Ark. Mat., **52** (2014), 113–134
DOI: 10.1007/s11512-012-0172-x

Bad boundary behavior in star-invariant subspaces I

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Abstract. We discuss the boundary behavior of functions in star-invariant subspaces $(BH^2)^\perp$, where B is a Blaschke product. Extending some results of Ahern and Clark, we are particularly interested in the growth rates of functions at points of the spectrum of B where B does not admit a derivative in the sense of Carathéodory.

1. Introduction

For a Blaschke product B with zeros $\{\lambda_n\}_{n\geq 1}\subset\mathbb{D}=\{z:|z|<1\}$, repeated according to multiplicity, let us recall the following theorem of Ahern and Clark [1] about the "good" non-tangential boundary behavior of functions in the model spaces $(BH^2)^\perp:=H^2\ominus BH^2$ [7] of the Hardy space H^2 of $\mathbb{D}$ ([4] and [6]).

Theorem 1.1. ([1]) *For a Blaschke product B with zeros $\{\lambda_n\}_{n\geq 1}$ and $\zeta\in\mathbb{T}:=\partial\mathbb{D}$, the following are equivalent*:

(1) *Every $f\in(BH^2)^\perp$ has a non-tangential limit at ζ, i.e.,*

$$f(\zeta):=\measuredangle\lim_{\lambda\to\zeta} f(\lambda) \text{ exists.}$$

(2) *B has an angular derivative in the sense of Carathéodory at ζ, i.e.,*

$$\measuredangle\lim_{z\to\zeta} B(z)=\eta\in\mathbb{T} \quad \text{and} \quad \measuredangle\lim_{z\to\zeta} B'(z) \text{ exists.}$$

(3) *The following condition holds*

$$\sum_{n\geq 1}\frac{1-|\lambda_n|}{|\zeta-\lambda_n|^2}<\infty. \tag{1.1}$$

The first author was supported by ANR FRAB: ANR-09-BLAN-0058-02. The authors would like to thank the referee for some useful comments and a simplification.

(4) *The family of reproducing kernels for* $(BH^2)^\perp$

$$k_\lambda^B(z) := \frac{1-\overline{B(\lambda)}B(z)}{1-\bar{\lambda}z}$$

is uniformly norm-bounded in each fixed Stolz domain

$$\Gamma_{\alpha,\zeta} := \left\{ z \in \mathbb{D} : \frac{|z-\zeta|}{1-|z|} < \alpha \right\}, \quad \alpha \in (1,\infty).$$

We point out three things here. First, the equivalence of conditions (2) and (3) of this theorem is a classical result of Frostman [5]. Moreover, condition (1) implies condition (4) by the uniform boundedness principle, while the reverse implication follows from the Banach–Alaoglu theorem. Second, this theorem can be extended to characterize the existence of non-tangential boundary limits of the derivatives (up to a given order) of functions in $(BH^2)^\perp$ as well as the boundary behavior of functions in $(IH^2)^\perp$, where I is a general inner function [1]. Third, there is a version of this result for various types of *tangential* boundary behavior of $(BH^2)^\perp$ functions ([2] and [9]). Of course there is the well-known result (see e.g. [7, p. 78]) which says that every $f \in (BH^2)^\perp$ has an analytic continuation across the complement of the accumulation points of the zeros of B.

In this paper we consider the growth of functions in $(BH^2)^\perp$ at the points $\zeta \in \mathbb{T}$ where (1.1) fails. Thus, as in the title of this paper, we are looking at the "bad" boundary behavior of functions from $(BH^2)^\perp$. First observe that every function $f \in H^2$ satisfies

$$|f(\lambda)| = o\left(\frac{1}{\sqrt{1-|\lambda|}}\right), \quad \lambda \in \Gamma_{\alpha,\zeta}, \tag{1.2}$$

and this growth is, in a sense, maximal. As seen in the Ahern–Clark theorem, functions in $(BH^2)^\perp$ can be significantly better behaved depending on the distribution of the zeros of B. We are interested in examining Blaschke products for which the growth rates for functions in $(BH^2)^\perp$ are somewhere between the Ahern–Clark situation, where every function has a non-tangential limit, and the maximal allowable growth in (1.2).

To explain this a bit more, let $\zeta = 1$ and observe that

$$|f(\lambda)| = |\langle f, k_\lambda^B \rangle| \leq \|f\| \left(\frac{1-|B(\lambda)|^2}{1-|\lambda|^2}\right)^{1/2}, \quad f \in (BH^2)^\perp \text{ and } \lambda \in \mathbb{D}. \tag{1.3}$$

In the above, $\|\cdot\|$ denotes the usual norm in H^2. So, in order to give an upper estimate of the admissible growth in a Stolz domain $\Gamma_{\alpha,1}$, we have to control $\|k_\lambda^B\|$ which ultimately involves getting a handle on how fast $|B(\lambda)|$ goes to 1 in $\Gamma_{\alpha,1}$.

Of course the subtlety occurs when

$$\measuredangle \lim_{z\to 1} B(z) = \eta \in \mathbb{T}$$

which is implied by the Frostman condition ([3] and [5])

$$\sum_{n\geq 1} \frac{1-|\lambda_n|}{|1-\lambda_n|} < \infty. \tag{1.4}$$

Observe the power 1 in the denominator in (1.4) compared with the square in the Ahern–Clark condition (1.1).

The main results of this paper will be non-tangential growth estimates of functions in $(BH^2)^{\perp}$ via non-tangential growth estimates of the norms of the kernel functions. Our main results (Theorems 3.1, 3.6, and 4.1) will be estimates of the form

$$\|k_r^B\| \asymp h(r), \quad r \to 1^-,$$

for some $h\colon [0,1)\to\mathbb{R}_+$ which depends on the position of the zeros of the Blaschke product B near 1. This will, of course via (1.3), yield the estimate

$$|f(r)| \lesssim h(r), \quad f\in (BH^2)^{\perp} \text{ and } r\to 1^-.$$

To get a handle on the sharpness of this growth estimate, we will show (Theorem 3.4) that for every $\varepsilon>0$, there exists an $f\in(BH^2)^{\perp}$ satisfying

$$|f(r)| \gtrsim \frac{h(r)}{\log^{1+\varepsilon} h(r)}, \quad r\to 1^-. \tag{1.5}$$

(All logarithms appearing in this paper should be understood in base 2.)

While this estimate might not be optimal, it allows us to show that a certain sequence of reproducing kernels cannot form an unconditional sequence (see Section 5).

Though a general result will be discussed in Section 4, the two basic types of Blaschke sequences $\{\lambda_n\}_{n\geq 1}$ for which we can get concise estimates of $\|k_r^B\|$, are

$$\lambda_n = (1-x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n \downarrow 0, \tag{1.6}$$

which approaches 1 very tangentially, and

$$\lambda_n = (1-\theta_n^2) e^{i\theta_n}, \quad 0<\theta_n<1 \text{ and } \sum_{n\geq 1} \theta_n < \infty, \tag{1.7}$$

which approaches 1 along an oricycle. For example, when $x_n=1/n$ in (1.6), we have the upper estimate (see Example 3.3(1))

$$|f(r)| \lesssim \sqrt{\log\log\frac{1}{1-r}}, \quad r\to 1^-,$$

for all $f\in(BH^2)^{\perp}$. This estimate is optimal in the sense of (1.5).

Picking $\theta_n = 1/n^\alpha$, $\alpha > 1$, in (1.7), we have the estimate (see Example 3.7(1))

$$|f(r)| \lesssim \frac{1}{(1-r)^{1/2\alpha}}, \quad r \to 1^-.$$

Compare these two results to the growth rate in (1.2) of a generic H^2 function.

This is the first of two papers on "bad" boundary behavior of $(IH^2)^\perp$ functions near a fixed point on the circle, where I is inner. In this paper we consider the case when I is a Blaschke product giving exact estimates on the norm of the reproducing kernel. The next paper will consider the case when I is a general inner function providing only upper estimates.

2. What can be expected

We have already mentioned that every $f \in H^2$ satisfies

$$|f(\lambda)| = o\left(\frac{1}{\sqrt{1-|\lambda|}}\right), \quad \lambda \in \Gamma_{\alpha,\zeta}. \tag{2.1}$$

The little-oh condition in (2.1) is, in a sense, sharp since one can construct suitable outer functions whose non-tangential growth gets arbitrarily close to that in (2.1).

Contrast this with the following result which shows that functions in certain $(BH^2)^\perp$ spaces can*not* reach the maximal growth in (2.1). Recall that a sequence $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$ is *interpolating* if

$$H^2|\Lambda = \left\{ \{a_n\}_{n\geq 1} : \sum_{n \geq 1} (1-|\lambda_n|^2)|a_n|^2 < \infty \right\},$$

where $X|\Lambda = \{\{f(\lambda_n)\}_{n\geq 1} : f \in X\}$ for a space X of holomorphic functions on $\mathbb{D}$.

Proposition 2.1. ([10]) *Let B be a Blaschke product whose zeros λ_n form an interpolating sequence and tend non-tangentially to 1. Then for $\{\varepsilon_n\}_{n\geq 1}$ there exists $f \in (BH^2)^\perp$ with*

$$|f(\lambda_n)| = \varepsilon_n \frac{1}{\sqrt{1-|\lambda_n|}} \quad \text{for all } n \in \mathbb{N}$$

if and only if $\{\varepsilon_n\}_{n\geq 1} \in \ell^2$.

Strictly speaking this result is stated in H^2 (and for arbitrary interpolating sequences), but since functions in BH^2 vanish on Λ, we obviously have $(BH^2)^\perp|\Lambda = H^2|\Lambda$.

A central result in our discussion is the following lemma.

Lemma 2.2. *If B is a Blaschke product with zeros $\lambda_n = r_n e^{i\theta_n}$ and*

$$\measuredangle \lim_{z\to 1} B(z) = \eta \in \mathbb{T},$$

then

$$\|k_r^B\|^2 \asymp \sum_{n\geq 1} \frac{1-r_n^2}{|1-\bar{\lambda}_n r|^2}, \quad r \in (0,1).$$

(The estimate extends naturally to a Stolz angle.)

Proof. Since $\measuredangle \lim_{z\to 1} B(z) = \eta \in \mathbb{T}$, the zeros of B (after some point) cannot lie in $\Gamma_{\alpha,1}$. Thus if

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z},$$

then

$$\inf_{n\geq 1} |b_{\lambda_n}(r)| \geq \delta > 0$$

and so

$$\log \frac{1}{|b_\lambda(r)|^2} \asymp 1 - |b_{\lambda_n}(r)|^2.$$

Use the well-known identity

$$1-|b_{\lambda_n}(r)|^2 = \frac{(1-r^2)(1-|\lambda_n|^2)}{|1-r\bar{\lambda}_n|^2},$$

to get

$$\log \frac{1}{|B(r)|^2} = \sum_{n\geq 1} \log \frac{1}{|b_{\lambda_n}(z)|^2} \asymp \sum_{n\geq 1} \frac{(1-|\lambda_n|^2)(1-|r|^2)}{|1-\bar{\lambda}_n r|^2} \asymp (1-r^2) \sum_{n\geq 1} \frac{1-r_n^2}{|1-\bar{\lambda}_n r|^2}.$$

Since $|B(r)| \to 1$ when $r \to 1^-$ the latter quantity goes to 0 and so

$$\|k_r^B\|^2 = \frac{1-|B(r)|^2}{1-r^2} \asymp -\frac{\log |B(r)|^2}{1-r^2} \asymp \sum_{n\geq 1} \frac{1-r_n^2}{|1-\bar{\lambda}_n r|^2}. \qquad \square$$

3. Key examples

We will prove a general growth result in Theorem 4.1. But just to give a more tangible approach to the subject, let us begin by obtaining growth estimates of functions in $(BH^2)^\perp$ for Blaschke products B whose zeros are

$$\lambda_n = (1-x_n 2^{-2n})e^{i2^{-n}}, \quad x_n \downarrow 0,$$

which approaches 1 very tangentially, or

$$\lambda_n = (1-\theta_n^2)e^{i\theta_n}, \quad 0<\theta_n<1 \text{ and } \sum_{n\geq 1}\theta_n<\infty,$$

which (essentially) approaches 1 along an oricycle.

3.1. First class of examples

$\Lambda=\{\lambda_k\}_{k\geq 1}$ with $\lambda_k=r_k e^{i\theta_k}$ and

$$1-r_k = x_k\theta_k^2, \quad \theta_k = \frac{1}{2^k} \text{ and } k\in\mathbb{N}. \tag{3.1}$$

We will suppose that $\{x_n\}_{n\geq 1}$ is a sequence of positive numbers satisfying

$$\frac{x_{n+1}}{x_n} \leq q < 2.$$

This in particular implies that $0\leq x_n \lesssim q^n \ll 2^n$ (however, in our examples below we will be essentially interested in examples for which $x_n\to 0$ and $n\to\infty$). Note that

$$1-r_k = x_k\theta_k^2 \lesssim \left(\frac{q}{4}\right)^k = \theta_k^\alpha,$$

where $\alpha=\log(4/q)>\log 2=1$, so that Λ goes tangentially to 1. Again, we will be in particular interested in sequences with $x_k\downarrow 0$. In this situation, the faster x_k decreases to zero, the more tangential the sequence Λ. The condition on $\{x_n\}_{n\geq 1}$ also implies that

$$\sum_{n\geq 1}(1-|\lambda_n|) = \sum_{n\geq 1}(1-r_n) = \sum_{n\geq 1}\frac{x_n}{2^{2n}} \lesssim \sum_{n\geq 1}\left(\frac{q}{4}\right)^n < \infty,$$

and so Λ is indeed a Blaschke sequence.

We will need the well-known Pythagorean-type result: If $\lambda=re^{i\theta}$, $r\in(0,1)$ and $\rho\in(0,1]$, then

$$|1-\rho\lambda|^2 \asymp (1-\rho r)^2+\theta^2 \asymp ((1-\rho r)+\theta)^2, \quad \rho\approx 1,\ r\approx 1 \text{ and } \theta\approx 0. \tag{3.2}$$

Observe that using (3.2) and $0 \le x_k\theta_k^2 \ll \theta_k$, we get

$$|1-\lambda_k| \asymp \sqrt{(1-r_k)^2+\theta_k^2} \asymp (1-r_k)+\theta_k \asymp x_k\theta_k^2+\theta_k \asymp \theta_k.$$

Hence

$$\sum_{n\ge 1} \frac{1-|\lambda_n|}{|1-\lambda_n|} \asymp \sum_{n\ge 1} \frac{1-r_n}{\theta_k} = \sum_{n\ge 1} \theta_n x_n < \infty$$

and so condition (1.4) is satisfied thus ensuring $\measuredangle \lim_{z\to 1} B(z) = \eta \in \mathbb{T}$. Similarly,

$$\sum_{n\ge 1} \frac{1-|\lambda_n|}{|1-\lambda_n|^2} \asymp \sum_{n\ge 1} x_n.$$

In light of the Ahern–Clark result (1.1), we will be interested in the "bad behavior" scenario when $\sum_{n\ge 1} x_n = \infty$.

Theorem 3.1. *For a sequence of positive numbers* $\{x_n\}_{n\ge 1}$ *with*

$$\frac{x_{n+1}}{x_n} \le q < 2,$$

consider the Blaschke product whose zeros are

$$\lambda_n = (1-x_n 2^{-2n})e^{i2^{-n}}.$$

Set

$$\sigma_N := \sum_{n=1}^{N} x_n,$$

and let φ_0 *be continuous on* $\mathbb{R}_+$, *piecewise linear, and such that* $\varphi_0(N) = \sigma_N$. *Define* φ *by*

$$\varphi(y) := \varphi_0\left(\log \frac{1}{1-y}\right).$$

Then

$$\|k_z^B\| \asymp \sqrt{\varphi(|z|)}, \quad z \in \Gamma_{\alpha,1},$$

and so every $f \in (BH^2)^\perp$ *satisfies*

$$|f(z)| \lesssim \sqrt{\varphi(|z|)}, \quad z \in \Gamma_{\alpha,1}.$$

The function φ_0 is an increasing function which is concave (convex) when $\{x_n\}_{n\ge 1}$ is decreasing (increasing).

A direct consequence of this theorem is the following result.

Corollary 3.2. *For every concave function $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ increasing to infinity, there exists a Blaschke product B, whose zeros accumulate at 1, such that*

$$\|k_z^B\| \asymp \sqrt{\psi(|z|)}, \quad z \in \Gamma_{\alpha,1},$$

where

$$\psi(y) = \psi_0\left(\log\left(\frac{1}{1-y}\right)\right), \quad y \in (0,1).$$

Proof. It suffices to pick

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}$$

with $x_n = \sigma_{n+1} - \sigma_n$ and $\sigma_n = \psi_0(n)$. Since ψ_0 is increasing and concave, the sequence $\{x_n\}_{n\geq 1}$ is positive and decreasing which in particular gives $x_{n+1}/x_n \leq q < 2$. Let φ_0 be the function constructed from $\{x_n\}_{n\geq 1}$ as in the theorem which is also concave. It is easy to see that $\varphi_0 \asymp \psi_0$, and the theorem allows us to conclude. □

Before discussing the proof, here are two concrete examples showing how the growth slows down when approaching the Ahern–Clark situation, i.e., the summability of the sequence $\{x_n\}_{n\geq 1}$.

Example 3.3. (1) If B is a Blaschke product whose zeros are

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n = \frac{1}{n},$$

then

$$\sigma_N = \sum_{n=1}^{N} \frac{1}{n} \asymp \log N$$

and so every $f \in (BH^2)^\perp$ satisfies the growth condition

$$|f(r)| \lesssim \sqrt{\log\log\frac{1}{1-r}}, \quad r \to 1^-.$$

(2) If the zeros of B are

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n = \frac{1}{n \log n},$$

then $\sigma_N \asymp \log\log N$ and so every $f \in (BH^2)^\perp$ satisfies

$$|f(r)| \lesssim \sqrt{\log\log\log\frac{1}{1-r}}, \quad r \to 1^-.$$

Proof of Theorem 3.1. We have already mentioned that

$$0 \le x_n \lesssim q^n \ll 2^n.$$

Set $\rho_N = 1-2^{-N}$ and $\theta_k = 2^{-k}$. Using (3.2) we have

$$\begin{aligned}|1-\rho_N\lambda_k|^2 &\asymp (\theta_k+(1-\rho_N r_k))^2 = (\theta_k+(1-\rho_N(1-x_k\theta_k^2)))^2\\ &= (\theta_k+(1-\rho_N)+\rho_N x_k\theta_k^2)^2,\end{aligned}$$

and since by the above observation $x_k\theta_k^2 \ll \theta_k$ when $k\to\infty$, we get

$$|1-\rho_N\lambda_k|^2 \asymp (\theta_k+(1-\rho_N))^2. \tag{3.3}$$

Hence

$$\frac{1-r_k^2}{|1-\rho_N\lambda_k|^2} \asymp \frac{x_k\theta_k^2}{(\theta_k+(1-\rho_N))^2} = \frac{x_k\theta_k^2}{(\theta_k+\theta_N)^2} \asymp \begin{cases} \dfrac{x_k\theta_k^2}{\theta_k^2}, & \text{if } k\le N,\\[2mm] \dfrac{x_k\theta_k^2}{\theta_N^2}, & \text{if } k> N,\end{cases} \tag{3.4}$$

$$\asymp \begin{cases} x_k, & \text{if } k\le N,\\ \dfrac{{}_k\theta_k^2}{\theta_N^2}, & \text{if } k> N.\end{cases}$$

Thus we can split the sum in Lemma 2.2 into two parts

$$\|k_{\rho_N}^B\|^2 \asymp \sum_{k\ge 0} \frac{1-r_k^2}{|1-\rho_N\lambda_k|^2} \asymp \sum_{k\le N} x_k + 2^{2N}\sum_{k\ge N+1} x_k\theta_k^2.$$

The first term is exactly σ_N. For the second term, observe that for $k\ge N+1$,

$$x_k\theta_k^2 = \prod_{l=N+1}^{k-1} \frac{x_{l+1}\theta_{l+1}^2}{x_l\theta_l^2} x_{N+1}\theta_{N+1}^2 \le x_{N+1}\theta_{N+1}^2\left(\frac{q}{4}\right)^{k-1-N} \le \frac{x_{N+1}}{2^{2N}}2^{-(k-1-N)}$$

which yields

$$2^{2N}\sum_{k\ge N+1} x_k\theta_k^2 \lesssim x_{N+1} \le q x_N \le q\sigma_N.$$

These estimates immediately give us the required estimate for $\rho_N = 1-1/2^N$,

$$\|k_{\rho_N}^B\|^2 \asymp \sigma_N = \varphi_0(N) = \varphi(\rho_N).$$

In order to get the same estimate for $z\in\Gamma_{\alpha,1}$ we need the following well-known result

$$|b_\lambda(\mu)| \le \varepsilon < 1 \quad\Longrightarrow\quad \frac{1-\varepsilon}{1+\varepsilon} \le \frac{|1-\bar\lambda z|}{|1-\bar\mu z|} \le \frac{1+\varepsilon}{1-\varepsilon},\ z\in\mathbb{D}. \tag{3.5}$$

Now let $z\in\Gamma_{\alpha,1}$ and suppose that $|z|>\frac{1}{2}$. Then there exists an N such that

$$|b_z(\rho_N)|=|b_z(1-2^{-N})|\le\delta<1$$

(where δ only depends on the opening of the Stolz angle). Hence

$$\|k_z^B\|^2\asymp\sum_{n\ge1}\frac{1-r_n^2}{|1-\bar\lambda_n z|^2}\asymp\sum_{n\ge1}\frac{1-r_n^2}{|1-\bar\lambda_n\rho_N|^2}\asymp\|k_{\rho_N}\|^2, \tag{3.6}$$

and so

$$\|k_z^B\|^2\asymp\|k_{\rho_N}^B\|^2\asymp\sigma_N.$$

Clearly

$$1\le\frac{\sigma_{n+1}}{\sigma_n}\le1+\frac{x_{n+1}}{\sigma_n}\le1+\frac{x_{n+1}}{x_n}\le1+q,$$

so that $\sigma_N\asymp\sigma_{N+1}\asymp\sigma_{N-1}$. Hence, by the construction of φ_0, we also have

$$\varphi_0(x)\asymp\varphi_0(N)=\sigma_N,\quad N-1\le x\le N+1.$$

Taking into account that $\rho_{N-1}\le|z|\le\rho_{N+1}$, we get

$$\|k_z^B\|^2\asymp\|k_{\rho_N}^B\|^2\asymp\sigma_N\asymp\varphi(|z|).$$

This completes our proof. $\square$

We would now like to consider the sharpness of the growth in Theorem 3.1.

Theorem 3.4. *Suppose B is a Blaschke product whose zeros satisfy the conditions of Theorem 3.1. Then for every $\varepsilon>0$ there exists an $f\in(BH^2)^\perp$ such that*

$$|f(z)|\gtrsim\sqrt{\frac{\varphi(|z|)}{\log^{1+\varepsilon}\varphi(|z|)}},\quad z\in\Gamma_{\alpha,1}. \tag{3.7}$$

An immediate consequence of this result is the following corollary.

Corollary 3.5. *For every concave function $\psi_0:\mathbb{R}_+\to\mathbb{R}_+$ increasing to infinity, there exists a Blaschke product B whose zeros accumulate at 1 and such that for every $\varepsilon>0$ there exists an $f\in(BH^2)^\perp$ with*

$$|f(z)|\gtrsim\sqrt{\frac{\psi(|z|)}{\log^{1+\varepsilon}\psi(|z|)}},\quad z\in\Gamma_{\alpha,1},$$

where $\psi(y)=\psi_0(\log(1/(1-y)))$, $y\in(0,1)$, while $\|k_z^B\|\asymp\sqrt{\psi(|z|)}$.

Proof of Theorem 3.4. Functions in $(BH^2)^\perp$ behave rather nicely if the sequence Λ is interpolating. To see that Λ is interpolating, recall that

$$\sup_{k\geq 1}\frac{x_{k+1}}{x_k}\leq q<2.$$

Hence

$$|b_{r_k}(r_{k+1})|\asymp\frac{x_k2^{-2k}-x_{k+1}2^{-2(k+1)}}{x_k2^{-2k}+x_{k+1}2^{-2(k+1)}}=\frac{1-\frac{1}{4}x_{k+1}/x_k}{1+\frac{1}{4}x_{k+1}/x_k}\geq\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}.$$

Thus the sequence of moduli is pseudo-hyperbolically separated which implies that the sequence of moduli is interpolating—as will be the one spread out by the arguments, i.e., Λ.

Now, since Λ is an interpolating sequence, we also know that the normalized reproducing kernels

$$K_n:=\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}=\frac{\sqrt{1-|\lambda_n|^2}}{1-\bar{\lambda}_n z},\quad n\in\mathbb{N},$$

form an unconditional basis for $(BH^2)^\perp$. This is essentially a result by Shapiro and Shields [10], see also [8, Section 3] and in particular [8, Exercise C3.3.3(c)]. Hence for every $f\in(BH^2)^\perp$, there is a sequence $\alpha:=\{\alpha_n\}_{n\geq 1}\in\ell^2$ such that

$$f_\alpha(z):=\sum_{n\geq 1}\alpha_n\frac{k_{\lambda_n}(z)}{\|k_{\lambda_n}\|}=\sum_{n\geq 1}\alpha_n\frac{\sqrt{1-r_n^2}}{1-r_ne^{-i\theta_n}z}. \tag{3.8}$$

We will examine this series for $z=r\in[0,1)$ (it could be necessary at some point to require $r\geq r_0>0$). In what follows we will assume that $\alpha_n>0$. Note that the argument of $1-rr_ne^{-i\theta_n}$ is positive (this is γ_n in Figure 1).

Hence, for fixed $\rho_N=1-2^{-N}$, we have

$$\begin{aligned}|f_\alpha(\rho_N)|\geq|\operatorname{Im}f_\alpha(\rho_N)|&=-\sum_{n\geq 1}\alpha_n\operatorname{Im}\left(\frac{\sqrt{1-r_n^2}}{1-\rho_Nr_ne^{-i\theta_n}}\right)\\&\geq-\sum_{n=1}^{N}\alpha_n\operatorname{Im}\left(\frac{\sqrt{1-r_n^2}}{1-\rho_Nr_ne^{-i\theta_n}}\right).\end{aligned} \tag{3.9}$$

In order to consider the last sum appearing in (3.9), we will first show that for $1\leq n\leq N$ the argument of $1-e^{-i\theta_n}\rho_Nr_n$ is uniformly close to $\pi/2$ (or at least from a certain n_0 on), meaning that $1-e^{-i\theta_n}\rho_Nr_n$ points in a direction uniformly close to the positive imaginary axis (this is actually clear from the tangential convergence

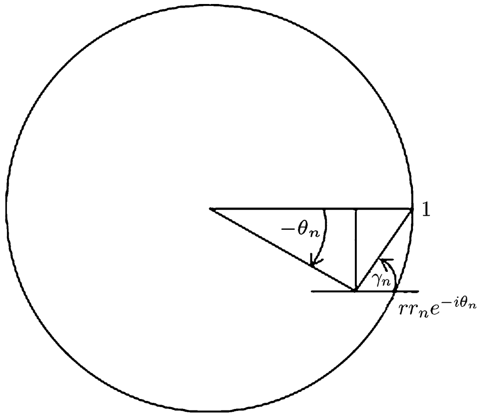

Figure 1. Angles.

of the sequence). To give a little argument, set $\gamma_n = \arg(1-\rho_N r_n e^{-i\theta_n})$. Then for sufficiently big $n \le N$,

$$\tan\gamma_n = \frac{r_n\rho_N \sin\theta_n}{1-r_n\rho_N\cos\theta_n} \asymp \frac{\theta_n}{1-(1-x_n\theta_n^2)(1-\theta_N)\big(1-\frac{1}{2}\theta_n^2+o(\theta_n^2)\big)} \ge \frac{\theta_n}{1-(1-\theta_N)} \ge 1.$$

Hence the argument of $1-\rho_N r_n e^{-i\theta_n}$ is uniformly bounded away from zero and less than $\pi/2$ so that

$$1 \ge \sin\arg(1-\rho_N r_n e^{-i\theta_n}) \ge \eta > 0.$$

In particular, for $1 \le n \le N$,

$$\left|\operatorname{Im}\frac{1}{1-\rho_N r_n e^{-i\theta_n}}\right| \asymp \frac{1}{|1-\rho_N r_n e^{-i\theta_n}|} \asymp \frac{1}{\theta_n+(1-\rho_N)} \asymp \frac{1}{\theta_n}.$$

This implies that

$$|f_\alpha(\rho_N)| \ge \sum_{n=1}^N \alpha_n\sqrt{1-r_n^2}\left|\operatorname{Im}\frac{1}{1-\rho_N r_n e^{-i\theta_n}}\right| \asymp \sum_{n=1}^N \alpha_n \frac{\sqrt{x_n}\theta_n}{\theta_n} = \sum_{n=1}^N \alpha_n\sqrt{x_n}.$$

Let us discuss the following choice

$$\alpha_n := \sqrt{\frac{x_n}{\sigma_n \log^{1+\varepsilon}\sigma_n}}.$$

We need to show two things (i) we get the desired lower estimate in the statement of the theorem; and (ii) $\{\alpha_n\}_{n\ge1} \in \ell^2$. Let us begin with the lower estimate. Observe that σ_N is increasing and so

$$\sum_{n=1}^{N}\alpha_n\sqrt{x_n}=\sum_{n=1}^{N}\frac{\sqrt{x_n}}{\sqrt{\sigma_n\log^{1+\varepsilon}\sigma_n}}\sqrt{x_n}=\sum_{n=1}^{N}\frac{x_n}{\sqrt{\sigma_n\log^{1+\varepsilon}\sigma_n}}$$
$$\geq\frac{1}{\sqrt{\sigma_N\log^{1+\varepsilon}\sigma_N}}\sum_{n=1}^{N}x_n=\frac{\sigma_N}{\sqrt{\sigma_N\log^{1+\varepsilon}\sigma_N}}=\sqrt{\frac{\sigma_N}{\log^{1+\varepsilon}\sigma_N}}.$$

This proves that

$$|f(\rho_N)|\gtrsim\sqrt{\frac{\sigma_N}{\log^{1+\varepsilon}\sigma_N}}.$$

To get the desired inequality in (3.7) (i.e., replace ρ_N with $z\in\Gamma_{\alpha,1}$), apply the argument used to prove (3.6).

To show that $\{\alpha_n\}_{n\geq1}\in\ell^2$, observe that

$$\sum_{n=1}^{N}\alpha_n^2=\sum_{n=1}^{N}\frac{x_n}{\sigma_n\log^{1+\varepsilon}\sigma_n}=\sum_{n=1}^{N}\frac{\sigma_n-\sigma_{n-1}}{\sigma_n\log^{1+\varepsilon}\sigma_n},$$

where we set $\sigma_0=1$ (since $\sigma_N\uparrow\infty$, we can assume that $\sigma_1>1$). This is a lower Riemann sum for the integral

$$\int_{\sigma_0}^{\sigma_N}\frac{1}{t\log^{1+\varepsilon}t}\,dt$$

which has a limit as $N\to\infty$. This completes our proof. □

Without going into cumbersome technical details, here is another remark on the optimality of Theorem 3.4. We are interested in the following question: For which sequences $\varepsilon_n\downarrow0$ does there exist a sequence $\{\alpha_n\}_{n\geq1}\in\ell^2$ such that

$$\sum_{n=1}^{N}\alpha_n\sqrt{x_n}=\varepsilon_N\sigma_N? \tag{3.10}$$

For example, when $x_n\equiv1$ (Theorem 3.4 is valid in this setting) we have $\sigma_N=N$ and the question becomes: For which sequences $\varepsilon_n\downarrow0$ does there exist a sequence $\{\alpha_n\}_{n\geq1}\in\ell^2$ such that

$$\sum_{n=1}^{N}\alpha_n=\varepsilon_N\sqrt{N}\,? \tag{3.11}$$

It is possible to show that, in this case, we can take α_n to be

$$\alpha_n=\varepsilon_n\sqrt{n}-\varepsilon_{n-1}\sqrt{n-1},$$

which, since $\{\alpha_n\}_{n\geq 1}\in\ell^2$, yields

$$\sum_{n\geq 1}\frac{\varepsilon_n^2}{n}=\sum_{n\geq 1}\frac{\varepsilon_n}{\sigma_n}<\infty.$$

So, for instance, if we were to choose $\varepsilon_n=1/\log^{\alpha} n$, then we would need $\alpha>\frac{1}{2}$ which is, in a sense, optimal in view of the preceding corollary.

A crucial point in this discussion is the fact that $\{\varepsilon_n\}_{n\geq 1}$ is a *decreasing* sequence.

3.2. Second class of examples

In the preceding class of examples from (3.1), we slowed down the growth of functions in $(BH^2)^{\perp}$ by controlling the "tangentiality" of the sequence (given by the speed of convergence to zero of x_n). Our second class of examples are of the type

$$\lambda_n=r_n e^{i\theta_n},\quad 0<\theta_n<1,\ 1-r_n=\theta_n^2 \text{ and } \sum_{n\geq 1}\theta_n<\infty, \tag{3.12}$$

where θ_n can be adjusted to control the growth speed of $(BH^2)^{\perp}$-functions. Asymptotically, this sequence is in the oricycle $\left\{z\in\mathbb{D}:|z-\frac{1}{2}|=\frac{1}{2}\right\}$. We also note that

$$\sum_{n\geq 1}(1-|\lambda_n|)=\sum_{n\geq 1}\theta_n^2<\infty$$

so indeed $\{\lambda_n\}_{n\geq 1}$ is a Blaschke sequence. Moreover,

$$\sum_{n\geq 1}\frac{1-|\lambda_n|}{|1-\lambda_n|}\asymp\sum_{n\geq 1}\frac{\theta_n^2}{\theta_n}=\sum_{n\geq 1}\theta_n<\infty \tag{3.13}$$

and so, by (1.4), $\lim_{r\to 1^-} B(r)=\eta\in\mathbb{T}$. Still further, we have

$$\sum_{n\geq 1}\frac{1-|\lambda_n|}{|1-\lambda_n|^2}\asymp\sum_{n\geq 1}\frac{\theta_n^2}{\theta_n^2}=\infty$$

so $\{\lambda_n\}_{n\geq 1}$ does not satisfy the hypothesis (1.1) of the Ahern–Clark theorem. Thus we can expect bad behavior of functions from $(BH^2)^{\perp}$.

As in (3.2), we have

$$\frac{1-|\lambda_k|^2}{|1-r\lambda_k|^2}\asymp\frac{1-r_k}{(1-r)^2+\theta_k^2}=\frac{\theta_k^2}{(1-r)^2+\theta_k^2}\asymp\begin{cases}1, & \text{if } 1-r\leq\theta_k,\\ \dfrac{\theta_k^2}{(1-r)^2}, & \text{if } 1-r>\theta_k.\end{cases}$$

Using again Lemma 2.2, the splitting gives

$$\|k_r^B\|^2 \asymp \sum_{k\geq 1} \frac{1-|\lambda_k|^2}{|1-r\lambda_k|^2} \asymp \sum_{k:(1-r)\leq\theta_k} 1 + \frac{1}{(1-r)^2} \sum_{k:(1-r)>\theta_k} \theta_k^2. \tag{3.14}$$

Theorem 3.6. *Let $\{\sigma_N\}_{N\geq 1}$ be a sequence of positive numbers strictly increasing to infinity such that*

$$\sigma_{N+1} \leq 2^\beta \sigma_N, \quad N\in\mathbb{N}, \tag{3.15}$$

for some $\beta\in(0,1)$. Then there exists a sequence $\{\theta_k\}_{k\geq 1}\in\ell^1$ such that

$$\|k_{\rho_N}^B\| \asymp \sqrt{\sigma_N},$$

where B is the Blaschke product whose zeros are $\Lambda=\{\lambda_k\}_{k\geq 1}$ and

$$\lambda_k = r_k e^{i\theta_k} \quad \textit{and} \quad 1-r_k=\theta_k^2.$$

Proof. Let $\{\sigma_N\}_{N\geq 1}$ be as in the theorem and let

$$\psi\colon [0,\infty) \longrightarrow [0,\infty)$$

be a continuous increasing function such that

$$\psi(N) = \sigma_N, \quad N\in\mathbb{N}. \tag{3.16}$$

We could, for example, choose ψ to be the continuous piecewise affine function defined at the nodes by (3.16). Since ψ is continuous and strictly increasing to infinity on $[0,\infty)$, it has an inverse function ψ^{-1}. Set

$$\theta_k = 2^{-\psi^{-1}(k)}, \quad k\in\mathbb{N}.$$

Let us consider the first sum in (3.14) (with $r=\rho_N$),

$$\sum_{k:(1-\rho_N)\leq\theta_k} 1 = \sum_{k:1/2^N\leq 1/2^{\psi^{-1}(k)}} 1 = \sum_{k:\psi^{-1}(k)\leq N} 1 = \sum_{k\leq\psi(N)} 1 = \psi(N) = \sigma_N.$$

We have to consider the second sum in (3.14),

$$\frac{1}{(1-\rho_N)^2} \sum_{k:(1-\rho_N)>\theta_k} \theta_k^2 = 2^{2N} \sum_{k:\psi^{-1}(k)\geq N+1} 2^{-2\psi^{-1}(k)} = 2^{2N} \sum_{k\geq\psi(N+1)} 2^{-2\psi^{-1}(k)}.$$

Since $\sigma_n = \psi(n)$, equivalently $\psi^{-1}(\sigma_n) = n$, we have

$$\begin{aligned} \sum_{k\geq\psi(N+1)} 2^{-2\psi^{-1}(k)} &= \sum_{n\geq N+1} \sum_{k=\sigma_n}^{\sigma_{n+1}-1} 2^{-2\psi^{-1}(k)} \\ &\leq \sum_{n\geq N+1} (\sigma_{n+1}-\sigma_n) 2^{-2\psi^{-1}(\sigma_n)} \\ &\leq \sum_{n\geq N+1} \frac{1}{2^{2n}}\sigma_{n+1} \\ &\leq 2^\beta \sum_{n\geq N+1} \frac{1}{2^{2n}}\sigma_n. \end{aligned} \tag{3.17}$$

Now, setting $u_n = \sigma_n/2^{2n}$, we get $v_n = u_{n+1}/u_n \leq 2^{\beta-2} < 1$, from which standard arguments give

$$\sum_{n\geq N+1} \frac{\sigma_n}{2^{2n}} \lesssim \frac{\sigma_N}{2^{2N}}. \tag{3.18}$$

Hence

$$2^{2N} \sum_{k\geq\psi(N+1)} 2^{-2\psi^{-1}(k)} \lesssim \sigma_N.$$

So, according to (3.14),

$$\sigma_N \leq \underbrace{\sum_{k:(1-r)\leq\theta_k} 1 + \frac{1}{(1-r)^2} \sum_{k:(1-r)>\theta_k} \theta_k^2}_{\asymp \|k_{\rho_N}^B\|^2} \lesssim \sigma_N + \sigma_N.$$

It remains to show that $\{\theta_n\}_{n\geq1} \in \ell^1$ (in order to satisfy the Frostman condition (3.13)). As in (3.17) we see that

$$\sum_{k\geq\sigma_1} \theta_k = \sum_{k\geq\sigma_1} 2^{-\psi^{-1}(k)} = \sum_{n\geq1} \sum_{k\geq\sigma_n}^{\sigma_{n+1}-1} 2^{-\psi^{-1}(k)} \leq 2^\beta \sum_{n\geq1} \frac{\sigma_n}{2^n}$$

which converges due do to (3.15) and the fact that $\beta \in (0,1)$. This completes the proof. □

Example 3.7. Here is a list of examples on how one applies our estimates.

(1) Let $\sigma_N = 2^{N/\alpha}$, $N = 1, 2, \ldots$, where $\alpha > 1$ (this is needed for (3.15)). Then we can choose $\psi(t) = 2^{t/\alpha}$. Hence

$$\theta_k = 2^{-\psi^{-1}(k)} = 2^{-\alpha \log k} = \frac{1}{k^\alpha}.$$

With this choice of arguments, we get

$$\|k_{\rho_N}^B\| \asymp 2^{N/2\alpha} = \frac{1}{(1-\rho_N)^{1/2\alpha}},$$

which, by similar arguments as given earlier (see the proof of Theorem 3.1), can be extended to every $r \in (0,1)$, i.e.,

$$|f(r)| \lesssim \frac{1}{(1-r)^{1/2\alpha}}, \quad f \in (BH^2)^\perp.$$

We thus obtain all power growths beyond the limiting case $\frac{1}{2}$.

(2) Let $\sigma_N = N^\alpha$, $N=1,2,...$, where $\alpha>0$. Then we can choose $\psi(t)=t^\alpha$. Hence

$$\theta_k = 2^{-\psi^{-1}(k)} = 2^{-k^{1/\alpha}},$$

and, with this choice of arguments, we get

$$\|k_{\rho_N}^B\| \asymp N^{\alpha/2} = \left(\log \frac{1}{1-\rho_N}\right)^{\alpha/2}.$$

Thus, as in the previous example, we get

$$|f(r)| \lesssim \left(\log \frac{1}{1-r}\right)^{\alpha/2}, \quad f \in (BH^2)^\perp.$$

In the special case $\alpha=2$ we obtain logarithmic growth.

(3) Let $\sigma_N = \log^2 N$, $N=2,3,...$. Then we can choose $\psi(t)=\log^2 t$. Hence

$$\theta_k = 2^{-\psi^{-1}(k)} = 2^{-2^{\sqrt{k}}}.$$

With this choice of arguments, we get, for large enough N,

$$\|k_{\rho_N}^B\| \asymp \log N = \log \log \frac{1}{1-\rho_N},$$

and so

$$|f(r)| \lesssim \log \log \frac{1}{1-r}, \quad f \in (BH^2)^\perp.$$

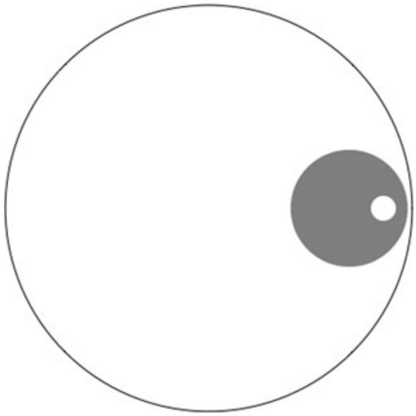

Figure 2. An example of a domain $\Gamma_n^{N,1}$.

4. A general growth result for $(BH^2)^\perp$

It turns out that growth results can be phrased in terms of a more general result. In fact our first class of examples can be deduced from such a general result (see Remark 4.2).

We will start by introducing a growth parameter associated with a Blaschke sequence $\Lambda=\{\lambda_n\}_{n\geq 1}\subset\mathbb{D}$ and a boundary point $\zeta\in\mathbb{T}$. Let us again set

$$\rho_N := 1-\frac{1}{2^N}, \quad N\in\mathbb{N}.$$

For every $N\in\mathbb{N}$ and $n\in\mathbb{Z}$, set

$$\Gamma_n^{N,\zeta} := \left\{ z\in\mathbb{D} : \frac{1-|z|^2}{|\zeta-\rho_N z|^2} \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \right\}. \tag{4.1}$$

This is a kind of pseudo-hyperbolic annulus (see Figure 2). A routine computation shows that

$$\frac{1-|z|^2}{|\zeta-\rho z|^2} = c \quad \Longleftrightarrow \quad \left| z-\frac{c\rho}{1+c\rho^2}\zeta\right|^2 = \frac{1-c(1-\rho^2)}{(1+c\rho^2)^2}.$$

From here observe that necessarily $c\leq 1/(1-\rho^2)$ which means that $\Gamma_n^{N,\zeta}$ is empty when

$$\frac{1}{2^{n+1}} \geq \frac{1}{1-\rho_N^2} \geq \frac{1}{2(1-\rho_N)} = 2^{N-1}.$$

We therefore assume that $n\geq -N$.

For simplicity, we will assume from now on that $\zeta=1$ and set

$$\Gamma_n^N := \Gamma_n^{N,1}.$$

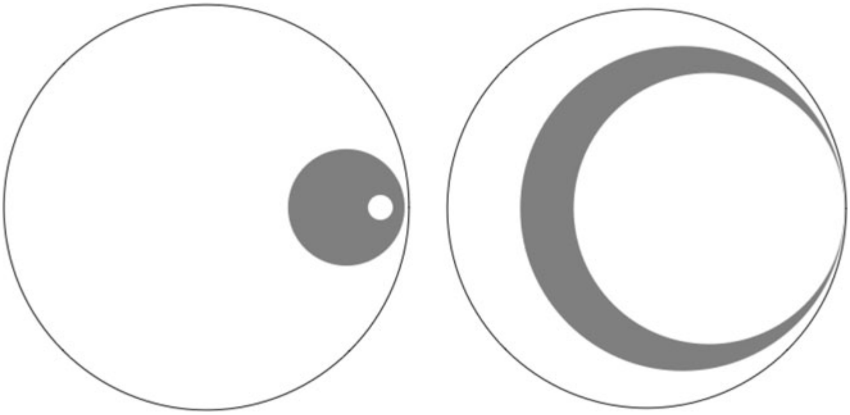

Figure 3. The domains Γ_n^N, $-N\le n$, cover $\mathbb{D}$.

Define

$$\alpha_{N,n} := \#(\Lambda\cap\Gamma_n^N)$$

(the number of points in $\Lambda\cap\Gamma_n^N$) along with the growth parameter

$$\sigma_N^\Lambda := \sum_{n\in\mathbb{Z}} \frac{\alpha_{N,n}}{2^n} = \sum_{n\ge -N} \frac{\alpha_{N,n}}{2^n}.$$

For each $\lambda\in\Lambda\cap\Gamma_n^N$ we have, by definition (see (4.1)),

$$\frac{1}{2^n} \asymp \frac{1-|\lambda|^2}{|1-\rho_N\lambda|^2}$$

and so, since there are $\alpha_{N,n}$ points in $\Lambda\cap\Gamma_n^N$, we have

$$\sum_{n\ge -N} \frac{1}{2^n}\#(\Lambda\cap\Gamma_n^N) \asymp \sum_{n\ge -N}\sum_{\lambda\in\Lambda\cap\Gamma_n^N} \frac{1-|\lambda|^2}{|1-\rho_N\lambda|^2}.$$

But since $\{\Gamma_n^N\}_{n\ge -N}$ is a partition of $\mathbb{D}$ (see Figure 3) we get

$$\sum_{n\ge -N}\sum_{\lambda\in\Lambda\cap\Gamma_n^N} \frac{1-|\lambda|^2}{|1-\rho_N\lambda|^2} = \sum_{n\ge 1} \frac{1-|\lambda_n|^2}{|1-\rho_N\lambda_n|^2}.$$

Putting this all together we arrive at

$$\sigma_N^\Lambda \asymp \sum_{n\ge 1} \frac{1-|\lambda_n|^2}{|1-\rho_N\lambda_n|^2}. \tag{4.2}$$

Combine (4.2) with Lemma 2.2 to get the two-sided estimate

$$\sigma_N^\Lambda \asymp \|k_{\rho_N}^B\|^2. \tag{4.3}$$

Note that if the zeros $\{\lambda_n\}_{n\ge1}$ of B satisfy the Ahern–Clark condition (1.1) then, by Theorem 1.1, the sequence $\{\|k_{\rho_N}^B\|\}_{N\ge1}$ is uniformly bounded and, by (4.3), so is $\{\sigma_N^\Lambda\}_{N\ge1}$.

To discuss the case when $\{\sigma_N^\Lambda\}_{N\ge1}$ is unbounded, we will impose the mild regularity condition

$$0 < m := \inf_N \frac{\sigma_{N+1}^\Lambda}{\sigma_N^\Lambda} \le M := \sup_N \frac{\sigma_{N+1}^\Lambda}{\sigma_N^\Lambda} < \infty. \tag{4.4}$$

In Section 3, this condition was automatically satisfied by $\sigma_N = \sum_{k=1}^N x_k$.

Let us associate with σ_N^Λ the functions φ_0 and φ as in Theorem 3.1. Then, from (4.3) we deduce the following result in the same way as Theorem 3.1.

Theorem 4.1. *Let $\Lambda = \{\lambda_n\}_{n\ge1} \subset \mathbb{D}$ be a Blaschke sequence with associated growth sequence $\sigma^\Lambda = \{\sigma_N^\Lambda\}_{N\ge1}$ at $\zeta = 1$ satisfying (4.4) and B be the Blaschke product with zeros Λ. Then*

$$\|k_z^B\| \asymp \sqrt{\varphi(|z|)}, \quad z \in \Gamma_{\alpha,1}.$$

Consequently, every $f \in (BH^2)^\perp$ satisfies

$$|f(z)| = |\langle f, k_z\rangle| \lesssim \sqrt{\varphi(|z|)}, \quad z \in \Gamma_{\alpha,1}.$$

Remark 4.2. It turns out that for the sequences discussed in Theorem 3.1 we have

$$\sigma_N^\Lambda \asymp \sigma_N = \sum_{k=1}^N x_k.$$

The details are somewhat cumbersome so we will not give them here.

5. A final remark on unconditional bases

Since a central piece of our discussion was the behavior of the reproducing kernels $k_{\rho_N}^B$, where B is the Blaschke product with the zero sequence discussed in Section 3 and $\rho_N = 1 - 1/2^N$, one could ask whether or not $\{k_{\rho_N}^B\}_{N\ge1}$ forms an unconditional bases (or sequence) for $(BH^2)^\perp$.

To this end, let $K_n = k_{\rho_n}^B / \|k_{\rho_n}^B\|$ and $G = (\langle K_n, K_k\rangle)_{n,k}$ be the associated Gram matrix. Suppose that $\{K_n\}_{n\ge1}$ were an unconditional basis (or sequence) for

$(BH^2)^\perp$. In this case, it is well known (see e.g. [8, Exercise C3.3.1(d)]) that G represents an isomorphism from ℓ^2 onto ℓ^2. It follows from the unconditionality of $\{K_n\}_{n\geq 1}$ that every $f\in(BH^2)^\perp$ (or every f in the span of $\{K_n\}_{n\geq 1}$) can be written as

$$f=f_\alpha:=\sum_{n\geq 1}\alpha_n K_n, \quad \alpha=\{\alpha_n\}_{n\geq 1}\in\ell^2,$$

with $\|f_\alpha\|^2\asymp\sum_{n\geq 1}|\alpha_n|^2<\infty$. As before we want to estimate $f=f_\alpha$ at ρ_N. Indeed,

$$f_\alpha(\rho_N)=\sum_{n\geq 1}\alpha_n\frac{k^B_{\rho_n}(\rho_N)}{\|k^B_{\rho_n}\|}=\|k^B_{\rho_N}\|\sum_{n\geq 1}\alpha_n\frac{\langle k^B_{\rho_n},k^B_{\rho_N}\rangle}{\|k^B_{\rho_n}\|\,\|k^B_{\rho_N}\|}=\|k^B_{\rho_N}\|(G\alpha)_N,$$

where $\beta:=G\alpha\in\ell^2$

After these general considerations suppose now that we were in the situation of Theorem 3.4. In particular for $\varepsilon>0$ there is a function f_α with

$$|f_\alpha(\rho_N)|\gtrsim\sqrt{\frac{\sigma_N}{\log^{1+\varepsilon}\sigma_N}}$$

(we refer to that theorem for notation). Since by Theorem 3.1 we have

$$\|k^B_{\rho_N}\|\asymp\sqrt{\sigma_N},$$

we would thus have

$$\beta_N:=\frac{|f_\alpha(\rho_N)|}{\|k^B_{\rho_N}\|}\asymp\frac{|f_\alpha(\rho_N)|}{\sqrt{\sigma_N}}\gtrsim\frac{1}{\log^{(1+\varepsilon)/2}\sigma_N}.$$

However, for instance, choosing $x_n=1/n$ yields $\sigma_N\simeq\log N$, in which case

$$\left\{\frac{1}{\log^{(1+\varepsilon)/2}\sigma_N}\right\}_{N\geq 1}$$

is obviously not in ℓ^2. (Actually one can also choose $x_n=1$ to get a sequence $\{\beta_N\}_{N\geq 1}\notin\ell^2$.) As a result, we can conclude that in the above examples $\{k^B_{\rho_N}\}_{N\geq 1}$ cannot be an unconditional basis for $(BH^2)^\perp$ (nor an unconditional sequence since the functions in Theorem 3.4 were constructed using the reproducing kernels, so they belong to the space spanned by $\{K_n\}_{n\geq 1}$).

It should be noted that the problem of deciding whether or not a sequence of reproducing kernels forms an unconditional basis (or sequence) for a model space is a difficult problem related to the Carleson condition and the invertibility of Toeplitz operators. We do not want to go into details here, but the situation becomes even more difficult in our context where $\overline{\lim}_{N\to\infty}|B(\rho_N)|=1$. See [8, Chapter D4] for more about this.

References

1. Ahern, P. R. and Clark, D. N., Radial limits and invariant subspaces, *Amer. J. Math.* **92** (1970), 332–342.
2. Cargo, G. T., Angular and tangential limits of Blaschke products and their successive derivatives, *Canad. J. Math.* **14** (1962), 334–348.
3. Collingwood, E. F. and Lohwater, A. J., *The Theory of Cluster Sets*, Cambridge University Press, Cambridge, 1966.
4. Duren, P. L., *Theory of* H^p *Spaces*, Academic Press, New York, 1970.
5. Frostman, O., Sur les produits de Blaschke, *Kungl. Fysiografiska Sällskapets i Lund Förhandlingar* **12** (1942), 169–182.
6. Garnett, J., *Bounded Analytic Functions*, Revised 1st ed., Springer, New York, 2007.
7. Nikolski, N. K., *Treatise on the Shift Operator*, Springer, Berlin, 1986.
8. Nikolski, N. K., *Operators, Functions, and Systems: An Easy Reading* **2**, Amer. Math. Soc., Providence, RI, 2002.
9. Protas, D., Tangential limits of functions orthogonal to invariant subspaces, *Trans. Amer. Math. Soc.* **166** (1972), 163–172.
10. Shapiro, H. S. and Shields, A. L., On some interpolation problems for analytic functions, *Amer. J. Math.* **83** (1961), 513–532.

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Received February 28, 2012
published online October 6, 2012