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Riesz transform characterization of Hardy spaces associated with Schrödinger operators with compactly supported potentials

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Abstract. Let $L=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^d$, $d\geq 3$. We assume that V is a nonnegative, compactly supported potential that belongs to $L^p(\mathbb{R}^d)$, for some $p>d/2$. Let K_t be the semigroup generated by $-L$. We say that an $L^1(\mathbb{R}^d)$-function f belongs to the Hardy space H^1_L associated with L if $\sup_{t>0}|K_t f|$ belongs to $L^1(\mathbb{R}^d)$. We prove that $f\in H^1_L$ if and only if $R_j f\in L^1(\mathbb{R}^d)$ for $j=1, ..., d$, where $R_j=(\partial/\partial x_j)L^{-1/2}$ are the Riesz transforms associated with L.

1. Introduction

Let $u(x,y)=u(x+iy)$ be a real-valued harmonic function in the upper half-plane $\{z=x+iy:y>0\}$. It was proved in Burkholder, Gundy, and Silverstein [2] that for $0<p<\infty$ the function u is a real part of a holomorphic function $F(z)=u(z)+iv(z)$ which satisfies the H^p property

$$\sup_{y>0}\int_{\mathbb{R}}|F(x+iy)|^p\,dx<\infty$$

if and only if the maximal function $u^*(x)=\sup_{|x'-x|<y}|u(x'+iy)|$ belongs to $L^p(\mathbb{R})$. In Fefferman–Stein [7] the authors present other characterizations of the H^p property by means of real analysis. These characterizations lead to the notion of classical real Hardy spaces $H^p(\mathbb{R}^d)$. Let us recall two equivalent characterizations of $H^1(\mathbb{R}^d)$ presented in [7]. An $L^1(\mathbb{R}^d)$-function f is an element of the real Hardy space $H^1(\mathbb{R}^d)$ if and only if the maximal function $\mathcal{M}_{\sqrt{-\Delta}}f(x)=\sup_{t>0}\left|\exp\left(-t\sqrt{-\Delta}\right)f(x)\right|$ belongs to $L^1(\mathbb{R}^d)$, where $\exp\left(-t\sqrt{-\Delta}\right)$ denotes the Poisson semigroup. Equivalently, one

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can take the maximal function $\mathcal{M}_{-\Delta}f(x)$ from the heat semigroup P_t in the definition of the Hardy space $H^1(\mathbb{R}^d)$, that is, $\mathcal{M}_{-\Delta}f(x)=\sup_{t>0}|P_tf(x)|$, where here and subsequently

$$P_tf(x)=f*P_t(x) \quad \text{and} \quad P_t(x)=(4\pi t)^{-d/2}\exp\left(-\frac{|x|^2}{4t}\right).$$

The second characterization of $H^1(\mathbb{R}^d)$ is given by boundedness of certain singular integrals. Let

$$\begin{aligned}\mathcal{R}_jf(x)&=c_d\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{x_j-y_j}{|x-y|^{d+1}}f(y)\,dy\\ &=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\int_{\mathbb{R}^d}\frac{\partial}{\partial x_j}P_t(x-y)f(y)\,dy\frac{dt}{\sqrt{t}},\end{aligned}\tag{1.1}$$

$j=1,2,...,d$, be the classical Riesz transforms on $\mathbb{R}^d$. Clearly, for $f\in L^1(\mathbb{R}^d)$ the limits in (1.1) exist in the sense of distributions and define $\mathcal{R}_jf$ as distributions. It was proved in [7] (see also [9]) that an $L^1(\mathbb{R}^d)$-function f is in the classical Hardy space $H^1(\mathbb{R}^d)$ if and only if $\mathcal{R}_jf\in L^1(\mathbb{R}^d)$ for $j=1,...,d$. Moreover,

$$\|f\|_{L^1(\mathbb{R}^d)}+\sum_{j=1}^d\|\mathcal{R}_jf\|_{L^1(\mathbb{R}^d)},\tag{1.2}$$

defines a norm in the space $H^1(\mathbb{R}^d)$, which is equivalent to the norms $\|\mathcal{M}_{-\Delta}f\|_{L^1(\mathbb{R}^d)}$ and $\|\mathcal{M}_{-\sqrt{-\Delta}}f\|_{L^1(\mathbb{R}^d)}$. In addition, in the case of $d=1$, if $f\in H^1(\mathbb{R})$, then $u^*\in L^1(\mathbb{R})$, where $u(x+iy)=\exp\big(-y\sqrt{-\Delta}\big)f(x)$, $y>0$. Conversely, every harmonic function u with the property that $u^*\in L^1(\mathbb{R})$ is obtained in this way. In this case, the conjugate function v is of the form $v(x+iy)=\exp\big(-y\sqrt{-\Delta}\big)g(x)$ with $g=-\pi^{-1/2}\mathcal{R}f$.

In this paper we consider a Schrödinger operator $L=-\Delta+V$ on $\mathbb{R}^d$, $d\geq 3$. We assume that V is a nonnegative function, $\operatorname{supp}V\subseteq B(0,1)=\{x\in\mathbb{R}:|x|<1\}$, and $V\in L^p(\mathbb{R}^d)$ for some $p>d/2$. Let $\{K_t\}_{t>0}$ be the semigroup of linear operators generated by $-L$. Since $V\geq 0$, by the Feynman–Kac formula, we have

$$0\leq K_t(x,y)\leq P_t(x-y),\tag{1.3}$$

where $K_t(x,y)$ is the integral kernel of the semigroup $\{K_t\}_{t>0}$. Let

$$\mathcal{M}_Lf(x)=\sup_{t>0}|K_tf(x)|.$$

We say that an $L^1(\mathbb{R}^d)$-function f belongs the Hardy space H^1_L if

$$\|f\|_{H^1_L}=\|\mathcal{M}_Lf\|_{L^1(\mathbb{R}^d)}<\infty.$$

For $j=1,\dots,d$ let us define the *Riesz transforms* R_j associated with L by setting

$$R_j f=\sqrt{\pi}\frac{\partial}{\partial x_j}L^{-1/2}f=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\frac{\partial}{\partial x_j}K_t f\frac{dt}{\sqrt{t}}, \tag{1.4}$$

where the limit is understood in the sense of distributions. The fact that for any $f\in L^1(\mathbb{R}^d)$ the operators

$$R_j^{\varepsilon} f=\int_{\varepsilon}^{1/\varepsilon}\frac{\partial}{\partial x_j}K_t f\frac{dt}{\sqrt{t}} \tag{1.5}$$

are well defined and the limits $\lim_{\varepsilon\to 0} R_j^{\varepsilon} f$ exist in the sense of distributions will be discussed below.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $f\in L^1(\mathbb{R}^d)$. Then f is in the Hardy space H_L^1 if and only if $R_j f\in L^1(\mathbb{R}^d)$ for every $j=1,\dots,d$. Moreover, there exists $C>0$ such that*

$$\frac{1}{C}\|f\|_{H_L^1}\le\|f\|_{L^1(\mathbb{R}^d)}+\sum_{j=1}^{d}\|R_j f\|_{L^1(\mathbb{R}^d)}\le C\|f\|_{H_L^1}. \tag{1.6}$$

The Hardy spaces H_L^1 associated with the Schrödinger operators L with compactly supported potentials were studied in [6]. It was proved there that the elements of the space H_L^1 admit special atomic decompositions. Furthermore, the space H_L^1 is isomorphic to the classical Hardy space $H^1(\mathbb{R}^d)$. To be more precise, let

$$\Gamma(x,y)=\int_0^\infty K_t(x,y)\,dt \quad\text{and}\quad \Gamma_0(x,y)=-\int_0^\infty P_t(x,y)\,dt,$$

and set

$$L^{-1}f(x)=\int_{\mathbb{R}^d}\Gamma(x,y)f(y)\,dy \quad\text{and}\quad \Delta^{-1}f(x)=\int_{\mathbb{R}^d}\Gamma_0(x,y)f(y)\,dy.$$

The operators $I-V\Delta^{-1}$ and $I-VL^{-1}$ are bounded and invertible on $L^1(\mathbb{R}^d)$, and

$$I=(I-V\Delta^{-1})(I-VL^{-1})=(I-VL^{-1})(I-V\Delta^{-1}).$$

Moreover,

$$(I-VL^{-1}):H_L^1\longrightarrow H^1(\mathbb{R}^d)$$

is an isomorphism (whose inverse is $I-V\Delta^{-1}$) and

$$\|(I-VL^{-1})f\|_{H^1(\mathbb{R}^d)}\simeq\|f\|_{H_L^1} \tag{1.7}$$

for $f\in H^1_L$ (see [6, Corollary 3.17]). Therefore, in order to prove Theorem 1.1, it suffices to show that $\mathcal{R}_j(I-VL^{-1})-R_j$, $j=1,2,...,d$, are bounded operators on $L^1(\mathbb{R}^d)$.

The proof of the relevant atomic decompositions presented in [6] was based on the following identity

$$(1.8)\quad P_t(I-VL^{-1})=K_t-\int_0^t (P_t-P_{t-s})VK_s\,ds-\int_t^\infty P_tVK_s\,ds=K_t-W_t-Q_t,$$

which comes from the perturbation formula

$$P_t=K_t+\int_0^t P_{t-s}VK_s\,ds.$$

Formula (1.8) will be used also here in the analysis of the integral (1.5) for large t, while for small t we shall use its slightly different version, namely

$$(1.9)\qquad P_t(I-VL^{-1})=K_t+\int_0^t P_{t-s}VK_s\,ds-P_tVL^{-1}=K_t+\widetilde{W}_t-\widetilde{Q}_t.$$

Boundedness of Riesz transforms for Schrödinger operators on L^p-spaces attracted attention of many authors. We refer the reader to [1], [4], and [8] for background and references. Under very relaxed assumptions on $\mathcal{V}\geq 0$, the weak type (1,1) inequality

$$(1.10)\qquad \|\nabla f\|_{1,\infty}+\|\mathcal{V}^{1/2}f\|_{1,\infty}\leq C\|(-\Delta+\mathcal{V})^{1/2}f\|_{L^1(\mathbb{R}^d)},\quad f\in C_c^\infty(\mathbb{R}^d),$$

is an unpublished result of Ouhabaz (see also [8] for a proof based on finite speed propagation of the wave equation).

In the case of Schrödinger operators with potentials $\mathcal{V}\geq 0$, $\mathcal{V}\not\equiv 0$, satisfying the reverse Hölder inequality with the exponent $d/2$ (which clearly implies that $\operatorname{supp}\mathcal{V}=\mathbb{R}^d$), Riesz transform characterizations of the relevant Hardy spaces $H^1_{-\Delta+\mathcal{V}}$ were obtained in [5].

2. Auxiliary estimates

In this section we will use notation $f_t(x)=t^{-d/2}f\big(x/\sqrt{t}\big)$.

For $f\in L^1(\mathbb{R}^d)$ and $0<\varepsilon<1$ we define *truncated Riesz transforms* by setting

$$R_j^\varepsilon f=\int_\varepsilon^{1/\varepsilon}\frac{\partial}{\partial x_j}K_tf\,\frac{dt}{\sqrt{t}}\quad\text{and}\quad \mathcal{R}_j^\varepsilon f=\int_\varepsilon^{1/\varepsilon}\frac{\partial}{\partial x_j}P_tf\,\frac{dt}{\sqrt{t}}.$$

Let

$$G(x,y)=\int_0^\infty K_t(x,y)\frac{dt}{\sqrt{t}} \quad \text{and} \quad G_0(x,y)=\int_0^\infty P_t(x-y)\frac{dt}{\sqrt{t}}.$$

Then

$$G(x,y)\le G_0(x,y)=c|x-y|^{-d+1}$$

and, consequently, for φ from the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ we have

$$\lim_{\varepsilon\to 0}\langle R_j^\varepsilon f,\varphi\rangle=-\int_{\mathbb{R}^{2d}} G(x,y)f(y)\frac{\partial}{\partial x_j}\varphi(x)\,dy\,dx.$$

Hence $R_j f$ is a well-defined distribution and

$$|\langle R_j f,\varphi\rangle|\le C\|f\|_{L^1(\mathbb{R}^d)}\left(\left\|\frac{\partial}{\partial x_j}\varphi\right\|_{L^1(\mathbb{R}^d)}+\left\|\frac{\partial}{\partial x_j}\varphi\right\|_{L^\infty(\mathbb{R}^d)}\right).$$

Using (1.8) and (1.9) we write

$$R_j^\varepsilon f=\mathcal{R}_j^\varepsilon(I-VL^{-1})f-\widetilde{\mathcal{W}}_j^\varepsilon f+\widetilde{\mathcal{Q}}_j^\varepsilon f+\mathcal{W}_j^\varepsilon f+\mathcal{Q}_j^\varepsilon f, \tag{2.1}$$

where $\mathcal{Q}_j^\varepsilon$, $\widetilde{\mathcal{Q}}_j^\varepsilon$, $\mathcal{W}_j^\varepsilon$, and $\widetilde{\mathcal{W}}_j^\varepsilon$ are operators with the integral kernels

$$\mathcal{Q}_j^\varepsilon(x,y)=\int_1^{1/\varepsilon}\frac{\partial}{\partial x_j}Q_t(x,y)\,\frac{dt}{\sqrt{t}} \quad \text{and} \quad \widetilde{\mathcal{Q}}_j^\varepsilon(x,y)=\int_\varepsilon^1\frac{\partial}{\partial x_j}\widetilde{Q}_t(x,y)\,\frac{dt}{\sqrt{t}},$$

$$\mathcal{W}_j^\varepsilon(x,y)=\int_1^{1/\varepsilon}\frac{\partial}{\partial x_j}W_t(x,y)\,\frac{dt}{\sqrt{t}} \quad \text{and} \quad \widetilde{\mathcal{W}}_j^\varepsilon(x,y)=\int_\varepsilon^1\frac{\partial}{\partial x_j}\widetilde{W}_t(x,y)\,\frac{dt}{\sqrt{t}},$$

respectively. We shall prove that $\mathcal{Q}_j^\varepsilon$, $\mathcal{W}_j^\varepsilon$, and $\widetilde{\mathcal{W}}_j^\varepsilon$ converge in the norm-operator topology on $L^1(\mathbb{R}^d)$, while $\widetilde{\mathcal{Q}}_j^\varepsilon$ converges strongly on $L^1(\mathbb{R}^d)$ as ε tends to 0.

Lemma 2.1. *The operators $\mathcal{Q}_j^\varepsilon$ converge in the norm-operator topology on $L^1(\mathbb{R}^d)$ as $\varepsilon\to 0$.*

Proof. There exists $\phi\in\mathcal{S}(\mathbb{R}^d)$, $\phi\ge 0$, such that

$$\left|\frac{\partial}{\partial x_j}P_t(x-z)\right|\le t^{-1/2}\phi_t(x-z). \tag{2.2}$$

On the other hand, by (1.3), $K_s(z,y)\leq Cs^{-d/2}$. Hence for $0<\varepsilon_2<\varepsilon_1<1$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{Q}_j^{\varepsilon_1}(x,y)-\mathcal{Q}_j^{\varepsilon_2}(x,y)|\,dx \\ &\leq C\int_{\mathbb{R}^d}\int_{1/\varepsilon_1}^{1/\varepsilon_2}\int_t^\infty\int_{\mathbb{R}^d} t^{-1/2}\phi_t(x-z)V(z)s^{-d/2}\,dz\,ds\frac{dt}{\sqrt{t}}\,dx\\ &\leq C\int_{1/\varepsilon_1}^{1/\varepsilon_2} t^{-d/2}\,dt\|V\|_{L^1(\mathbb{R}^d)},\end{aligned}\tag{2.3}$$

which tends to zero uniformly with respect to y as $\varepsilon_1,\varepsilon_2\to 0$. □

Lemma 2.2. *The operators $\mathcal{W}_j^\varepsilon$ converge in the norm-operator topology on $L^1(\mathbb{R}^d)$ as $\varepsilon\to 0$.*

Proof. The proof borrows ideas from [6]. Let $0<\varepsilon_2<\varepsilon_1<\frac{1}{2}$. Then

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{W}_j^{\varepsilon_1}(x,y)-\mathcal{W}_j^{\varepsilon_2}(x,y)|\,dx \\ &\leq \int_{\mathbb{R}^d}\int_{1/\varepsilon_1}^{1/\varepsilon_2}\int_0^{t^{8/9}}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}(P_t(x-z)-P_{t-s}(x-z))\right|V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx\\ &\quad+\int_{\mathbb{R}^d}\int_{1/\varepsilon_1}^{1/\varepsilon_2}\int_{t^{8/9}}^{t}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}(P_t(x-z)-P_{t-s}(x-z))\right|V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx\\ &=\mathcal{W}'(y)+\mathcal{W}''(y).\end{aligned}$$

Observe that there exists $\phi\in\mathcal{S}(\mathbb{R})$, $\phi\geq 0$, such that for $0<s<t^{8/9}$,

$$\left|\frac{\partial}{\partial x_j}(P_t(x-z)-P_{t-s}(x-z))\right|\leq st^{-3/2}\phi_t(x-z).$$

Since $sK_s(z,y)\leq Cs^{(2-d)/2}\exp(-c|z-y|^2/s)\leq C|z-y|^{2-d}$, we have that

$$\begin{aligned}\mathcal{W}'(y)&\leq\int_{\mathbb{R}^d}\int_{1/\varepsilon_1}^{1/\varepsilon_2}\int_0^{t^{8/9}}\int_{\mathbb{R}^d} st^{-2}\phi_t(x-z)V(z)K_s(z,y)\,dz\,ds\,dt\,dx\\ &\leq\int_{1/\varepsilon_1}^{1/\varepsilon_2} t^{-10/9}\,dt\int_{\mathbb{R}^d}V(z)|z-y|^{2-d}\,dz\leq C\varepsilon_1^{1/9}\end{aligned}\tag{2.4}$$

uniformly in y. The last inequality is a simple consequence of the Hölder inequality and the assumption $p>d/2$.

For $t^{8/9}<s<t$ we have $K_s(z,y)\leq C\,s^{-d/2}\leq C\,t^{-4d/9}$. Using (2.2) we get that

$$\begin{aligned}\mathcal{W}''(y)&\leq C\int_{\mathbb{R}^d}\int_{1/\varepsilon_1}^{1/\varepsilon_2}\int_{t^{8/9}}^{t}\int_{\mathbb{R}^d}(t^{-1/2}\phi_t(x-z)+(t-s)^{-1/2}\phi_{t-s}(x-z))\\&\qquad\times V(z)t^{-4d/9}\,dz\,ds\frac{dt}{\sqrt{t}}\,dx\\&\leq C\|V\|_{L^1(\mathbb{R}^d)}\int_{1/\varepsilon_1}^{1/\varepsilon_2}t^{-4d/9}\,dt\\&\qquad+C\|V\|_{L^1(\mathbb{R}^d)}\int_{1/\varepsilon_1}^{1/\varepsilon_2}t^{-4d/9-1/2}\int_0^t(t-s)^{-1/2}\,ds\,dt\\&\leq C\varepsilon_1^{4d/9-1}\end{aligned}\tag{2.5}$$

uniformly in y. Now the lemma follows from (2.4) and (2.5). □

Lemma 2.3. *There exists a limit of the operators* $\widetilde{\mathcal{W}}_j^{\varepsilon}$ *in the norm-operator topology on* $L^1(\mathbb{R}^d)$ *as* $\varepsilon\to 0$.

Proof. Let $0<\varepsilon_1<\varepsilon_2<1$. Applying (2.2) we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d}&|\widetilde{\mathcal{W}}_j^{\varepsilon_2}(x,y)-\widetilde{\mathcal{W}}_j^{\varepsilon_2}(x,y)|\,dx\\&\leq\int_{\mathbb{R}^d}\int_{\varepsilon_1}^{\varepsilon_2}\int_0^t\int_{\mathbb{R}^d}(t-s)^{-1/2}\phi_{t-s}(x-z)V(z)K_s(z,y)\,dz\,ds\frac{dt}{\sqrt{t}}\,dx\\&=\int_{\mathbb{R}^d}\int_{\varepsilon_1}^{\varepsilon_2}\int_0^{t/2}\int_{\mathbb{R}^d}\ldots+\int_{\mathbb{R}^d}\int_{\varepsilon_1}^{\varepsilon_2}\int_{t/2}^{t}\int_{\mathbb{R}^d}\ldots\\&=\widetilde{\mathcal{W}}'(y)+\widetilde{\mathcal{W}}''(y).\end{aligned}\tag{2.6}$$

If $0<s<t/2$, then, of course, $(t-s)^{-1/2}\leq Ct^{-1/2}$. Note that

$$\int_0^t K_s(z,y)\,ds\leq C|z-y|^{2-d}\exp\left(-\frac{|z-y|^2}{8t}\right)\leq t\psi_t(z-y)$$

for some $\psi\in L^{p'}(\mathbb{R}^d)$, $\psi\geq 0$ (p' denotes the Hölder conjugate exponent to p). Hence,

$$\begin{aligned}\widetilde{\mathcal{W}}'(y)&\leq C\int_{\varepsilon_1}^{\varepsilon_2}\int_0^{t/2}\int_{\mathbb{R}^d}t^{-1}V(z)K_s(z,y)\,dz\,ds\,dt\leq C\int_{\varepsilon_1}^{\varepsilon_2}\int_{\mathbb{R}^d}V(z)\psi_t(z-y)\,dz\,dt\\&\leq C\int_{\varepsilon_1}^{\varepsilon_2}\|V\|_p\|\psi_t\|_{p'}\,dt\leq C\int_{\varepsilon_1}^{\varepsilon_2}t^{-d/2p}\|\psi_1\|_{p'}\,dt\leq C\varepsilon_2^{1-d/2p}\end{aligned}\tag{2.7}$$

uniformly in y.

If $t/2 \le s \le t$, then there exists $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\varphi \ge 0$, such that $K_s(z,y) \le \varphi_t(z-y)$. Therefore

$$\begin{aligned}\widetilde{W}''(y) &\le \int_{\varepsilon_1}^{\varepsilon_2}\int_{t/2}^{t}\int_{\mathbb{R}^d}(t-s)^{-1/2}V(z)K_s(z-y)\,dz\,ds\frac{dt}{\sqrt{t}}\\ &\le C\int_{\varepsilon_1}^{\varepsilon_2}\int_{\mathbb{R}^d}\int_0^{t/2}(st)^{-1/2}V(z)\varphi_t(z-y)\,ds\,dz\,dt\\ &\le C\int_{\varepsilon_1}^{\varepsilon_2}\|V\|_p\|\varphi_t\|_{p'}\,dt\\ &\le C\varepsilon_2^{1-d/2p} \qquad (2.8)\end{aligned}$$

uniformly in y. Now the lemma is a consequence (2.7)–(2.8). □

Lemma 2.4. *Assume that $f \in L^1(\mathbb{R}^d)$. Then the limit $F = \lim_{\varepsilon\to 0}\widetilde{\mathcal{Q}}_j^{\varepsilon} f$ exists in the $L^1(\mathbb{R}^d)$-norm. Moreover, $\|F\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^1(\mathbb{R}^d)}$ with C independent of f.*

Proof. Of course, for any fixed $y \in \mathbb{R}^d$, the function $z \mapsto U(z,y) = V(z)\Gamma(z,y)$ is supported in the unit ball and $\|U(z,y)\|_{L^r(dz)} \le C_r$ for fixed $r \in [1, dp/(dp+d-2p))$ with C_r independent of y. The last statement follows from (1.3) and the Hölder inequality. Let

$$\begin{aligned}H_j^{\varepsilon}(x,z) &= \int_{\varepsilon}^{1}\frac{\partial}{\partial x_j}P_t(x-z)\frac{dt}{\sqrt{t}},\\ H_j^{\varepsilon}g(x) &= \int_{\mathbb{R}^d}H_j^{\varepsilon}(x,z)g(z)\,dz,\\ H_j^{*}g(x) &= \sup_{0<\varepsilon<1}|H_j^{\varepsilon}g(x)|.\end{aligned}$$

It follows from the theory of singular integral convolution operators (see, e.g., [3, Chapter 4]) that for $1<r<\infty$ there exists C_r such that

$$\|H_j^{*}g\|_{L^r(\mathbb{R}^d)} \le C_r\|g\|_{L^r(\mathbb{R}^d)} \quad \text{for } g \in L^r(\mathbb{R}^d) \qquad (2.9)$$

and $\lim_{\varepsilon\to 0}H_j^{\varepsilon}g(x) = H_j g(x)$ a.e. and in the $L^r(\mathbb{R}^d)$-norm.

Note that $\widetilde{\mathcal{Q}}_j^{\varepsilon}(x,y) = H_j^{\varepsilon}U(\,\cdot\,,y)(x)$. Hence, there exists a function $\widetilde{\mathcal{Q}}_j(x,y)$ such that $\lim_{\varepsilon\to 0}\widetilde{\mathcal{Q}}_j^{\varepsilon}(x,y) = \widetilde{\mathcal{Q}}_j(x,y)$ a.e. and

$$\sup_y \int_{\mathbb{R}^d}\sup_{0<\varepsilon<1}|\widetilde{\mathcal{Q}}_j^{\varepsilon}(x,y)|^r\,dx \le C_r' \quad \text{for } 1<r<\frac{dp}{dp+d-2p}. \qquad (2.10)$$

Fix $N>d$. Since $|H_j^\varepsilon(x,z)|\le C_N|x-z|^{-N}$ for $|x-z|>1$,

$$(2.11)\quad |\widetilde{\mathcal{Q}}_j^\varepsilon(x,y)|=\left|\int_{|z|\le 1} H_j^\varepsilon(x,z)U(z,y)\,dz\right|\le C_N|x|^{-N} \quad \text{for } |x|>2 \text{ and } y\in\mathbb{R}^d.$$

The Hölder inequality combined with (2.10) and (2.11) implies that

$$(2.12)\qquad \sup_y \int_{\mathbb{R}^d} \sup_{0<\varepsilon<1} |\widetilde{\mathcal{Q}}_j^\varepsilon(x,y)|\,dx\le C$$

and

$$(2.13)\qquad \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |\widetilde{\mathcal{Q}}_j^\varepsilon(x,y)-\widetilde{\mathcal{Q}}_j(x,y)|\,dx=0 \quad \text{for every } y.$$

Now the lemma can be easily concluded from (2.12), (2.13), and Lebesgue's dominated convergence theorem. □

3. Proof of the main theorem

Recall that $I-VL^{-1}$ is an isomorphism in $L^1(\mathbb{R}^d)$. Consider $f\in L^1(\mathbb{R}^d)$. Using (2.1) and Lemmas 2.1–2.4 we get that R_jf belongs to $L^1(\mathbb{R}^d)$ if and only if

$$\mathcal{R}_j(I-VL^{-1})f\in L^1(\mathbb{R}^d)$$

and

$$\|f\|_{L^1(\mathbb{R}^d)}+\sum_{j=1}^d \|R_jf\|_{L^1(\mathbb{R}^d)}\sim \|(I-VL^{-1})f\|_{L^1(\mathbb{R}^d)}+\sum_{j=1}^d \|\mathcal{R}_j(I-VL^{-1})f\|_{L^1(\mathbb{R}^d)}.$$

Applying the characterization of the classical Hardy space $H^1(\mathbb{R}^d)$ by means of the Riesz transforms $\mathcal{R}_j$ (see (1.2)) and (1.7) we obtain Theorem 1.1.

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