* $\widetilde{\mathrm{Kos}}(\mathcal{E}/k)_n$ *are acyclic for $n>0$ and* $\mathrm{Kos}(\mathcal{E}/k)_n\to\widetilde{\mathrm{Kos}}(\mathcal{E}/k)_n$ *is an isomorphism for any $n\geq 0$.*

Let us denote by $\overline{\mathcal{K}}_{p,n}$ the kernels of the morphisms i_D in the Koszul complex $\mathrm{Kos}(\mathcal{E}/k)_n$; that is,

$$\overline{\mathcal{K}}_{p,n}:=\mathrm{Ker}\left([\Omega^p_{\mathcal{B}/k}]_n\longrightarrow[\Omega^{p-1}_{\mathcal{B}/k}]_n\right)$$

Theorem 3.11. *Let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on a k-scheme $(X,\mathcal{O})$ and $\mathbb{P}=\operatorname{Proj} S^{\cdot}\mathcal{E}\xrightarrow{\pi} X$ the corresponding projective bundle.*

Let n be a positive integer. One has:

1. $R^i\pi_*\Omega^p_{\mathbb{P}/k}=\Omega^{p-i}_{X/k}$.
2. $$R^i\pi_*\Omega^p_{\mathbb{P}/k}(n)=\begin{cases}0 & \text{for } i\neq 0\\ \overline{\mathcal{K}}_{p,n} & \text{for } i=0\end{cases}$$

and, if X is a $\mathbb{Q}$-scheme, then one has an isomorphism (of k-modules, not of $\mathcal{O}$-modules)

$$\overline{\mathcal{K}}_{p,n}\oplus\overline{\mathcal{K}}_{p-1,n}=[\Omega^p_{\mathcal{B}/k}]_n$$

3. $R^i\pi_*\Omega^p_{\mathbb{P}/k}(-n)=0$ *for* $i\neq r$ *and* $R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)$ *is locally isomorphic to*

$$\bigoplus_{q=0}^{p}\Omega^{p-q}_{X/k}\otimes\mathcal{K}^*_{r-q,n}$$

Moreover, if X is a $\mathbb{Q}$-scheme, then one has an isomorphism (of k-modules, not of $\mathcal{O}$-modules)

$$R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)\oplus R^r\pi_*\Omega^{p-1}_{\mathbb{P}/k}(-n)=R^r\pi_*\widetilde{\Omega}^p_{\mathcal{B}/k}(-n)$$

4. *If X is a smooth k-scheme (of relative dimension d), then*

$$R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)=\overline{\mathcal{K}}^*_{d+r-p,n}\otimes\Omega^d_{X/k}$$

and, if X is a $\mathbb{Q}$-scheme, then one has an isomorphism (of k-modules, not of $\mathcal{O}$-modules)

$$R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)\oplus R^r\pi_*\Omega^{p-1}_{\mathbb{P}/k}(-n)=\left[\Omega^{d+\bar{p}}_{\mathcal{B}/k}\right]^*_n\otimes\Omega^d_{X/k}$$

Proof. If $\mathcal{E}$ is trivial, then $\Omega_{\mathbb{P}/k}=\pi^*\Omega_{X/k}\oplus\Omega_{\mathbb{P}/X}$, so $\Omega^p_{\mathbb{P}/k}=\bigoplus_{q=0}^p\pi^*\Omega^q_{X/k}\otimes\Omega^{p-q}_{\mathbb{P}/X}$ and (1)–(3) follow from Theorem 3.4 in this case. Since $\mathcal{E}$ is locally trivial, we obtain the vanishing statements of (1)–(3).

(1) The exact sequences $0\to\Omega^p_{\mathbb{P}/k}\to\widetilde{\Omega}^p_{\mathcal{B}/k}\to\Omega^{p-1}_{\mathbb{P}/k}\to 0$ induce morphisms

$$\pi_*\Omega^{p-i}_{\mathbb{P}/k}\longrightarrow R^1\pi_*\Omega^{p-i+1}_{\mathbb{P}/k}\longrightarrow\dots\longrightarrow R^i\pi_*\Omega^p_{\mathbb{P}/k}$$

whose composition with the natural morphism $\Omega^{p-i}_{X/k}\to\pi_*\Omega^{p-i}_{\mathbb{P}/k}$ gives a morphism $\Omega^{p-i}_{X/k}\to R^i\pi_*\Omega^p_{\mathbb{P}/k}$. This morphism is an isomorphism because it is locally so.

(2) The exact sequence $0\to\Omega^p_{\mathbb{P}/k}(n)\to\widetilde{\Omega}^p_{\mathcal{B}/k}(n)\to\widetilde{\Omega}^{p-1}_{\mathcal{B}/k}(n)$ induces, taking direct image, the isomorphism $\pi_*\Omega^p_{\mathbb{P}/k}(n)=\overline{\mathcal{K}}_{p,n}$.

(4) follows from (2) and (relative) Grothendieck duality. Indeed, notice that, under the smoothness hypothesis, $R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)$ is locally free, by (3). Hence, if

suffices to compute its dual. This is given by duality: the relative dualizing sheaf is $\Omega^r_{\mathbb{P}/X}$ and one has isomorphisms $\Omega^{d+r}_{\mathbb{P}/k}=\Omega^r_{\mathbb{P}/X}\otimes\pi^*\Omega^d_{X/k}$ and $\mathcal{H}om\,(\Omega^p_{\mathbb{P}/k},\Omega^{d+r}_{\mathbb{P}/k})=\Omega^{d+r-p}_{\mathbb{P}/k}$; then:

$$\begin{aligned}\left[R^r\pi_*\Omega^p_{\mathbb{P}/k}(-n)\right]^* &= \pi_*\mathcal{H}om_{\,\mathbb{P}}(\Omega^p_{\mathbb{P}/k}(-n),\Omega^r_{\mathbb{P}/k})\\ &= \pi_*[\mathcal{H}om_{\,\mathbb{P}}(\Omega^p_{\mathbb{P}/k}(-n),\Omega^{d+r}_{\mathbb{P}/k})\otimes\pi^*(\Omega^d_{X/k})^*]\\ &= (\pi_*\widetilde{\Omega}^{d+r-p}_{\mathbb{P}/k}(n))\otimes(\Omega^d_{X/k})^* \overset{(2)}{=} \overline{\mathcal{K}}_{d+r-p,n}\otimes(\Omega^d_{X/k})^*\end{aligned}$$

Finally, the statements of (2)–(4) regarding the case of a $\mathbb{Q}$-scheme follow from Corollary 2.6. □

Remark 3.12. For $n=1$ a little more can be said (as Verdier does): The natural morphism $\Omega^p_{X/k}\otimes\mathcal{E}\to\pi_*\Omega^p_{\mathbb{P}/k}(1)$ is an isomorphism. Indeed, the exact sequence

$$0\longrightarrow\Omega_{X/k}\otimes\mathcal{B}\longrightarrow\Omega_{\mathcal{B}/k}\longrightarrow\Omega_{\mathcal{B}/\mathcal{O}}\longrightarrow 0$$

induces for each p an exact sequence

$$0\longrightarrow\Omega^p_{X/k}\otimes\mathcal{B}\longrightarrow\Omega^p_{\mathcal{B}/k}\longrightarrow\Omega^{p-1}_{\mathcal{B}/k}\otimes\Omega_{\mathcal{B}/\mathcal{O}}\longrightarrow\Omega^{p-2}_{\mathcal{B}/k}\otimes S^2\Omega_{\mathcal{B}/\mathcal{O}}\longrightarrow\dots$$

and taking degree 1, an exact sequence

$$0\longrightarrow\Omega^p_{X/k}\otimes\mathcal{E}\longrightarrow[\Omega^p_{\mathcal{B}/k}]_1\longrightarrow\Omega^{p-1}_{X/k}\otimes\mathcal{E}\longrightarrow 0$$

On the other hand, taking π_* in the exact sequence

$$0\longrightarrow\Omega^p_{\mathbb{P}/k}(1)\longrightarrow\widetilde{\Omega}^p_{\mathcal{B}/k}(1)\longrightarrow\Omega^{p-1}_{\mathbb{P}/k}(1)\longrightarrow 0$$

gives the exact sequence

$$0\longrightarrow\pi_*\Omega^p_{\mathbb{P}/k}(1)\longrightarrow[\Omega^p_{\mathcal{B}/k}]_1\longrightarrow\pi_*\Omega^{p-1}_{\mathbb{P}/k}(1)\longrightarrow 0$$

Thus, the isomorphism $\Omega^p_{X/k}\otimes\mathcal{E}\to\pi_*\Omega^p_{\mathbb{P}/k}(1)$ is proved by induction on p.

Remark 3.13. It is known (see [1] or [6]) that $\mathbb{R}\pi_*\Omega^p_{\mathbb{P}/k}$ is decomposable, i.e., one has an isomorphism in the derived category $\mathbb{R}\pi_*\Omega^p_{\mathbb{P}/k}=\bigoplus_{i=0}^r\Omega^{p-i}_{X/k}[-i]$. Let us see that, for $p\in[0,r]$, this is a consequence of Theorem 2.1 and Proposition 3.9. Indeed, by Theorem 2.1, one has the exact sequence

$$0\longrightarrow\Omega^p_{\mathbb{P}/k}\longrightarrow\widetilde{\Omega}^p_{\mathcal{B}/k}\longrightarrow\widetilde{\Omega}^{p-1}_{\mathcal{B}/k}\longrightarrow\dots\longrightarrow\widetilde{\Omega}_{\mathcal{B}/k}\longrightarrow\mathcal{O}_{\mathbb{P}}\longrightarrow 0$$

and, by Proposition 3.9, $\widetilde{\Omega}^{p-i}_{\mathcal{B}/k}$ are π_*-acyclic for any $i\geq 0$ and $\pi_*\widetilde{\Omega}^{p-i}_{\mathcal{B}/k}=\Omega^{p-i}_{X/k}$. Then

$$\mathbb{R}\pi_*\Omega^p_{\mathbb{P}/k}\equiv 0\longrightarrow\Omega^p_{X/k}\longrightarrow\Omega^{p-1}_{X/k}\longrightarrow\dots\longrightarrow\Omega_{X/k}\longrightarrow\mathcal{O}\longrightarrow 0$$

and, since the differential $i_D\colon\Omega^j_{X/k}\to\Omega^{j-1}_{X/k}$ is null, we obtain the result.

The decomposability of $\mathbb{R}\pi_*\Omega^p_{\mathbb{P}/k}$ implies an isomorphism

$$H^q(\mathbb{P},\Omega^p_{\mathbb{P}/k}) = \bigoplus_{i=0}^{r} H^{q-i}(X,\Omega^{p-i}_{X/k})$$

For the twisted case we have the following:

Corollary 3.14. *Let X be a proper scheme over a field k of characteristic zero. Let $\mathcal{E}$ be a locally free module on X of rank $r+1$ and $\mathbb{P}=\operatorname{Proj} S^{\cdot}\mathcal{E}$ the associated projective bundle. Then, for any positive integer n, one has*:

1. $\dim_k H^q(\mathbb{P},\Omega^p_{\mathbb{P}/k}(n)) = \sum_{i=0}^{p}(-1)^i \dim_k H^q(X,[\Omega^{p-i}_{\mathcal{B}/k}]_n)$.
2. *If X is smooth over k of dimension d, then*

$$\dim_k H^q(\mathbb{P},\Omega^p_{\mathbb{P}/k}(-n)) = \sum_{i=0}^{d+r-p} (-1)^i \dim_k H^{d+r-q}(X,[\Omega^{d+r-p-i}_{\mathcal{B}/k}]_n)$$

Proof. (1) By Corollary 2.6,

$$H^q(\mathbb{P},\Omega^p_{\mathbb{P}/k}(n)) \oplus H^q(\mathbb{P},\Omega^{p-1}_{\mathbb{P}/k}(n)) = H^q(\mathbb{P},\widetilde{\Omega}^p_{\mathcal{B}/k}(n))$$

and $H^q(\mathbb{P},\widetilde{\Omega}^p_{\mathcal{B}/k}(n)) = H^q(X,[\Omega^p_{\mathcal{B}/k}]_n)$ by Proposition 3.9. Conclusion follows.

(2) follows from (1) and duality. □

References

1. Berthelot, P. and Illusie, L., Classes de Chern en cohomologie cristalline, *C. R. Acad. Sci. Paris Sér. A–B* **270** (1970), A1750–A1752, *C. R. Acad. Sci. Paris Sér. A–B* **270** (1970), A1695–A1697.
2. Bott, R., Homogeneous vector bundles, *Ann. of Math.* (2) **66** (1957), 203–248.
3. Bourbaki, N., *Éléments de mathématique. Algèbre*, *Chapitre* 10. *Algèbre homologique*, Masson, Paris, 1980.
4. Deligne, P., Groupes de monodromie en géométrie algébrique. II, in *Séminaire de Géométrie Algébrique du Bois-Marie* 1967–1969 (*SGA* 7 *II*), Lecture Notes in Mathematics **340**, Springer, Berlin, 1973.
5. Giral, J. M. and Planas-Vilanova, F., A note on the acyclicity of the Koszul complex of a module, *Ark. Mat.* **45** (2007), 273–278.
6. Gros, M., Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique, *Mém. Soc. Math. Fr.* (*N.S.*) **21** (1985), 87.
7. Grothendieck, A., Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes, *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 222.
8. Grothendieck, A., Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I, *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 167.

9. Micali, A., Sur les algèbres universelles, *Ann. Inst. Fourier* (*Grenoble*) **14** (1964), 33–87.
10. Quillen, D., *Homology of Commutative Rings*, MIT, Cambridge, MA, 1967, mimeographed notes.
11. Sancho de Salas, F., Residues of a Pfaff system relative to an invariant subscheme, *Trans. Amer. Math. Soc.* **352** (2000), 4019–4035.
12. Verdier, J. L., Le théorème de Le Potier. Différents aspects de la positivité, in *Sém. Géom. Anal., École Norm. Sup., Paris*, 1972–1973, Astérisque **17**, pp. 68–78, Soc. Math. France, Paris, 1974.

Björn Andreas
Mathematisches Institut
Freie Universität Berlin
Berlin
Germany
andreasb@math.fu-berlin.de

Darío Sánchez Gómez
Departamento de Matemáticas and
Instituto Universitario de Física
Fundamental y Matemáticas (IUFFyM)
Universidad de Salamanca
Salamanca
Spain
dario@usal.es

Fernando Sancho de Salas
Departamento de Matemáticas and
Instituto Universitario de Física
Fundamental y Matemáticas (IUFFyM)
Universidad de Salamanca
Salamanca
Spain
fsancho@usal.es

Received January 4, 2016
published online July 27, 2016