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Quasiconformality, homeomorphisms between metric measure spaces preserving quasiminimizers, and uniform density property

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Abstract. We characterize quasiconformal mappings as those homeomorphisms between two metric measure spaces of locally bounded geometry that preserve a class of quasiminimizers. We also consider quasiconformal mappings and densities in metric spaces and give a characterization of quasiconformal mappings in terms of the uniform density property introduced by Gehring and Kelly.

1. Introduction

Let $1<p<\infty$ and let $\Omega\subset\mathbb{R}^n$ be a non-empty open set. A function $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer*, $Q\geq 1$, related to the index p in Ω if

$$\int_{\phi\neq 0}|\nabla u|^p\,dx\leq Q\int_{\phi\neq 0}|\nabla(u+\phi)|^p\,dx \tag{1}$$

for all $\phi\in W^{1,p}_0(\Omega)$. Quasiminimizers were introduced by Giaquinta–Giusti [12] and [13], and, for instance, solutions to the equation

$$\operatorname{div}\mathcal{A}(x,\nabla u)=0$$

are known to be quasiminimizers. The mapping $\mathcal{A}$ belongs to the set $\mathbf{A}_p$, and for $1<p<\infty$, this set is the collection of all mappings satisfying the Carathéodory conditions and the standard structural assumption, see, e.g., Heinonen–Kilpeläinen–Martio [17, p. 56].

Let $\mathcal{A}$ and $\mathcal{A}^*$ belong to $\mathbf{A}_p$. Following Heinonen–Kilpeläinen–Martio [16], we say that a continuous mapping $f\colon\Omega\to\mathbb{R}^n$ is an $(\mathcal{A}^*,\mathcal{A})$-harmonic morphism if

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$u\circ f$ is $\mathcal{A}^*$-harmonic in $f^{-1}(\Omega')$ whenever u is $\mathcal{A}$-harmonic in Ω'. Further, f is an $\mathbf{A}_p$-harmonic morphism if f is an $(\mathcal{A}^*,\mathcal{A})$-harmonic morphism for some $\mathcal{A}^*$ and $\mathcal{A}$.

It is a well-known fact that in the plane any continuous mapping which is a harmonic morphism for the Laplace equation, i.e. an $(\mathcal{A},\mathcal{A})$-harmonic morphism with $\mathcal{A}(x,h)=h$, is necessarily conformal, see Gehring–Haahti [10]. In potential theory, the first paper to study harmonic morphisms systematically is the article by Constantinescu and Cornea [7]. In higher dimensions and for general p the problem of determining harmonic morphisms is more challenging. It is known that in the Sobolev borderline case $p=n$ quasiregular mappings provide examples of $\mathbf{A}_p$-harmonic morphisms. This was shown by Granlund–Lindqvist–Martio in [14]. On the other hand, the main result by Heinonen–Kilpeläinen–Martio in [16, Theorem 4.1] shows that every sense-preserving $\mathbf{A}_n$-harmonic morphism is a quasiregular mapping. In particular, every homeomorphic $\mathbf{A}_n$-harmonic morphism is quasiconformal, i.e. a homeomorphic quasiregular mapping. In [16] the authors also studied some basic properties of such morphisms and examined the case $1<p<n$. However, very little is known in the case when $p>n$.

In this paper we consider an analytic characterization (Theorem 4.1) and a geometric characterization (Theorem 6.5) of quasiconformal mappings.

In the first part of the paper (Theorem 4.1) we consider transformations which preserve a class of quasiminimizers. Holopainen and Shanmugalingam [22] have shown that quasiconformal mappings between metric measure spaces that satisfy certain bounds on their mass and geometry (these metric spaces are said to be of locally q-bounded geometry; see Definition 2.4) preserve quasiminimizers. These spaces are discussed in more detail in Sections 2 and 3. Heinonen and Koskela [19] developed the foundations of the theory of quasiconformal maps in such metric spaces. In this paper we show that if a homeomorphism between two metric measure spaces of locally q-bounded geometry preserves the class of quasiminimizers associated with the natural dimension $p=q$ of the space, then, under some additional assumptions on this homeomorphism, it is quasiconformal. That is, every homeomorphic quasiharmonic morphism is a quasiconformal mapping. We also establish a few properties of quasiharmonic morphisms in general. Since harmonic functions as in Heinonen–Kilpeläinen–Martio [16] are contained in the class of quasiminimizers our results seem to be new even in the Euclidean setting; also cf. Capogna–Cowling [5, Theorem 5.7] and Herron–Koskela [21]. Results in [21] dealing with preservation of Sobolev classes require additional conditions on the domain which are stronger than our requirement of local q-bounded geometry, a condition automatically holding in all Euclidean domains. Hence our characterization of quasiconformal mappings in terms of preserving Sobolev classes is more general even in the Euclidean setting.

The above-mentioned characterization of quasiconformal mappings is purely analytical in nature. In the second part of this note (Theorem 6.5) we consider a purely geometric characterization of quasiconformal mappings by studying metric quasiconformal mappings and densities along the lines of Gehring and Kelly [11]. We generalize a result of [11] which states that points of density are preserved under quasiconformal mappings, and we show that if a homeomorphism is absolutely continuous on q-modulus almost every curve then quasiconformality and the uniform density property are indeed equivalent. In the Euclidean setting, the uniform density property for the inverse function implies absolute continuity on almost every line parallel to the coordinate axes. This together with a suitable integrability property is enough to guarantee that a function is in the correct Sobolev class. Thus in the Euclidean setting the a priori assumption of absolute continuity on q-modulus almost every curve is not needed. The proof in [11] uses projections along coordinate axes and exploits the linear structure of the ambient space, and so the proof of [11] cannot be directly extended to general metric spaces. We instead show that the uniform density property for the inverse is enough to imply that the homeomorphism is absolutely continuous on 1-modulus almost every curve. This is a strictly weaker property than absolute continuity on q-modulus almost every curve. Nevertheless, if the space supports a $(1,1)$-Poincaré inequality, then absolute continuity on 1-modulus almost every curve together with some integrability conditions to be specified in Section 6 is enough to show that the homeomorphism is in the correct Sobolev space, and thus we may replace the extra assumption of absolute continuity with the condition that the metric measure space supports the strongest possible Poincaré inequality—the $(1,1)$-Poincaré inequality. Observe that Euclidean spaces automatically support such a Poincaré inequality.

This paper is organized as follows. In Section 2 we introduce metric measure spaces of locally q-bounded geometry and provide a definition of quasiminimizers. In Section 3 we discuss definitions of quasiconformal mappings in metric spaces and recall a few well-known facts. In Sections 4 and 5 we study connections between quasiconformal mappings and quasiminimizers. Finally, in Section 6 we study quasiconformal mappings in connection with the uniform density property of Gehring–Kelly [11].

2. Metric spaces

The theory of quasiminimizers fits naturally into the study of analysis in metric spaces.

We follow Heinonen and Koskela [19] in introducing upper gradients as follows.

Definition 2.1. Let (X,d_X) and (Y,d_Y) be metric spaces. A non-negative Borel function g on X is an *upper gradient* of a function $f\colon X\to Y$ if for all curves $\gamma\colon[0,l_\gamma]\to X$,

$$d_Y(f(\gamma(0)),f(\gamma(l_\gamma)))\le\int_\gamma g\,ds, \tag{2}$$

where ds denotes arc length. If g is a non-negative measurable function on X and if (2) holds for p-almost every curve, $1\le p<\infty$, then g is a *p-weak upper gradient* of f.

By saying that (2) holds for p-almost every curve we mean that this inequality fails only for a curve family Γ with p-modulus $\operatorname{Mod}_p\Gamma$ equal to zero; recall that the p-modulus of a curve family Γ is the number

$$\operatorname{Mod}_p\Gamma=\inf\int_X\rho^p\,d\mu,$$

the infimum being taken over all Borel functions $\rho\colon X\to[0,\infty]$ that are admissible for Γ, that is, $\int_\gamma\rho\,ds\ge1$ for each rectifiable $\gamma\in\Gamma$. For basic properties of the p-modulus we refer the interested reader to [19].

The notion of p-weak upper gradients was introduced by Koskela and MacManus [28], where it was also shown that if $g\in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j\to g$ in $L^p(X)$. If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f\in L^p(X)$ in the sense that for every p-weak upper gradient $g\in L^p(X)$ of f, $g_f\le g$ a.e., see Shanmugalingam [31, Corollary 3.7]. While the results in [28] and [31] are formulated for real-valued functions and their (p-weak) upper gradients, they are applicable for metric space valued functions and their upper gradients; the proofs of these results require only the manipulation of (p-weak) upper gradients, which are real-valued.

We consider the following version of Sobolev spaces on the metric space X due to Shanmugalingam in [30].

Definition 2.2. Whenever $u\in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)}=\left(\int_X|u|^p\,d\mu+\inf_g\int_X g^p\,d\mu\right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X)=\{u:\|u\|_{N^{1,p}(X)}<\infty\}/\sim,$$

where $u\sim v$ if and only if $\|u-v\|_{N^{1,p}(X)}=0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice [30]. A function $u \in N^{1,p}(X)$ is said to be quasicontinuous, if there exists an open set $G \subset X$ with arbitrarily small Sobolev p-capacity (see Section 2.1) such that the restriction of u to $X \setminus G$ is continuous. Note that under our assumptions (see below) all functions in $N^{1,p}(X)$ will be quasicontinuous by Björn–Björn–Shanmugalingam [4], and Newtonian functions have Lebesgue points outside a set of zero Sobolev p-capacity by Kinnunen–Latvala [25]. If $u, v \in N^{1,p}(X)$ and $u=v$ μ-a.e. then $u \sim v$ [30]. Moreover, if $u \in N^{1,p}(X)$ then $u \sim v$ if and only if $u=v$ p-quasieveryhere (p-q.e. for short), i.e., outside a set of zero Sobolev p-capacity. Section 3 in Heinonen–Koskela–Shanmugalingam–Tyson [20] gives the definition of, and has a detailed discussion on, the Sobolev space $N^{1,p}(X,Y)$, where Y is a metric measure space.

Definition 2.3. Let $1 \le p < \infty$. We say that X supports a *local* (*weak*) $(1,p)$-*Poincaré inequality* if there exist constants $C>0$ and $\lambda \ge 1$ such that each point $x \in X$ has a neighborhood U such that for all balls B with $\lambda B \subset U$, all integrable functions f on λB and all upper gradients g of f,

$$\fint_B |f - f_B| \, d\mu \le C (\operatorname{diam} B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}, \tag{3}$$

where $f_B := \fint_B f \, d\mu := \int_B f \, d\mu / \mu(B)$.

In the definition of $(1,p)$-Poincaré inequality we can equivalently assume that g is a p-weak upper gradient—see the comments above.

In this paper a metric space (X,d) is assumed to be complete and equipped with a *locally Ahlfors q-regular measure* μ, i.e. there exists a constant $C \ge 1$ so that each point $x_0 \in X$ has a neighborhood U such that for all $x \in U$ and $r>0$ with $B(x,r) \subset U$,

$$\frac{1}{C} r^q \le \mu(B(x,r)) \le C r^q.$$

Here $B(x,r) = \{y \in X : d(x,y) < r\}$ and $q>0$ is a fixed constant. Moreover we require the measure to support a local $(1,q)$-Poincaré inequality. *In what follows, q will always refer to this mass bound exponent.*

Definition 2.4. A metric measure space X is said to be of *locally q-bounded geometry*, with $q>1$, if X is a separable, path-connected, locally compact metric space equipped with a locally Ahlfors q-regular measure that admits a local $(1,q)$-Poincaré inequality.

Note that these assumptions imply uniform local linear connectivity in a neighborhood of each point in X; see [19, Section 3].

Let Ω be a connected open set in X, and E and F be two disjoint non-empty compact sets in Ω. The *q-capacity* of the triple $(E,F;\Omega)$ is defined to be the (possibly infinite) number

$$\operatorname{Cap}_q(E,F;\Omega)=\inf\int_\Omega g_u^q\,d\mu,$$

where the infimum is taken over minimal q-weak upper gradients g_u of all functions u in Ω with the property that $u|_E=1$, $u|_F=0$, and $0\le u\le 1$.

Finally, we say that u is a *q-potential* of the condenser (E,F) with respect to Ω if it is q-harmonic (see below) in $\Omega\setminus(E\cup F)$ with boundary data $u=0$ on E and $u=1$ on F taken in the sense of Newtonian spaces. If a q-potential exists it is unique and we have

$$\operatorname{Cap}_q(E,F;\Omega)=\int_\Omega g_u^q\,d\mu;$$

the q-potential exists if $\operatorname{Cap}_q(E,F;\Omega)<\infty$, see, e.g., [22, Lemma 3.3].

Throughout the rest of the paper we will assume that X and Y are two metric measure spaces of locally q-bounded geometry.

2.1. Quasiminimizers

In metric spaces the natural counterpart to $|\nabla u|$ is g_u. Observe that in metric spaces we have no natural counterpart to the vector ∇u, only to the scalar $|\nabla u|$ (see however Cheeger [6]). In metric spaces we replace the Sobolev space $W^{1,q}$ by the Newtonian space $N^{1,q}$. Given a non-empty open set Ω, a function $u\in N^{1,q}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer*, $Q\ge 1$, in Ω if

$$\int_{\phi\neq 0} g_u^q\,d\mu\le Q\int_{\phi\neq 0} g_{u+\phi}^q\,d\mu \tag{4}$$

for all $\phi\in\operatorname{Lip}_c(\Omega)=\{f\in\operatorname{Lip}(X):\ \operatorname{supp}(f)\Subset\Omega\}$.

Our definition of quasiminimizers is one of several equivalent possibilities, see A. Björn [2].

By Giaquinta–Giusti [13], a Q-quasiminimizer in a Euclidean space can be modified on a set of measure zero so that it becomes locally Hölder continuous. See Kinnunen–Shanmugalingam [26] for the metric space analog. A Harnack inequality holds true for Q-quasiminimizers, see [26]; the Euclidean space analog is due to DiBenedetto and Trudinger [9]. A continuous Q-quasiminimizer is said to be a *Q-quasiharmonic* function, and a 1-quasiharmonic function is *q-harmonic*.

We will need the following removability result. Here C_q is the Sobolev capacity, see Heinonen–Kilpeläinen–Martio [17] for a definition in $\mathbb{R}^n$ and, for instance, [3] for a definition in the metric space case.

Theorem 2.5. (A. Björn [3]) *Let $E\subset\Omega$ be a relatively closed subset with $C_q(E)=0$. Assume that u is bounded and Q-quasiharmonic in $\Omega\setminus E$. Then u has a Q-quasiharmonic extension U to Ω given by*

$$U(x)=\operatorname*{ess\,lim\,inf}_{\Omega\setminus E\ni y\to x} u(y).$$

3. Definitions of quasiconformality

We define quasiconformality in the metric setting following Heinonen and Koskela [18]. A homeomorphism $f\colon X\to Y$ between two metric spaces X and Y is *quasiconformal*, or *K-quasiconformal*, $K\geq 1$, if

$$K(f,x)=\limsup_{r\to 0}\frac{\sup_{d_X(x,y)=r} d_Y(f(x),f(y))}{\inf_{d_X(x,y)=r} d_Y(f(x),f(y))}\leq K<\infty$$

for every $x\in X$. A homeomorphism $f\colon X\to Y$ is *η-quasisymmetric* if there is a homeomorphism $\eta\colon[0,\infty)\to[0,\infty)$ such that for each $t>0$,

$$d_X(x,a)\leq t d_X(x,b) \quad \text{implies} \quad d_Y(f(x),f(a))\leq \eta(t) d_Y(f(x),f(b))$$

for each triple x,a,b of points in X. The question of which quasiconformal maps are quasisymmetric was studied in [19, Theorems 4.7 and 4.9]. In the above definition, the lim sup can be relaxed to lim inf and exceptional sets can be allowed. For more details we refer the interested reader to Balogh–Koskela–Rogovin [1].

Theorem 3.1. (Heinonen–Koskela [19] and Heinonen–Koskela–Shanmugalingam–Tyson [20]) *Let $f\colon X\to Y$ be a homeomorphism. Then the following are equivalent:*

(1) *There exists K such that f is K-quasiconformal;*

(2) *There exists η such that f is locally η-quasisymmetric;*

(3) *There exists K such that $f\in N^{1,q}_{\mathrm{loc}}(X,Y)$ and $L_f(x)^q\leq K J_f(x)$ for a.e. $x\in X$;*

(4) *There exists a constant C such that for every collection Γ of curves in X,*

$$\frac{1}{C}\operatorname{Mod}_q\Gamma\leq\operatorname{Mod}_q f(\Gamma)\leq C\operatorname{Mod}_q\Gamma.$$

Moreover, if any of the above conditions holds for f, then f is absolutely continuous in measure and absolutely continuous along q-modulus a.e. curve in X, and the inverse f^{-1} is also quasiconformal.

In the above theorem

$$L_f(x) := \limsup_{r\to 0} \sup_{d_X(x,y)\le r} \frac{d_Y(f(x),f(y))}{r}$$

is the *maximal stretching* of f, and the volume derivative, or the *generalized Jacobian*, is defined as

$$J_f(x) := \limsup_{r\to 0} \frac{\mu_Y(f(B(x,r))}{\mu_X(B(x,r))}.$$

By the Lebesgue–Radon–Nikodym theorem (see, e.g. Mattila [29]) the lim sup in the definition of J_f can be replaced by lim for μ_X-a.e. $x\in X$. Moreover, since f is absolutely continuous in measure, for every measurable subset E of X,

$$\int_E J_f \, d\mu_X = \mu_Y(f(E)).$$

By *absolute continuity of f in measure* we mean that f satisfies *Luzin's condition* (N): if $E\subset X$ satisfies $\mu_X(E)=0$, then $\mu_Y(f(E))=0$. Recall that f is absolutely continuous on q-modulus a.e. curve if the collection of rectifiable curves in X for which the following condition does not hold has q-modulus zero: for every $\varepsilon>0$ there exists $\delta>0$ such that for every measurable subset A of X one has

$$\mathcal{H}^1(|\gamma|\cap A) < \delta \quad \text{implies} \quad \mathcal{H}^1(f(|\gamma|\cap A)) < \varepsilon,$$

or equivalently, $f\circ\gamma$ satisfies Luzin's condition (N) with respect to the 1-dimensional Lebesgue measure. Here $|\gamma|$ denotes the point set $\gamma([0,1])\subset X$ of a (rectifiable) curve $\gamma\colon [0,1]\to X$.

Quasiconformal mappings between metric spaces of locally q-bounded geometry preserve the Newtonian space $N^{1,q}$.

Corollary 3.2. (Heinonen–Koskela–Shanmugalingam–Tyson [20]) *Let $f\colon X\to Y$ be a quasiconformal mapping, and $u\in N^{1,q}_{\mathrm{loc}}(Y)$. Then $u\circ f\in N^{1,q}_{\mathrm{loc}}(X)$ and for every relatively compact open set $\Omega\subset X$,*

$$\int_\Omega g_{u\circ f}^q \, d\mu_X \le C \int_{f(\Omega)} g_u^q \, d\mu_Y,$$

where $C\ge 1$ depends only on the constants of quasiconformality of f and the data associated with X and Y.

Metric quasiconformal mappings also preserve the class of quasiminimizers.

Proposition 3.3. (Holopainen–Shanmugalingam [22]) *Let $f\colon X\to Y$ be a quasiconformal mapping. Then f preserves the class of quasiminimizers, associated with index q, on relatively compact domains.*

Observe that in the preceding proposition, if u is a Q-quasiminimizer, then $u\circ f$ is a Q'-quasiminimizer, with Q' depending solely on Q, the quasiconformality constants of f, and the data associated with X and Y.

4. Quasiconformal mappings and quasiminimizers

As stated in previous sections, we assume that the metric measure spaces X and Y are of locally q-bounded geometry. The following theorem gives an analytic characterization of quasiconformal mappings.

Theorem 4.1. *Let $f\colon X\to Y$ be a homeomorphism. Then f is a quasiconformal mapping if and only if $f\in N^{1,q}_{\mathrm{loc}}(X,Y)$ and one of the following two conditions holds*:

(1) *For all $u\in N^{1,q}_{\mathrm{loc}}(Y)$ and every relatively compact open set $\Omega\subset X$,*

$$\int_\Omega g_{u\circ f}^q\,d\mu_X \le C\int_{f(\Omega)} g_u^q\,d\mu_Y,$$

where $C\ge 1$ depends only on the data associated with X and Y.

(2) *For each $Q\ge 1$ there exists $1\le Q'<\infty$ such that whenever u is a Q-quasiminimizer in a relatively compact open set $\Omega_0\subset Y$, $u\circ f$ is a Q'-quasiminimizer in $f^{-1}(\Omega_0)$.*

Furthermore, if one of the two conditions holds, then both conditions hold.

Proof. The necessity of both conditions follows from Theorem 3.1, Corollary 3.2, and Proposition 3.3.

To prove sufficiency of condition (1), fix $x_0\in X$ and choose $r>0$ small enough so that $\overline{B}(x_0,10r)$ is compact and balls inside $B(x_0,10r)$ have the Ahlfors q-regularity and the $(1,q)$-Poincaré inequality properties and in addition $X\setminus B(x_0,10r)$ is nonempty. Let x_m and x_M be two points in the compact set

$$S(x_0,r):=\{x\in X: d_X(x_0,x)=r\}$$

such that

$$L(x_0,r):=d_Y(f(x_0),f(x_M))=\max_{x\in S(x_0,r)} d_Y(f(x_0),f(x)),$$

and

$$l(x_0,r):=d_Y(f(x_0),f(x_m))=\min_{x\in S(x_0,r)} d_Y(f(x_0),f(x)).$$

We need to show that for sufficiently small r the ratio of $L(x_0,r)$ and $l(x_0,r)$ is bounded from above by a number that is independent of x_0.

Choose r to be small enough so that there exists a point $x_3\in X\setminus B(x_0,2r)$ with $f(x_3)\in B(f(x_0),4L(x_0,r))\setminus\overline{B}(f(x_0),2L(x_0,r))$.

By the fact that X and Y support a local Poincaré inequality we know that they are locally quasiconvex; see for example Hajłasz–Koskela [15, Proposition 4.5] or [27, Theorem 3.3]. Hence Y is bi-Lipschitz equivalent to a local geodesic space, and so we can assume that Y is geodesic. Let $\alpha\colon [0,1]\to Y$ be a geodesic curve joining $f(x_0)$ to $f(x_m)$ and $\beta\colon [0,1]\to Y$ be a curve joining $f(x_M)$ to $f(x_3)$ in

$$B(f(x_0),4C'L(x_0,r))\setminus B\biggl(f(x_0),\frac{L(x_0,r)}{C'}\biggr),$$

where $C'>1$ depends only on the constants in the Poincaré inequality on Y and on q. Such a curve β exists by [27, Theorem 3.3]. Hence, we have that $l_\alpha=l(x_0,r)$ and l_β is comparable to $L(x_0,r)$, where l_α and l_β denote the lengths of α and β, respectively.

If $l(x_0,r)\geq L(x_0,r)/2C'$ we have $L(x_0,r)/l(x_0,r)\leq 2C'$, and so we may assume that $l(x_0,r)<L(x_0,r)/2C'$; in which case $|\alpha|$ and $|\beta|$ are disjoint. Let $\Gamma(\alpha,\beta)$ denote all rectifiable curves joining $|\alpha|$ to $|\beta|$ in Y and let $\widehat{\alpha}=f^{-1}\circ\alpha$ and $\hat{\beta}=f^{-1}\circ\beta$ be the corresponding curves in X. By Kallunki–Shanmugalingam [23] we obtain

$$\operatorname{Mod}_q(\Gamma(\alpha,\beta))=\operatorname{Cap}_q(|\alpha|,|\beta|;Y).$$

Let u be the q-potential of the condenser $(|\alpha|,|\beta|)$ with respect to the ball $B:=B(f(x_0),10CL(x_0,r))\subset Y$. Then condition (1) implies that

$$\operatorname{Cap}_q(|\alpha|,|\beta|;B)=\int_B g_u^q\,d\mu_Y\geq C\int_{f^{-1}(B)} g_{u\circ f}^q\,d\mu_X.$$

If v is the q-potential of $(|\widehat{\alpha}|,|\hat{\beta}|)$ in $f^{-1}(B)$, then

$$\operatorname{Cap}_q(|\widehat{\alpha}|,|\hat{\beta}|;f^{-1}(B))=\int_{f^{-1}(B)} g_v^q\,d\mu_X\leq\int_{f^{-1}(B)} g_{u\circ f}^q\,d\mu_X.$$

Thus it follows that

$$\operatorname{Cap}_q(|\widehat{\alpha}|,|\hat{\beta}|;f^{-1}(B))\leq C\operatorname{Cap}_q(|\alpha|,|\beta|;B)$$

which is by [23] and Heinonen–Koskela [19, Proposition 2.17] equivalent to

$$\operatorname{Mod}_q(\Gamma(\widehat{\alpha},\hat{\beta}))\leq C\operatorname{Mod}_q(\Gamma(\alpha,\beta)). \tag{5}$$

The radius r was chosen so that $x_3 \in X \setminus B(x_0, 2r)$, therefore, we have that $\min\{\operatorname{diam}(|\widehat{\alpha}|), \operatorname{diam}(|\hat{\beta}|)\} \geq r$ and $\operatorname{dist}(|\widehat{\alpha}|, |\hat{\beta}|) \leq 2r$. Under our assumptions on X it is a Loewner space (see, e.g., [19]), thus there exists a decreasing homeomorphism $\psi_X : (0, \infty) \to (0, \infty)$ so that

$$\operatorname{Mod}_q(\Gamma(\widehat{\alpha}, \hat{\beta})) \geq \psi_X\left(\frac{\operatorname{dist}(|\widehat{\alpha}|, |\hat{\beta}|)}{\min\{\operatorname{diam}(|\widehat{\alpha}|), \operatorname{diam}(|\hat{\beta}|)\}}\right) \geq \psi_X(2) > 0.$$

Plugging this into the modulus estimate (5), we have the following lower bound for the modulus

$$\operatorname{Mod}_q(\Gamma(\alpha, \beta)) \geq \frac{\psi_X(2)}{C} > 0. \tag{6}$$

Let us define

$$\rho(y) = \frac{\widetilde{C}\chi_A}{d(f(x_0), y)},$$

where $A = \overline{B}(f(x_0), L(x_0, r)/C') \setminus B(f(x_0), l(x_0, r))$ and $\widetilde{C}$ is to be fixed. Let $B_i = B(f(x_0), 2^{-i}L(x_0, r)/C')$, $i = 0, 1, ..., k_0$, where k_0 is chosen so that $2^{-k_0}L(x_0, r)/C' \leq l(x_0, r) < 2^{-k_0+1}L(x_0, r)/C'$. We want to estimate the size of $\Gamma(\alpha, \beta)$ in terms of q-modulus using ρ. We first show that ρ is an admissible function for computing the q-modulus of $\Gamma(\alpha, \beta)$. To do so, take $\gamma \in \Gamma(\alpha, \beta)$. By our assumption that $l(x_0, r) < L(x_0, r)/2C'$, every curve that connects α to β must intersect both spheres $S(x_0, l(x_0, r))$ and $S(x_0, L(x_0, r)/C')$. We obtain

$$\int_\gamma \rho\, ds \geq \sum_{i=0}^{k_0-1} \int_{\gamma_i} \rho\, ds \geq \sum_{i=0}^{k_0-1} \frac{C'\widetilde{C} l_{\beta_i}}{2^{-i}L(x_0, r)} \geq \frac{\widetilde{C}k_0}{2} \geq \frac{\widetilde{C}}{2} \log_2 \frac{L(x_0, r)}{C'l(x_0, r)},$$

where $\gamma_i := \gamma|_{B_i \setminus \overline{B}_{i+1}}$ and β_i is a subcurve of γ lying in $B_i \setminus B_{i+1}$ and connecting ∂B_i to ∂B_{i+1}. It follows that ρ is admissible for $\Gamma(\alpha, \beta)$ if we choose

$$\widetilde{C} \approx \left(\log_2 \frac{L(x_0, r)}{C'l(x_0, r)}\right)^{-1}.$$

Hence we get that

$$\begin{aligned}\operatorname{Mod}_q(\Gamma(\alpha, \beta)) &\leq \int_A \rho^q\, d\mu = \sum_{i=0}^{k_0-1} \int_{B_i \setminus \overline{B}_{i+1}} \rho^q\, d\mu \leq \sum_{i=0}^{k_0-1} \frac{(C')^q \widetilde{C}^q \mu(B_i)}{(2^{-i-1}L(x_0, r))^q} \\ &\leq C\left(\log_2 \frac{L(x_0, r)}{C'l(x_0, r)}\right)^{-q} k_0 \leq C\left(\log_2 \frac{L(x_0, r)}{C'l(x_0, r)}\right)^{1-q},\end{aligned}$$

where the constant C does not depend on x_0 or r. Hooking this up with the modulus estimate (6) we obtain that

$$\log_2 \frac{L(x_0,r)}{C'l(x_0,r)} \le \left(\frac{C}{\psi_X(2)}\right)^{1/(q-1)}. \tag{7}$$

From (7) it follows that the distortion of f at an arbitrary point $x_0 \in X$ is

$$\limsup_{r\to 0} \frac{\sup_{d_X(x_0,x)=r} d_Y(f(x_0),f(x))}{\inf_{d_X(x_0,x)=r} d_Y(f(x_0),f(x))} \le C' \exp\left(\left(\frac{C}{\psi_X(2)}\right)^{1/(q-1)}\right) < \infty,$$

and hence, f is quasiconformal.

The proof of the sufficiency of condition (2) is analogous to above, with minor modifications as follows: In this case, the proof can be modified by pointing out that if u is the q-potential of the condenser $(|\alpha|,|\beta|)$ with respect to the ball B, then u is a 1-quasiminimizer in $B\setminus(|\alpha|\cup|\beta|)$ and so $u\circ f$ is a Q'-quasiminimizer in $f^{-1}(B)\setminus(|\widehat{\alpha}|\cup|\hat{\beta}|)$, and thus the comparison between the two q-capacities still holds. □

5. Properties of quasiharmonic morphisms

In this section we do not assume that $f\colon X\to Y$ is a homeomorphism.

Proposition 5.1. (Radó property) *Let $f\colon X\to Y$ be continuous and non-constant such that the following two conditions hold*:

(1) *$f\in N^{1,q}_{\mathrm{loc}}(X,Y)$;*

(2) *for all $u\in N^{1,q}_{\mathrm{loc}}(Y)$ and relatively compact open sets $\Omega\subset X$,*

$$\int_\Omega g^q_{u\circ f}\, d\mu_X \le C \int_{f(\Omega)} g^q_u\, d\mu_Y,$$

where $C\ge 1$ depends only on the data associated with X and Y.

Then for every $y\in Y$, $f^{-1}(y)$ is of zero q-capacity, and so has empty interior and is totally disconnected.

Proof. We may assume that $y\in f(X)$. Since f is non-constant there exists $z\in Y$ such that $f(z)\ne f(y)$. Moreover, because $\{y\}$ is of zero q-capacity, $\chi_{\{y\}}\in N^{1,q}(Y)$ with

$$\int_Y g^q_{\chi_{\{y\}}}\, d\mu_Y = 0.$$

Assumption 2 therefore implies that

$$\int_Y g^q_{\chi_{\{y\}}\circ f}\, d\mu_X = 0,$$

and that $\chi_{\{y\}}\circ f=\chi_{f^{-1}(y)}$ is q-q.e. constant in X, i.e., constant outside a set of zero q-capacity, by the facts that X is connected and locally supports a $(1,q)$-Poincaré inequality (local Poincaré inequality by itself implies that $\chi_{\{y\}}\circ f$ is locally q-q.e. constant). Since f is continuous and $Y\setminus\{y\}$ is open, $f^{-1}(Y\setminus\{y\})$ is open and also non-empty (recall that $z\in f(X)\setminus\{y\}$). Therefore,

$$X\setminus f^{-1}(y)=f^{-1}(f(X)\setminus\{y\})=f^{-1}(Y\setminus\{y\})$$

is non-empty and open. Thus

$$C_q(X\setminus f^{-1}(y))>0.$$

Hence $\chi_{\{y\}}\circ f$ must be 0 q-q.e. in X, i.e. $C_q(f^{-1}(y))=0$. $\square$

Theorem 5.2. *Let $f\colon X\to Y$ be continuous and the following three conditions hold*:

(1) $f\in N^{1,q}_{\mathrm{loc}}(X,Y)$;

(2) *for all $u\in N^{1,q}_{\mathrm{loc}}(Y)$ and relatively compact open sets $\Omega\subset X$,*

$$\int_\Omega g^q_{u\circ f}\, d\mu_X \le C\int_{f(\Omega)} g^q_u\, d\mu_Y,$$

where $C\ge 1$ depends only on the data associated with X and Y;

(3) *for some $Q\ge 1$ there exists $1\le Q'<\infty$ such that whenever u is a Q-quasiminimizer in a relatively compact domain $\Omega_0\subset Y$, $u\circ f$ is a Q'-quasiminimizer in $f^{-1}(\Omega_0)$.*

Then f is an open mapping or constant.

The following proof closely follows the proof of Theorem 2.1 in Heinonen–Kilpeläinen–Martio [16].

Proof. We may assume that f is non-constant. Fix $x_0\in X$ and assume, on the contrary, that there is a sequence of radii $r_i\to 0$ such that $f(B(x_0,r_i))$ is not a neighborhood of $f(x_0)$. Since f is non-constant there exists at least one point in $f(X)\setminus\{f(x_0)\}$. Choose a ball $B=B(x_0,r)$ such that $f(x_0)\in\partial f(B)$ and that $f(X)\setminus f(\overline{B})$ is not empty.

Since $f(x_0)\in\partial f(B)$, there exists a sequence $\{y_i\}_{i=1}^\infty$ of points from $Y\setminus f(B)$ such that $\lim_{i\to\infty} y_i=f(x_0)$. For each i let G_i be a singular function in Ω_0 with singularity at y_i; more precisely, let $G_i \in N^{1,q}_{\mathrm{loc}}(Y\setminus\{y_i\})$ be q-harmonic and positive in $\Omega_0\setminus\{y_i\}\subset Y$, where Ω_0 is a relatively compact subset such that $C_q(Y\setminus\Omega_0)>0$, $f(\overline{B})\subset\Omega_0$, G_i is 0 q-q.e. outside Ω_0, and $\lim_{x\to y_i} G_i(x)=\infty$; see Holopainen–Shanmugalingam [22] for the existence of such functions.

By Proposition 5.1, $f^{-1}(f(x_0))$ is of zero q-capacity, and thus has empty interior. Hence there exists $z_0\in B$ such that $f(z_0)\neq f(x_0)$ and $f(z_0)\in f(B)\subset f(\overline{B})\subset\Omega_0$. Since $f(z_0)\neq y_i$, we may rescale G_i so that for each i we have $G_i(f(z_0))=1$. This rescaled sequence is also denoted G_i.

From a scale invariant Harnack inequality, see Kinnunen–Shanmugalingam [26], it follows that for each relatively compact set $K\subset\Omega_0\setminus\{f(x_0)\}$ there exists i_0 such that the sequence G_i, $i\geq i_0$, is a uniformly bounded family of q-harmonic functions, and hence equicontinuous on K, and we may select a subsequence, still denoted by G_i, which converges locally uniformly to a q-harmonic function G_∞ in $\Omega_0\setminus\{f(x_0)\}$, see Shanmugalingam [32, Theorem 1.2]. The limit function G_∞ is positive in $\Omega_0\setminus\{f(x_0)\}$, and G_∞ is not constant as $G_\infty(f(z_0))=1$ and $G_\infty(z)\to 0$ as $\Omega_0\ni z\to\partial\Omega_0$, see the construction in [22]. By Theorem 2.5 there exists a singularity at $f(x_0)$ since G_∞ is a non-constant q-harmonic function with zero boundary values. Moreover, by the estimates in [22] for a singular function near its singularity, we may deduce that G_∞ indeed has an infinite limit at $f(x_0)$ and hence is a singular function in Ω_0 with singularity at $f(x_0)$.

Each G_i is a positive q-harmonic function in $\Omega_0\setminus\{y_i\}$. Thus by condition (3) the pull back $u_i=G_i\circ f$ is a positive Q'-quasiminimizer in $f^{-1}(\Omega_0\setminus\{y_i\})$ so, in particular, in B (recall that $y_i\notin f(B)$ so $f^{-1}(y_i)\notin B$). Arguing as above, we extract a subsequence of u_i which converges locally uniformly to a positive Q'-quasiminimizer u_∞ in B with $u_i(z_0)=1$. As a quasiminimizer in B, u_∞ is Hölder continuous and, in particular, finite-valued in B. Moreover,

$$u_\infty(x_0)=\lim_{i\to\infty} u_i(x_0)=\lim_{i\to\infty} G_i(f(x_0)),$$

but by Danielli–Garofalo–Marola [8]

$$G_i(f(x_0))\approx \mathrm{Cap}_q(\overline{B}(y_i,d_Y(y_i,f(x_0))),Y\setminus\Omega_0;Y)^{1/(1-q)},$$

which indicates that $u_\infty(x_0)=\infty$. This is a contradiction. □

6. Quasiconformal mappings and densities

As before, in this section X and Y are metric measure spaces of locally q-bounded geometry.

Let $E\subset X$ be a measurable set. For $x\in X$, we call

$$\overline{D}(E,x)=\limsup_{r\to 0}\frac{\mu_X(E\cap B(x,r))}{\mu_X(B(x,r))}$$

the *upper density* of E at x and

$$\underline{D}(E,x)=\liminf_{r\to 0}\frac{\mu_X(E\cap B(x,r))}{\mu_X(B(x,r))}$$

the *lower density* of E at x. When $\overline{D}(E,x)=\underline{D}(E,x)$, the common value $D(E,x)$ is called the *density* of E at x.

It is well-known, by a result of Gehring and Kelly in [11], that if D and D' are two domains in $\mathbb{R}^n$, $n\geq 2$, $f\colon D\to D'$ is a quasiconformal mapping, and $E\subset D$ is measurable, then

$$D(f(E),f(x))=1 \quad \text{if and only if} \quad D(E,x)=1$$

for *all* $x\in D$. That is, points of density are preserved under quasiconformal mappings. We show that this holds true also for metric quasiconformal mappings.

Theorem 6.1. *If $f\colon X\to Y$ is a K-quasiconformal mapping and if $E\subset X$ is measurable, then*

$$\overline{D}(f(E),f(x))\leq b\overline{D}(E,x)^a$$

and

$$\underline{D}(f(E),f(x))\geq 1-b(1-\underline{D}(E,x))^a$$

for all $x\in X$. Here a and b are fixed positive constants.

We postpone the proof of Theorem 6.1 until after the proof of Theorem 6.2.

6.1. Uniform density property

Suppose that $f\colon X\to Y$ is a homeomorphism. For each $x\in X$ and each ball $B_1\subset X$ with center at x let B_2 denote the largest open ball in $f(B_1)$ with center at $f(x)$. We say that f has a *uniform density property* if there exists a continuous function $\varphi\colon [0,\infty)\to[0,\infty)$ with $\varphi(0)=0$ such that for each $x\in X$ and each sufficiently small B_1 with center at x,

$$\frac{\mu_Y(f(E)\cap B_2)}{\mu_Y(B_2)}\leq\varphi\biggl(\frac{\mu_X(E\cap B_1)}{\mu_X(B_1)}\biggr) \tag{8}$$

for every measurable $E\subset X$.

It readily follows that a function f with a uniform density property also satisfies Luzin's condition (N), that is, f is absolutely continuous in measure.

Theorem 6.2. *If $f\colon X\to Y$ is a K-quasiconformal mapping, then f has a uniform density property with $\varphi(t)=bt^a$ for some positive constants a and b.*

Proof. Fix $x\in X$ and choose $B_1=B(x,r)\subset X$. By Corollary 7.14 in Heinonen–Koskela [19],

$$\fint_{B_1} J_f^p\,d\mu_X \le C\biggl(\fint_{B_1} J_f\,d\mu_X\biggr)^p = C\biggl(\frac{\mu_Y(f(B_1))}{\mu_X(B_1)}\biggr)^p \tag{9}$$

for some $p>1$. Let E be a measurable subset of X. By the absolute continuity in measure of f, Hölder's inequality, and (9), we have

$$\begin{aligned}\mu_Y(f(E)\cap B_2) &\le \int_{E\cap B_1} J_f\,d\mu_X \le \biggl(\int_{B_1} J_f^p\,d\mu_X\biggr)^{1/p}\mu_X(E\cap B_1)^{(p-1)/p}\\ &\le C\mu_X(B_1)^{1/p}\frac{\mu_Y(f(B_1))}{\mu_X(B_1)}\mu_X(E\cap B_1)^{(p-1)/p}.\end{aligned}$$

Hence

$$\frac{\mu_Y(f(E)\cap B_2)}{\mu_Y(f(B_1))} \le C\biggl(\frac{\mu_X(E\cap B_1)}{\mu_X(B_1)}\biggr)^{(p-1)/p}.$$

By the quasiconformality (and hence the quasisymmetry) of f and Ahlfors regularity of the spaces,

$$\mu_Y(f(B_1)) \le C\mu_Y(B_2).$$

Thus f satisfies (8) with $\varphi(t)=Ct^{(p-1)/p}$. □

Proof of Theorem 6.1. The first inequality follows from Theorem 6.2, since as the radius of B_1 converges to 0, so does the radius of B_2. The second inequality of the theorem follows from the relation

$$\underline{D}(E,x)+\overline{D}(X\setminus E,x)=1$$

by applying the first inequality to $X\setminus E$. □

6.2. Characterization for quasiconformality

Recall that given a continuous map $f\colon X\to Y$, the *generalized Jacobian* of f at $x\in X$ is defined by

$$J_f(x)=\limsup_{r\to 0}\frac{\mu_Y(f(B(x,r)))}{\mu_X(B(x,r))}.$$

It follows from Mattila [29, p. 36, Theorem 2.12] applied to the pull-back measure $\nu(E)=\mu_Y(f(E))$, that for μ_X-a.e. $x\in X$,

$$J_f(x)<\infty,$$

and that for each measurable set $E\subset X$

$$\int_E J_f(x)\,d\mu_X \le \mu_Y(f(E)), \tag{10}$$

with equality if f is absolutely continuous in measure. While the results of [29] are stated and proved in the Euclidean setting, the proofs relevant to the above extend directly to metric measure spaces whose measure μ_X is locally doubling (locally Ahlfors regular spaces satisfy this condition).

We have the following characterization for quasiconformal mappings in terms of the uniform density property. In $\mathbb{R}^n$ the result was proved by Gehring and Kelly in [11, Theorem 3].

Theorem 6.3. *A homeomorphism $f\colon X\to Y$ is quasiconformal if and only if it is absolutely continuous on q-modulus a.e. curve and f^{-1} has a uniform density property.*

Remark 6.4. Recall that f is quasiconformal if and only if f^{-1} is. Hence, by this symmetry we could equivalently state the preceding theorem: A homeomorphism $f\colon X\to Y$ is quasiconformal if and only if f^{-1} is absolutely continuous on q-modulus a.e. curve and f has a uniform density property. Moreover, Theorem 6.5 could be stated as follows: Suppose that Y satisfies a local $(1,1)$-Poincaré inequality. Then a homeomorphism $f\colon X\to Y$ is quasiconformal if and only if f has a uniform density property.

Proof. The necessity follows from Theorems 3.1 and 6.2 and the fact that if f is quasiconformal then so is f^{-1}. To prove sufficiency, we show that if f is absolutely continuous on q-modulus a.e. curve and f^{-1} has a uniform density property then f is quasiconformal.

Let $x\in X$ and

$$K_0(x):=\limsup_{r\to 0} K_0(x,r):=\limsup_{r\to 0}\frac{\mu_Y(B')}{\mu_Y(f(B))},$$

where $B=B(x,r)$ and B' denotes the smallest open ball with center at $f(x)$ containing $f(B)$. Let $B_1=B'$, $B_2=B$ and $E=f(B)$. For sufficiently small r the uniform density property for f^{-1} implies that

$$1\le\varphi\biggl(\frac{\mu_Y(f(B))}{\mu_Y(B')}\biggr)$$

and hence that

$$\frac{\mu_Y(f(B))}{\mu_Y(B')} \geq \sigma := \inf\{s>0\colon \varphi(s)\geq 1\} > 0.$$

So

$$K_0(x) \leq K := 1/\sigma \tag{11}$$

for each $x\in X$, where K is a positive constant which depends only on φ. By first applying (11) and then the discussion preceding (10), we obtain that

$$\begin{aligned}\limsup_{r\to 0} \frac{\mu_Y(B')}{\mu_X(B)} &= \limsup_{r\to 0} \frac{\mu_Y(B')}{\mu_Y(f(B))}\frac{\mu_Y(f(B))}{\mu_X(B)} \\ &\leq K \limsup_{r\to 0}\frac{\mu_Y(f(B))}{\mu_X(B)} = KJ_f(x) < \infty\end{aligned}$$

for μ_X-a.e. $x\in X$. It follows from the Ahlfors regularity of X and Y that

$$L_f(x)^q \leq C\limsup_{r\to 0}\frac{\mu_Y(B')}{\mu_X(B)} \leq CKJ_f(x) \tag{12}$$

for μ_X-a.e. in X. Because f is absolutely continuous on q-modulus a.e. curve in X, we may take L_f as a q-weak upper gradient for f, and $L_f\in L^q_{\text{loc}}(X)$ by inequalities (10) and (12). As X supports a local $(1,q)$-Poincaré inequality, it follows that $f\in N^{1,q}_{\text{loc}}(X,Y)$ by [20, Theorem 6.11]. Thus by (12), f is quasiconformal with constant CK. $\square$

6.3. Absolute continuity

In Euclidean spaces, the equivalence of the uniform density property and quasiconformality can be shown without the extra assumption of absolute continuity of f on q-modulus a.e. curve. In $\mathbb{R}^n$ the ACL-property of f follows directly from the uniform density property which, in turn, can be shown by an adaptation of Väisälä [33, Theorem 31.2]. The proof of [33, Theorem 31.2] cannot be modified to our setting mainly because the linear structure of $\mathbb{R}^n$ plays an important role in the proof. If we, however, assume in addition that X (or, by symmetry, Y) satisfies a $(1,1)$-Poincaré inequality we are able to remove the extra assumption also in the metric setting. The following is the main result of this section.

Theorem 6.5. *Suppose that X satisfies a local $(1,1)$-Poincaré inequality. Then a homeomorphism $f\colon X\to Y$ is quasiconformal if and only if f^{-1} has a uniform density property.*

The assumption of a stronger $(1,1)$-Poincaré inequality is necessary for the next lemma, which shows that it is enough to have absolute continuity only on 1-modulus a.e. curve in order to show that a function is in $N^{1,q}_{\mathrm{loc}}(X,Y)$.

Lemma 6.6. *Suppose that X satisfies a local $(1,1)$-Poincaré inequality. Let $f\in N^{1,1}_{\mathrm{loc}}(X,Y)$ be continuous. Then if both f and a 1-weak upper gradient are L^q_{loc}-integrable, it follows that $f\in N^{1,q}_{\mathrm{loc}}(X,Y)$.*

If we remove the condition that f is continuous, we can show that there exists a function $\tilde f\in N^{1,q}_{\mathrm{loc}}(X,Y)$ such that $f=\tilde f$ μ_X-a.e. in X.

The proof of the preceding lemma is similar to that of Lemma 5.1 in Kinnunen–Korte–Shanmugalingam–Tuominen [24]. The crux of the proof is to use the discrete convolution of f. In fact, [24, Lemma 5.1] is stated and proved for real-valued functions, but the argument continues to work when the target is Y. It suffices to consider the case in which the target is the Banach space ℓ^∞. Recall that any separable metric space can be isometrically embedded into ℓ^∞. In this case, the discrete convolution can be carried out. Furthermore, as in Heinonen–Koskela–Shanmugalingam–Tyson [20] we define

$$N^{1,q}(X,Y)=\{f\in N^{1,q}(X,\ell^\infty(Y)) : f(x)\in Y \text{ for } q\text{-q.e. } x\in X\},$$

and similarly for $N^{1,q}_{\mathrm{loc}}(X,Y)$. We leave the details of the proof to the reader.

Note that the previous lemma does not hold in general without a local $(1,1)$-Poincaré inequality. For a counterexample, see Remark 5.2 in [24].

Theorem 6.5 follows from Theorem 6.3 and the following two lemmas.

Lemma 6.7. *Let $f\colon X\to Y$ be a homeomorphism such that f^{-1} satisfies the uniform density property. Let γ be a rectifiable curve in X such that*

$$\liminf_{r\to 0}\frac{\mu_Y(f(|\gamma|_r))}{r^{q-1}}<\infty, \tag{13}$$

where $|\gamma|_r$ denotes the r-neighborhood of γ. Then f is absolutely continuous on γ.

Proof. Let Ω be some bounded set containing γ. Let F be a compact subset of $|\gamma|$ and set

$$F_k=\{x\in F : K_0(x,r)\le 2K \text{ for all } 0<r<1/k\},$$

where K is as in (11). Then $F=\bigcup_{k=1}^\infty F_k$ and $F_k\subset F_{k+1}$ for all $k\in\mathbb{N}$. By the continuity of f, F_k is compact for every k.

Fix k and $t>0$. Let $0<r<1/k$ be small enough so that F_k can be covered by balls $B_i=B(x_i,r)$, $i=1,2,...,p$, with $rp<C\mathcal{H}^1(F_k)$ for some constant C that only depends on the doubling constant, and so that for every $x\in B_i$,

$$d(f(x),f(x_i))<\tfrac{1}{2}t.$$

We can do this because as f is continuous and F_k is compact, f is uniformly continuous on F_k. Moreover, we can choose the balls B_i such that $x_i\in F_k$ for every i and

$$\sum_{i=1}^{p}\chi_{B_i}<C,$$

with C depending only the doubling constant of μ. Let $\widetilde{B}_i$ be the smallest ball with center at $f(x_i)$ and containing $f(B_i)$. Then by Ahlfors regularity of Y and (11), we have for sufficiently small $r>0$,

$$\operatorname{diam}(\widetilde{B}_i)^q\approx\mu_Y(\widetilde{B}_i)\le 2K\mu_Y(f(B_i)).$$

Notice that, by the choice of r, the diameter of $\widetilde{B}_i$ is not greater than t. Therefore, by Hölder's inequality, we can estimate the Hausdorff content of $f(F_k)$ as follows:

$$\begin{aligned}\mathcal{H}^1_t(f(F_k))^q &\le \left(\sum_{i=1}^{p}\operatorname{diam}(\widetilde{B}_i)\right)^q \le p^{q-1}\sum_{i=1}^{p}\operatorname{diam}(\widetilde{B}_i)^q\\ &\le CKp^{q-1}\sum_{i=1}^{p}\mu_Y(f(B_i))\le CK\frac{\mathcal{H}^1(F_k)^{q-1}}{r^{q-1}}\mu_Y(f(|\gamma|_r)).\end{aligned}$$

Here we used the bounded overlap property of the balls B_i and the fact that f is a homeomorphism when obtaining the last inequality above. Next, by letting $r\to 0$ so that (13) is satisfied, then letting $t\to 0$ and using (13), we obtain that

$$\mathcal{H}^1(f(F_k))<C\mathcal{H}^1(F_k)^{(q-1)/q},$$

with C independent of k. Thus it follows that

$$\mathcal{H}^1(f(F))<C\mathcal{H}^1(F)^{(q-1)/q}.$$

Since the above inequality holds for all compact subsets F of $|\gamma|$, the function f is absolutely continuous on γ. □

Lemma 6.8. *Let $f\colon X\to Y$ be a homeomorphism such that f^{-1} satisfies the uniform density property. Let Γ be the family of rectifiable curves γ such that*

$$\liminf_{r\to 0}\frac{\mu_Y(f(|\gamma|_r))}{r^{q-1}}=\infty. \tag{14}$$

Then $\operatorname{Mod}_1(\Gamma)=0$.

Proof. Let Ω be a relatively compact domain in X. Let $\Gamma_{M,\varepsilon}$ be the family of rectifiable curves γ in Ω such that its length $l_\gamma > \varepsilon$ and the quotient in (14) is larger than M for all $r<2\varepsilon$. Then, with the constant $C=10^q$, the function

$$\rho_{M,\varepsilon}(x) = \frac{C}{M}\frac{\mu_Y(f(B(x,\varepsilon)))}{\varepsilon^q}$$

is lower semicontinuous and hence is a Borel measurable function, and furthermore it is an admissible test function for the 1-modulus of $\Gamma_{M,\varepsilon}$; to see this, note that if $\gamma \in \Gamma_{M,\varepsilon}$, then we can break γ into subcurves γ_j, $j=1,...,n$, such that the length of γ_j is between $\frac{1}{10}\varepsilon$ and $\frac{1}{8}\varepsilon$. Then

$$\int_\gamma \rho_{M,\varepsilon}\,ds = \sum_{j=1}^n \int_{\gamma_j} \rho_{M,\varepsilon}\,ds.$$

For $j=1,...,n$ we fix $x_j \in |\gamma_j|$. For each $x \in |\gamma_j|$, we have by the monotonicity of measures and the fact that $B\big(x_j, \frac{1}{4}\varepsilon\big) \subset B(x,\varepsilon)$,

$$\rho_{M,\varepsilon}(x) = \frac{C}{M\varepsilon^q}\mu_Y(f(B(x,\varepsilon))) \ge \frac{C}{M\varepsilon^q}\mu_Y\big(f\big(B\big(x_j,\tfrac{1}{4}\varepsilon\big)\big)\big).$$

Hence

$$\int_{\gamma_j} \rho_{M,\varepsilon}\,ds \ge \frac{C}{M\varepsilon^q}\mu_Y\big(f\big(B\big(x_j,\tfrac{1}{4}\varepsilon\big)\big)\big) l_{\gamma_j} \ge \frac{C}{10M\varepsilon^{q-1}}\mu_Y\big(f\big(B\big(x_j,\tfrac{1}{4}\varepsilon\big)\big)\big).$$

It follows that

$$\int_\gamma \rho_{M,\varepsilon}\,ds \ge \frac{C}{10M\varepsilon^{q-1}}\sum_{j=1}^n \mu_Y\big(f\big(B\big(x_j,\tfrac{1}{4}\varepsilon\big)\big)\big) \ge \frac{C}{10M\varepsilon^{q-1}}\mu_Y\bigg(f\bigg(\bigcup_{j=1}^n B\big(x_j,\tfrac{1}{4}\varepsilon\big)\bigg)\bigg).$$

If $z\in X$ such that $\operatorname{dist}(z,|\gamma|) < \frac{1}{10}\varepsilon$, then there is some $j \in \{1,...,n\}$ and a point $x \in |\gamma_j|$ such that $d(z,x) < \frac{1}{10}\varepsilon$, in which case we have $d(x_j,z) \le d(x_j,x)+d(x,z) < \frac{1}{8}\varepsilon + \frac{1}{10}\varepsilon < \frac{1}{4}\varepsilon$, and so it follows that $|\gamma|_{\frac{1}{10}\varepsilon} \subset \bigcup_{j=1}^n B\big(x_j,\frac{1}{4}\varepsilon\big)$. Therefore, by the choice of C in the definition of $\rho_{M,\varepsilon}$,

$$\int_\gamma \rho_{M,\varepsilon}\,ds \ge \frac{C}{10M}\frac{\mu_Y(f(|\gamma|_{\frac{1}{10}\varepsilon}))}{\varepsilon^{q-1}} \ge \frac{C}{10M}\frac{M}{10^{q-1}} = 1.$$

Thus we get the estimate

$$\operatorname{Mod}_1(\Gamma_{M,\varepsilon}) \le \int_\Omega \rho_{M,\varepsilon}\,d\mu_X.$$

Since μ is doubling we can cover Ω by countably many balls $\{B(x_i,\varepsilon)\}_{i=1}^\infty$, $x_i \in \Omega$, such that $\sum_{i=1}^\infty \chi_{B(x_i,2\varepsilon)} \le C$ with the constant C independent of ε. If

$x \in B(x_i, \varepsilon)$, then $B(x,\varepsilon) \subset B(x_i, 2\varepsilon)$, and so by the bounded overlap property above (and by extending $\rho_{M,\varepsilon}$ by zero outside of Ω)

$$\begin{aligned}\int_\Omega \rho_{M,\varepsilon}\,d\mu_X &\le \sum_{i=1}^\infty \int_{B(x_i,\varepsilon)} \rho_{M,\varepsilon}\,d\mu_X \le \frac{C}{M\varepsilon^q}\sum_{i=1}^\infty \mu_Y(f(B(x_i,2\varepsilon)))\mu_X(B(x_i,\varepsilon)) \\ &\le \frac{C}{M}\sum_{i=1}^\infty \mu_Y(f(B(x_i,2\varepsilon))) \le \frac{C}{M}\mu_Y(f(\Omega_{2\varepsilon})),\end{aligned}$$

where $\Omega_{2\varepsilon}$ is the 2ε-neighborhood of Ω. Now by first letting $\varepsilon \to 0$ and then $M \to \infty$, it follows that $\operatorname{Mod}_1(\Gamma)=0$. □

Proof of Theorem 6.5. If f is absolutely continuous on a rectifiable curve γ, then

$$d_Y(f(x), f(y)) \le \int_\gamma L_f\,ds,$$

where x and y are the endpoints of γ, and L_f is the maximal stretching of f defined in Section 3. By Lemmas 6.7 and 6.8, the function f is absolutely continuous on 1-modulus a.e. curve. This implies that L_f is a 1-weak upper gradient of f. It follows from the proof of Theorem 6.3 that L_f is locally L^q-integrable and therefore also locally L^1-integrable. Thus $f \in N^{1,1}_{\rm loc}(X,Y)$. By Lemma 6.6 (or Heinonen–Koskela–Shanmugalingam–Tyson [20]), this implies that $f \in N^{1,q}_{\rm loc}(X,Y)$. The rest of the proof follows as in the proof of Theorem 6.3. □

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