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Boundary integral operators and boundary value problems for Laplace's equation

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Abstract. In this paper, we define boundary single and double layer potentials for Laplace's equation in certain bounded domains with d-Ahlfors regular boundary, considerably more general than Lipschitz domains. We show that these layer potentials are invertible as mappings between certain Besov spaces and thus obtain layer potential solutions to the regularity, Neumann, and Dirichlet problems with boundary data in these spaces.

1. Introduction

In this note we study layer potentials for Laplace's equation on the boundaries of certain bounded domains with d-Ahlfors regular boundary in $\mathbf{R}^n$, $n\geq 3$. As an application of our results, we obtain layer potential solutions to the regularity, Neumann, and Dirichlet problems for the Laplacian with boundary data in certain Besov spaces. We remark that in Lipschitz domains, there is an extensive literature concerning solution of the regularity, Neumann, and Dirichlet problems by way of layer potentials (with boundary data in L^p) for classical linear elliptic partial differential equations arising in mathematical physics, e.g., Laplace's equation, Maxwell's equation, Stokes and Láme systems of equations (see [2], [3], [4], [14] and [18]). More recently layer potential solutions to these problems have been studied for Laplace's equation in domains beyond Lipschitz domains and in Lipschitz domains with boundary data in certain Besov spaces (see [7] and [13] for references). To compare our results with those cited above, we shall need some notation. Let $X=(X_1,...,X_n)$ denote a point in $\mathbf{R}^n$, let $|X|$ be the standard Euclidean norm of X and for given $r>0$, set $B(X,r)=\{Y\in\mathbf{R}^n:|Y-X|<r\}$.

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Let $d(E,F)=\inf\{|X-Y|:X\in E \text{ and } Y\in F\}$ denote the Euclidean distance between $E,F\subset\mathbf{R}^n$ and let diam $E=\sup\{|X-Y|:X,Y\in E\}$ be the diameter of E. Given $k>0$, define Hausdorff k-measure on $\mathbf{R}^n$, denoted H^k, as follows: For fixed $0<\delta<r_0$ and $E\subseteq\mathbf{R}^n$, let $L(\delta)=\{B(Z_i,r_i)\}_{i=1}^{\infty}$ be such that $E\subseteq\bigcup_{i=1}^{\infty}B(Z_i,r_i)$ and $0<r_i<\delta$, $i=1,2,\dots$. Set

$$\phi_\delta(E)=\inf_{L(\delta)}\sum_{i=1}^{\infty}r_i^k.$$

Then

$$H^k(E)=\lim_{\delta\to 0}\phi_\delta(E).$$

If $1\le q\le\infty$, let L^q be the usual Lebesgue space of qth power integrable functions h on $\mathbf{R}^n$ with norm denoted by $\|h\|_{L^q}$. Let $W^{1,q}$ be the Sobolev space of functions $f\colon\mathbf{R}^n\to\mathbf{R}$ with distributional gradient $\nabla f=(f_{x_1},\dots,f_{x_n})$, both of which are qth power integrable on $\mathbf{R}^n$. Let

$$\|f\|_{W^{1,q}}=\|f\|_{L^q}+\big\||\nabla f|\big\|_{L^q}$$

be the norm of f in $W^{1,q}$. $L^q(O)$ and $W^{1,q}(O)$ are defined similarly whenever O is an open set. Let $\|\cdot\|_{L^q(O)}$ and $\|\cdot\|_{W^{1,q}(O)}$ denote the norms in these spaces and let $C_0^\infty(O)$ be the class of infinitely differentiable functions with compact support in O. Let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the $W^{1,q}(O)$ norm. If $1\le q<n$ and $q^*=nq/(n-q)$, we let $R^{1,q}$ be the Riesz potential space consisting of real-valued functions f on $\mathbf{R}^n$ with distributional gradients and norm

$$\|f\|_{R^{1,q}}=\|f\|_{L^{q^*}}+\big\||\nabla f|\big\|_{L^q}<\infty.$$

Recall that a measurable function $\omega\colon\mathbf{R}^n\to[0,\infty]$ is an A_2 weight provided there is a number C, $0<C<\infty$, such that

$$\int_{B(Z,\rho)}\omega\,dX\int_{B(Z,\rho)}\frac{1}{\omega}\,dX\le CH^n(B(Z,\rho))^2.$$

The least such C for which the above inequality holds is denoted by $\|\hat{\omega\|}$ and is called the A_2 constant for ω. Let L^2_ω be the space of Lebesgue measurable functions that are square integrable with respect to $\omega\,dX$ and with norm denoted by $\|\cdot\|_{L^2_\omega}$. Throughout this paper we assume that Ω is an open set and $\partial\Omega\subset\mathbf{R}^n$ is a bounded $d=(d_1,\dots,d_N)$-*Ahlfors regular set.* That is, we make the following assumption.

Assumption A1. (a) *$\partial\Omega=\bigcup_{i=1}^N E_i$, where $E_i\subset\mathbf{R}^n$, $1\le i\le N<\infty$, is compact.*

(b) *There is an $r_1>0$ with $d(E_i,E_j)>r_1$ whenever $i\ne j$ and $1\le i,j\le N$.*

(c) *There exist $c_1<1\le c_2$ and d_i, $1\le i\le N$, such that $n-2<d_i<n$ and $c_1r^{d_i}\le H^{d_i}(B(X,r)\cap E_i)\le c_2r^{d_i}$ whenever $X\in E_i$ and $0<r<r_1$.*

We note that if $N=1$, then our definition agrees with the definition of a d-set in [10]. In our theorems involving double layer potentials we also require Assumptions A2 and A3.

Assumption A2. *Let G be either Ω or $\mathbf{R}^n \setminus \bar{\Omega}$. There exists $\sigma_0 > 0$ such that if $q \in [2-\sigma_0, 2+\sigma_0]$ and $v \in W^{1,q}$ with $v=a=\text{constant}$ in G, then $v(X)=a$ for H^{d_i}-almost every $X \in E_i$ and $1 \le i \le N$.*

Assumption A3. *Let G be either Ω or $\mathbf{R}^n \setminus \bar{\Omega}$. There exists c_3 and c_4, $0 < c_3, c_4 < \infty$, such that the following is true whenever ω is an A_2 weight with $\|\omega\hat{\|} \le c_3$. Let f be in $W^{1,1}(O)$ whenever $O \subset G$ is a bounded open set. Then f has a locally integrable extension $\hat{f}$ to $\mathbf{R}^n$ with distributional derivative $\nabla \hat{f}$. Moreover,*

$$\|\nabla \hat{f}\|^2_{L^2_\omega} \le c_4 \int_G |\nabla f|^2 \omega \, dX.$$

Note that the above inequality holds trivially if the right-hand side is infinite. Also if G is bounded, then $f \in W^{1,1}(G)$. Next, given p, $1<p<\infty$, let $L^p(E_i)$, $1 \le i \le N$, be the Lebesgue space of pth power integrable functions g on E_i with

$$\|g\|^p_{L^p(E_i)} = \int_{E_i} |g|^p \, dH^{d_i} < \infty.$$

If $f\colon \partial\Omega \to \mathbf{R}$ and $f|_{E_i} \in L^p(E_i)$, $1 \le i \le N$, set

$$\|f\|_{L^p(\partial\Omega)} = \sum_{i=1}^N \|f|_{E_i}\|_{L^p(E_i)}.$$

If $1<p<\infty$, $0<s_i<1$ and $1 \le i \le N$, let $\widetilde{B}^{p,s_i}(E_i)$ be the Besov space of H^{d_i} measurable functions f on E_i with $\|f\|_{\tilde{B}^{p,s_i}(E_i)} < \infty$, where

$$\|f\|_{\tilde{B}^{p,s_i}(E_i)} = \left(\int_{E_i} \int_{E_i} \frac{|f(P)-f(Q)|^p}{|P-Q|^{s_i p + d_i}} \, dH^{d_i}(P) \, dH^{d_i}(Q) \right)^{1/p} + \|f\|_{L^p(E_i)}.$$

If $f\colon \partial\Omega \to \mathbf{R}$ and $f|_{E_i} \in \widetilde{B}^{p,s_i}(E_i)$ for $1 \le i \le N$ we put $s=(s_1, ..., s_N)$ and write

$$\|f\|_{B^{p,s}(\partial\Omega)} = \sum_{i=1}^N \|f|_{E_i}\|_{\tilde{B}^{p,s_i}(E_i)}.$$

We note that $B^{p,s}(\partial\Omega)$ is a Banach space. Let $B^{p,s}_*(\partial\Omega)$ denote the space of bounded linear functionals on $B^{p,s}(\partial\Omega)$. Given $\theta \in B^{p,s}_*(\partial\Omega)$ and $f \in B^{p,s}(\partial\Omega)$ let $\langle \theta, f \rangle$ be the duality pairing between a Besov space and its dual (see [11] for further descriptions of this pairing).

We now introduce the layer potentials we shall consider. Fix p, $1<p<\infty$, and let $p'=p/(p-1)$, $\alpha_i=1-(n-d_i)/p$ and $\beta_i=1-(n-d_i)/p'$ for $1\le i\le N$. Put $\alpha=(\alpha_1,...,\alpha_N)$ and $\beta=(\beta_1,...,\beta_N)$. If $\phi\in B_*^{p',\beta}(\partial\Omega)$ and $\partial\Omega$ satisfies Assumption A1 set

$$\mathcal{S}\phi(X)=\langle\phi,\Gamma(X-\cdot)\rangle,\quad X\in\mathbf{R}^n\setminus\partial\Omega, \tag{1.1}$$

where

$$\Gamma(X)=-\frac{1}{(n-2)\omega_n}\frac{1}{|X|^{n-2}}$$

is the fundamental solution of Laplace's equation in $\mathbf{R}^n$.

If Ω is a bounded connected open set (i.e., a domain) for which $\partial\Omega$ satisfies Assumption A1, we put $\Omega_+=\Omega$ and $\Omega_-=\mathbf{R}^n\setminus\bar{\Omega}$, and for $f\in B^{p,\alpha}(\partial\Omega)$, set

$$\mathcal{K}^{\pm}f(X)=\int_{\Omega_{\mp}}\nabla_Y\Gamma(Y-X)\cdot\nabla F(Y)\,dY,\quad X\in\mathbf{R}^n, \tag{1.2}$$

where $F\in W^{1,p}$ is an extension of f. We remark that $\mathcal{K}^{\pm}f$ does not depend on the particular extension of f, as will follow from Lemma 2.3 and Proposition 2.2. Also, we observe that $\mathcal{S}\phi$ is harmonic in $\mathbf{R}^n\setminus\partial\Omega$ and $\mathcal{K}^{\pm}f$ are harmonic in $\Omega_{\pm}$. Our boundary layer potentials are defined by

$$S\phi=\mathcal{S}\phi|_{\partial\Omega}\quad\text{and}\quad T_{\pm}f=\mathcal{K}^{\pm}f|_{\partial\Omega}. \tag{1.3}$$

We refer to $\mathcal{S}\phi$ as the *single layer potential* of ϕ in $\mathbf{R}^n$. Also, $\mathcal{K}^{\pm}f$ are called the *double layer potentials* of f in $\mathbf{R}^n$. Finally define $T_{\pm}^*\colon B_*^{p,\alpha}(\partial\Omega)\to B_*^{p,\alpha}(\partial\Omega)$ by $\langle T_{\pm}^*\psi,f\rangle=\langle\psi,T_{\pm}f\rangle$ whenever $\psi\in B_*^{p,\alpha}(\partial\Omega)$ and $f\in B^{p,\alpha}(\partial\Omega)$.

Our main results in this paper are stated as follows.

Theorem 1.1. *Let $\partial\Omega$ satisfy Assumption A1. There exists $\varepsilon_0>0$, depending only on $d=(d_1,...,d_n)$, c_1, c_2, r_1, N, n and $\operatorname{diam}\partial\Omega$, such that if $2-\varepsilon_0\le p\le 2+\varepsilon_0$, then $S\colon B_*^{p',\beta}(\partial\Omega)\to B^{p,\alpha}(\partial\Omega)$ is one-to-one, bounded, and onto, and thus invertible.*

Theorem 1.2. *Let $\Omega\subset\mathbf{R}^n$ be a bounded domain satisfying Assumption A1 as well as Assumptions A2 and A3 with $G=\mathbf{R}^n\setminus\bar{\Omega}$. There exists $\varepsilon_1>0$ depending on the same quantities as ε_0 in Theorem 1.1 and also on σ_0, c_3 and c_4, such that if $p\in[2-\varepsilon_1,2+\varepsilon_1]$, then $T_+\colon B^{p,\alpha}(\partial\Omega)\to B^{p,\alpha}(\partial\Omega)$ is one-to-one, bounded, and onto, and thus invertible.*

Theorem 1.3. *Let* $\Omega\subset\mathbf{R}^n$ *be a bounded domain satisfying Assumption* A1 *as well as Assumptions* A2 *and* A3 *with* $G=\Omega$. *There exists* $\varepsilon_2>0$ *depending on the same quantities as* ε_1 *in Theorem* 1.2 *such that for* $p\in[2-\varepsilon_2, 2+\varepsilon_2]$, $T_-^*\colon \widehat{B}_*^{p,\alpha}(\partial\Omega)\to \widehat{B}_*^{p,\alpha}(\partial\Omega)$ *is one-to-one, bounded, and onto, and thus invertible.*

In Theorem 1.3, $\widehat{B}_*^{p,\alpha}(\partial\Omega)=\{\phi\in B_*^{p,\alpha}(\partial\Omega):\langle\phi,1\rangle=0\}$. We note that in Remark 5.5 at the end of Section 5 we shall define the weak normal derivative, $\partial u/\partial\mathbf{n}\in\widehat{B}_*^{p,\alpha}(\partial\Omega)$, of a harmonic functions u defined on Ω, with $|\nabla u|\in L^{p'}(\Omega)$. If $\phi\in\widehat{B}_*^{p,\alpha}(\partial\Omega)$ and $u=S\phi|_\Omega$, then it turns out that $\phi\to\partial u/\partial\mathbf{n}=T_-^*\phi$.

Using this remark and Theorems 1.1–1.3 we easily obtain Theorems 1.4–1.6.

Theorem 1.4. *Let* $\partial\Omega$ *satisfy Assumption* A1. *If* $2-\varepsilon_0\le p\le 2+\varepsilon_0$, *then given* $f\in B^{p,\alpha}(\partial\Omega)$, *there is a unique* $\phi\in B_*^{p',\beta}(\partial\Omega)$ *with* $\|\phi\|_{B_*^{p',\beta}(\partial\Omega)}\le c\|f\|_{B^{p,\alpha}(\partial\Omega)}$ *and the property that if* $u=\mathcal{S}\phi$, *then*

$$\begin{cases}\Delta u=0 & \textit{in } \mathbf{R}^n\setminus\partial\Omega,\\ u=f & \textit{on } \partial\Omega,\end{cases}$$

where $c>0$ *has the same dependence as* ε_0.

Theorem 1.5. *Let* G, Ω *and* ε_1 *be as in Theorem* 1.2. *If* $2-\varepsilon_1\le p\le 2+\varepsilon_1$, *then given* $f\in B^{p,\alpha}(\partial\Omega)$, *there is a unique* $h\in B^{p,\alpha}(\partial\Omega)$ *for which* $u=\mathcal{K}^+h$ *satisfies*

$$\begin{cases}\Delta u=0 & \textit{in } \Omega,\\ u=f & \textit{on } \partial\Omega,\\ \|h\|_{B^{p,\alpha}(\partial\Omega)}\le c\|f\|_{B^{p,\alpha}(\partial\Omega)},\end{cases}$$

where $c>0$ *has the same dependence as* ε_2.

Theorem 1.6. *Let* G, Ω *and* ε_2 *be as in Theorem* 1.3. *If* $2-\varepsilon_2\le p\le 2+\varepsilon_2$, *then given* $\psi\in\widehat{B}_*^{p,\alpha}(\partial\Omega)$, *there is a unique harmonic* $\hat{u}$ *in* $\mathbf{R}^n\setminus\partial\Omega$, *satisfying*

$$\begin{cases}\hat{u}=\mathcal{S}\phi & \textit{for some } \phi\in\widehat{B}_*^{p,\alpha}(\partial\Omega),\\ \dfrac{\partial u}{\partial\mathbf{n}}=\psi & \textit{if } u=\hat{u}|_\Omega,\\ \|\phi\|_{B_*^{p,\alpha}(\partial\Omega)}\le c\|\psi\|_{B_*^{p,\alpha}(\partial\Omega)},\end{cases}$$

where $c>0$ *has the same dependence as* ε_2.

Remark 1.7. Theorems 1.4–1.6 can be thought of as weak versions for Besov spaces of the regularity, Neumann, and Dirichlet problems with boundary data in a

Besov space. For Lipschitz domains it follows from the results in [13] that analogues of Theorems 1.4–1.6 hold for boundary data in a variety of other Besov and Hardy spaces. On the other hand, as mentioned earlier, the domains we consider are considerably more general than Lipschitz domains or even those considered in [7]. For example Theorem 1.4 holds when $\partial\Omega$ is a finite union of Cantor sets (with the proper dimension) and fractal surfaces. Theorems 1.5 and 1.6 are less general. For example it follows easily from Propositions 2.1 and 2.2 that necessarily $\partial\Omega=\partial(\mathbf{R}^n\setminus\bar{\Omega})$ when Assumption A2 holds and $G=\mathbf{R}^n\setminus\bar{\Omega}$. Thus in this case $N=1$ and $n>d_1\geq n-1$. Assumption A3 further restricts the class of admissible domains. Still there are numerous non-Lipschitz fractal-type surfaces satisfying these requirements, as we point out in Section 6.

As for the plan of this paper, in Section 2, we prove a trace lemma which together with theorems from [9] enables us to define our single layer potentials on $\mathbf{R}^n\setminus\partial\Omega$ under Assumption A1. In this section we also study harmonic functions with Lipschitz boundary values on a portion of a d-Ahlfors regular boundary and state a Whitney-type extension theorem. In Sections 3, 4 and 5 we prove Theorems 1.1, 1.2 and 1.3, respectively. In Section 6 we discuss Assumptions A2 and A3 and indicate some domains for which Theorems 1.1–1.3 are valid.

2. Preliminary reductions

In the sequel we let $c\geq 1$ denote a positive constant "depending only on the data", not necessarily the same at each occurrence. By this phrase we include dependence on c_1, c_2, $d=(d_1,\dots,d_N)$, N, r_1 and $\operatorname{diam}\partial\Omega$, and if explicitly stated, p, throughout Sections 2 and 3. In Sections 4 and 5 we also allow dependence on c_3, c_4 and $H^n(\Omega)$. In the proof of Theorems 1.1–1.3 we shall need the following extension and restriction theorems.

Proposition 2.1. (Extension theorem) *Let Ω be a bounded domain satisfying Assumption A1 and p be fixed, $n-\min\{d_1,\dots,d_N\}<p<\infty$. Then for all $f\in B^{p,\alpha}(\partial\Omega)$ there exists $F\in W^{1,p}$ such that $F|_{\partial\Omega}=f$ and*

$$\|F\|_{W^{1,p}}\leq c\|f\|_{B_{p,\alpha}(\partial\Omega)},$$

where c depends on the data (including p).

Proposition 2.2. (Restriction theorem) *Let Ω be a bounded domain satisfying Assumption A1 and p be as in Proposition 2.1. Then the operator*

$$\mathcal{R}\colon W^{1,p}\longrightarrow B^{p,\alpha}(\partial\Omega)$$

defined by $\mathcal{R}(F)=F|_{\partial\Omega}$ *is bounded. That is, there is a positive constant* $c\geq 1$, *having the same dependence as in Proposition* 2.1, *such that*

$$\|\mathcal{R}(F)\|_{B^{p,\alpha}(\partial\Omega)}\leq c\|F\|_{W^{1,p}}.$$

Propositions 2.1 and 2.2 are proved in [9] when $N=1$. It is easily seen that Propositions 2.1 and 2.2 follow from the just cited $N=1$ case. Indeed to get Proposition 2.1 we extend $f|_{E_i}$ to $f_i\in W^{1,p}$ for $1\leq i\leq n$ where $\|f_i\|_{W^{1,p}}\leq c\|f|E_i\|_{B^{p,\alpha_i}(E_i)}$. Let $0\leq\psi_i\in C_0^\infty(\mathbf{R}^n)$ with support contained in $\{X:d(X,E_i)<r_1\}$, $|\nabla\psi_i|\leq c/r_1$ and $\psi_i\equiv 1$ on E_i for $1\leq i\leq n$. If $F=\sum_{i=1}^N f_i\psi_i$, it is easily checked that Proposition 2.1 holds for this F. Proposition 2.2 follows from applying the $N=1$ case to each $E_i, 1\leq i\leq N$. We note that since $\partial\Omega$ is bounded, we may assume that F in Propositions 2.1 and 2.2 has compact support. We shall also need the following lemma.

Lemma 2.3. *Let* $\partial\Omega$ *satisfy Assumption* A1 *and* p *be fixed and satisfy* $n-\min\{d_1,\dots,d_n\}<p<\infty$. *Suppose that* $F\in W^{1,p}$ *with* $F=a=$*constant* H^{d_i}*-almost everywhere on* E_i *for* $1\leq i\leq N$. *Then given* $\varepsilon>0$ *there exists* $g\in W^{1,p}$ *with compact support,* $g=a$ *in a neighborhood of* $\partial\Omega$, *and* $\|g-F\|_{W^{1,p}}<\varepsilon$.

Proof. We remark that Lemma 2.3 is perhaps implied by the results in [9] or [10], although we could not find any direct reference. Also if we knew that $F\equiv 0$ almost everywhere with respect to a certain Riesz p-capacity (defined below), then Lemma 2.3 would follow from [1], Section 9.2. Since this also is not apparent to the authors we give a proof of Lemma 2.3. In the proof c may also depend on p. To begin, given a bounded set $\widehat{E}\subset\mathbf{R}^n$ and $1<p<\infty$ define the *outer Riesz capacity* of $\widehat{E}$ by $\gamma_p(\widehat{E})=\inf\int_{\mathbf{R}^n}|\nabla\theta|^p\,dX$, where the infimum is taken over all $\theta\in C_0^\infty(\mathbf{R}^n)$ with $\theta\equiv 1$ on $\widehat{E}$. It is well known (see [1], Chapter 5) that for $1<p\leq n$,

$$\gamma_p(\widehat{E})=0 \quad\Longrightarrow\quad H^{n-p+\varepsilon}(\widehat{E})=0 \quad\text{whenever } \varepsilon>0. \tag{2.1}$$

If $p>n$, then nonempty sets have positive capacity. Let F be as in Lemma 2.3 for fixed $p, n-\min\{d_1,\dots,d_N\}<p<\infty$, and let $F_{B(X,r)}$ be the average of F on $B(X,r)$. Then F can be defined almost everywhere on $\mathbf{R}^n$, with respect to the γ_p-capacity (see [1] or [17]), by $F(X)=\lim_{X\to 0}F_{B(X,r)}$. If E denotes the set where this limit does not exist, then from (2.1) it follows that $H^{d_i}(E\cap E_i)=0$ for $p\leq n$, while $E=\varnothing$ when $p>n$. Let $X\in E_i$, $0<r\leq r_1/100$ and

$$I_1(\chi|\nabla F|)(Y)=\int_{B(X,r)}|\nabla F(Z)|\,|Z-Y|^{1-n}\,dZ,\quad Y\in B(X,r),$$

where χ denotes the characteristic function of $B(X,r)$. Approximating F by C^∞ functions and taking limits it follows once again from Sobolev-type estimates and arguments involving γ_p that

$$|F(Y)-F_{B(X,r)}|\le cI_1(\chi|\nabla F|)(Y) \tag{2.2}$$

for H^{d_i}-almost every $Y\in B(X,r/2)$. Let μ be r^{-d_i} times the H^{d_i} measure on $\partial\Omega\cap B(X,r/2)$ and set

$$I_1\mu(Y)=\int_{\mathbf{R}^n}|Y-Z|^{1-n}\,d\mu(Z).$$

Using that $F\equiv a$ H^{d_i}-almost everywhere on ∂E_i, as well as Assumption A1, and integrating (2.2) with respect to μ we deduce from Hölder's inequality that

$$\begin{aligned}|(F-a)_{B(X,r)}| &\le c\int_{B(X,r)}|\nabla F(Z)|I_1\mu(Z)\,dZ \\ &\le c^2\big\|\chi|\nabla F|\big\|_{L^p}\|\chi I_1\mu\|_{L^{p'}}\le c^3\big\|\chi|\nabla F|\big\|_{L^p}r^{1-n/p}.\end{aligned} \tag{2.3}$$

For $p\ge n$, (2.3), is a consequence of theorems of Sobolev and Morrey. Equation (2.3) for $p<n$ follows from Hölder's inequality and the fact that (see [1], Section 4.5)

$$\|I_1\mu\|_{L^{p'}}^{p'}\approx\int_{\mathbf{R}^n}W(Y)\,d\mu(Y), \tag{2.4}$$

where W is the Wolff potential defined by

$$W(X)=\int_0^\infty[t^{p-n}\mu(B(X,t))]^{1/(p-1)}\frac{dt}{t}.$$

Indeed from Assumption A1 and the definition of μ we find that $W(X)\le cr^{(p-n)/(p-1)}$ whenever $X\in\mathbf{R}^n$. Using this inequality in (2.4) we deduce first that

$$\|I_1\mu\|_{L^{p'}}\le cr^{1-n/p}$$

and thereupon that (2.3) is true.

From (2.3) we get upon raising both sides to the pth power and then dividing by r^{-p} that

$$r^{-p}|(F-a)_{B(X,r)}|^p\le c(|\nabla F|^p)_{B(X,r)}. \tag{2.5}$$

Next given η, $0<\eta\ll r_1$, we define $O_j=\{X\in\mathbf{R}^n:d(X,\partial\Omega)<j\eta\}$, $j=1,2$. Let $\zeta\in C_0^\infty(O_2)$ with $\zeta\equiv1$ on O_1 and $|\nabla\zeta|\le c/\eta$. Put $\hat g=F-(F-a)\zeta$. We shall show that if $\eta=\eta(\varepsilon)>0$ is small enough then

$$\|\hat g-F\|_{W^{1,p}}\le\tfrac{1}{2}\varepsilon. \tag{2.6}$$

Indeed, from the definition of $\hat{g}$ and our choice of ζ, we have that

$$\|\hat{g}-F\|^p_{W^{1,p}} \le c\int_{O_2} (|\nabla F|^p+\eta^{-p}|F-a|^p)\,dX. \tag{2.7}$$

To estimate the right-hand side of this equation we first use a well-known covering lemma to get a covering $\{B(X_i,10\eta)\}_{i=1}^\infty$ of O_2 with centers in O_2 and the property that the balls $\{B(X_i,\eta)\}_{i=1}^\infty$ are pairwise disjoint. Let Z_i be a point in $\partial\Omega$ with $|X_i-Z_i|=d(X_i,\partial\Omega)$ and let $O_3=\{X: d(X,\partial\Omega)<12\eta\}$. Then

$$\begin{aligned}\int_{O_2} |F(X)-a|^p\,dX &\le \sum_{i=1}^\infty \int_{B(Z_i,12\eta)} |F(X)-a|^p\,dX \\ &\le c\sum_{i=1}^\infty \int_{B(Z_i,12\eta)} |F(X)-F_{B(Z_i,12\eta)}|^p\,dX \\ &\qquad +c\eta^n\sum_{i=1}^\infty |(F-a)_{B(Z_i,12\eta)}|^p \\ &= J_1+J_2.\end{aligned} \tag{2.8}$$

J_1 can be estimated using Poincaré's inequality. We get

$$J_1\le c\sum_{i=1}^\infty \eta^p\int_{B(Z_i,12\eta)} |\nabla F|^p\,dX \le c^2\eta^p\int_{O_3}|\nabla F|^p\,dX, \tag{2.9}$$

where to get the last inequality we observed that each point in $\bigcup_{i=1}^\infty B(Z_i,12\eta)$ lies in at most c of the balls $\{B(Z_i,12\eta)\}_{i=1}^\infty$, as follows from a "volume" argument using the disjointness of $\{B(X_i,\eta)\}_{i=1}^\infty$. To estimate J_2 we use (2.5) to get

$$\begin{aligned}\eta^n\sum_{i=1}^\infty |(F-a)_{B(Z_i,12\eta)}|^p &\le c\eta^p\sum_{i=1}^\infty \int_{B(Z_i,12\eta)} |\nabla F(X)|^p\,dX \\ &\le c^2\eta^p\int_{O_3}|\nabla F(X)|^p\,dX.\end{aligned} \tag{2.10}$$

Using (2.8)–(2.10) in (2.7), we get (2.6) for $\eta=\eta(\varepsilon)$ sufficiently small, since $F\in W^{1,p}$. Finally let $\psi\in C_0^\infty(B(0,2R))$ with $\psi\equiv1$ on $B(0,R)$ and $|\nabla\psi|\le c/R$. Let $g=(\hat{g}\psi)_\delta$ denote convolution of $\hat{g}\psi$ with an approximate identity whose support is contained in $B(0,\delta)$. If R is large enough and $\delta>0$ small enough we obtain from standard properties of mollifiers that $\|g-\hat{g}\|_{W^{1,p}}<\varepsilon/2$. Using this inequality in (2.6) we conclude the validity of Lemma 2.3. □

Next we prove the following lemma.

Lemma 2.4. *Suppose that* v *is harmonic in* $B(\widehat{X},4\rho)\setminus\partial\Omega$, *where* $\widehat{X}\in\partial\Omega$, $0<\rho<r_1/100$ *and* r_1 *is as in Assumption* A1. *Let* $\zeta\in C_0^\infty(B(\widehat{X},3\rho))$ *with* $\zeta\equiv1$ *on* $B(\widehat{X},2\rho)$ *and* $\||\nabla\zeta|\|_{L^\infty}\leq1000/\rho$. *Assume that* $(v-F)\zeta\in W_0^{1,2}(B(\widehat{X},3\rho)\setminus\partial\Omega)$, *where* $F\colon\mathbf{R}^n\to\mathbf{R}$ *is in* $W^{1,q}$. *Then there exists* $\delta>0$, *depending only on the data, such that if* $2<q\leq2+\delta$, *then* $\|v\|_{W_1^q(B(\widehat{X},\rho)\setminus\partial\Omega)}<\infty$. *Moreover,*

$$\int_{B(\widehat{X},\rho)}|\nabla v|^q\,dX\leq c\rho^{n(1-q/2)}\biggl(\int_{B(\widehat{X},2\rho)}|\nabla v|^2\,dX\biggr)^{q/2}+c\int_{B(\widehat{X},2\rho)}|\nabla F|^q\,dX.$$

Proof. To prove Lemma 2.4, we show that if $Y\in B(\widehat{X},\rho)$ and $0<r\leq\rho/100$, then

$$\int_{B(Y,r)}|\nabla v|^2\,dX \leq\frac{1}{100^n}\int_{B(Y,12r)}|\nabla v|^2\,dX+cr^{n(b-2)/b}\biggl(\int_{B(Y,12r)}|\nabla v|^b\,dX\biggr)^{2/b}+cM, \tag{2.11}$$

where $b=\frac{6}{5}$ for $n=3,4$ and $b=2-4/n$ for $n\geq5$. Also, $M=\int_{B(Y,12r)}|\nabla F|^2\,dX$. Lemma 2.4 then follows from this reverse Hölder-type inequality and an argument originally due to Gehring (see [5]). Thus we prove only (2.11).

We consider two cases. If $0<r\leq d(Y,\partial\Omega)/2$, then (2.11) follows from standard estimates for harmonic functions in balls. If $d(Y,\partial\Omega)\leq2r$, $\widehat{Y}\in\partial\Omega$ and $|\widehat{Y}-Y|=d(Y,\partial\Omega)$, then

$$B(Y,t)\subset B(\widehat{Y},3t)\subset B(Y,6t)\quad\text{whenever } t\geq r.$$

Hence it suffices to prove that (2.11) holds with Y, r and $12r$, replaced by $\widehat{Y}$, $3r$ and $6r$, respectively. To this end, let $0\leq\psi\in C_0^\infty(B(\widehat{Y},6r))$ with $\psi\equiv1$ on $B(\widehat{Y},3r)$ and $|\nabla\psi|\leq c/r$. If $w=(v-F)\psi^2$, then from the hypotheses of Lemma 2.4 we see that

$$\int_{B(\widehat{Y},6r)}\nabla v\cdot\nabla w\,dX=0, \tag{2.12}$$

where $\cdot$ denotes the standard inner product on $\mathbf{R}^n$. Using (2.12) and Cauchy's inequality with ε's, we obtain that

$$\int_{B(\widehat{Y},3r)}|\nabla v|^2\,dX\leq cM+\frac{c}{r^2}\int_{B(\widehat{Y},6r)}G^2\,dX, \tag{2.13}$$

where we have put $G=v-F$. From (2.5) with $a=0$, $p=b$ and r, X and F replaced by $6r$, $\widehat{Y}$ and G, respectively, we get that

$$|G_{B(\widehat{Y},6r)}|\leq cr^{1-n/b}\|\widetilde{\chi}|\nabla G|\|_{L^b}, \tag{2.14}$$

where $\widetilde{\chi}$ denote the characteristic function of $B(\widehat{Y},6r)$. From (2.14) and Poincaré's inequality, we deduce that

$$\begin{aligned}\frac{1}{r^2}\int_{B(\widehat{Y},6r)}|G|^2\,dX &\le \frac{4}{r^2}\int_{B(\widehat{Y},6r)}|G-G_{B(\widehat{Y},6r)}|^2\,dX+cr^{n-2}(G_{B(\widehat{Y},6r)})^2\\ &\le \frac{c'}{r^2}\int_{B(\widehat{Y},6r)}|v-v_{B(\widehat{Y},6r)}|^2\,dX+c'M \\ &\quad +c'r^{n-2n/b}\big\|\widetilde{\chi}|\nabla v|\big\|^2_{L^b},\end{aligned}\tag{2.15}$$

where c' depends only on the data. Putting (2.15) in (2.13) we find that

$$\begin{aligned}\int_{B(\widehat{Y},3r)}|\nabla v|^2\,dX &\le \frac{c}{r^2}\int_{B(\widehat{Y},6r)}|v-v_{B(\widehat{Y},6r)}|^2\,dX\\ &\quad +cM+cr^{n-2n/b}\big\|\widetilde{\chi}|\nabla v|\big\|^2_{L^b}.\end{aligned}\tag{2.16}$$

Next we note from (2.2) with $F=v$ and r, Y and X replaced by $4r$, X and $\widehat{Y}$, respectively, that

$$|v(X)-v_{B(\widehat{Y},6r)}|\le cI_1(\widetilde{\chi}|\nabla v|)(X)\quad \text{whenever } X\in B(\widehat{Y},6r).\tag{2.17}$$

Also (see [1], Proposition 3.1.2), we have

$$I_1(\widetilde{\chi}|\nabla v|)(X)\le c\big\|\widetilde{\chi}|\nabla v|\big\|^{2/n}_{L^2}\widehat{M}(\widetilde{\chi}|\nabla v|)(X)^{1-2/n}\quad \text{for } X\in B(\widehat{Y},6r),\tag{2.18}$$

where

$$\widehat{M}k(X)=\sup_{r>0}\frac{1}{H^n(B(X,r))}\int_{B(X,r)}|k|\,dX\tag{2.19}$$

denotes the Hardy–Littlewood maximal function of a locally integrable function k on $\mathbf{R}^n$. Squaring both sides of (2.17) and integrating over $B(\widehat{Y},4r)$, we deduce from (2.18) and the Hardy–Littlewood maximal theorem (see [15]) that

$$\begin{aligned}\frac{1}{r^2}\int_{B(\widehat{Y},6r)}|v-v_{B(\widehat{Y},6r)}|^2\,dX &\le \frac{c}{r^2}\big\|\widetilde{\chi}|\nabla v|\big\|^{4/n}_{L^2}\int_{B(\widehat{Y},6r)}\widehat{M}(\widetilde{\chi}|\nabla v|)(X)^{2-4/n}\,dX\\ &\le cr^{\lambda}\big\|\widetilde{\chi}|\nabla v|\big\|^{4/n}_{L^2}\left(\int_{B(\widehat{Y},6r)}|\nabla v|^b\,dX\right)^{(2n-4)/nb},\end{aligned}\tag{2.20}$$

where $\lambda=(n-2)(b-2)/b$. The right-hand side of (2.20) can be estimated using Young's inequality with η's. Doing this we find that

$$(2.21)\quad r^{\lambda}\big\|\widetilde{\chi}|\nabla v|\big\|_{L^2}^{4/n}\left(\int_{B(\widehat{Y},6r)}|\nabla v|^b\,dX\right)^{(2n-4)/nb} \le c\eta^{n/2}\big\|\widetilde{\chi}|\nabla v|\big\|_{L^2}^2+c\eta^{-n/(n-2)}r^{n-2n/b}\big\|\widetilde{\chi}|\nabla v|\big\|_{L^b}^2.$$

Combining (2.20) and (2.21), and using the resulting inequality in (2.16) we conclude for sufficiently small $\eta>0$ that

$$(2.22)\qquad \int_{B(\widehat{Y},3r)}|\nabla v|^2\,dX\le\frac{1}{100^n}\int_{B(\widehat{Y},6r)}|\nabla v|^2\,dX+cr^{n-2n/b}\big\|\widetilde{\chi}|\nabla v|\big\|_{L^b}^2+cM.$$

In view of our earlier remarks we now conclude the validity of Lemma 2.4. □

Finally in this section we state the following result.

Lemma 2.5. *Given $k\in W^{1,p}\cup R^{1,p}$ and $\lambda>0$ there exists a Lipschitz function θ on $\mathbf{R}^n$ with $\theta(x)=k(x)$ for H^n-almost every x of $L(\lambda)=\{y:\widehat{M}(|\nabla k|)(y)\le\lambda\}$ and $\big\||\nabla\theta|\big\|_{L^\infty}\le c\lambda$.*

Proof. Note from the definition of $\widehat{M}(|\nabla k|)$ in (2.19) that $L(\lambda)$ is closed. Also, $L(\lambda)\neq\varnothing$ since $M(|\nabla k|)\in L^p$ by the Hardy–Littlewood maximal theorem. Now for almost every $X,Y\in L(\lambda)$ if $r=2|X-Y|$, then

$$(2.23)\qquad |k(X)-k(Y)|\le cI_1(\chi|\nabla k|)(X)+cI_1(\chi|\nabla k|)(Y),$$

where we have used (2.2) with F replaced by k. One can write the integral involving I_1 as a sum and make simple estimates to show (see [1])

$$(2.24)\quad \begin{aligned} I_1(\chi|\nabla k|)(X)+I_1(\chi|\nabla k|)(Y) &\le c|X-Y|(\widehat{M}(|\nabla k|)(X)+\widehat{M}(|\nabla k|)(Y))\\ &\le c^2\lambda|X-Y|, \end{aligned}$$

since $X,Y\in L(\lambda)$. From (2.23) and (2.24) we conclude that k agrees H^n-almost everywhere on $L(\lambda)$ with a Lipschitz function on $L(\lambda)$ having norm $\le c\lambda$. Existence of θ now follows from applying the Whitney extension theorem to the Lipschitz function on $L(\lambda)$ (see [15], Chapter VI). □

3. Proof of Theorem 1.1

In the proof of Theorem 1.1 we assume that $p>n-\min\{d_1,\dots,d_N\}$ and that $|p-2|\le\delta$. Initially we allow $\delta>0$ to vary but shall later fix δ to be a small positive number satisfying several conditions. We then put $\varepsilon_0=\delta$. Since the Laplacian is invariant under translations we assume, as we may, that $0\in\partial\Omega$. Let $p'=p/(p-1)$, $\alpha_i=1-(n-d_i)/p$ and $\beta_i=1-(n-d_i)/p'$ for $1\le i\le N$, and set $\alpha=(\alpha_1,...,\alpha_N)$ and $\beta=(\beta_1,...,\beta_N)$. As in (1.1) and (1.3), we put

$$\mathcal{S}\phi(X)=\langle\phi,\Gamma(X-\cdot)\rangle,\quad X\in\mathbf{R}^n,$$

and $S\phi=\mathcal{S}\phi|_{\partial\Omega}$ whenever $\phi\in B_*^{p',\beta}(\partial\Omega)$. We first prove the following lemma.

Lemma 3.1. *If* $\phi\in B_*^{p',\beta}(\partial\Omega)$, *then* $\mathcal{S}\phi\in R^{1,p}$ *and* $S\phi\in B^{p,\alpha}(\partial\Omega)$ *with*

$$\|\mathcal{S}\phi\|_{R^{1,p}}+\|S\phi\|_{B^{p,\alpha}(\partial\Omega)}\le c\|\phi\|_{B_*^{p',\beta}(\partial\Omega)}.$$

Proof. If $X\in\mathbf{R}^n\setminus\bar\Omega$, then it follows easily from linearity of ϕ, Taylor's theorem with remainder, and a difference quotient argument that

$$D^\lambda\mathcal{S}\phi(X)=\langle\phi,D_X^\lambda\Gamma(X-\cdot)\rangle,\quad\text{where } D_X^\lambda=\frac{\partial^{|\lambda|}}{\partial X_1^{\lambda_1}...\partial X_n^{\lambda_n}}\tag{3.1}$$

and λ is a multi-index. As Γ is harmonic in $\mathbf{R}^n\setminus\{0\}$, it follows that $\mathcal{S}\phi$ is harmonic in $\mathbf{R}^n\setminus\partial\Omega$. Let $F\in C_0^\infty(\mathbf{R}^n)$ and set $O_\varepsilon=\{x\in\mathbf{R}^n:d(x,\partial\Omega)>\varepsilon\}$ for $\varepsilon>0$ while $O_0=\mathbf{R}^n$. We note that if χ_ε is the characteristic function of O_ε and

$$I_2(\chi_\varepsilon F)(X)=\int\chi_\varepsilon(Y)F(Y)\Gamma(X-Y)\,dY\quad\text{for }\varepsilon\ge0,$$

then from well-known properties of Riesz potentials (see [1], Section 1) we have that

$$\|I_2(\chi_\varepsilon F)\|_{L^{\tilde q}}+\big\||\nabla I_2(\chi_\varepsilon F)|\big\|_{L^{p'}}\le c\|F\|_{L^q},\tag{3.2}$$

where $1/\tilde q=1/q-2/n$ and $1/q=1/p'+1/n$. We also note for $\varepsilon>0$ that

$$\int_{O_\varepsilon}\mathcal{S}\phi F\,dX=\int_{O_\varepsilon}\langle\phi,\Gamma(X-\cdot)\rangle F(X)\,dX=\langle\phi,I_2(\chi_\varepsilon F)|_{\partial\Omega}\rangle\tag{3.3}$$

as follows from writing the left-hand integral as a limit of Riemann sums and using the linearity of ϕ. To estimate the right-hand term in (3.3) let R_0 be the smallest positive number ≥1 such that $\partial\Omega\subset\overline{B}(0,R_0)$ and let $\zeta\in C_0^\infty(B(0,4R_0))$ with $\zeta\equiv1$ on $B(0,2R_0)$ and $|\nabla\zeta|\le1000/R_0$. Then $\zeta I_2(\chi_\varepsilon F)\in W^{1,p'}$ and from Proposition 2.2 we have that

$$\begin{aligned}|\langle\phi,I_2(\chi_\varepsilon F)|_{\partial\Omega}\rangle|&\le\|\phi\|_{B_*^{\beta,p'}(\partial\Omega)}\|I_2(\chi_\varepsilon F)|_{\partial\Omega}\|_{B^{\beta,p'}(\partial\Omega)}\\&\le c\|\phi\|_{B_*^{\beta,p'}(\partial\Omega)}\|\zeta I_2(\chi_\varepsilon F)\|_{W^{1,p'}}.\end{aligned}\tag{3.4}$$

Using (3.2) and Hölder's inequality we deduce first that

$$\|\zeta I_2(\chi_\varepsilon F)\|_{W^{1,p'}} \le c\|F\|_{L^q}$$

and thereupon from (3.3) and (3.4) that for $\varepsilon>0$,

$$\left|\int_{O_\varepsilon} \mathcal{S}\phi F\,dX\right| \le c\|\phi\|_{B_*^{\beta,p'}(\partial\Omega)}\|F\|_{L^q}. \tag{3.5}$$

As $C_0^\infty(\mathbf{R}^n)$ is dense in L^q it follows from a duality argument that

$$\|\mathcal{S}\phi\|_{L^{p^*}(O_\varepsilon)} \le c\|\phi\|_{B_*^{\beta,p'}(\partial\Omega)}, \tag{3.6}$$

where $p^*=np/(n-p)$. Since c is independent of ε we conclude that (3.6) holds with $\varepsilon=0$. We now take limits in (3.3). Using (3.6), Proposition 2.2 and the fact that $\zeta I_2(\chi_\varepsilon F)\to\zeta I_2 F$ pointwise and in $W^{1,p'}$ we deduce that

$$\int_{\mathbf{R}^n} \mathcal{S}\phi F\,dX = \langle\phi, I_2F|_{\partial\Omega}\rangle. \tag{3.7}$$

Similarly for F, O_ε and ζ as above we find for $1\le i\le n$ and $\varepsilon>0$ that

$$\int_{O_\varepsilon} \frac{\partial\mathcal{S}\phi}{\partial X_i} F\,dX = -\left\langle \phi, \left.\frac{\partial I_2(\chi_\varepsilon F)}{\partial X_i}\right|_{\partial\Omega}\right\rangle. \tag{3.8}$$

From Calderón–Zygmund singular integral estimates we have that

$$\left\|\zeta\frac{\partial I_2(\chi_\varepsilon F)}{\partial X_i}\right\|_{W^{1,p'}} \le c\|F\|_{L^{p'}}, \tag{3.9}$$

where c is independent of $\varepsilon\ge0$. Using Proposition 2.2 once again it follows that

$$\left|\int_{O_\varepsilon} \frac{\partial\mathcal{S}\phi}{\partial X_i} F\,dX\right| \le c\|\phi\|_{B_*^{\beta,p'}(\partial\Omega)}\|F\|_{L^{p'}}. \tag{3.10}$$

From duality and (3.10) we conclude first that

$$\left\|\chi_\varepsilon\frac{\partial\mathcal{S}\phi}{\partial X_i}\right\|_{L^p} \le c\|\phi\|_{B_*^{\beta,p'}(\partial\Omega)}, \tag{3.11}$$

where c is independent of ε. Second, letting $\varepsilon\to0$ we get (3.11) when $\varepsilon=0$.

It remains to show that $\partial\mathcal{S}\phi/\partial X_i$ is the distributional derivative of $\mathcal{S}\phi$. To this end we note again from Calderón–Zygmund singular integral theory that $\zeta\partial I_2(\chi_\varepsilon F)/\partial X_i$ converges to $\zeta\partial I_2F/\partial X_i$ in $W^{1,p'}$. Using this fact, Proposition 2.2 and (3.10) in (3.8) we obtain that

$$\int_{\mathbf{R}^n} \frac{\partial\mathcal{S}\phi}{\partial X_i} F\,dX = -\left\langle\phi, \left.\frac{\partial I_2F}{\partial X_i}\right|_{\partial\Omega}\right\rangle = -\left\langle\phi, I_2\left(\frac{\partial F}{\partial X_i}\right)\Big|_{\partial\Omega}\right\rangle, \tag{3.12}$$

where the last equality follows from integration by parts. Finally from (3.7) with F replaced by $\partial F/\partial X_i$ for $1\leq i\leq n$ we see that

$$\int_{\mathbf{R}^n} \mathcal{S}\phi \frac{\partial F}{\partial X_i}\, dX = \left\langle \phi, I_2\left(\frac{\partial F}{\partial X_i}\right)\bigg|_{\partial\Omega}\right\rangle. \tag{3.13}$$

Hence $\partial\mathcal{S}\phi/\partial X_i$, $1\leq i\leq n$, is the distributional derivative of $\mathcal{S}\phi$. This fact, Proposition 2.2 applied to $\zeta\mathcal{S}\phi$, and (3.6), (3.11) with $\varepsilon=0$, imply Lemma 3.1. ☐

To begin the proof of Theorem 1.1 we observe from Lemma 3.1 and Proposition 2.2 that S is a bounded linear operator from $B_*^{p',\beta}(\partial\Omega)$ into $B^{p,\alpha}(\partial\Omega)$. To show that S is one-to-one we prove the following lemma.

Lemma 3.2. *There exists $\delta>0$ such that if $|p-2|\leq\delta$ and $\phi\in B_*^{p',\beta}(\partial\Omega)$, with $S\phi=0$, then $\phi=0$.*

Proof. As in Lemma 3.1 we assume that $0\in\partial\Omega$ and $\partial\Omega\subset\overline{B}(0,R_0)$. We first prove Lemma 3.2 when $p'\leq 2$. In this case given $\rho>2R_0$, choose $\sigma\in C_0^\infty(B(0,2\rho))$ with $\sigma\equiv 1$ on $B(0,\rho)$ and $\big\||\nabla\sigma|\big\|_{L^\infty}\leq c/\rho$. Then from Lemma 3.1, the hypotheses of Lemma 3.2, and Hölder's inequality we see that $\sigma\mathcal{S}\phi\in W^{1,p}$ with trace 0 on $\partial\Omega$. In view of Lemma 2.3 it follows that we can approximate this function in the $W^{1,p}$ norm by functions in $C_0^\infty(\mathbf{R}^n\setminus\partial\Omega)$. This fact, the fact that $p>2$, and harmonicity of $\mathcal{S}\phi$ in $\mathbf{R}^n\setminus\partial\Omega$ imply that

$$\int_{\mathbf{R}^n} \nabla\mathcal{S}\phi\cdot\nabla(\sigma\mathcal{S}\phi)\, dX = 0. \tag{3.14}$$

Equation (3.14) and the usual estimates involving Cauchy's inequality with ε's yield

$$\int_{B(0,\rho)} |\nabla\mathcal{S}\phi|^2\, dX \leq \frac{c}{\rho^2}\int_{B(0,2\rho)\setminus B(0,\rho)} |\mathcal{S}\phi|^2\, dX. \tag{3.15}$$

Now from linearity of ϕ we see there exists ρ^* with

$$|\mathcal{S}\phi(X)| \leq c|X|^{2-n}\|\phi\|_{B_*^{p',\beta}} \quad \text{for } |X|>\rho^*. \tag{3.16}$$

Using (3.16) to estimate the right-hand side in (3.15) and letting $\rho\to\infty$, we conclude first from (3.15) that

$$\int_{\mathbf{R}^n} |\nabla\mathcal{S}\phi|^2\, dX = 0$$

and thereupon from (3.16) that $\nabla\mathcal{S}\phi\equiv 0$ in $\mathbf{R}^n\setminus\partial\Omega$. Finally replacing F by $\partial F/\partial X_i$ in (3.12) and summing the resulting expression, we get that

$$\int_{\mathbf{R}^n} \nabla\mathcal{S}\phi\cdot\nabla F\, dX = -\langle\phi, I_2(\Delta F)\rangle = -\langle\phi, F|_{\partial\Omega}\rangle, \tag{3.17}$$

where we have used the fact that $F = I_2(\Delta F)$ when $F \in C_0^\infty(\mathbf{R}^n)$ (see [15]). From Proposition 2.2 and an approximation argument we see that (3.17) holds whenever $F \in W^{1,p'}$ with compact support. From Proposition 2.1, (3.17) and $\nabla \mathcal{S}\phi \equiv 0$ in $\mathbf{R}^n \setminus \partial\Omega$, we conclude that $\phi = 0$. Hence Lemma 3.2 is valid when $p' \le 2$.

If $p' > 2$, and $\lambda > 0$ is fixed, we use Lemma 2.5 with $k = \mathcal{S}\phi$ to get $\theta \in W^{1,\infty}$ with $\big\| |\nabla\theta| \big\|_{L^\infty} \le c\lambda$ and $\theta = \mathcal{S}\phi$ for H^n-almost every $X \in L(\lambda) = \{X : \widehat{M}(|\nabla \mathcal{S}\phi|)(X) \le \lambda\}$. For fixed $\rho > R_0$ let $\hat{u} = \hat{u}(\,\cdot\,, \rho)$ be the unique harmonic function in $B(0, 2\rho) \setminus \partial\Omega$ with $\hat{u} - \sigma\theta \in W_0^{1,2}(B(0,2\rho) \setminus \partial\Omega)$. Here σ is as in (3.14). Existence of $\hat{u}$ follows from the usual minimizing argument involving the Dirichlet integral and the fact that $\gamma_2(\partial\Omega) > 0$ (see [1]). From the maximum principle for harmonic functions we see that

$$\hat{u}(X) \le C|X|^{2-n} \quad \text{in } B(0,2\rho) \setminus B(0, 2R_0), \tag{3.18}$$

where C is independent of ρ. Using (3.18), properties of harmonic functions, and the fact that $W_0^{1,2}(B(0,2\rho))$ is reflexive, we deduce that $\hat{u}(\,\cdot\,,\rho) \to u$ as $\rho \to \infty$, where u satisfies:

$$u \text{ is harmonic in } \mathbf{R}^n \setminus \partial\Omega, \tag{3.19a}$$

$$(u - \theta)\sigma \in W_0^{1,2}(B(0,2\rho) \setminus \partial\Omega) \quad \text{whenever } \rho > R_0, \tag{3.19b}$$

$$u(X) \le C|X|^{2-n} \qquad \text{in } \mathbf{R}^n \setminus B(0, 2R_0). \tag{3.19c}$$

From (3.19), Lemma 2.4, compactness of $\partial\Omega$ and Sobolev's theorem we see that $\sigma u \in W^{1,q}$ for some $q > 2$ depending only on the data. Also from (3.19b) and Proposition 2.2 we see that $(u - \theta)\sigma = 0$ on $\partial\Omega$ in the sense of Lemma 2.3. From these facts and Lemma 2.3, we conclude that there exists a sequence of $C_0^\infty(\mathbf{R}^n \setminus \partial\Omega)$ functions converging to $(u - \theta)\sigma$ in the norm of $W^{1,q}$. Using these functions as test functions and taking a limit, we see from Hölder's inequality and Lemma 3.1, that

$$\int_{\mathbf{R}^n} \nabla \mathcal{S}\phi \cdot \nabla(\sigma(u - \theta))\, dX = 0 \tag{3.20}$$

provided $\delta > 0$ is small enough and $2 - \delta \le p < 2$. Thus,

$$\int_{\mathbf{R}^n} \nabla \mathcal{S}\phi \cdot [\nabla(u - \theta)]\sigma \, dX = -\int_{\mathbf{R}^n \setminus B(0,\rho)} (\nabla \mathcal{S}\phi \cdot \nabla\sigma)(u - \theta)\, dX. \tag{3.21}$$

From (3.16), (3.19c) and properties of harmonic functions it also follows that there exists ρ^* with

$$|\nabla \mathcal{S}\phi(X)| + |\nabla u(X)| \le c(\|\phi\|_{B_*^{p',\beta}} + C)|X|^{1-n} \quad \text{in } \mathbf{R}^n \setminus B(0, 2\rho^*). \tag{3.22}$$

Next note from (3.16) that for ρ^* large enough

$$\theta(X)=\mathcal{S}\phi(X) \quad \text{in } \mathbf{R}^n\setminus B(0,2\rho^*). \tag{3.23}$$

From (3.16), (3.19c) and (3.23), it is easily seen that the right-hand side of (3.21) tends to 0 as $\rho\to\infty$. Likewise from (3.22) and (3.23) the left-hand side of (3.21) converges in L^1 to

$$\int_{\mathbf{R}^n} \nabla\mathcal{S}\phi\cdot[\nabla(u-\theta)]\,dX. \tag{3.24}$$

Since $|\nabla u|, |\nabla\theta|\in L^q$ for some $q>2$, it follows that

$$\int_{\mathbf{R}^n} \nabla\mathcal{S}\phi\cdot\nabla u\,dX=\int_{\mathbf{R}^n} \nabla\mathcal{S}\phi\cdot\nabla\theta\,dX. \tag{3.25}$$

Now we can use Lemma 2.3 applied to $F=\mathcal{S}\phi$, harmonicity of u in $\mathbf{R}^n\setminus\partial\Omega$, and (3.16), (3.19c) and (3.22), to conclude that the left-hand side of (3.25) is zero. Hence

$$\int_{\mathbf{R}^n} \nabla\mathcal{S}\phi\cdot\nabla\theta\,dX=0. \tag{3.26}$$

From (3.26) and Lemma 2.5 it follows that

$$T_1(\lambda)=\int_{L(\lambda)} |\nabla\mathcal{S}\phi|^2\,dX\le c\lambda\int_{\mathbf{R}^n\setminus L(\lambda)} |\nabla\mathcal{S}\phi|\,dX=T_2(\lambda). \tag{3.27}$$

Multiplying the left-hand side of (3.27) by λ^{p-3} and integrating over $\lambda\in(0,\infty)$ we have

$$\begin{aligned}\int_0^\infty \lambda^{p-3}T_1(\lambda)\,d\lambda &= \int_{\mathbf{R}^n} |\nabla\mathcal{S}\phi|^2\left(\int_{\widehat{M}(|\nabla\mathcal{S}\phi|)}^\infty \lambda^{p-3}\,d\lambda\right)dX\\ &=\frac{1}{2-p}\int_{\mathbf{R}^n} \widehat{M}(|\nabla\mathcal{S}\phi|)^{p-2}|\nabla\mathcal{S}\phi|^2\,dX.\end{aligned} \tag{3.28}$$

Similarly,

$$\begin{aligned}\int_0^\infty \lambda^{p-3}T_2(\lambda)\,d\lambda &= c\int_{\mathbf{R}^n} |\nabla\mathcal{S}\phi|\left(\int_0^{\widehat{M}(|\nabla\mathcal{S}\phi|)} \lambda^{p-2}\,d\lambda\right)\\ &=\frac{c}{p-1}\int_{\mathbf{R}^n} \widehat{M}(|\nabla\mathcal{S}\phi|)^{p-1}|\nabla\mathcal{S}\phi|\,dX.\end{aligned} \tag{3.29}$$

From (3.27)–(3.29) we see that

$$I=\int_{\mathbf{R}^n} \widehat{M}(|\nabla\mathcal{S}\phi|)^{p-2}|\nabla\mathcal{S}\phi|^2\,dX\le c\delta\int_{\mathbf{R}^n} \widehat{M}(|\nabla\mathcal{S}\phi|)^{p-1}|\nabla\mathcal{S}\phi|\,dX=c\delta J. \tag{3.30}$$

We note that if $|2-p|\leq\frac{1}{4}$, then $\widehat{M}(|\nabla\mathcal{S}\phi|)^{p-2}$ is an A_2 weight (see [16], Chapter V) with A_2 constant depending only on n. Using this fact and properties of A_2 weights we find that

$$K=\int_{\mathbf{R}^n}\widehat{M}(\nabla\mathcal{S}\phi|)^p\,dX\leq cI. \tag{3.31}$$

Also, trivially $J\leq K$. In view of (3.30) it follows that $K\leq c\delta K$ where c depends only on the data. Hence $K\equiv0$ for $\delta>0$ small enough, depending only on the data, which implies as earlier that $\mathcal{S}\phi\equiv0$. Thus Lemma 3.2 is valid if $\delta>0$ is small enough. ☐

Next we prove the following result.

Lemma 3.3. *There exists $\delta>0$ such that if $|p-2|\leq\delta$, then $S(B_*^{p',\beta}(\partial\Omega))$ is closed.*

Proof. Since S is continuous it is easily seen that Lemma 3.3 follows from the inequality

$$\|S\phi\|_{B^{p,\alpha}(\partial\Omega)}\geq\eta\|\phi\|_{B_*^{p',\beta}(\partial\Omega)} \tag{3.32}$$

for some $\eta>0$ and all $\phi\in B_*^{p',\beta}(\partial\Omega)$. The proof of (3.32) is by contradiction. Otherwise, there exists $\phi_m\in B_*^{p',\beta}(\partial\Omega)$, $m=1,2,\ldots$, with

$$\|\phi_m\|_{B_*^{p',\beta}(\partial\Omega)}=1 \quad\text{and}\quad \mathcal{S}\phi_m\to0,\ \text{as } m\to\infty,\ \text{in } B^{p,\alpha}(\partial\Omega). \tag{3.33}$$

Again we consider two cases. If $p\geq2$, we can put $\phi=\phi_m$ and $F=\sigma\mathcal{S}\phi_m$ in (3.17). Here $\sigma\in C_0^\infty(B(0,2\rho))$ is as in (3.14). Using (3.16), (3.22) and letting $\rho\to\infty$ it follows as earlier that

$$\int_{\mathbf{R}^n}|\nabla\mathcal{S}\phi_m|^2\,dX=-\langle\phi_m,S\phi_m\rangle\leq c\|S\phi_m\|_{B^{p,\alpha}(\partial\Omega)}\to0 \quad\text{as } m\to\infty. \tag{3.34}$$

Here we have used the fact that

$$B^{p,\alpha}(\partial\Omega)\subset B^{p',\beta}(\partial\Omega) \quad\text{when } p\geq2 \text{ with } \|\cdot\|_{B^{p',\beta}(\partial\Omega)}\leq c\|\cdot\|_{B^{p,\alpha}(\partial\Omega)}. \tag{3.35}$$

If $p>2$ we note that (3.34) and (3.16) yield that

$$\mathcal{S}\phi_m\to0 \quad\text{uniformly in } O_\varepsilon, \tag{3.36}$$

where O_ε was define below (3.1). From Proposition 2.1 and Lemma 3.1, we deduce for $p>2$ the existence of $F_m\in W^{1,p}$, $m=1,2,\ldots$, with compact support and such that $F_m|_{\partial\Omega}=S\phi_m$ and

$$\|F_m\|_{W^{1,p}}\leq c\|S\phi_m\|_{B^{p,\alpha}(\partial\Omega)}\to0 \quad\text{as } m\to\infty. \tag{3.37}$$

Given $\widehat{X}\in\partial\Omega$, let ρ and ζ be as in Lemma 2.4 and note from Lemma 2.3 that $(\mathcal{S}\phi_m-F_m)\zeta\in W_0^{1,p}[B(\widehat{X},3\rho)\setminus\partial\Omega]$ whenever $\widehat{X}\in\partial\Omega$. Thus we can apply Lemma 2.4 with $v=\mathcal{S}\phi_m$ and $p=q$, to conclude for $\delta>0$ small enough that

$$\int_{B(\widehat{X},\rho)}|\nabla\mathcal{S}\phi_m|^p\,dX\le c\rho^{n(1-p/2)}\biggl(\int_{B(\widehat{X},2\rho)}|\nabla\mathcal{S}\phi_m|^2\,dX\biggr)^{p/2}+c\int_{B(\widehat{X},2\rho)}|\nabla F_m|^p\,dX. \tag{3.38}$$

In view of (3.38), (3.34) and (3.37) it follows that $\int_{B(\widehat{X},\rho)}|\nabla\mathcal{S}_m|^p\,dX\to 0$ as $m\to\infty$.

Next from arbitrariness of $\widehat{X}\in\partial\Omega$ and compactness of $\partial\Omega$, we see for $\varepsilon>0$ small enough that $\int_{\Omega\setminus O_\varepsilon}|\nabla\mathcal{S}_m|^p\,dX\to 0$ as $m\to\infty$. Finally, this limit, (3.36), properties of harmonic functions and (3.22) imply for $2<p\le 2+\delta$ that

$$\bigl\|\,|\nabla\mathcal{S}\phi_m|\,\bigr\|_{L^p}\to 0\quad\text{as } m\to\infty. \tag{3.39}$$

If $p=2$, then (3.39) follows from (3.34). From (3.39) we can easily get a contradiction to (3.33) when $2\le p\le 2+\delta$. In fact if $g\in B^{\beta,p'}(\partial\Omega)$ and $G\in W^{1,p'}$ are as in Proposition 2.1 with compact support and $G=g$ on $\partial\Omega$, then from (3.17) we see that

(3.40)
$$\langle\phi_m,g\rangle=-\int_{\mathbf{R}^n}\nabla\mathcal{S}\phi_m\cdot\nabla G\,dX\le c\bigl\|\,|\nabla\mathcal{S}\phi_m|\,\bigr\|_{L^p}\|g\|_{B^{p',\beta}(\partial\Omega)}\le\frac{1}{2}\|g\|_{B^{p',\beta}(\partial\Omega)}$$

for m large enough independent of $g\in B^{\beta,p'}(\partial\Omega)$. We have reached a contradiction since $\|\phi_m\|_{B_*^{p',\beta}(\partial\Omega)}=1$. From this contradiction we obtain first (3.32) and after that Lemma 3.3 when $2\le p\le 2+\delta$.

If $2-\delta\le p<2$, suppose $\phi\in B_*^{p',\beta}(\partial\Omega)$, $\psi\in B_*^{p,\alpha}(\partial\Omega)$, and that σ is as in (3.14). Putting $F=\sigma\mathcal{S}\psi$ in (3.17), using Lemma 3.1, (3.16) and (3.22), and letting $\rho\to\infty$ it follows now from standard arguments that

$$\int_{\mathbf{R}^n}\nabla\mathcal{S}\phi\cdot\nabla\mathcal{S}\psi\,dX=-\langle\phi,S\psi\rangle=-\langle\psi,S\phi\rangle, \tag{3.41}$$

where the last equality follows from interchanging the roles of ϕ and ψ. We now also interchange the roles of p and p' in the earlier case proved of Lemma 3.3. Thus

$$\|\psi\|_{B_*^{p,\alpha}(\partial\Omega)}\le c\|S\psi\|_{B^{p',\beta}(\partial\Omega)}. \tag{3.42}$$

From (3.41) and (3.42) we find that

$$|\langle\phi_m,S\psi\rangle|=|\langle\psi,S\phi_m\rangle|\le c\|S\phi_m\|_{B^{p,\alpha}(\partial\Omega)}\|S\psi\|_{B^{p',\beta}(\partial\Omega)}\to 0\quad\text{as } m\to\infty. \tag{3.43}$$

We conclude from (3.43) (as in (3.40)), that if $S(B^{p,\alpha}_*(\partial\Omega))$ is dense in $B^{p',\beta}(\partial\Omega)$ with respect to the norm of this space, then again we have reached a contradiction to (3.33). Otherwise, it follows from the Hahn–Banach theorem and (3.41) that there exists $\phi\in B^{p',\beta}_*$, $\phi\not\equiv 0$, with

$$0=\langle\phi,S\psi\rangle=\int_{\mathbf{R}^n}\nabla\mathcal{S}\phi\cdot\nabla\mathcal{S}\psi\,dX\quad\text{whenever }\psi\in B^{p,\alpha}_*(\partial\Omega). \tag{3.44}$$

To get a contradiction, we essentially repeat the argument after (3.15) with a few twists. Given $\lambda>0$ construct θ relative to $k=\mathcal{S}\phi$ and λ, as in Lemma 2.5. This construction is permissible due to Lemma 3.1. Next construct u relative to θ, satisfying (3.19). Then $|\nabla u|,|\nabla\theta|\in L^{p'}$ provided $2<p'\le 2+\delta$. Using this fact, (3.19), (3.22) and harmonicity of $\mathcal{S}\phi$ in $\mathbf{R}^n\setminus\partial\Omega$, we get as in (3.25) that

$$\int_{\mathbf{R}^n}\nabla\mathcal{S}\phi\cdot\nabla u\,dX=\int_{\mathbf{R}^n}\nabla\mathcal{S}\phi\cdot\nabla\theta\,dX. \tag{3.45}$$

Now if $h\in B^{p,\alpha}(\partial\Omega)$ then from Proposition 2.1 there is an extension H of h with compact support and such that $\|H\|_{W^{1,p}}\le c\|h\|_{B^{p,\alpha}(\partial\Omega)}$. Hence

$$\left|\int_{\mathbf{R}^n}\nabla u\cdot\nabla H\,dX\right|\le c\|\nabla u\|_{L^{p'}}\|h\|_{B^{p,\alpha}(\partial\Omega)}. \tag{3.46}$$

Since $|\nabla u|\in L^{p'}$ it follows from (3.46) and (3.17), that if ψ is defined by

$$\langle\psi,h\rangle=\int_{\mathbf{R}^n}\nabla u\cdot\nabla H\,dX\quad\text{for all }h\in B^{p,\alpha}(\partial\Omega), \tag{3.47}$$

then $\psi\in B^{p,\alpha}_*(\partial\Omega)$ and

$$\int_{\mathbf{R}^n}\nabla u\cdot\nabla H\,dX=-\int_{\mathbf{R}^n}\nabla\mathcal{S}\psi\cdot\nabla H\,dX. \tag{3.48}$$

Also, if $h=S\phi$ then we can argue as earlier using Lemmas 2.3 and 3.1, (3.48), (3.44) and (3.45) to deduce that

$$0=\int_{\mathbf{R}^n}\nabla\mathcal{S}\psi\cdot\nabla\mathcal{S}\phi\,dX=-\int_{\mathbf{R}^n}\nabla u\cdot\nabla\mathcal{S}\phi\,dX=-\int_{\mathbf{R}^n}\nabla\theta\cdot\nabla\mathcal{S}\phi\,dX. \tag{3.49}$$

Armed with (3.49) we can now repeat verbatim the argument after (3.26) to get $S\phi\equiv 0$. From Lemma 3.2 it follows that $\phi\equiv 0$. We have reached a contradiction to our assumption that $\phi\not\equiv 0$. The proof of Lemma 3.3 is now complete. □

We complete the proof of Theorem 1.1 with the following lemma.

Lemma 3.4. *If* $|p-2|\le\delta$ *and* $\delta>0$ *is small enough, depending only on the data, then* S *maps* $B_*^{p',\beta}(\partial\Omega)$ *onto* $B^{p,\alpha}(\partial\Omega)$.

Proof. The proof of Lemma 3.4 is by contradiction. Otherwise it follows from Lemma 3.3 and the Hahn–Banach theorem that there exists $\psi\in B_*^{p,\alpha}(\partial\Omega)$, $\psi\not\equiv 0$, with $\langle\psi, S\phi\rangle=0$ whenever $\phi\in B_*^{\beta,p'}(\partial\Omega)$. For $2<p\le 2+\delta$ the argument from (3.44) to the end of Lemma 3.3 gives a contradiction. If $2-\delta\le p\le 2$, we can use (3.35) with p and p' interchanged and argue as in (3.34) to get first that $\int_{\mathbf{R}^n}|\nabla\mathcal{S}\psi|^2\,dX=0$ and second that $\psi\equiv 0$. In either case we have reached a contradiction. Thus Lemma 3.4 is true. Finally, invertibility follows from Lemmas 3.2–3.4 and (3.32). In fact it is well known that a one-to-one, onto linear operator is invertible. ☐

4. Proof of Theorem 1.2

In the proofs of Theorems 1.2 and 1.3 we assume that $\Omega=\Omega_+$ is a bounded domain with $0\in\Omega\subset B(0,R_0)$ and $\Omega_-=\mathbf{R}^n\setminus\bar{\Omega}_+$.

Recall from (1.2) and (1.3), that the *double layer* and *boundary double layer* potentials are defined for $f\in B^{p,\alpha}(\partial\Omega)$ by

$$\begin{aligned}\mathcal{K}^{\pm}f(X)&=\int_{\Omega_\mp}\nabla_Y\Gamma(Y-X)\cdot\nabla F(Y)\,dY,\quad X\in\mathbf{R}^n,\\ T_\pm f&=\mathcal{K}^\pm f|_{\partial\Omega},\end{aligned}\tag{4.1}$$

where $F\in W^{1,p}$ with compact support in $\mathbf{R}^n$ and $F|_{\partial\Omega}=f$. Existence of one such F is a consequence of Proposition 2.1. Using Calderón–Zygmund theory and properties of Riesz potentials we also deduce that $\mathcal{K}^\pm f\in R^{1,p}$ with

$$\|\mathcal{K}^\pm f\|_{R^{1,p}}\le c\|\nabla F\|_{L^p}.\tag{4.2}$$

We now show that $T_\pm f$ is independent of the choice of F. Indeed, suppose that $F,\widetilde{F}\in W^{1,p}$ and $f=\widetilde{F}|_{\partial\Omega}=F|_{\partial\Omega}\in B^{p,\alpha}(\partial\Omega)$. Then $G=F-\widetilde{F}$ has trace 0 so by Lemma 2.3, given $\varepsilon>0$ there exists $g\in W^{1,p}(\mathbf{R}^n)\cap C_0^\infty(\mathbf{R}^n\setminus\partial\Omega)$ with $\|g-G\|_{W^{1,p}}<\varepsilon$. Let $\zeta\in C_0^\infty(B(0,2R_0))$ with $\zeta\equiv 1$ on $B(0,R_0)$ and $|\nabla\zeta|\le c/R_0$. We note that $T_\pm g\equiv 0$ on $\partial\Omega$ as follows easily from integration by parts in the integral defining $\mathcal{K}^\pm g$. Using this fact, (4.2) and Proposition 2.2, we deduce that

$$\begin{aligned}\|T_\pm G\|_{B^{p,\alpha}(\partial\Omega)}&=\|T_\pm(g-G)\|_{B^{p,\alpha}(\partial\Omega)}\le c\|\zeta\mathcal{K}^\pm(g-G)\|_{W^{1,p}}\\ &\le c^2\|\mathcal{K}^\pm(g-G)\|_{R^{1,p}}\le c^3\|\nabla g-\nabla G\|_{L^p}\le c^3\varepsilon.\end{aligned}\tag{4.3}$$

Letting $\varepsilon\to 0$ we get $T_\pm G=0$ in $B^{p,\alpha}(\partial\Omega)$. Hence $T_\pm F=T_\pm\widetilde{F}$ and $T_\pm$ is well defined on $B^{p,\alpha}(\partial\Omega)$. Next for given $f\in B^{p,\alpha}(\partial\Omega)$ we choose F as in Proposition 2.1 with support in $B(0,2R_0)$ and use the same argument as in (4.3) to get

$$\|T_\pm f\|_{B^{p,\alpha}(\partial\Omega)}\le c\|\zeta\mathcal{K}^\pm f\|_{W^{1,p}}\le c^2\|\nabla F\|_{L^p}\le c^3\|f\|_{B^{p,\alpha}(\partial\Omega)}. \tag{4.4}$$

From (4.4) we see that $T_\pm$ is a bounded linear operator from $B^{p,\alpha}(\partial\Omega)\to B^{p,\alpha}(\partial\Omega)$. Now let $f\in B^{p,\alpha}(\partial\Omega)$ and $\psi\in B^{p,\alpha}_*(\partial\Omega)$. From Theorem 1.1 we see, for $\delta>0$ small enough, that there exists $\phi\in B^{p',\beta}(\partial\Omega)$ with $S\phi=f$. Arguing as in the proof of Theorem 1.1 we find as in (3.41) that

$$\langle\psi,T_\pm(S\phi)\rangle=\int_{\Omega_\mp}\nabla\mathcal{S}\psi\cdot\nabla\mathcal{S}\phi\,dX=\langle\phi,T_\pm(S\psi)\rangle. \tag{4.5}$$

Now assume that Assumption A3 holds with $G=\Omega_-$ or $G=\Omega_+$. Given p with $|p-2|\le\frac14$, let v be as in Assumption A3 with $v=f$. Assume also that $|\nabla v|\in L^p(G)$. Let $\hat v$ be the extension of v to $\mathbf{R}^n$ guaranteed by Assumption A3. Put $h=|\nabla\hat v|$ when $G=\Omega_-$ and $h=|\nabla\hat v|\chi$ when $G=\Omega_+$ where χ is the characteristic function of $B(0,\rho)$ for some $\rho>R_0$. Once again we observe that $(\widehat{M}h)^{p-2}$ is an A_2 weight when $|p-2|\le\frac14$. Using this observation, properties of A_2 weights, Assumption A3, and either Young's inequality with ε's or $\widehat{M}h\ge h$, we see that

$$\begin{aligned}\int_{\mathbf{R}^n}(\widehat{M}h)^p\,dX&\le c\int_{\mathbf{R}^n}(\widehat{M}h)^{p-2}|\nabla v|^2\,dX\le c^2\int_G(\widehat{M}h)^{p-2}|\nabla\hat v|^2\,dX\\&\le\frac12\int_{\mathbf{R}^n}(\widehat{M}h)^p\,dX+c'\int_G|\nabla v|^p\,dX,\end{aligned} \tag{4.6}$$

where c' depends only on the data. Subtracting the first term on the lower right-hand side of (4.6) from the left-hand side we get that

$$\int_{\mathbf{R}^n}(\widehat{M}h)^p\,dX\le c\int_G|\nabla v|^p\,dX.$$

This equality and (4.6) imply for $G=\Omega_-$ that

$$\int_{\mathbf{R}^n}|\nabla\hat v|^p\,dX\approx\int_{\mathbf{R}^n}\widehat{M}(|\nabla\hat v|)^{p-2}|\nabla\hat v|^2\,dX\approx\int_G|\nabla v|^p\,dX, \tag{4.7}$$

where $\approx$ means that the ratio of any two quantities is bounded above and below by constants depending only on the data. If $G=\Omega_+$, then (4.7) is also valid as we deduce from letting $\rho\to\infty$ in the above inequalities and using the monotone convergence theorem.

We now begin the proof of Theorem 1.2. Fix p, $|p-2|\le\frac{1}{4}$, and let $V^{1,p}(\Omega_-)$ be the space of all locally integrable functions v on Ω_- with distributional gradient ∇v satisfying $\lim_{\rho\to\infty} v_{B(0,\rho)}=0$ and $\|v\|_{V^{1,p}(\Omega_-)}=\big\||\nabla v|\big\|_{L^p(\Omega_-)}$. Recall that $v_{B(0,\rho)}$ is the average of v on $B(0,\rho)$. Let $\hat v$ be the extension of v to $\mathbf{R}^n$ given by Assumption A3. From (4.7) and Sobolev-type estimates we see that $\hat v\in R^{1,p}$ with

$$\frac{1}{c}\|v\|_{V^{1,p}(\Omega_-)}\le\|\hat v\|_{R^{1,p}}\le c\|v\|_{V^{1,p}(\Omega_-)}.$$

Thus $V^{1,p}$ is a reflexive Banach space. The following lemma will play a key role in our proof of Theorem 1.2.

Lemma 4.1. *There are $\delta>0$ and $c\ge 1$, depending only on the data, such that if $|p-2|\le\delta$ and $v\in V^{1,p}(\Omega_-)$, then there is $\tilde u\in C_0^\infty(\mathbf{R}^n)$ with $\bar u=\tilde u|_{\Omega_-}$, $\|\bar u\|_{V^{1,p'}(\Omega_-)}\le c$ and*

$$\|v\|_{V^{1,p}(\Omega_-)}\le\int_{\Omega_-}\nabla\bar u\cdot\nabla v\,dX. \tag{4.8}$$

Proof. If $v=0$ set $\tilde u=0$. Otherwise, from the linearity we may assume that $\|v\|_{V^{1,p}(\Omega_-)}=1$. Let $\hat v$ denote the extension of v to $\mathbf{R}^n$ guaranteed by Assumption A3. Given $\eta>0$ we claim that there exists $\widehat w\in C_0^\infty(\mathbf{R}^n)$ such that if $w=\widehat w|_{\Omega_-}$ then

(4.9)
(a) $\|w-v\|_{V^{1,p}(\Omega_-)}\le\eta$,
(b) (4.7) is valid with v, $\hat v$ and G replaced by w, $\widehat w$ and Ω_-, respectively.

To prove (4.9) we note that if σ and ρ are as in (3.14) and $v^*=(\hat v-\hat v_{B(0,2\rho)})\sigma$ then v^* converges to $\hat v$ in the norm of $V^{1,p}$ as $\rho\to\infty$. To prove this note, we could for example use (2.2) with $F=v$, $X=0$ and $r=2\rho$. Writing the resulting integral on the right-hand side of (2.2) as a sum one gets as in (2.24) that

$$|\hat v(Y)-\hat v_{B(0,2\rho)}|\le c\rho\widehat M(|\nabla\hat v|)(Y)\quad\text{whenever } Y\in B(0,2\rho).$$

Our note follows easily from this inequality and the Hardy–Littlewood maximal theorem. Regularizing v^*, we see that there exists a sequence, $v_j\in C_0^\infty(\mathbf{R}^n)$, $j=1,2,\ldots$, converging to $\hat v$ pointwise and in the norm of $V^{1,p}$. Clearly (a) of (4.9) is valid if we take $\widehat w=v_j$ and j is large enough. Moreover, using (4.6) for $\hat v$ and v, the Fatou lemma, and the fact that

$$M(|\nabla(v'-v'')|)\le M(|\nabla v'|)+M(|\nabla v''|)\quad\text{whenever } v',v''\in\{v_j,\hat v\colon j=1,2,\ldots\},$$

we get that

$$\begin{aligned}\limsup_{j\to\infty}\int_{\mathbf{R}^n}\widehat{M}(|\nabla v_j|)^p\,dX &\le c\liminf_{j\to\infty}\int_{\Omega_-}\widehat{M}(|\nabla\ v_j|)^{p-2}|\nabla v_j|^2\,dX\\ &\le \frac{1}{2}\limsup_{j\to\infty}\int_{\mathbf{R}^n}\widehat{M}(|\nabla v_j|)^p\,dX + c'\liminf_{j\to\infty}\int_{\Omega_-}|\nabla v_j|^p\,dX.\end{aligned}$$

It follows from this inequality and the Hardy–Littlewood maximal theorem that we can also choose $\widehat{w}=v_j$ in (b) of (4.9) when j is large enough.

To continue the proof of Lemma 4.1 we suppose that η is a small positive number and $\widehat{w}$ has been chosen relative to η. First suppose that $0<2-p\le\frac{1}{4}$. Let $\lambda_0>0$ be the largest number such that $\widehat{M}(|\nabla\widehat{w}|)\ge 2\lambda_0$ on the support of $\widehat{w}$. Construct $\theta=\theta(\,\cdot\,,\lambda)$ relative to $\widehat{w}$ and λ as in Lemma 2.5. Put

$$u=(2-p)\sum_{m=m_0}^{\infty}2^{m(p-2)}\theta(\,\cdot\,,2^m),$$

where m_0 is the largest integer such that $2^{m_0}\le\lambda_0$. We note that since $M(|\nabla\widehat{w}|)$ is bounded, we have $\theta(\,\cdot\,,\lambda)=\widehat{w}$ for large λ. Also, $\theta(\,\cdot\,,\lambda)\equiv 0$ in a neighborhood of ∞ independent of $\lambda\ge\lambda_0/2$. From these remarks it is easily seen that u is Lipschitz and for almost every X,

$$\nabla u(X)=\nabla\widehat{w}(X)(2-p)\sum_{m\in\Lambda(X)}2^{m(p-2)}+E(X), \tag{4.10}$$

where $\Lambda(X)$ denotes the set of all integers $m\ge m_0$ with $2^m\ge M(|\nabla\widehat{w}|)(X)$. Moreover, if $\Lambda_1(X)$ denotes all integers $\ge m_0$ that are not in $\Lambda(X)$, then

$$|E(X)|\le c(2-p)\sum_{m\in\Lambda_1}2^{m(p-1)}\le c'(2-p)M(|\nabla\widehat{w}|)(X)^{p-1}. \tag{4.11}$$

Finally if $h(X)=(2-p)\sum_{m\in\Lambda(X)}2^{m(p-2)}$ then

$$\begin{aligned} &h\le cM(|\nabla\widehat{w}|)^{p-2} &&\text{almost everywhere on } \mathbf{R}^n,\\ &h\ge\frac{1}{c}M(|\nabla\widehat{w}|)^{p-2} &&\text{almost everywhere on the support of } \widehat{w}.\end{aligned} \tag{4.12}$$

These inequalities can be proved by comparing $h(X)$ with $\int_{\widehat{M}(|\nabla\widehat{w}|)(X)}^{\infty}\lambda^{p-3}\,d\lambda$. From (4.10)–(4.12), we conclude first that $|\nabla u|\le cM(|\nabla\widehat{w}|)^{p-1}$ so from the Hardy–Littlewood maximal theorem and (a) of (4.9),

$$\|u\|_{V^{1,p'}(\Omega_-)}\le c\|w\|^{p-1}_{V^{1,p}(\Omega_-)}\le 2c. \tag{4.13}$$

Second, from (4.10)–(4.12) we get

$$\int_{\Omega_-} \nabla u \cdot \nabla w \, dX \geq \frac{1}{c} \int_{\Omega_-} |\nabla w|^2 \widehat{M}(|\nabla \widehat{w}|)^{2-p} \, dX \\ -c(2-p) \int_{\mathbf{R}^n} M(|\nabla \widehat{w}|)^p \, dX. \tag{4.14}$$

From (4.13), (4.14) and (4.9), we see first that (4.8) holds with $\bar{u}$ and v replaced by u and w, respectively, provided $\delta > 0$ is small. Secondly, choosing η small enough, depending only on the data, we find that this display holds for u and v. Finally, as in the approximation of $\hat{v}$ by v_j, we can approximate u by $\tilde{u} \in C_0^\infty(\mathbf{R}^n)$ in such a way that Lemma 4.1 is valid when $2-\delta \leq p < 2$.

The case $p=2$ of Lemma 4.1 is easily handled so we assume that $2 < p \leq 2+\delta$. Let $V_*^{1,p}(\Omega_-)$ be the space of bounded linear functions on $V^{1,p}(\Omega_-)$ and let $\Gamma \subset V_*^{1,p}(\Omega_-)$ be the set of all linear functionals ψ which can be written in the form

$$\langle \psi, v \rangle = \int_{\Omega_-} \nabla u \cdot \nabla v \, dX, \ \ v \in V^{1,p}(\Omega_-), \quad \text{where } u \in V^{1,p'}(\Omega_-). \tag{4.15}$$

We claim that

$$\Gamma = V_*^{1,p}(\Omega_-). \tag{4.16}$$

Once (4.16) is proved we can use the Hahn–Banach theorem to get, for $v \in V^{1,p}(\Omega_-)$ with $\|v\|_{V^{1,p}(\Omega_-)} = 1$, a linear functional ψ as in (4.15) with $\|\psi\|_{V_*^{1,p}(\Omega_-)} = 1$ and

$$1 = \langle \psi, v \rangle = \int_{\Omega_-} \nabla u \cdot \nabla v \, dX. \tag{4.17}$$

Also, since $p' < 2$ we can apply the previous case with p and p' interchanged to conclude that $\|u\|_{V^{1,p'}} \leq c$. As in the case $2-\delta \leq p < 2$, we can then extend u to $\hat{u}$ as in Assumption A3 and after that approximate $\hat{u}$ by a $C_0^\infty(\mathbf{R}^n)$ function in such a way that Lemma 4.1 holds. Thus to complete the proof of Lemma 4.1 when $2 < p \leq 2+\delta$, it suffices to prove (4.16). To do this given $u \in V^{1,p'}(\Omega_-)$ let $\Lambda(u)$ be the bounded linear functional on $V^{1,p}(\Omega)$ defined in (4.15). From Hölder's inequality we see that $\Lambda \colon V^{1,p'}(\Omega_-) \to V_*^{1,p}(\Omega_-)$ is a bounded linear operator with norm ≤ 1. From the $2-\delta \leq p < 2$ case of Lemma 4.1 with p and p' interchanged it is easily seen that

$$\frac{1}{c} \|u\|_{V^{1,p'}} \leq \|\Lambda(u)\|_{V_*^{1,p}} \leq \|u\|_{V^{1,p'}}. \tag{4.18}$$

Clearly (4.18) implies that $\Gamma = \Lambda(V^{1,p'}(\Omega_-))$ is closed in $V_*^{1,p}(\Omega_-)$. If (4.16) is false, it follows from an argument involving the Hahn–Banach theorem and reflexivity of $V^{1,p}(\Omega_-)$ that there exists $v \in V^{1,p}(\Omega_-), v \not\equiv 0$, with

$$\int_{\Omega_-} \nabla u \cdot \nabla v \, dX = 0 \quad \text{for all } u \in V^{1,p'}(\Omega_-). \tag{4.19}$$

It is easily seen that (4.19) implies that v is harmonic in Ω_-. Using subharmonicity of $|\nabla v|$, $v\in V^{1,p}$, and Hölder's inequality one sees for some constant C that

$$|\nabla v(X)|\le C|X|^{-n/p} \tag{4.20}$$

for $|x|\ge 2R_0$. Using (4.20), the mean-value theorem, and $\lim_{\rho\to\infty} v_{B(0,\rho)}=0$, it follows that $v(X)\to 0$ as $|X|\to\infty$. This fact and either the Kelvin transformation or the Poisson integral formula for $\mathbf{R}^n\setminus \overline{B}(0,2R_0)$ imply for some constant C' that

$$|v(X)|+|X|\,|\nabla v(X)|\le C'|X|^{2-n} \quad \text{for } |X|\ge 2R_0. \tag{4.21}$$

Armed with (4.21) we can now argue as earlier to get a contradiction. That is, let σ and ρ be as in (3.14) and set $u=v\sigma$. Then $u\in V^{1,p'}(\Omega_-)$, and from (4.19) and (4.21) it follows that

$$\int_{B(0,\rho)} |\nabla v|^2\,dX \le \frac{c}{\rho^2}\int_{B(0,2\rho)\setminus \overline{B}(0,\rho)} v^2\,dX \to 0 \quad \text{as } \rho\to\infty.$$

Thus $v\equiv 0$ in Ω_-, which is a contradiction. We conclude that (4.16) and Lemma 4.1 are true when $2<p\le 2+\delta$ provided $\delta>0$ is sufficiently small, depending only on the data. ☐

We continue the proof of Theorem 1.2 with the following lemma.

Lemma 4.2. *There exists $\delta>0$ such that if $|p-2|\le\delta$ and $f\in B^{p,\alpha}(\partial\Omega)$ with $T_+f=0$ then $f=0$.*

Proof. Let $\phi\in B^{p',\beta}(\partial\Omega)$ with $f=S\phi$. From (4.5) we see that

$$\langle\psi, T_+f\rangle = \int_{\Omega_-} \nabla\mathcal{S}\phi\cdot\nabla\mathcal{S}\psi\,dX = 0 \quad \text{whenever } \psi\in B^{p,\alpha}_*(\partial\Omega). \tag{4.22}$$

Also, from Lemma 3.1 we find that $v=\mathcal{S}\phi\in V^{1,p}(\Omega_-)$ so by Lemma 4.1 there exists $\tilde u\in C_0^\infty(\mathbf{R}^n)$ with $\tilde u|_{\Omega_-}=\bar u$, $\|\bar u\|_{V^{1,p'}(\Omega_-)}\le c$ and

$$\|\mathcal{S}\phi\|_{V^{1,p}(\Omega_-)} \le \int_{\Omega_-} \nabla\mathcal{S}\phi\cdot\nabla\bar u\,dX. \tag{4.23}$$

Choose $\psi\in B^{p,\alpha}(\partial\Omega)$ with $S\psi=\tilde u|_{\partial\Omega}$. This choice is possible as we see from Proposition 2.2 and Theorem 1.1. Using once again Lemma 2.3, the fact that $\tilde u$ has compact support, and the decay of $\mathcal{S}\psi$ near ∞ given by (3.16) and (3.22), we conclude that

$$\int_{\Omega_-} \nabla\mathcal{S}\phi\cdot(\nabla\mathcal{S}\psi-\nabla\bar u)\,dX = 0. \tag{4.24}$$

Combining (4.22)–(4.24) we have that

$$\int_{\Omega_-} |\nabla\mathcal{S}\phi|^p\,dX=0,$$

which in view of (3.16) implies that $\mathcal{S}\phi=0$ in Ω_-. Using Assumption A2 we see that $f=S\phi=0$. ☐

Next we prove the following result.

Lemma 4.3. *There exists $\delta>0$ such that if $|p-2|\le\delta$ then $T_+(B^{p,\alpha}(\partial\Omega))$ is closed.*

Proof. As in (3.32) it is easily seen that Lemma 4.3 follows once we show the existence of $\eta>0$ so that

$$\|T_+f\|_{B^{p,\alpha}(\partial\Omega)}\ge\eta\|f\|_{B^{p,\alpha}(\partial\Omega)}. \tag{4.25}$$

To prove (4.25) we again argue by contradiction. Otherwise there exist functions $f_m\in B^{p,\alpha}(\partial\Omega)$, $m=1,2,\dots$, with

$$\|f_m\|_{B^{p,\alpha}(\partial\Omega)}=1 \quad\text{and}\quad \|T_+f_m\|_{B^{p,\alpha}(\partial\Omega)}\to0 \text{ as } m\to\infty. \tag{4.26}$$

Choose $\phi_m\in B^{p',\beta}(\partial\Omega)$ with $\mathcal{S}\phi_m|_{\partial\Omega}=f_m$, $m=1,2,\dots$, and note from (4.5) and (4.26) that

$$\int_{\Omega_-}\nabla\mathcal{S}\phi_m\cdot\nabla\mathcal{S}\psi\,dX=\langle\psi,T_+f_m\rangle\to0\quad\text{as } m\to\infty, \tag{4.27}$$

whenever $\psi\in B^{p,\alpha}_*(\partial\Omega)$. As in Lemma 4.2 we set $v=\mathcal{S}\phi_m$ and choose $\tilde u$ and $\bar u$ as in Lemma 4.1 relative to v. We then find $\psi\in B^{p,\alpha}(\partial\Omega)$ with $S\psi=\tilde u|_{\Omega_-}$. Arguing as in (4.22)–(4.24) it follows that

$$\begin{aligned}\int_{\Omega_-}|\nabla\mathcal{S}\phi_m|^p\,dX&\le\int_{\Omega_-}\nabla\mathcal{S}\phi_m\cdot\nabla\bar u\,dX\\&=\int_{\Omega_-}\nabla\mathcal{S}\phi_m\cdot\nabla\mathcal{S}\psi\,dX\to0\quad\text{as } m\to\infty.\end{aligned} \tag{4.28}$$

Let $\widehat{\mathcal{S}}\phi_m$ be the extension of $\mathcal{S}\phi_m|_{\Omega_-}$ to $\mathbf{R}^n$ guaranteed by Assumption A3. Then from (4.28) and (4.7) we deduce that

$$\big\||\nabla\widehat{\mathcal{S}}\phi_m|\big\|_{L^p}\le c\big\||\nabla\mathcal{S}\phi_m|\big\|_{L^p(\Omega_-)}\to0\quad\text{as } m\to\infty. \tag{4.29}$$

From Assumption A2 applied to $\widehat{\mathcal{S}}\phi_m-\mathcal{S}\phi_m$ with $G=\Omega_-$, we see that $\widehat{\mathcal{S}}\phi_m|_{\partial\Omega}=f_m$. Using this fact and applying Proposition 2.2 to $\sigma\widehat{\mathcal{S}}\phi_m$ (σ and ρ as in (3.14)) we get upon letting $\rho\to\infty$ that

$$\|f_m\|_{B^{p,\alpha}(\partial\Omega)}\le c\big\||\nabla\widehat{\mathcal{S}}\phi_m|\big\|_{L^p}\to 0 \quad \text{as } m\to\infty. \tag{4.30}$$

Equation (4.30) contradicts (4.26). Thus Lemma 4.3 is true. □

To complete the proof of Theorem 1.2 we prove the following lemma.

Lemma 4.4. *If $|p-2|\le\delta$ and $\delta>0$ is small enough, depending only on the data, then T_+ maps $B^{p,\alpha}(\partial\Omega)$ onto $B^{p,\alpha}(\partial\Omega)$.*

Proof. The proof of Lemma 4.4 is also by contradiction. Otherwise it follows from Lemma 4.3, the Hahn–Banach theorem, and Theorem 1.1 that there exists $\psi\in B^{p,\alpha}_*(\partial\Omega)$, $\psi\not\equiv 0$, with $\langle\psi,T_+(S\phi)\rangle=0$ whenever $\phi\in B^{p',\beta}_*(\partial\Omega)$. From (4.5) we see that

$$0=\langle\psi,S\phi\rangle=\int_{\Omega_-}\nabla\mathcal{S}\psi\cdot\nabla\mathcal{S}\phi\,dX \quad \text{whenever } \phi\in B^{p',\beta}_*(\partial\Omega). \tag{4.31}$$

Also using Lemma 4.1 and arguing as in (4.22)–(4.24) we obtain, for some $\phi\in B^{p',\beta}_*(\partial\Omega)$ and c depending only on the data, that

$$\int_{\Omega_-}|\nabla\mathcal{S}\psi|^p\,dX\le c\int_{\Omega_-}\nabla\mathcal{S}\psi\cdot\nabla\mathcal{S}\phi\,dX. \tag{4.32}$$

Equations (4.31) and (4.32) yield first that $\int_{\Omega_-}|\nabla\mathcal{S}\psi|^p\,dX=0$ and then from (3.16) that $\mathcal{S}\psi=0$ on Ω_-. Using Assumption A2 we conclude that $S\psi=0$ which in view of Theorem 1.1 is a contradiction to our assumption that $\psi\not\equiv 0$. □

5. Proof of Theorem 1.3

Recall from Section 1, as well as Theorem 1.1, that if $\psi\in B^{p,\alpha}_*(\partial\Omega)$ and $\phi\in B^{p',\beta}_*(\partial\Omega)$, then

$$\langle T_-^*\psi,S\phi\rangle=\langle\psi,T_-(S\phi)\rangle \tag{5.1}$$

and that $\widehat{B}^{p,\alpha}_*(\partial\Omega)=\{\psi\in B^{p,\alpha}_*(\partial\Omega):\langle\psi,1\rangle=0\}$. We first claim that

(5.2) T_-^* is a bounded linear operator from $\widehat{B}^{p,\alpha}_*(\partial\Omega)$ into $\widehat{B}^{p,\alpha}_*(\partial\Omega)$.

To prove claim (5.2) given $f \in B^{p,\alpha}(\partial\Omega)$ choose $\phi \in B_*^{p',\beta}(\partial\Omega)$ with $S\phi = f$. Existence of ϕ follows from Theorem 1.1. If $f=1$, then from Lemma 2.3 we may approximate $\sigma\mathcal{S}\phi$ (σ as in (3.14)) arbitrarily closely in the norm of $W^{1,p}$ by $C_0^\infty(\mathbf{R}^n)$ functions which are 1 in a neighborhood of $\partial\Omega$. Using this fact, (5.1), (4.5) and Lemma 3.1, we find that

$$\langle T_-^*\psi, S\phi\rangle = \int_{\Omega_+} \nabla\mathcal{S}\phi\cdot\nabla\mathcal{S}\psi\, dX = 0, \tag{5.3}$$

whenever $\psi \in \widehat{B}_*^{p,\alpha}(\partial\Omega)$. From (5.3) we conclude that T_-^* maps $\widehat{B}_*^{p,\alpha}(\partial\Omega)$ into $\widehat{B}_*^{p,\alpha}(\partial\Omega)$. Boundedness of T_-^* follows from (5.1) and (4.4). Thus claim (5.2) is true.

We follow the same proof scheme as in Theorem 1.2.

Lemma 5.1. *There are $\delta > 0$ and $c \geq 1$, depending only on the data, such that if $|p-2| \leq \delta$ the following statement is true. Given $v \in W^{1,p}(\Omega_+)$, there exists $\tilde{u} \in C_0^\infty(\mathbf{R}^n)$ with $\bar{u} = \tilde{u}|_{\Omega_+}$, $\big\||\nabla\bar{u}|\big\|_{L^{p'}(\Omega_+)} \leq c$ and*

$$\big\||\nabla v|\big\|_{L^p(\Omega_+)} \leq \int_{\Omega_+} \nabla\bar{u}\cdot\nabla v\, dX.$$

Proof. To prove Lemma 5.1 for $2-\delta \leq p \leq 2$, we simply copy the proof of Lemma 4.1 with Ω_- replaced by Ω_+. To prove this lemma for $2 < p \leq 2+\delta$ we introduce for $2-\sigma_0 \leq q \leq 2+\sigma_0$ the space $U^{1,q}(\Omega_+)$ of integrable functions v on Ω_+ with distributional gradient ∇v satisfying

$$|\nabla v| \in L^q(\Omega_+) \quad \text{and} \quad \int_{\Omega_+} v\, dX = 0.$$

Given $v \in U^{1,q}(\Omega_+)$, let $\hat{v}$ denote the extension of v to $\mathbf{R}^n$ provided for in Assumption A3. From (4.7) and Poincaré's inequality we see that

$$\begin{aligned} |\hat{v}_{B(0,R_0)}|^q &= |\hat{v}_{B(0,R_0)} - v_{\Omega_+}|^q \leq \frac{c}{H^n(\Omega_+)} \int_{\Omega_+} |v(Y) - v_{B(0,R_0)}|^q\, dY \\ &\leq \frac{c}{H^n(\Omega_+)} R_0^q \int_{B(0,R_0)} |\nabla\hat{v}|^q\, dX \leq c' \int_{\Omega_+} |\nabla v|^q\, dX. \end{aligned} \tag{5.4}$$

Using (5.4), Poincaré's inequality and (4.7) we deduce that $U^{1,q}$ is a reflexive Banach space with norm $\|v\|_{U^{1,q}(\Omega_+)} = \|\nabla v\|_{L^q(\Omega_+)}$. In fact,

$$\frac{1}{c}\|v\|_{U^{1,q}(\Omega_+)} \leq \|v\|_{W^{1,q}(\Omega_+)} \leq c\|v\|_{U^{1,q}(\Omega_+)}.$$

Let $U^{1,q}_*(\Omega_+)$ denote bounded linear functionals on $U^{1,q}(\Omega_+)$. If $2<p\le 2+\delta$ we let $\widetilde{\Gamma}\subset U^{1,p}_*(\Omega_+)$ denote all linear functionals ψ which can be written in the form

$$\langle\psi,v\rangle=\int_{\Omega_+}\nabla u\cdot\nabla v\,dX,\;\; v\in U^{1,p}(\Omega_+),\quad \text{where } u\in U^{1,p'}(\Omega_+).$$

We claim that

$$\widetilde{\Gamma}=U^{1,p}_*(\Omega_+). \tag{5.5}$$

Once (5.5) is proved we can argue as in the discussion after (4.16) to get Lemma 5.1. Thus we shall only prove (5.5). To do this we argue by contradiction. Repeating the argument after (4.16) we find that if (5.5) is false, then there exists $v\in U^{1,p}(\Omega_+)$, $v\not\equiv 0$, with

$$\int_{\Omega_+}\nabla u\cdot\nabla v\,dX=0\quad \text{for all } u\in U^{1,p'}(\Omega_+). \tag{5.6}$$

Choosing $u=v$ in (5.6) it follows that

$$\int_{\Omega_+}|\nabla v|^2\,dX=0$$

so that v is constant in Ω_+. Finally $v\equiv 0$ in Ω_+ since $v_{\Omega_+}=0$. From this contradiction we conclude Lemma 5.1 when $2<p\le 2+\delta$. □

We continue the proof of Theorem 1.3 with the following lemma.

Lemma 5.2. *There exists $\delta>0$ such that if $|p-2|\le\delta$ and $\psi\in\widehat{B}^{p,\alpha}_*(\partial\Omega)$ with $T^*_-\psi=0$ then $\psi=0$.*

Proof. Given $f\in B^{p,\alpha}(\partial\Omega)$ choose $\phi\in B^{p',\beta}_*(\partial\Omega)$ with $f=S\phi$. From (4.5), (5.1) and Theorem 1.1 we see that

$$0=\langle T^*_-\psi,f\rangle=\int_{\Omega_+}\nabla\mathcal{S}\psi\cdot\nabla\mathcal{S}\phi\,dX\quad \text{whenever } \phi\in B^{p',\beta}_*(\partial\Omega). \tag{5.7}$$

Now from Lemma 5.1 with p and p' interchanged and Lemma 3.1, there exists $\tilde{u}\in C^\infty_0(\mathbf{R}^n)$ with $\bar{u}=\tilde{u}|_{\Omega_+}$, $\big\||\nabla\bar{u}|\big\|_{L^p(\Omega_+)}\le c$, and

$$\big\||\nabla\mathcal{S}\psi|\big\|_{L^{p'}(\Omega_+)}\le\int_{\Omega_+}\nabla\mathcal{S}\psi\cdot\nabla\bar{u}\,dX. \tag{5.8}$$

Choose ϕ as above so that $\tilde{u}|_{\partial\Omega}=S\phi$. Then from Lemma 2.3 we conclude that

$$\int_{\Omega_+} \nabla\mathcal{S}\psi\cdot(\nabla\bar{u}-\nabla\mathcal{S}\phi)\,dX=0. \tag{5.9}$$

Equations (5.7)–(5.9) imply that $\nabla\mathcal{S}\psi\equiv0$ in Ω_+. Thus $\mathcal{S}\psi=a=$constant in Ω_+ so by Assumption A2, $S\psi=a$. If $2-\delta\le p\le2$, it follows from (3.41) and (3.35) with p and p' interchanged that

$$0=\langle\psi,S\psi\rangle=\int_{\mathbf{R}^n}|\nabla\mathcal{S}\psi|^2\,dX, \tag{5.10}$$

so from (3.16), $\mathcal{S}\psi\equiv0$. In view of Theorem 1.1 we have $\psi\equiv0$. If $2<p\le2+\delta$ we observe from Theorem 1.1 that there exists $\tilde{\psi}\in B^{p',\alpha}_*(\partial\Omega)$ with $S\tilde{\psi}=a$. From uniqueness in Theorem 1.1 and (3.35) it follows that $\tilde{\psi}|_{B^{p,\alpha}(\partial\Omega)}=\psi$. Moreover, $\mathcal{S}\psi=\mathcal{S}\tilde{\psi}$ so (5.10) is also valid when $2<p\le2+\delta$ and once again, $\psi\equiv0$. ☐

To continue the proof of Theorem 1.3 we have the following result.

Lemma 5.3. *There exists $\delta>0$ such that if $|p-2|\le\delta$ then $T^*_-(\widehat{B}^{p,\alpha}_*(\partial\Omega))$ is closed.*

Proof. As earlier it is easily seen that Lemma 5.3 follows once we show the existence of $\eta>0$ so that

$$\|T^*_-\psi\|_{B^{p,\alpha}_*(\partial\Omega)}\ge\eta\|\psi\|_{B^{p,\alpha}_*(\partial\Omega)}. \tag{5.11}$$

To prove (5.11) we again argue by contradiction. Otherwise there exist functions $\psi_m\in\widehat{B}^{p,\alpha}_*(\partial\Omega)$, $m=1,2,\ldots$, with

$$\|\psi_m\|_{B^{p,\alpha}_*(\partial\Omega)}=1 \text{ and } \|T^*_-\psi_m\|_{B^{p,\alpha}_*(\partial\Omega)}\to0, \quad \text{as } m\to\infty. \tag{5.12}$$

Using (5.12), Lemma 5.1, Theorem 1.1, (4.5) and (5.1), as in Lemma 5.2, we find that

$$\int_{\Omega_+}|\nabla\mathcal{S}\psi_m|^{p'}\,dX\to0 \quad \text{as } m\to\infty.$$

From (4.7) it follows that

$$\int_{\mathbf{R}^n}|\nabla\widehat{\mathcal{S}}\psi_m|^{p'}\,dX\to0 \quad \text{as } m\to\infty,$$

where $\widehat{\mathcal{S}}\psi_m$ denotes the extension of $\mathcal{S}\psi_m|_{\Omega_+}$ in Assumption A3. From Proposition 2.2 and Assumption A2 applied to $\mathcal{S}\psi_m-\widehat{\mathcal{S}}\psi_m$ in Ω_+, it follows that $\|S\psi_m\|_{B^{p',\beta}(\partial\Omega)}\to0$ as $m\to\infty$. Since from Theorem 1.1,

$$\|\psi_m\|_{B^{p,\alpha}_*(\partial\Omega)}\le c\|S\psi_m\|_{B^{p',\beta}(\partial\Omega)},$$

we have reached a contradiction to (5.12). Thus Lemma 5.3 is true. □

To complete the proof of Theorem 1.3 we prove the following lemma.

Lemma 5.4. *If* $|p-2|\leq\delta$ *and* $\delta>0$ *is small enough, depending only on the data, then* T_-^* *maps* $\widehat{B}_*^{p,\alpha}(\partial\Omega)$ *onto* $\widehat{B}_*^{p,\alpha}(\partial\Omega)$.

Proof. The proof of Lemma 5.4 is once again by contradiction. Otherwise it follows from the Hahn–Banach theorem and reflexivity of $B^{p,\alpha}(\partial\Omega)$ that there exists $f\in B^{p,\alpha}(\partial\Omega)$, $f\not\equiv$constant, with

$$\langle T_-^*\tau, f\rangle = \langle \tau, T_- f\rangle = 0 \quad \text{for all } \tau\in\widehat{B}_*^{p,\alpha}(\partial\Omega). \tag{5.13}$$

Choose $\theta\in B_*^{p,\alpha}(\partial\Omega)$ so that $S\theta=1$. From Theorem 1.1 and a uniqueness argument, as in Lemma 5.2, we see that $\mathcal{S}\theta\in R^{1,2}$ and

$$\langle \theta, 1\rangle = \langle \theta, S\theta\rangle = \int_{\mathbf{R}^n} |\nabla\mathcal{S}\theta|^2\, dX \neq 0. \tag{5.14}$$

Also from Lemma 2.3 we see that

$$\int_{\Omega_+} |\nabla\mathcal{S}\theta|^2\, dX = 0,$$

so $\mathcal{S}\theta\equiv1$ in Ω_+. It follows from this fact and (4.5) that if $f=S\phi$, $\phi\in B_*^{p',\beta}(\partial\Omega)$, then

$$\langle T_-^*\theta, f\rangle = \langle \theta, T_- f\rangle = \int_{\Omega_+} \nabla\mathcal{S}\theta\cdot\nabla\mathcal{S}\phi\, dX = 0. \tag{5.15}$$

Finally we note from (5.14) that if $\psi\in B_*^{p,\alpha}(\partial\Omega)$, then

$$\psi = \tau + \frac{\langle\psi, 1\rangle}{\langle\theta, 1\rangle}\theta, \tag{5.16}$$

where $\tau\in\widehat{B}_*^{p,\alpha}(\partial\Omega)$. Using (5.13), (5.15), (5.16) and (4.5) we see that

$$0 = \langle T_-^*\psi, f\rangle = \langle \psi, T_- f\rangle = \int_{\Omega_+} \nabla\mathcal{S}\psi\cdot\nabla\mathcal{S}\phi\, dX = 0 \quad \text{for all } \psi\in B_*^{p,\alpha}(\partial\Omega). \tag{5.17}$$

Due to (5.17) we can now apply the same argument as in Lemma 5.2 with ψ and ϕ interchanged. Doing this, we conclude that $f=S\phi\equiv$constant. From this contradiction we get first Lemma 5.4 and then Theorem 1.3. □

Remark 5.5. Given p, $1<p<\infty$, let u be harmonic in Ω with $|\nabla u|\in L^{p'}(\Omega)$. Define a linear functional $\partial u/\partial\mathbf{n}$ on $B^{p,\alpha}(\partial\Omega)$ by

$$\left\langle \frac{\partial u}{\partial\mathbf{n}}, f\right\rangle = \int_\Omega \nabla u\cdot\nabla F\, dX,$$

where $f\in B^{p,\alpha}(\partial\Omega)$ and $F\in W^{1,p}$ is the extension of f in Proposition 2.1. Using Hölder's inequality and Proposition 2.1 we see that

$$\left\|\frac{\partial u}{\partial\mathbf{n}}\right\|_{B^{p,\alpha}_*(\partial\Omega)} \le c\big\||\nabla u|\big\|_{L^{p'}(\Omega)}.$$

If $u=\mathcal{S}\psi|_\Omega$, then from (4.5) and (5.1) we see that $\partial u/\partial\mathbf{n}=T^*_-\psi$.

6. Domains which satisfy Assumptions A1–A3

In this section we discuss Assumptions A1–A3. We begin with a class of domains first considered in [8].

A connected open set G is said to be an (A,r_0) *uniform domain* if given $X_1,X_2\in G$ with $|X_1-X_2|<r_0$, there is a rectifiable curve γ: $[0,1]\to G$ with $\gamma(0)=X_1$, $\gamma(1)=X_2$, and

$$\text{(6.1)}\qquad \begin{aligned} &\text{(a)}\quad H^1(\gamma)\le A|X_1-X_2|,\\ &\text{(b)}\quad \min\{H^1(\gamma([0,t])),H^1(\gamma([t,1]))\}\le Ad(\gamma(t),\partial\Omega)\quad \text{for } t\in[0,1].\end{aligned}$$

We remark that our definition of an (A,r_0) uniform domain is slightly different but equivalent to the $(1/A,r_0)$ uniform domain defined in [8] (see [6]). For short we say that G is a *uniform domain* if (6.1) holds for some (A,r_0). We first prove the next lemma.

Lemma 6.1. *Let Ω be a bounded domain satisfying Assumption A1 and let $p>n-\min\{d_i:1\le i\le N\}$. If either $G=\Omega$ or $G=\mathbf{R}^n\setminus\overline{\Omega}$ is a uniform domain, then Assumption A2 holds for G.*

Proof. In this section we let c denote a positive constant which may depend on r_0, A, n and Ω, not necessarily the same at each occurrence. We first prove Assumption A2 when Ω is a uniform domain. Suppose $v\in W^{1,p}$ and $v=a=$constant on Ω. Let $X\in\partial\Omega$ and $0<r<\frac{1}{2}\min\{r_0,\operatorname{diam}\Omega\}$. Then it is easily seen that (6.1) and connectivity of Ω imply the existence of $W=W(X,r)$ and $c\ge4$ with $W\in\Omega\cap B(X,r/2)$

and $d(W,\partial\Omega)\geq r/c$. Using this fact and integrating $v=F$ in (2.2) over $Y\in B(W,r/c)$ we deduce that

$$|a-v_{B(X,r)}|\leq cr|\nabla v|_{B(X,r)}\leq cr(|\nabla v|^p_{B(X,r)})^{1/p}$$

or equivalently,

$$r^{n-p}|a-v_{B(X,r)}|^p\leq c\int_{B(X,r)}|\nabla v|^p\,dX. \tag{6.2}$$

Given $\varepsilon>0$ let $K(\varepsilon)\subset\partial\Omega$ be the set of points $X\in\partial\Omega$ where

$$\limsup_{r\to 0}|a-v_{B(X,r)}|>\varepsilon.$$

Using a well-known covering theorem, (6.2), and the definition of Hausdorff measure it is easily seen that

$$H^{n-p}(K(\varepsilon))\leq c\varepsilon^{-p}\int_{\mathbf{R}^n}|\nabla v|^p\,dX.$$

From this inequality and $p>n-\min\{d_i:1\leq i\leq N\}$ we conclude that if $K=\bigcup_{\varepsilon>0}K(\varepsilon)$, then $H^{d_i}(K\cap E_i)=0$ whenever $1\leq i\leq N$. Thus Assumption A2 holds with $G=\Omega$ when Ω is an (A,r_0) uniform domain. To prove (6.1) when $\mathbf{R}^n\setminus\bar{\Omega}$ is a uniform domain observe that our definition of uniform requires $\mathbf{R}^n\setminus\bar{\Omega}$ to be connected. Thus $\partial\Omega=\partial(\mathbf{R}^n\setminus\bar{\Omega})$. With this observation the proof is essentially unchanged. We omit the details. □

We also prove the following result.

Lemma 6.2. *Let Ω be a bounded domain. If $G=\Omega$ or $G=\mathbf{R}^n\setminus\bar{\Omega}$ is a uniform domain, then Assumption A3 holds for G.*

Proof. Again we shall just prove Lemma 6.2 when $G=\Omega$. We assume as we may that $0\in\Omega$ and R_0 is the smallest positive number for which $\Omega\subset B(0,R_0)$. We note that since Ω is bounded, connected, and satisfies (6.1) for some (A,r_0) it follows from a compactness argument that in fact Ω is a (b,∞) uniform domain (see [6]), where b now depends on A, n and Ω. Following Jones [8] we let $\{Q_j=Q_j(X_j,r_j)\}_{j=1}^{\infty}$ be a Whitney decomposition of $\mathbf{R}^n\setminus\partial\Omega$ into open cubes with center at X_j and side length r_j satisfying

$$\begin{aligned}
&(\alpha)\quad \bigcup_j\bar{Q}_j=\mathbf{R}^n\setminus\partial\Omega;\\
&(\beta)\quad Q_j\cap Q_i=\varnothing\quad\text{when } i\neq j;\\
&(\gamma)\quad 10^{-2n}d(Q_j,\partial\Omega)\leq r_j\leq 10^{-n}d(Q_j,\partial\Omega).
\end{aligned}\tag{6.3}$$

Let $L_1=\{Q_j:\bar{Q}_j\subset\Omega\}$ and $L_2=\{Q_j:\bar{Q}_j\subset\mathbf{R}^n\setminus\bar{\Omega}\}$. The same argument as in Lemma 6.1 shows that if $Q_i=Q_i(X_i,r_i)\in L_2$ and $0<r_i<2R_0$, then we can choose $Q_i'=Q_j(X_j,r_j)\in L_1$ with

$$\max\{r_i,r_j,|X_i-X_j|\}\le c\min\{r_i,r_j,|X_i-X_j|\}. \tag{6.4}$$

We call Q_i' the *reflection* of Q_i in $\partial\Omega$. If $r_i\ge 2R_0$ we set $Q_i'=\widetilde{Q}$, where $\widetilde{Q}$ is a fixed cube in L_1 with side length $\ge R_0/c$. Next given $Q_i\in L_2$ let $\Lambda(i)=\{j:Q_j\cap\bar{Q}_i\ne\varnothing\}$ and let K_i be the interior of $\bigcup_{j\in\Lambda(i)}\bar{Q}_j$. Let $\{\phi_i\}_{i=1}^{\infty}$ be a partition of unity for $\mathbf{R}^n\setminus\bar{\Omega}$, with ϕ_i adapted to $Q_i\in L_2$. That is,

$$\begin{aligned} &\text{(i)} && 0\le\phi_i\in C_0^\infty(K_i) && \text{with } |\nabla\phi_i|\le c/r_i;\\ &\text{(ii)} && \phi_i=\text{constant}\ge\frac{1}{c} && \text{on } Q_i;\\ &\text{(iii)} && \sum_{i=1}^{\infty}\phi_i(X)=1 && \text{whenever } X\in\mathbf{R}^n\setminus\bar{\Omega}.\end{aligned} \tag{6.5}$$

Let $f\in W^{1,1}(\Omega)$. Define $\hat{f}$ on $\mathbf{R}^n\setminus\partial\Omega$ by $\hat{f}=f$ on Ω and

$$\hat{f}(X)=\sum_{i=1}^{\infty}\phi_i(X)f_{Q_i'}\quad \text{when } X\in\mathbf{R}^n\setminus\bar{\Omega}.$$

In this display Q_i' is the reflection of $Q_i\in L_2$ in $\partial\Omega$ and $f_{Q_i'}$ denotes the average of f on Q_i'. From (6.3) and (6.5), we see there exists $\hat{c}\ge 1$ such that

$$\hat{f}\equiv f_{\tilde{Q}}\quad \text{in } \mathbf{R}^n\setminus\bar{B}(0,\hat{c}R_0). \tag{6.6}$$

In [8] it is shown that $H^n(\partial\Omega)=0$ and that $\hat{f}\in W^{1,1}(B(0,\rho))$ for each $\rho>0$. It remains to prove the inequality involving A_2 weights in Assumption A3. To do this, we observe as in [8], Lemma 2.8, that if $Q_i\in L_2$ and $j\in\Lambda(i)$, then it follows from the uniform condition in (6.1) that there is a chain of cubes $Q_1^*,Q_2^*,\dots,Q_m^*$ in L_1 with $m\le c$ such that $Q_1^*=Q_i'$, $Q_m^*=Q_j'$ and $Q_k^*\cap Q_{k+1}^*\ne\varnothing$ for $1\le k\le m-1$. Using (2.2) in balls containing successive cubes, (6.4), and the triangle inequality we find that

$$|f_{Q_i'}-f_{Q_j'}|\le cr_i^{1-n}\int_{O_{i,j}}|\nabla f|\,dX, \tag{6.7}$$

where $O_{i,j}$ is an open set with $Q_i',Q_j'\subset O_{i,j}$ and the property that

$$\frac{1}{c}r_i\le\min\{\operatorname{diam} O_{i,j},d(O_{i,j},\partial\Omega)\}\le\max\{\operatorname{diam} O_{i,j},d(O_{i,j},\partial\Omega)\}\le cr_i. \tag{6.8}$$

Let $O_i = \bigcup_{j\in\Lambda(i)} O_{i,j}$ and let Θ be the set of all $O_i \subset \Omega$ corresponding to a $Q_i \in L_2$ with $Q_i \cap B(0, 2\hat{c}R_0) \neq \varnothing$. From (6.3) and (6.8) we see for $X \in \Omega$ that

$$\sum_{O_i\in\Theta} \chi_{O_i}(X) \le c, \tag{6.9}$$

where χ_{O_i} is the characteristic function of O_i. Finally we note from (6.5), (6.7) and the definition of $\hat{f}$ that for $X \in Q_i \in L_2$, $Q_i \cap B(0, 2\hat{c}R_0) \neq \varnothing$, we have

$$\frac{1}{c}|\nabla \hat{f}(X)| \le \frac{1}{r_i} \sum_{j\in\Lambda(i)} |f_{Q'_i} - f_{Q'_j}| \le \frac{c}{r_i^n} \int_{O_i} |\nabla f|\, dX \tag{6.10}$$

while $\nabla \hat{f} = 0$ in $\mathbf{R}^n \setminus \overline{B}(0, \hat{c}R_0)$ due to (6.6). Now suppose that ω is an A_2 weight on $\mathbf{R}^n$. Then from (6.8)–(6.10), and Hölder's inequality we conclude that

$$\begin{aligned} \int_{\mathbf{R}^n\setminus\bar{\Omega}} \omega |\nabla \hat{f}|^2\, dX &= \sum_{Q_i\in L_2} \int_{Q_i} \omega |\nabla \hat{f}|^2\, dX \\ &\le c \sum_{Q_i\in L_2} \omega(Q_i) \left(\frac{1}{r_i^n} \int_{O_i} |\nabla f|\, dX \right)^2 \\ &\le c^2 \|\omega\hat{\|} \sum_{Q_i\in L_2} \int_{O_i} |\nabla f|^2 \omega\, dX \\ &\le c^3 \|\omega\hat{\|} \int_{\Omega} |\nabla f|^2 \omega\, dX. \end{aligned} \tag{6.11}$$

In (6.11) we have used the doubling property of ω. From (6.11) we conclude the validity of Lemma 6.2 when Ω is a uniform domain. ☐

Lemmas 6.1 and 6.2 are easily used to identify bounded domains $\Omega \subset \mathbf{R}^3$ satisfying the hypotheses of Theorems 1.1–1.3. These domains can have fractal boundaries of Hausdorff dimension larger than $n-1$. For example Wolff snowflakes constructed in [19] have this property and satisfy the hypotheses of Theorems 1.1–1.3, as we see from Lemmas 6.1 and 6.2.

Concluding remarks We remark that Assumption A2 is a stability property of Sobolev functions defined in [1], Definition 11.1.7. Sufficient conditions for this stability property to hold are also given in [1], Theorem 11.4.1. Based on these conditions our intuition is that Lemma 6.1 remains valid for Ω without any uniform assumption. Also we believe that Lemma 6.1 is valid for $\mathbf{R}^n \setminus \bar{\Omega}$, without any uniform assumption, provided this set is connected. However, we have not been able to justify our intuition.

Assumption A3 implies a similar condition for A_p weights, $1<p<\infty$, as can be deduced from Proposition 2.17 in [7]. The authors consider it an interesting question whether Theorems 1.2 and 1.3 remain valid under more general conditions than the uniform assumption in Lemmas 6.1 and 6.2. For example can this uniform condition be replaced by a local John-type condition as in Definition 3.4 of [7] or more generally by the visual John boundary condition in Condition 4.1 of [12].

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