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On minimal exposed faces

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Abstract. In this paper we consider the problem of the non-empty intersection of exposed faces in a Banach space. We find a sufficient condition to assure that the non-empty intersection of exposed faces is an exposed face. This condition involves the concept of *inner point*. Finally, we also prove that every minimal face of the unit ball must be an extreme point and show that this is not the case at all for minimal exposed faces since we prove that every Banach space with dimension greater than or equal to 2 can be equivalently renormed to have a non-singleton, minimal exposed face.

1. Introduction

Before presenting the nature of the problems that we treat in this manuscript it is necessary that we recall some basic definitions and concepts (see [2] and [3]).

Definition 1.1. (Kreĭn and Milman, 1940; Phelps, 1989) Let X be a real Banach space, C be a convex subset of S_X, and $c \in \mathsf{S}_X$. Then:

1. C is said to be a *face* of B_X if it satisfies the extremal condition with respect to B_X, in other words, if $x, y \in \mathsf{B}_X$, $t \in (0,1)$, and $tx+(1-t)y \in C$, then $x, y \in C$.
2. c is said to be an *extreme point* of B_X if $\{c\}$ is a face of B_X.
3. C is said to be an *exposed face* of B_X if there exists $f \in \mathsf{S}_{X^*}$ such that $C = f^{-1}(1) \cap \mathsf{B}_X$.
4. c is said to be an *exposed point* if $\{c\}$ is an exposed face; the set of exposed points is usually denoted by $\exp(\mathsf{B}_X)$.
5. c is said to be a *smooth point* of B_X if there exists only one functional $f \in \mathsf{S}_{X^*}$ such that $f(x)=1$; the set of smooth points is usually denoted by $\mathrm{smo}(\mathsf{B}_X)$.

Obviously, every exposed face is a face, and the converse is not true in general. Another obvious fact is that the non-empty intersection of faces is always a face.

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The natural question here is whether the non-empty intersection of exposed faces is an exposed face. In the third section of this paper we find a sufficient condition for the non-empty intersection of exposed faces to be an exposed face. In [1] we solved this question in the negative as shown in the next result.

Theorem 1.2. (Aizupuru and García-Pacheco, 2008) *Let L be an uncountable, discrete topological space. Let $\hat{L}$ denote the one-point compactification of L. Then, the constant function $\mathbf{1}$ is not an exposed point of $\mathsf{B}_{\mathcal{C}(\hat{L})}$ but $\{\mathbf{1}\}$ is the intersection of the exposed faces of $\mathsf{B}_{\mathcal{C}(\hat{L})}$ that contain $\mathbf{1}$.*

The fourth section of this paper is about minimal faces and minimal exposed faces. We first want to clarify that by minimal (exposed) face we mean a minimal element of the set of proper (exposed) faces of the unit ball partially ordered by the inclusion. The first result that we prove in the fourth section is that every minimal face must be an extreme point. Finally, we show that this situation does not hold for minimal exposed faces, since we prove that every real Banach space of dimension greater than or equal to 2 admits an equivalent renorming whose unit ball has a minimal exposed face that is not a singleton. The second section is dedicated to characterizing minimal exposed faces.

2. Characterizations of minimal exposed faces

This section is for characterizing minimal exposed faces and provide the necessary tools and results that we will make use of in the further sections. Our first characterization of minimal exposed faces is about uniqueness of ω^*-exposed faces. We recall that an ω^*-*exposed face* is an exposed face of the unit ball of a dual Banach space determined by an ω^*-continuous functional. We also recall that $\mathsf{NA}(X)$ denotes the set of norm-attaining functionals on a Banach space X.

Theorem 2.1. *Let X be a real Banach space, and let $f \in \mathsf{S}_{X^*} \cap \mathsf{NA}(X)$. Then the following conditions are equivalent:*

1. *$f^{-1}(1) \cap \mathsf{B}_X$ is a minimal exposed face of B_X;*
2. *there exists only one ω^*-exposed face of B_{X^*} containing f.*

Proof. Assume first that $f^{-1}(1) \cap \mathsf{B}_X$ is a minimal exposed face of B_X. Let $x, y \in \mathsf{S}_X$ be such that $x^{-1}(1) \cap \mathsf{B}_{X^*}$ and $y^{-1}(1) \cap \mathsf{B}_{X^*}$ contain f. Assume that there exists $g \in (x^{-1}(1) \cap \mathsf{B}_{X^*}) \setminus (y^{-1}(1) \cap \mathsf{B}_{X^*})$. Then, we clearly have that

$$\left(\frac{f+g}{2}\right)^{-1}(1) \cap \mathsf{B}_X \subsetneq f^{-1}(1) \cap \mathsf{B}_X,$$

which is a contradiction. Conversely, assume that there exists only one ω^*-exposed face of B_{X^*} containing f. Let $g\in\mathsf{S}_{X^*}\cap\mathsf{NA}(X)$ be such that

$$g^{-1}(1)\cap\mathsf{B}_X\subsetneq f^{-1}(1)\cap\mathsf{B}_X.$$

Then, there exists $x\in(f^{-1}(1)\cap\mathsf{B}_X)\setminus(g^{-1}(1)\cap\mathsf{B}_X)$. Now, take any $y\in g^{-1}(1)\cap\mathsf{B}_X$. Obviously, $x^{-1}(1)\cap\mathsf{B}_{X^*}$ and $y^{-1}(1)\cap\mathsf{B}_{X^*}$ are different ω^*-exposed faces of B_{X^*} containing f. □

As a corollary of the previous result we obtain the following consequence.

Corollary 2.2. *Let X be a real Banach space. Then the following conditions are equivalent*:

1. *the exposed faces of B_X are pairwise disjoint*;
2. *the ω^*-exposed faces of B_{X^*} are pairwise disjoint.*

We also want to present another characterization of minimal exposed faces. This time we will involve the concept of smoothness.

Theorem 2.3. *Let X be a reflexive real Banach space. Let $f\in\mathsf{S}_{X^*}\cap\mathsf{NA}(X)$. Then the following conditions are equivalent*:

1. *$f^{-1}(1)\cap\mathsf{B}_X$ is a minimal exposed face of B_X*;
2. *f is a smooth point of* $\operatorname{span}\{f,g\}$ *for all $g\in\mathsf{S}_{X^*}$ such that $(g^{-1}(1)\cap\mathsf{B}_X)\cap(f^{-1}(1)\cap\mathsf{B}_X)\neq\varnothing$.*

Proof. Assume first that $f^{-1}(1)\cap\mathsf{B}_X$ is a minimal exposed face of B_X. Let $g\in\mathsf{S}_{X^*}$ be such that $(g^{-1}(1)\cap\mathsf{B}_X)\cap(f^{-1}(1)\cap\mathsf{B}_X)\neq\varnothing$. Since $f^{-1}(1)\cap\mathsf{B}_X$ is a minimal exposed face of B_X we deduce that $f^{-1}(1)\cap\mathsf{B}_X\subseteq g^{-1}(1)\cap\mathsf{B}_X$. Suppose to the contrary that f is not a smooth point of $Y:=\operatorname{span}\{f,g\}$. Then, there exists $h\in\mathsf{S}_{Y^*}$ such that $h(f)=1$ and $h(g)<1$. By the Hahn–Banach theorem we can extend h to a functional $x\in\mathsf{S}_X$. Then, $x\in f^{-1}(1)\cap\mathsf{B}_X\subseteq g^{-1}(1)\cap\mathsf{B}_X$ which is impossible.

Conversely, assume that f is a smooth point of $\operatorname{span}\{f,g\}$ for all $g\in\mathsf{S}_{X^*}$ such that $(g^{-1}(1)\cap\mathsf{B}_X)\cap(f^{-1}(1)\cap\mathsf{B}_X)\neq\varnothing$. If $f^{-1}(1)\cap\mathsf{B}_X$ is not a minimal exposed face of B_X then we can find $g\in\mathsf{S}_{X^*}\cap\mathsf{NA}(X)$ such that $g^{-1}(1)\cap\mathsf{B}_X\subsetneq f^{-1}(1)\cap\mathsf{B}_X$. Let $x\in g^{-1}(1)\cap\mathsf{B}_X$ and $y\in(f^{-1}(1)\cap\mathsf{B}_X)\setminus(g^{-1}(1)\cap\mathsf{B}_X)$. Clearly, $x|_Y$ and $y|_Y$ are different norm-1 functionals on $Y:=\operatorname{span}\{f,g\}$ attaining their norm at f. □

3. Minimal exposed faces and intersection of exposed faces

We will base our results in this section upon the concept of *inner point*, which we will introduce in the next definition.

Definition 3.1. Let X be a real vector space. Let M be a subset of X. We say that a point $x \in M$ is an *inner point* of M if x belongs to the interior of every maximal segment of M containing x. The set of inner points of M is denoted by $I(M)$.

This concept will help us find a sufficient condition for the non-empty intersection of exposed faces to be an exposed face. However, we first need the following *key* lemma.

Lemma 3.2. *Let X be a real Banach space. If $f \in \mathsf{S}_{X^*}$ is an inner point of a convex subset $D \subset \mathsf{S}_{X^*}$, then $f^{-1}(1) \cap \mathsf{B}_X \subseteq g^{-1}(1) \cap \mathsf{B}_X$ for all $g \in D$. In particular, if $f \in \mathsf{NA}(X)$, then $D \subset \mathsf{NA}(X)$.*

Proof. Let $f \neq g \in D \cap \mathsf{NA}(X)$. Since f is an inner point of D, there exists $h \in D$ such that $f \in (h, g)$. Therefore, $f^{-1}(1) \cap \mathsf{B}_X \subseteq g^{-1}(1) \cap \mathsf{B}_X$. □

Now we are in the right position to state and prove our sufficient condition.

Theorem 3.3. *Let X be a real Banach space. Assume that every ω^*-closed face of S_{X^*} has inner points. Then, the non-empty intersection of any family of exposed faces of B_X is an exposed face of B_X.*

Proof. Let $\{f_i\}_{i \in I} \subset \mathsf{S}_{X^*}$ such that $C := \bigcap_{i \in I} f_i^{-1}(1) \cap \mathsf{B}_X \neq \varnothing$. Obviously,

$$\{f_i\}_{i \in I} \subseteq D := \bigcap_{x \in C} x^{-1}(1) \cap \mathsf{B}_{X^*}.$$

Observe that

$$D = \{h \in \mathsf{S}_{X^*} : h^{-1}(1) \cap \mathsf{B}_X \supseteq C\}.$$

By hypothesis, $I(D) \neq \varnothing$. Let $f \in I(D)$. According to Lemma 3.2 we have that $f^{-1}(1) \cap \mathsf{B}_X \subseteq f_i^{-1}(1) \cap \mathsf{B}_X$ for all $i \in I$. This concludes the proof. □

To conclude this section we will make use of Theorem 2.1 and Lemma 3.2 once more to represent minimal exposed faces in terms of inner points.

Lemma 3.4. *Let X be a real Banach space. If $f \in \mathsf{S}_{X^*}$ is an inner point of a maximal ω^*-exposed face D of B_{X^*}, then $f^{-1}(1) \cap \mathsf{B}_X$ is a minimal exposed face of B_X and*

$$I(D) \subseteq \{g \in \mathsf{S}_{X^*} : f^{-1}(1) \cap \mathsf{B}_X = g^{-1}(1) \cap \mathsf{B}_X\} \subseteq D.$$

As a consequence,

$$I(\{g\in \mathsf{S}_{X^*}: f^{-1}(1)\cap \mathsf{B}_X = g^{-1}(1)\cap \mathsf{B}_X\}) = I(D)\neq\varnothing.$$

Proof. Let C be another ω^*-exposed face of B_{X^*} containing f. Since f is an inner point of D, we must have that $D\subseteq C$. However, D is maximal among the ω^*-exposed faces of B_{X^*}, so $C=D$. According to Theorem 2.1 we deduce that $f^{-1}(1)\cap\mathsf{B}_X$ is a minimal exposed face of B_X. The rest is a consequence of Lemma 3.2. □

Remark 3.5. Observe that in the previous lemma we cannot assure that

$$I(D)=\{g\in \mathsf{S}_{X^*}: f^{-1}(1)\cap \mathsf{B}_X = g^{-1}(1)\cap \mathsf{B}_X\}.$$

Indeed, if X is a non-smooth real Banach space such that X^* is smooth, then we have that every $x\in\mathsf{S}_X$ is an exposed point of B_X and thus a minimal exposed face. However, if $x\in\mathsf{S}_X$ is not a smooth point of B_X then $D:=x^{-1}(1)\cap\mathsf{B}_{X^*}$ satisfies, in virtue of Lemma 3.2, that

$$I(D)\subsetneq D=\{g\in \mathsf{S}_{X^*}: g^{-1}(1)\cap \mathsf{B}_X=\{x\}\}.$$

The natural question that arises now is the following.

Question 3.6. Let X be a real Banach space. Let C be a minimal exposed face of B_{X^*}. Does $\{g\in\mathsf{S}_{X^*}: g^{-1}(1)\cap\mathsf{B}_X=C\}$ have inner points?

We will show that the answer to the previous question is negative.

Theorem 3.7. *Let K be an infinite, compact Hausdorff topological space. Let* $\mathbf{1}$ *denote the constant function on K equal to* 1. *Then,* $\mathbf{1}^{-1}(1)\cap\mathsf{B}_{\mathcal{C}(K)^*}$ *is free of inner points.*

Proof. It is clear that $\mathbf{1}^{-1}(1)$ is the set of probability measures on K, so given a probability measure μ it suffices to find one other probability measure τ so that the affine line $(1-t)\mu+t\nu$ cannot be extended to any negative t. For this it is sufficient to have τ singular to μ. However, we will give now an explicit construction of τ. We will show that, given $\mu\in\mathbf{1}^{-1}(1)\cap\mathsf{B}_{\mathcal{C}(K)^*}$, there exists $\tau\in\mathbf{1}^{-1}(1)\cap\mathsf{B}_{\mathcal{C}(K)^*}$ such that $t\mu+(1-t)\tau\notin\mathbf{1}^{-1}(1)\cap\mathsf{B}_{\mathcal{C}(K)^*}$ if $t>1$. Since K is infinite, we have that $\sum_{n=1}^{\infty}\mu(\{x_n\})=\mu(\{x_n: n\in\mathbb{N}\})<\infty$ for every sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq K$ such that $x_i\neq x_j$ for every $i\neq j$, and thus $\{\mu(\{x_n\})\}_{n\in\mathbb{N}}$ converges to 0. Let then

$$\tau:=\sum_{n=1}^{\infty}\frac{1}{2^n}\delta_{x_n},$$

where $\{x_n\}_{n\in\mathbb{N}}\subseteq K$ is any sequence satisfying that $x_i\neq x_j$ for every $i\neq j$ and $\{2^n\mu(\{x_n\})\}_{n\in\mathbb{N}}$ converges to 0. Observe that if $t>1$ then there exists $n\in\mathbb{N}$ such that

$$\mu(\{x_n\})<\frac{t-1}{2^n t}.$$

Then,

$$(t\mu+(1-t)\tau)(\{x_n\})<0,$$

so $t\mu+(1-t)\tau$ cannot be a probability measure. This proves that $\mathbf{1}^{-1}(1)\cap\mathsf{B}_{\mathcal{C}(K)^*}$ is free of inner points. $\square$

Example 3.8. (Counterexample to Question 3.6) Let K be any infinite, compact Hausdorff space such that $\mathbf{1}$ is an exposed point of $\mathsf{B}_{\mathcal{C}(K)}$ (for instance, take $K:=\beta\mathbb{N}$, so that $\mathcal{C}(K)=\ell_\infty$). We have that $\{\mathbf{1}\}$ is a minimal exposed face of $\mathsf{B}_{\mathcal{C}(K)}$ but $\mathbf{1}^{-1}(1)\cap\mathsf{B}_{\mathcal{C}(K)^*}$ is free of inner points in accordance with Theorem 3.7.

4. Minimal faces and minimal exposed faces

In this section we will study minimal exposed faces that are not singletons. To do this we first study the more general case of minimal faces. What we will first show is that every minimal face of the unit ball of a Banach space is actually a singleton, that is, an extreme point. We will rely on the following result (notice that the concept of face is purely algebraic so it can be similarly defined for real vector spaces, that is, C is a face of the convex subset M of the real vector space X if C is convex and satisfies the extremal condition with respect to M).

Theorem 4.1. *Let X be a real vector space. Let M be a convex subset of X with more than one point. Then, every $x\in M\setminus I(M)$ is contained in a proper face C of M.*

Proof. Notice that we can assume without any loss of generality that $x=0$. Consider

$$C=\bigcup\{S\subset M: S \text{ is a maximal segment of } M \text{ whose interior contains } 0\}.$$

Notice that if $C=\varnothing$, then 0 is an extreme point of M. On the other hand, since $0\in M\setminus I(M)$ there exists a maximal segment in M such that one of its endpoints is 0, and thus $C\neq M$. To see that C is convex, we take $x,y\in C$ and $t\in[0,1]$. There exists $\alpha<0$ such that $\alpha x,\alpha y\in M$. Then,

$$\alpha(tx+(1-t)y)=t(\alpha x)+(1-t)(\alpha y)\in M.$$

Thus $tx+(1-t)y\in C$. Finally, to see that C is a face of M we take $x,y\in M$ and $t\in(0,1)$ such that $tx+(1-t)y\in C$. There exists $\alpha<0$ such that $\alpha(tx+(1-t)y)\in M$. Then,

$$\frac{\alpha t}{1-\alpha(1-t)}x=\frac{-\alpha(1-t)}{1-\alpha(1-t)}y+\frac{1}{1-\alpha(1-t)}\alpha(tx+(1-t)y)\in M$$

and

$$\frac{\alpha(1-t)}{1-\alpha t}y=\frac{-\alpha t}{1-\alpha t}x+\frac{1}{1-\alpha t}\alpha(tx+(1-t)y)\in M.$$

This proves that $x,y\in C$. Observe that, by construction, $0\in I(C)$. □

The following is a lemma that we will use later on.

Lemma 4.2. *Let X be a real topological vector space. Let M be a bounded, closed convex subset of X. If C is a face of M, then $C\setminus I(C)\neq\varnothing$.*

Proof. If C is a singleton, then $I(C)=\varnothing$ by definition. Let then S be any non-trivial maximal segment of C. Since M is bounded and closed, there are $m\neq n\in M$ such that $[m,n]$ is a maximal segment of M containing S. Since C is a face of M we deduce that $S=[m,n]$. Now, $m,n\in C\setminus I(C)$. □

Now we are ready to prove that, in bounded and closed convex sets, minimal faces must be singletons.

Theorem 4.3. *Let X be a real topological vector space. Let M be a bounded, closed convex subset of X. If C is a minimal face of M, then C is a singleton.*

Proof. Assume that M is not a singleton. According to Lemma 4.2 there exists $c\in C\setminus I(C)$. Now, by Theorem 4.1 there exists a proper face D of C containing c. This is impossible. □

Corollary 4.4. *Let X be a real Banach space. If C is a minimal face of B_X, then C is a singleton.*

At this point, the question is if the same happens with minimal exposed faces, that is, if every minimal exposed face is an exposed point. Obviously the answer is negative as shown in the next remark.

Remark 4.5. Let X be a non-rotund, smooth real Banach space. Since X is non-rotund, there exists an exposed face in B_X that is not a singleton. And, since X is smooth, every exposed face is minimal.

Our aim now is to prove that every real Banach space of dimension greater than or equal to 2 is isomorphic to a Banach space whose unit ball has a minimal exposed face that is not a singleton. As the reader may notice, this renorming does not rely on the previous remark, since there are Banach spaces that do not admit an equivalent smooth norm. The following lemma will give us the key.

Lemma 4.6. *Let X and Y be real Banach spaces. If C is a maximal face of B_X, then $C\times\{0\}$ is a maximal face of $\mathsf{B}_{X\oplus_2 Y}$.*

Proof. Let $(x,y)\in\mathsf{S}_{X\oplus_2 Y}$ and $(c,0)\in C\times\{0\}$ be such that $(x+c/2,y/2)\in\mathsf{S}_{X\oplus_2 Y}$. Then,

$$\begin{aligned}\|x\| &= \frac{2+2\|x\|}{2}-1=\frac{\|x\|^2+\|y\|^2+\|c\|^2+2\|x\|\,\|c\|}{2}-1\\ &= \frac{(\|x\|+\|c\|)^2+\|y\|^2}{2}-1\geq\frac{\|x+c\|^2+\|y\|^2}{2}-1=\frac{4}{2}-1=1.\end{aligned}$$

This means that $\|x\|=1$ and $\|y\|=0$. Therefore, $x\in C$ since C is a maximal face of S_X. □

Finally, we are in the right position to finish this paper by presenting and proving the main result in this section.

Theorem 4.7. *Let X be a real Banach space with dimension greater than or equal to 2. Then, X can be equivalently renormed to have a minimal exposed face of B_X that is not a singleton.*

Proof. In the first place, if X has dimension 2 then X admits an equivalent, non-rotund, smooth norm. The unit ball of this norm admits non-singleton, minimal exposed faces. So, assume that X has dimension strictly greater than 2. We will show two different ways to renorm X:

1. Let Y be $\mathbb{R}^2$ endowed with the norm whose unit ball is the intersection of the two circles

$$\{(x,y)\in\mathbb{R}^2:(x+1)^2+y^2=4\}\quad\text{and}\quad\{(x,y)\in\mathbb{R}^2:(x-1)^2+y^2=4\}.$$

Now, let $Z^*:=Y\oplus_\infty\mathbb{R}$. We have that the segment $C:=\left[\left(0,\sqrt{3},-1\right),\left(0,\sqrt{3},1\right)\right]$ is a maximal face of B_{Z^*}. Let Z be a predual of Z^*. Then, X is clearly isomorphic to a space of the form $Z\oplus_2 M$, where M is another Banach space. Finally, X^* is then dual-isomorphic to $Z^*\oplus_2 M^*$. By applying Lemma 4.6 we deduce that C is actually a maximal face of $\mathsf{B}_{Z^*\oplus_2 M^*}$, and clearly $C\subset\mathsf{NA}(Z\oplus_2 M)$. In accordance with Lemma 3.4 we have that $f := \big(0, \sqrt{3}, 0\big)$ defines a minimal exposed face of $\mathsf{B}_{Z\oplus_2 M}$. Finally, observe that f is not a smooth point of Z^*, therefore $f^{-1}(1)\cap \mathsf{B}_{Z\oplus_2 M}$ cannot be a singleton.

2. Let Y be $\mathbb{R}^2$ with the unit ball being the convex hull of

$$\{(x,y)\in\mathbb{R}^2 : (x-1)^2+y^2\leq 1\} \quad \text{and} \quad \{(x,y)\in\mathbb{R}^2 : (x+1)^2+y^2\leq 1\}.$$

This space clearly has a minimal exposed face which is not a singleton (it is actually smooth but not rotund). It is clear that any 2-dimensional subspace of any Banach space is complemented. After renorming the 2-dimensional subspace and taking a $\oplus_2$ sum with the complement the new space is obtained. □

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