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Time regularity of the solutions to second order hyperbolic equations

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Abstract. We consider the Cauchy problem for a second order weakly hyperbolic equation, with coefficients depending only on the time variable. We prove that if the coefficients of the equation belong to the Gevrey class γ^{s_0} and the Cauchy data belong to γ^{s_1}, then the Cauchy problem has a solution in $\gamma^{s_0}([0,T^*];\gamma^{s_1}(\mathbb{R}))$ for some $T^*>0$, provided $1\le s_1\le 2-1/s_0$. If the equation is strictly hyperbolic, we may replace the previous condition by $1\le s_1\le s_0$.

1. Introduction

In this paper we are concerned with the Cauchy problem for the second order equation

$$\partial_t^2 u - a(t)\partial_x^2 u = b(t)\partial_x u + c(t)\partial_t u, \tag{1}$$

with initial data

$$u(0,x) = u_0(x) \quad \text{and} \quad \partial_t u(0,x) = u_1(x). \tag{2}$$

We say that the Cauchy problem (1)–(2) is *well-posed* in a function space X if for any initial data $u_0(x), u_1(x)$ in X there exists a unique solution $u\in\mathcal{C}^2([0,T];X)$ to (1). Here we deal with the solvability in the Gevrey spaces: $X=\gamma^s=\gamma^s(\mathbb{R})$ with $s\ge 1$. We recall that a function $f\colon \mathbb{R}\to\mathbb{R}$ is a *Gevrey function of order* s, if for any compact set $K\subset\mathbb{R}$ there exists two positive constants C_K and R_K such that

$$\sup_{x\in K} |\partial_x^j f(x)| \le C_K R_K^j j!^s \quad \text{for any } j\in\mathbb{N}.$$

In particular, when $s=1$, γ^1 is the space of analytic functions. When $s>1$, we denote by γ_0^s the subspace of γ^s formed by the functions with compact support.

By the Cauchy–Kovalevski theorem we know that the Cauchy problem (1)–(2) with analytic coefficients and data has a unique solution which is analytic in both t and x. This theorem can be applied to any type of equation, even if the coefficients depend also on x. Here we remark that the coefficients and also the solution are analytic in t. If the equation is of *hyperbolic* type, one can expect to relax some conditions. It is well known that the regularity of the coefficients influences the class in which the Cauchy problem (1)–(2) is well-posed.

Indeed, from [CJS] for the *weakly hyperbolic* case, that is $a(t) \geq 0$, if $a \in \mathcal{C}^{\varkappa}([0,T])$ and $b, c \in \mathcal{C}^0([0,T])$ then the Cauchy problem (1)–(2) is well-posed in γ^s for any $1 \leq s < 1+\varkappa/2$, provided that $\varkappa \leq 2$. In order to have the well-posedness in γ^s with $s>2$ a supplementary Levi condition is to be assumed. For instance, assuming

$$\int_0^T \frac{|b(s)|}{\sqrt{a(s)+\varepsilon}}\, ds \leq C\varepsilon^{-1/\eta} \quad \text{for any } \varepsilon \in]0,1], \tag{3}$$

for some $C, \eta > 0$, we get the well-posedness of (1)–(2) in γ^s for $1 \leq s < 1+\min(\varkappa, \eta)/2$ (see [CJS]). While, from [CDS] for the *strictly hyperbolic* case, that is $a(t) \geq \delta$ for some $\delta > 0$, if $a \in \mathcal{C}^{\varkappa}([0,T])$ and $b, c \in \mathcal{C}^0([0,T])$ then the Cauchy problem (1)–(2) is well-posed in γ^s for $1 \leq s < 1/(\varkappa - 1)$, provided that $\varkappa \leq 1$.

Furthermore, it is possible for the coefficient $a(t)$ to consider also Gevrey classes between the analytic class and $\mathcal{C}^{\varkappa}$. [CN] showed that if $a(t) \geq 0$ and $a \in \gamma^s([0,T])$ and $b=c=0$ then the Cauchy problem (1)–(2) is well-posed in an intermediate class between γ^s and $\mathcal{C}^\infty$.

For the regularity in t of the solution, the results of the well-posedness are involved with $\mathcal{C}^2$. On the other hand, the Cauchy–Kovalevski theorem gives analytic regularity in t of the solution. In this paper, our interest is to get the Gevrey regularity in t of the solution between the analytic class and $\mathcal{C}^2$. For this purpose, by [CJS] the coefficients would belong to Gevrey classes between the analytic class and $\mathcal{C}^{\varkappa}$. Thus, we assume that $a, b, c \in \gamma^{s_0}$, more precisely, that there exist $C_0 > 0$ and $R_0 > 0$ such that

$$\sup_{t \in [0,T]} [|\partial_t^j a(t)| + |\partial_t^j b(t)| + |\partial_t^j c(t)|] \leq C_0 R_0^j j!^{s_0} \quad \text{for any } j \in \mathbb{N}. \tag{4}$$

Then we can obtain the following theorem.

Theorem 1.1. *Let $s_0 \geq 1$. Assume that $a, b, c \in \gamma^{s_0}$ as in (4) and that $a(t) \geq 0$. If the initial data u_0, u_1 belong to γ^{s_1}, then the Cauchy problem (1)–(2) has a solution $u \in \gamma^{s_0}([0,T^*]; \gamma^{s_1}(\mathbb{R}))$, for some $T^* \in]0,T[$, provided*

$$1 \leq s_1 \leq 2 - \frac{1}{s_0}. \tag{5}$$

If (1) is *strictly hyperbolic*, we can improve the result of Theorem 1.1 as follows.

Theorem 1.2. *Let $s_0 \geq 1$. Assume that $a, b, c \in \gamma^{s_0}$ as in (4) and that $a(t) \geq \delta$ for some $\delta > 0$. If the initial data u_0, u_1 belong to γ^{s_1}, then the Cauchy problem (1)–(2) has a solution $u \in \gamma^{s_0}([0, T^*]; \gamma^{s_1}(\mathbb{R}))$, for some $T^* \in]0, T[$, provided*

$$1 \leq s_1 \leq s_0. \tag{6}$$

In both theorems we allow local solutions for some sufficiently small $T^* \in]0, T[$. Then, we see that our theorems with $s_0 = s_1 = 1$ just imply the local solvability in the analytic class of the Cauchy–Kovalevski theorem. If one expects global solutions for arbitrarily large $T > 0$, one may replace the assumptions (5) and (6) by

$$1 \leq s_1 < 2 - \frac{1}{s_0} \quad \text{and} \quad 1 \leq s_1 < s_0,$$

respectively. In particular, if the equation is restricted to be *strictly hyperbolic*, Theorem 1.2 states that the Cauchy–Kovalevski theorem can be relaxed to the version in the Gevrey classes. That is, we also get the following corollary.

Corollary 1.3. *Let $s \geq 1$. Assume that $a(t) \geq \delta$ for some $\delta > 0$. If the coefficients a, b, c and the initial data u_0, u_1 belong to γ^s, then the Cauchy problem (1)–(2) has a solution $u \in \gamma^s([0, T^*] \times \mathbb{R})$, for some $T^* \in]0, T[$.*

[Tah] also proved the same conclusion under the assumption of the existence of a $\mathcal{C}^\infty([0, T]; \gamma^s(\mathbb{R}))$ solution instead of the strictly hyperbolic assumption $a(t) \geq \delta$. The existence of a $\mathcal{C}^\infty([0, T]; \gamma^s(\mathbb{R}))$ solution to (1)–(2) could be deduced from [CDS] and [CJS]. In this paper, our approach is to show energy inequalities. However, since we have to show more regularity in t of the solution we are forced to modify standard energy methods. To this end we shall define a new type of *infinite-order energy* by which the Gevrey regularity in t can be derived. Usually infinite-order energies involve all the derivatives with respect to x and are employed to get Gevrey well-posedness results (see [D], [DS], [Ki], [Tag] etc.). Since we are interested in the Gevrey regularity in t of the solution, we introduce an infinite-order energy involving all the derivatives with respect to t.

This paper is organized as follows. In Section 2 we define the *infinite-order energy* and we derive an a priori estimate in the weakly hyperbolic case. In Section 3 we consider the strictly hyperbolic case: at first we derive an a priori estimate for the wave operator, then we show that any strictly hyperbolic operator can be transformed into a wave operator. Due to the dependence on all the time derivatives, we cannot immediately estimate the infinite-order energy at initial time using the initial conditions. However, using a majorizing argument of Gevrey type, we prove such an estimate in Section 4. In fact, all the proofs can be carried out for any dimensional space. Thus, our theorems and corollary hold also for the hyperbolic equation

$$\partial_t^2 u - \sum_{j,k=1}^{n} a_{jk}(t)\partial_{x_j}\partial_{x_k} u = \sum_{j=1}^{n} b_j(t)\partial_{x_j} u + c(t)\partial_t u \quad \text{in } [0,T]\times\mathbb{R}^n.$$

Notation. In the computations of the proofs, given two functions h and g, we often use the notation

$$h(t) \lesssim g(t)$$

which means that

$$h(t) \le C g(t)$$

for some constant C depending only on C_0 in (4).

2. Energy inequality of Theorem 1.1

We may only consider the case $s_1 > 1$ due to the Cauchy–Kovalevski theorem. We recall that (1) enjoys the finite speed of propagation property (see [CDS], [CJS]): if the initial data vanish in some interval then the solution vanishes in a triangle, whose slope is determined by the L^∞-norm of a. This allows us to assume, with no loss of generality, that $u_0, u_1 \in \gamma_0^{s_1}$ and $u \in \mathcal{C}^2([0,T^*];\gamma_0^{s_1}(\mathbb{R}))$ with $s_1 > 1$. Thus, by Fourier transform with respect to x, (1) is transformed into

$$\partial_t^2 v + a(t)\xi^2 v = i\xi b(t) v + c(t)\partial_t v, \tag{7}$$

where $v(t,\xi) := \mathcal{F}_{x\to\xi}(u(t,x))$.

In order to derive the Gevrey regularity of the solution with respect to time, we shall introduce a new sort of energy. Let us put

$$\begin{aligned} d_0^2 &:= |\partial_t v|^2 + a_\varepsilon \xi^2 |v|^2, \\ d_j^2 &:= |\partial_t^{j+1} v|^2 + a_\varepsilon \xi^2 |\partial_t^j v|^2 + j^{2(2-s_0)}\langle\xi\rangle^{2(2-1/s_1)}|\partial_t^{j-1} v|^2 && \text{for } j \ge 1, \\ e_j^2 &:= a_\varepsilon \xi^2 |\partial_t^j v|^2 && \text{for } j \ge 0, \end{aligned}$$

where

$$a_\varepsilon(t) := a(t) + \varepsilon, \quad \varepsilon := \langle\xi\rangle^{2/s_1 - 2} \quad \text{and} \quad \langle\xi\rangle := (1+\xi^2)^{1/2}. \tag{8}$$

From the definitions, we see that $|\partial_t^{j+1}v|$, $\sqrt{a_\varepsilon}|\xi|\,|\partial_t^j v|$, $j^{2-s_0}\langle\xi\rangle^{2-1/s_1}|\partial_t^{j-1}v|$ and $\langle\xi\rangle^{1/s_1}|\partial_t^j v|$ are dominated by d_j. Next, we define a finite sequence of energies $\{E_l\}_{l=1}^{N+1}$ by

$$E_l := \sum_{j=0}^{\infty}(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{(l-1)(1-1/s_1)}d_j\frac{r^j}{j!^{s_0}}, \quad \text{if } l=1,...,N,$$

and

$$E_{N+1} := \sum_{j=0}^{\infty}(j+1)^{N(1-s_0)}\langle\xi\rangle^{N(1-1/s_1)}e_j\frac{r^j}{j!^{s_0}},$$

where r is a sufficiently small positive constant such that $rR_0<\frac{1}{2}$ and

$$N := \left[\frac{s_0}{s_0-1}\right]+1. \tag{9}$$

Here $[a]$ denotes the largest integer which does not exceed a.

Remark 2.1. If we multiply any of the E_l by the weight $\exp(\rho\langle\xi\rangle^{1/s_1})$, by taking the supremum in ξ, we get a norm in $\gamma^{s_0}([0,T^*];\gamma_0^{s_1}(\mathbb{R}))$, for any $T^*\in]0,T]$. However, in our computations we may omit this weight and the supremum in ξ.

In this section we prove that we have the a priori estimates

$$E_l'(t) \lesssim \langle\xi\rangle^{1/s_1}(E_l(t)+E_{l+1}(t)), \qquad l=1,...,N,$$

$$E_{N+1}'(t) \lesssim \langle\xi\rangle^{1/s_1}(E_{N+1}(t)+E_1(t)),$$

thus, defining the *total energy*

$$\mathcal{E}(t) := \sum_{l=1}^{N+1}E_l(t),$$

we get

$$\mathcal{E}'(t) \lesssim \langle\xi\rangle^{1/s_1}\mathcal{E}(t).$$

2.1. Estimate of E_1

We shall devote ourselves to estimate $E_1 := \sum_{j=0}^{\infty} d_j r^j / j!^{s_0}$. Differentiating $d_0^2(t)$, and taking (7) into account, by standard calculations we get

$$\frac{d}{dt} d_0^2(t) \le \left(2\langle\xi\rangle^{1/s_1} + \frac{|a_\varepsilon'|}{a_\varepsilon} + 2\frac{|b|}{\sqrt{a_\varepsilon}} \right) d_0^2(t) + 2|c|\,|\partial_t v|^2,$$

which gives

$$\frac{d}{dt} d_0(t) \le \left(\langle\xi\rangle^{1/s_1} + \frac{1}{2}\frac{|a_\varepsilon'|}{a_\varepsilon} + \frac{|b|}{\sqrt{a_\varepsilon}} \right) d_0(t) + 2|c|\,|\partial_t v|.$$

As for $j \ge 1$, differentiating $d_j^2(t)$ we have

$$\begin{aligned} \frac{d}{dt} d_j^2(t) &= 2\,\mathrm{Re}(\partial_t^{j+2} v, \partial_t^{j+1} v) + a_\varepsilon' \xi^2 |\partial_t^j v|^2 \\ &\quad + 2a_\varepsilon \xi^2\,\mathrm{Re}(\partial_t^j v, \partial_t^{j+1} v) + 2j^{2(2-s_0)} \langle\xi\rangle^{2(2-1/s_1)}\,\mathrm{Re}(\partial_t^{j-1} v, \partial_t^j v). \end{aligned}$$

Using (7) and Leibniz's formula we get

$$\begin{aligned} \frac{d}{dt} d_j^2(t) &= -2\xi^2 \sum_{k=1}^{j} \binom{j}{k} \mathrm{Re}(\partial_t^k a \partial_t^{j-k} v, \partial_t^{j+1} v) + 2\xi \sum_{k=0}^{j} \binom{j}{k} \mathrm{Im}(\partial_t^k b \partial_t^{j-k} v, \partial_t^{j+1} v) \\ &\quad + 2 \sum_{k=0}^{j} \binom{j}{k} \mathrm{Re}(\partial_t^k c \partial_t^{j-k+1} v, \partial_t^{j+1} v) + a_\varepsilon' \xi^2 |\partial_t^j v|^2 \\ &\quad + 2\langle\xi\rangle^{2/s_1-2} \xi^2\,\mathrm{Re}(\partial_t^j v, \partial_t^{j+1} v) + 2j^{2(2-s_0)} \langle\xi\rangle^{2(2-1/s_1)}\,\mathrm{Re}(\partial_t^{j-1} v, \partial_t^j v) \\ &\le 2\xi^2 \sum_{k=1}^{j} \binom{j}{k} |\partial_t^k a|\,|\partial_t^{j-k} v|\,|\partial_t^{j+1} v| + 2|\xi| \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k b|\,|\partial_t^{j-k} v|\,|\partial_t^{j+1} v| \\ &\quad + 2 \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k c|\,|\partial_t^{j-k+1} v|\,|\partial_t^{j+1} v| + \xi^2 |a_\varepsilon'|\,|\partial_t^j v|^2 \\ &\quad + 2\langle\xi\rangle^{2/s_1} |\partial_t^j v|\,|\partial_t^{j+1} v| + 2j^{2(2-s_0)} \langle\xi\rangle^{2(2-1/s_1)} |\partial_t^{j-1} v|\,|\partial_t^j v|. \end{aligned}$$

This gives

$$\begin{aligned} \frac{d}{dt} d_j(t) &\le \left(\langle\xi\rangle^{1/s_1} + \frac{1}{2}\frac{|a_\varepsilon'|}{a_\varepsilon} + \frac{|b|}{\sqrt{a_\varepsilon}} \right) d_j(t) + \xi^2 \sum_{k=1}^{j} \binom{j}{k} |\partial_t^k a|\,|\partial_t^{j-k} v| \\ &\quad + |\xi| \sum_{k=1}^{j} \binom{j}{k} |\partial_t^k b|\,|\partial_t^{j-k} v| + \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k c|\,|\partial_t^{j-k+1} v| \\ &\quad + j^{2-s_0} \langle\xi\rangle^{2-1/s_1} |\partial_t^j v|. \end{aligned} \tag{10}$$

Hence

$$E_1' \le \left(\langle\xi\rangle^{1/s_1} + \frac{1}{2}\frac{|a_\varepsilon'|}{a_\varepsilon} + \frac{|b|}{\sqrt{a_\varepsilon}}\right)E_1(t) + \xi^2 \sum_{j=1}^{\infty} j|a'|\,|\partial_t^{j-1}v|\frac{r^j}{j!^{s_0}}$$

$$+\xi^2 \sum_{j=2}^{\infty}\sum_{k=2}^{j}\binom{j}{k}|\partial_t^k a|\,|\partial_t^{j-k}v|\frac{r^j}{j!^{s_0}} + |\xi|\sum_{j=1}^{\infty}\sum_{k=1}^{j}\binom{j}{k}|\partial_t^k b|\,|\partial_t^{j-k}v|\frac{r^j}{j!^{s_0}}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}|\partial_t^k c|\,|\partial_t^{j-k+1}v|\frac{r^j}{j!^{s_0}} + \sum_{j=0}^{\infty} j^{2-s_0}\langle\xi\rangle^{2-1/s_1}|\partial_t^j v|\frac{r^j}{j!^{s_0}}$$

$$= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.$$

Estimation of I. By Glaeser's inequality for a_ε, we see that

$$|a'| = |a_\varepsilon'| \lesssim \sqrt{a_\varepsilon}. \tag{11}$$

Hence we have

$$\frac{|a_\varepsilon'|}{a_\varepsilon} + \frac{|b|}{\sqrt{a_\varepsilon}} \lesssim \frac{1}{\sqrt{a_\varepsilon}} \le \frac{1}{\sqrt{\varepsilon}} = \langle\xi\rangle^{1-1/s_1} \le \langle\xi\rangle^{1/s_1}, \tag{12}$$

since $s_1 \le 2$. This gives

$$\mathrm{I} \lesssim \langle\xi\rangle^{1/s_1} E_1(t).$$

Estimation of II. Recalling Glaeser's inequality (11) again we have, letting $j=h+1$,

$$\mathrm{II} \lesssim \sum_{j=1}^{\infty} j^{1-s_0}\langle\xi\rangle d_{j-1}\frac{r^j}{(j-1)!^{s_0}} = \sum_{h=0}^{\infty}(h+1)^{1-s_0}\langle\xi\rangle d_h \frac{r^{h+1}}{h!^{s_0}} \le \langle\xi\rangle^{1/s_1}E_2(t).$$

Estimation of III. Using the Gevrey estimate of a and Fubini's formula for series

$$\mathrm{III} = \xi^2 \sum_{k=2}^{\infty}\sum_{j=k}^{\infty}\binom{j}{k}|\partial_t^k a|\,|\partial_t^{j-k}v|\frac{r^j}{j!^{s_0}}$$

$$\lesssim \langle\xi\rangle^2 \sum_{k=2}^{\infty}\sum_{j=k}^{\infty}\binom{j}{k}R_0^k k!^{s_0}|\partial_t^{j-k}v|\frac{r^j}{j!^{s_0}}$$

$$= \langle\xi\rangle^2 \sum_{k=2}^{\infty} (rR_0)^k \sum_{j=k}^{\infty} \binom{j}{k}^{1-s_0} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k)!^{s_0}}$$

$$\lesssim \sup_{k\geq 2} \langle\xi\rangle^2 \sum_{j=k}^{\infty} \binom{j}{k}^{1-s_0} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k)!^{s_0}},$$

since $\sum_{k=2}^{\infty}(rR_0)^k \leq 1$. Now, we have

$$\binom{j}{k} = \frac{j(j-1)...(j-k+3)(j-k+2)(j-k+1)}{k(k-1)...3\cdot 2\cdot 1} \geq \frac{1}{2}(j-k+1)^2, \quad \text{if } 2\leq k(\leq j),$$

hence, letting $j=h+k-1$,

$$\mathrm{III} \lesssim \sup_{k\geq 2} \langle\xi\rangle^2 \sum_{j=k}^{\infty} (j-k+1)^{2-s_0} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k+1)!^{s_0}}$$

$$= \langle\xi\rangle^2 \sum_{h=1}^{\infty} h^{2-s_0} |\partial_t^{h-1} v| \frac{r^{h-1}}{h!^{s_0}} \leq \sum_{h=1}^{\infty} \langle\xi\rangle^{1/s_1} d_h \frac{r^{h-1}}{h!^{s_0}} \lesssim \langle\xi\rangle^{1/s_1} E_1(t).$$

Estimation of IV. Using the Gevrey estimate of b and Fubini's formula for series

$$\mathrm{IV} \lesssim \langle\xi\rangle \sum_{j=1}^{\infty} \sum_{k=1}^{j} \binom{j}{k} R_0^k k!^{s_0} |\partial_t^{j-k} v| \frac{r^j}{j!^{s_0}}$$

$$= \langle\xi\rangle \sum_{k=1}^{\infty} (rR_0)^k \sum_{j=k}^{\infty} \binom{j}{k}^{1-s_0} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k)!^{s_0}}.$$

Now

$$\binom{j}{k} \geq j-k+1, \quad \text{if } 1\leq k(\leq j).$$

Hence, letting $j=h+k-1$,

$$\mathrm{IV} \lesssim \langle\xi\rangle \sum_{k=1}^{\infty} (rR_0)^k \sum_{j=k}^{\infty} (j-k+1)^{1-s_0} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k)!^{s_0}}$$

$$= \langle\xi\rangle \sum_{k=1}^{\infty} (rR_0)^k \sum_{h=1}^{\infty} h^{1-s_0} |\partial_t^{h-1} v| \frac{r^{h-1}}{(h-1)!^{s_0}}.$$

Using the geometric mean inequality:

$$h^{1-s_0} = h^{1-s_0} \langle\xi\rangle^{1/2-s_1/2} \langle\xi\rangle^{-1/2+s_1/2} \leq \tfrac{1}{2}(h^{2-2s_0}\langle\xi\rangle^{1-1/s_1} + \langle\xi\rangle^{1/s_1-1}),$$

we get

$$\begin{aligned}\mathrm{IV} &\lesssim \sum_{h=1}^{\infty} h^{2-s_0}\langle\xi\rangle^{2-1/s_1}|\partial_t^{h-1}v|\frac{r^{h-1}}{h!^{s_0}}+\sum_{h=1}^{\infty}\langle\xi\rangle^{1/s_1}|\partial_t^{h-1}v|\frac{r^{h-1}}{(h-1)!^{s_0}}\\ &\le \sum_{h=0}^{\infty} d_h\frac{r^h}{h!^{s_0}}+\sum_{h=1}^{\infty} d_{h-1}\frac{r^{h-1}}{(h-1)!^{s_0}}\lesssim E_1(t).\end{aligned}$$

Estimation of V. Using the Gevrey estimate of c, Fubini's formula for series and letting $j=h+k$ we obtain that

$$\begin{aligned}\mathrm{V} &\lesssim \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}R_0^k k!^{s_0}|\partial_t^{j-k+1}v|\frac{r^j}{j!^{s_0}}\\ &=\sum_{k=0}^{\infty}(rR_0)^k\sum_{j=k}^{\infty}\binom{j}{k}^{1-s_0}|\partial_t^{j-k+1}v|\frac{r^{j-k}}{(j-k)!^{s_0}}\\ &=\sum_{k=0}^{\infty}(rR_0)^k\sum_{h=0}^{\infty}\binom{h+k}{k}^{1-s_0}|\partial_t^{h+1}v|\frac{r^h}{h!^{s_0}}\\ &\lesssim\sum_{k=0}^{\infty}(rR_0)^k\sum_{h=0}^{\infty}|\partial_t^{h+1}v|\frac{r^h}{h!^{s_0}}\\ &\le\sum_{k=0}^{\infty}(rR_0)^k\sum_{h=0}^{\infty}d_h\frac{r^h}{h!^{s_0}}\\ &\le E_1(t),\end{aligned}$$

since $\binom{h+k}{k}\ge 1$ for any $k,h\in\mathbb{N}$.

Estimation of VI. We should distinguish three cases according to $s_0<2$, $s_0=2$ and $s_0>2$.

Case 1. $s_0<2$. Using Hölder's inequality

$$xy\le\frac{x^p}{p}+\frac{y^q}{q}\quad\text{if } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,$$

with $p=s_0/(2-s_0)$ and $q=s_0/(2s_0-2)$

$$\begin{aligned}j^{2-s_0}&=j^{2-s_0}\langle\xi\rangle^{-\alpha}\langle\xi\rangle^{\alpha}\\&\le\frac{2-s_0}{s_0}j^{s_0}\langle\xi\rangle^{-\alpha s_0/(2-s_0)}+\frac{2(s_0-1)}{s_0}\langle\xi\rangle^{\alpha s_0/(2s_0-2)},\end{aligned}$$

we have

$$\begin{aligned}\mathrm{VI}&\lesssim\langle\xi\rangle^{2-1/s_1-\alpha s_0/(2-s_0)}\sum_{j=1}^{\infty}|\partial_t^j v|\frac{r^j}{(j-1)!^{s_0}}+\langle\xi\rangle^{2-1/s_1+\alpha s_0/(2s_0-2)}\sum_{j=1}^{\infty}|\partial_t^j v|\frac{r^j}{j!^{s_0}}\\&\lesssim\langle\xi\rangle^{2-1/s_1-\alpha s_0/(2-s_0)}\sum_{j=1}^{\infty}d_{j-1}\frac{r^{j-1}}{(j-1)!^{s_0}}+\langle\xi\rangle^{2-2/s_1+\alpha s_0/(2s_0-2)}\sum_{j=1}^{\infty}d_j\frac{r^j}{j!^{s_0}}\\&\le(\langle\xi\rangle^{2-1/s_1-\alpha s_0/(2-s_0)}+\langle\xi\rangle^{2-2/s_1+\alpha s_0/(2s_0-2)})E_1(t).\end{aligned}$$

Choosing

$$\alpha:=\frac{2(s_0-1)(2-s_0)}{s_1s_0^2},$$

we get

$$\mathrm{VI}\lesssim\langle\xi\rangle^{\frac{2s_1s_0-3s_0+2}{s_1s_0}}E_1(t)\lesssim\langle\xi\rangle^{1/s_1}E_1(t),$$

since s_1 and s_0 satisfy (5).

Case 2. $s_0=2$. In this case we can easily estimate

$$\mathrm{VI}=\sum_{j=0}^{\infty}\langle\xi\rangle^{2-1/s_1}|\partial_t^j v|\frac{r^j}{j!^{s_0}}\le\langle\xi\rangle^{2-2/s_1}\sum_{j=0}^{\infty}d_j\frac{r^j}{j!^{s_0}}=\langle\xi\rangle^{2-2/s_1}E_1\le\langle\xi\rangle^{1/s_1}E_1,$$

since $s_1\le\frac{3}{2}$ according to (5) with $s_0=2$.

Case 3. $s_0>2$. In this case, Hölder's inequality with $p=2(s_0-1)/(s_0-2)$ and $q=2(s_0-1)/s_0$ gives us that

$$\begin{aligned}j^{2-s_0}&=j^{2-s_0}\langle\xi\rangle^{\beta}\langle\xi\rangle^{-\beta}\\&\le\frac{s_0-2}{2(s_0-1)}j^{2-2s_0}\langle\xi\rangle^{2\beta(s_0-1)/(s_0-2)}+\frac{s_0}{2(s_0-1)}\langle\xi\rangle^{-2\beta(s_0-1)/s_0}\end{aligned}$$

and we have

$$\begin{aligned}\mathrm{VI}\lesssim{}&\langle\xi\rangle^{2-1/s_1+2\beta(s_0-1)/(s_0-2)}\sum_{j=1}^{\infty}j^{2-s_0}|\partial_t^j v|\frac{r^j}{(j+1)!^{s_0}}\\&+\langle\xi\rangle^{2-1/s_1-2\beta(s_0-1)/s_0}\sum_{j=1}^{\infty}|\partial_t^j v|\frac{r^j}{j!^{s_0}}\end{aligned}$$

$$\lesssim \langle\xi\rangle^{2\beta(s_0-1)/(s_0-2)}\sum_{j=1}^{\infty} d_{j+1}\frac{r^{j+1}}{(j+1)!^{s_0}}+\langle\xi\rangle^{2-2/s_1-2\beta(s_0-1)/s_0}\sum_{j=1}^{\infty} d_j\frac{r^j}{j!^{s_0}}$$

$$\leq (\langle\xi\rangle^{2\beta(s_0-1)/(s_0-2)}+\langle\xi\rangle^{2-2/s_1-2\beta(s_0-1)/s_0})E_1(t).$$

Choosing

$$\beta:=\frac{1}{2}\left(1-\frac{1}{s_1}\right)\frac{(s_0-2)s_0}{(s_0-1)^2},$$

we get

$$\mathrm{VI}\lesssim\langle\xi\rangle^{(1-1/s_1)s_0/(s_0-1)}E_1(t)\leq\langle\xi\rangle^{1/s_1}E_1(t),$$

since s_1 and s_0 satisfy (5).

Final estimate. Finally we get

$$E_1'(t)\lesssim\langle\xi\rangle^{1/s_1}(E_1(t)+E_2(t)).$$

2.2. Estimate of E_l for $2\leq l\leq N$

From (10) we get

$$\begin{aligned}
E_l' \leq &\left(\langle\xi\rangle^{1/s_1}+\frac{1}{2}\frac{|a_\varepsilon'|}{a_\varepsilon}+\frac{|b|}{\sqrt{a_\varepsilon}}\right)E_l(t)\\
&+\sum_{j=1}^{\infty} j(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{(l-1)(1-1/s_1)}|a'|\,|\xi|\,|\partial_t^{j-1}v|\frac{r^j}{j!^{s_0}}\\
&+\sum_{j=2}^{\infty}\sum_{k=2}^{j}\binom{j}{k}(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{(l-1)(1-1/s_1)}|\partial_t^k a|\,|\partial_t^{j-k}v|\frac{r^j}{j!^{s_0}}\\
&+\sum_{j=1}^{\infty}\sum_{k=1}^{j}\binom{j}{k}(j+1)^{(l-1)(1-s_0)}\,\langle\xi\rangle^{(l-1)(1-1/s_1)}|\partial_t^k b|\,|\partial_t^{j-k}v|\frac{r^j}{j!^{s_0}}\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{(l-1)(1-1/s_1)}|\partial_t^k c|\,|\partial_t^{j-k+1}v|\frac{r^j}{j!^{s_0}}\\
&+\sum_{j=0}^{\infty} j^{2-s_0}(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{(l-1)(1-1/s_1)+2-1/s_1}|\partial_t^j v|\,\frac{r^j}{j!^{s_0}}\\
=\ &\mathrm{I}_l+\mathrm{II}_l+\mathrm{III}_l+\mathrm{IV}_l+\mathrm{V}_l+\mathrm{VI}_l.
\end{aligned}$$

The first term is estimated as before, whereas the weight

$$(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{(l-1)(1-1/s_1)}$$

in the last four terms does not play an important role. Proceeding as in the estimate of E_1 we get

$$\mathrm{I}_l+\mathrm{III}_l+\mathrm{IV}_l+\mathrm{V}_l+\mathrm{VI}_l \lesssim \langle\xi\rangle^{1/s_1} E_l(t).$$

We can estimate the second term as in the estimate of E_1: using (11), we get for $2\le l\le N-1$, letting $j=h+1$,

$$\begin{aligned}
\mathrm{II}_l &\lesssim \sum_{j=1}^{\infty} j^{1-s_0}(j+1)^{(l-1)(1-s_0)}\langle\xi\rangle^{1+(l-1)(1-1/s_1)} d_{j-1}\frac{r^j}{(j-1)!^{s_0}}\\
&= \sum_{h=0}^{\infty}(h+1)^{1-s_0}\,(h+2)^{(l-1)(1-s_0)}\langle\xi\rangle^{1+(l-1)(1-1/s_1)} d_h\frac{r^{h+1}}{h!^{s_0}}\\
&\lesssim \langle\xi\rangle^{1/s_1}\sum_{h=0}^{\infty}(h+1)^{l(1-s_0)}\langle\xi\rangle^{l(1-1/s_1)} d_h\frac{r^h}{h!^{s_0}}\\
&\le \langle\xi\rangle^{1/s_1} E_{l+1}.
\end{aligned}$$

Replacing d_{j-1} and d_h by e_{j-1} and e_h, respectively, in the above computation, we have the same inequality also for $l=N$. Thus, we get for $2\le l\le N$,

$$E_l'(t)\lesssim \langle\xi\rangle^{1/s_1}(E_l(t)+E_{l+1}(t)).$$

2.3. Estimate of E_{N+1}

We have

$$\frac{d}{dt}e_j^2(t) := a_\varepsilon'\xi^2|\partial_t^j v|^2+2a_\varepsilon\xi^2\,\mathrm{Re}(\partial_t^j v,\partial_t^{j+1}v)\le \frac{a_\varepsilon'}{a_\varepsilon}e_j^2(t)+2a_\varepsilon\xi^2|\partial_t^j v|\,|\partial_t^{j+1}v|.$$

Hence

$$e_j'(t)\le \frac{1}{2}\frac{|a_\varepsilon'|}{a_\varepsilon}e_j(t)+e_{j+1}(t).$$

This gives

$$E_{N+1}'(t)\le \frac{1}{2}\frac{|a_\varepsilon'|}{a_\varepsilon}E_{N+1}(t)+\sum_{j=0}^{\infty}(j+1)^{N(1-s_0)}\langle\xi\rangle^{N(1-1/s_1)}e_{j+1}(t)\frac{r^j}{j!^{s_0}} = \mathrm{I}_{N+1}+\mathrm{II}_{N+1}.$$

We can estimate I_{N+1} by Glaeser's inequality (11) as before. Thus we estimate II_{N+1}. We have

$$\mathrm{II}_{N+1} = \frac{1}{r}\sum_{j=1}^{\infty} j^{N(1-s_0)+s_0}\langle\xi\rangle^{N(1-1/s_1)}e_j(t)\frac{r^j}{j!^{s_0}}.$$

Note that by our choice of N the exponent $N(1-s_0)+s_0$ is negative. By Hölder's inequality, with $p=N(1-s_0)/(N(1-s_0)+s_0)>1$ and $q=N(1-1/s_0)$ we get

$$\begin{aligned} j^{N(1-s_0)+s_0} &= j^{N(1-s_0)+s_0}\langle\xi\rangle^{\delta}\langle\xi\rangle^{-\delta} \\ &\le \frac{N(1-s_0)+s_0}{N(1-s_0)} j^{N(1-s_0)}\langle\xi\rangle^{\delta N(1-s_0)/(N(1-s_0)+s_0)} \\ &\quad + \frac{s_0}{N(s_0-1)}\langle\xi\rangle^{-\delta N(1-1/s_0)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{II}_{N+1} &\lesssim \sum_{j=1}^{\infty} j^{N(1-s_0)}\langle\xi\rangle^{\delta p+N(1-1/s_1)} e_j(t)\frac{r^j}{j!^{s_0}} \\ &\quad + \sum_{j=1}^{\infty}\langle\xi\rangle^{-\delta N(1-1/s_0)+N(1-1/s_1)} e_j(t)\frac{r^j}{j!^{s_0}} \\ &\lesssim \langle\xi\rangle^{\delta p}E_{N+1}(t)+\langle\xi\rangle^{-\delta N(1-1/s_0)+N(1-1/s_1)}E_1(t), \end{aligned}$$

since $e_j\le d_j$. Taking

$$\delta := \frac{N(1-1/s_1)}{p+N(1-1/s_0)},$$

we have

$$\mathrm{II}_{N+1}\lesssim\langle\xi\rangle^{N(1-1/s_1)/N(1-1/s_0)}(E_1(t)+E_{N+1}(t))\lesssim\langle\xi\rangle^{(s_1-1)s_0/s_1(s_0-1)}\mathcal{E}(t).$$

Note that this estimate is independent of N as far as $N(1-s_0)+s_0$ is negative (cf. (9)).

Due to the assumption (5) we have

$$\frac{s_1-1}{s_1}\frac{s_0}{s_0-1}\le\frac{1}{s_1}.$$

Consequently

$$E'_{N+1}(t)\lesssim\langle\xi\rangle^{1/s_1}\mathcal{E}(t).$$

2.4. Estimate of the total energy

Combining the estimates for E_l, we get

$$\mathcal{E}'(t)\lesssim\langle\xi\rangle^{1/s_1}\mathcal{E}(t).$$

Thus Grönwall's lemma yields

$$\mathcal{E}(t)\le\exp(C'T^*\langle\xi\rangle^{1/s_1})\mathcal{E}(0). \tag{13}$$

3. Energy inequality of Theorem 1.2

We assume first that $a(t)\equiv 1$. In this case it is sufficient to consider only E_1 for the energy. Let us put

$$d_j^2 := |\partial_t^{j+1} v|^2 + \xi^2 |\partial_t^j v|^2 \quad \text{for } j \geq 0$$

and $E_1 := \sum_{j=0}^{\infty} d_j r^j / j!^{s_0}$. Differentiating d_j^2 and proceeding as before we get

$$\begin{aligned}
\frac{d}{dt} d_j^2(t) &= 2\,\mathrm{Re}(\partial_t^{j+2} v, \partial_t^{j+1} v) + 2\xi^2\,\mathrm{Re}(\partial_t^j v, \partial_t^{j+1} v) \\
&= 2\xi \sum_{k=0}^{j} \binom{j}{k} \mathrm{Im}(\partial_t^k b \partial_t^{j-k} v, \partial_t^{j+1} v) + 2 \sum_{k=0}^{j} \binom{j}{k} \mathrm{Re}(\partial_t^k c \partial_t^{j-k+1} v, \partial_t^{j+1} v) \\
&\leq 2|\xi| \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k b|\,|\partial_t^{j-k} v|\,|\partial_t^{j+1} v| + 2 \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k c|\,|\partial_t^{j-k+1} v|\,|\partial_t^{j+1} v|.
\end{aligned}$$

This gives

$$\frac{d}{dt} d_j(t) \leq |b| d_j(t) + |\xi| \sum_{k=1}^{j} \binom{j}{k} |\partial_t^k b|\,|\partial_t^{j-k} v| + \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k c|\,|\partial_t^{j-k+1} v|.$$

Hence

$$\begin{aligned}
E_1'(t) &\leq |b| E_1(t) + |\xi| \sum_{j=1}^{\infty} \sum_{k=1}^{j} \binom{j}{k} |\partial_t^k b|\,|\partial_t^{j-k} v| \frac{r^j}{j!^{s_0}} + \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{j}{k} |\partial_t^k c|\,|\partial_t^{j-k+1} v| \frac{r^j}{j!^{s_0}} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Using the Gevrey estimates of b as before we get

$$\mathrm{II} \lesssim \langle \xi \rangle \sum_{k=1}^{\infty} (rR_0)^k \sum_{j=k}^{\infty} \binom{j}{k}^{1-s_0} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k)!^{s_0}} \lesssim \langle \xi \rangle \sum_{j=k}^{\infty} |\partial_t^{j-k} v| \frac{r^{j-k}}{(j-k)!^{s_0}} \lesssim E_1(t),$$

since $\binom{j}{k} \geq 1$ if $1 \leq k (\leq j)$ and $|\xi|\,|\partial_t^{j-k} v| \leq d_{j-k}(t)$. Estimating III as before, we get

$$E_1'(t) \lesssim E_1(t).$$

Now we consider the non-constant coefficients case. The strict positivity of $a(t)$ allows us to make the change of variable

$$\tau = F(t) := \int_0^t \sqrt{a(s)}\, ds.$$

We remark that F^{-1} exists and belongs to γ^{s_0}. Then putting

$$u(t,x)=u(F^{-1}(\tau),x)=:\tilde{u}(\tau,x),\quad a(t)=a(F^{-1}(\tau))=:\tilde{a}(\tau),$$
$$b(t)=b(F^{-1}(\tau))=:\tilde{b}(\tau),\qquad c(t)=c(F^{-1}(\tau))=:\tilde{c}(\tau),$$

we get

$$\partial_\tau^2\tilde{u}-\partial_x^2\tilde{u}-\frac{\tilde{b}(\tau)}{\tilde{a}(\tau)}\partial_x\tilde{u}+\left(\frac{\tilde{a}'(\tau)}{2\tilde{a}^{3/2}(\tau)}-\frac{\tilde{c}(\tau)}{\tilde{a}^{1/2}(\tau)}\right)\partial_\tau\tilde{u}=0.$$

By the strict positivity of $a(t)$ we see that $u(t,x)$ has the same regularity as $\tilde{u}(\tau,x)$. Therefore we can reduce to the constant coefficient case and

$$\mathcal{E}(t)\lesssim\mathcal{E}(0) \tag{14}$$

holds with the energy $\mathcal{E}(t):=E_1(F(t))$.

Here we remark that this energy inequality is valid without the assumption (6).

4. Proofs of Theorems 1.1 and 1.2

We have derived the energy inequalities in the previous sections. To complete the proofs of Theorems 1.1 and 1.2, we need to estimate $\mathcal{E}(0)$.

To simplify the presentation we set

$$s:=\min(s_1,s_0),$$

so that all the functions involved belong to γ^s. Next, since we have assumed that $u_0,u_1\in\gamma_0^s$, we have

$$|\partial_x^k u_0(x)|+|\partial_x^k u_1(x)|\le C_1R_1^k k!^s\quad\text{for any } k\in\mathbb{N}\text{ and } x\in\mathbb{R},$$

for some constants C_1 and R_1. Our aim is now to prove that we have

$$|\partial_t^j\partial_x^k u(0,x)|\le C_2R_2^{j+k}j!^s k!^s\quad\text{for any } k,j\in\mathbb{N}\text{ and } x\in\mathbb{R}, \tag{15}$$

for some constants C_2 and R_2 depending on C_0, R_0, C_1 and R_1 (see (4)).

We fix some notation.

Notation. We write $\psi\prec\varphi$ if

$$|\partial_t^j\partial_x^k\psi(0,0)|\le\partial_t^j\partial_x^k\varphi(0,0)\quad\text{for any } j,k\in\mathbb{N}.$$

In particular, $\varphi\succ0$ means that

$$\partial_t^j\partial_x^k\varphi(0,0)\ge0\quad\text{for any } j,k\in\mathbb{N}.$$

The following lemma is easily proved by induction over the time derivatives.

Lemma 4.1. *Assume that* $a \prec \mathcal{A}$, $b \prec \mathcal{B}$, $c \prec \mathcal{C}$, $d \prec \mathcal{D}$, $u_0 \prec \mathcal{U}_0$, $u_1 \prec \mathcal{U}_1$, *and let* u *and* $\mathcal{U}$ *be the solutions of the problems*

$$\begin{cases} \partial_t^2 u - a\partial_x^2 u = b\partial_x u + c\partial_t u + du, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x), \end{cases} \quad \text{and} \quad \begin{cases} \partial_t^2 \mathcal{U} - \mathcal{A}\partial_x^2 \mathcal{U} = \mathcal{B}\partial_x \mathcal{U} + \mathcal{C}\partial_t \mathcal{U} + \mathcal{D}\mathcal{U}, \\ \mathcal{U}(0,x) = \mathcal{U}_0(x), \\ \partial_t \mathcal{U}(0,x) = \mathcal{U}_1(x). \end{cases}$$

Then $u \prec \mathcal{U}$.

We say that a sequence $\{c_j\}_{j\geq 0}$ of real numbers is a *Gevrey sequence of order* s if there exists $r>0$ such that

$$c_j = c_0 r^j j!^s \quad \text{for } j > 0.$$

We have the following Gevrey version of the classical Borel lemma [P, Theorem 2.1].

Lemma 4.2. *Let* $\{c_j\}_{j\geq 0}$ *be a Gevrey sequence of order* s. *Then there exists* $h(t) \in \gamma^s$ *such that*

$$\partial_t^j h(0) = c_j.$$

Let $\{A_j\}_{j\geq 0}$, $\{B_j\}_{j\geq 0}$ and $\{W_j\}_{j\geq 0}$ be Gevrey sequences of order s such that

$$|\partial_t^j a(t)| \leq A_j, \quad |\partial_t^j b(t)| + |\partial_t^j c(t)| \leq B_j, \quad |\partial_x^j u_0(x)| \leq W_j, \quad |\partial_x^j u_1(x)| \leq A_0 W_{j+1}.$$

Without loss of generality, we may suppose that $A_0 \geq 1$. Due to Lemma 4.2, we may associate three functions $\mathcal{A}(t), \mathcal{B}(t), \mathcal{W}(x) \in \gamma^s$ with these sequences so that

$$\partial_t^j \mathcal{A}(0) = A_j > 0, \quad \partial_t^j \mathcal{B}(0) = B_j > 0 \quad \text{and} \quad \partial_x^j \mathcal{W}(0) = W_j > 0.$$

Hence

$$a \prec \mathcal{A}, \quad b \prec \mathcal{B}, \quad u_0 \prec \mathcal{W} \quad \text{and} \quad u_1 \prec A_0 \mathcal{W}'. \tag{16}$$

Let $\mathcal{V}$ be the solution of the problem

$$\begin{cases} \partial_t \mathcal{V} - \mathcal{A}(t)\partial_x \mathcal{V} = \mathcal{B}(t)\mathcal{V}, \\ \mathcal{V}(0,x) = \mathcal{W}(x), \end{cases}$$

that is

$$\mathcal{V}(t,x) := \exp\left(\int_0^t \mathcal{B}(s)\, ds\right) \mathcal{W}\left(x + \int_0^t \mathcal{A}(s)\, ds\right).$$

$\mathcal{V}(t,x)$ is a Gevrey function, since it is the composition of Gevrey functions. Thus we have for some $C>0$ and $R>0$,

$$|\partial_t^j \partial_k^j \mathcal{V}(0,0)| \leq C R^{j+k} j!^s k!^s.$$

Moreover $\mathcal{V} \succ 0$. We also find that $\mathcal{V}$ solves the problem

$$\begin{cases} \partial_t^2 \mathcal{V} - \mathcal{A}^2 \partial_x^2 \mathcal{V} = [\mathcal{A}\mathcal{B} + \mathcal{A}'] \partial_x \mathcal{V} + \mathcal{B} \partial_t \mathcal{V} + \mathcal{B}' \mathcal{V}, \\ \mathcal{V}(0,x) = \mathcal{W}(x), \\ \partial_t \mathcal{V}(0,x) = A_0 \mathcal{W}'(x). \end{cases} \tag{17}$$

Now we compare problems (17) and

$$\begin{cases} \partial_t^2 u - a \partial_x^2 u = b \partial_x u + c \partial_t u, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x). \end{cases} \tag{18}$$

Note that, since $A_0 = \mathcal{A}(0) \geq 1$ we have $\mathcal{A} \prec \mathcal{A}^2$. Indeed

$$\partial_t^j \mathcal{A}^2(0) = \sum_{k=0}^{j} \binom{j}{k} \partial_t^k \mathcal{A}(0) \partial_t^{j-k} \mathcal{A}(0) \geq \partial_t^j \mathcal{A}(0) \mathcal{A}(0) \geq \partial_t^j \mathcal{A}(0).$$

Similarly we have $\mathcal{B} \prec \mathcal{A}\mathcal{B}$. We derive that each coefficient in (17) dominates the corresponding coefficient in (18). Applying Lemma 4.1 we get $u \prec \mathcal{V}$, which gives (15) for $x=0$. To obtain (15) for a generic x_0 we compare problem (17) with the problem

$$\begin{cases} \partial_t^2 u - a \partial_x^2 u = b \partial_x u + c \partial_t u, \\ u(0,x) = u_0(x+x_0), \\ \partial_t u(0,x) = u_1(x+x_0). \end{cases}$$

From (15), due to the Paley–Wiener theorem (see [R]), we derive

$$|\partial_t^j v(0,\xi)| \leq C_3 R_3^j j!^s \exp(C_4 \langle \xi \rangle^{1/s}) \quad \text{for any } \xi \in \mathbb{R},$$

which gives the thought for estimate

$$\mathcal{E}(0) \leq C_5 \exp(C_6 \langle \xi \rangle^{1/s}). \tag{19}$$

In the weakly hyperbolic case, combining (19) and (13) we have

$$\mathcal{E}(t) \leq \exp(C' T^* \langle \xi \rangle^{1/s_1}) \mathcal{E}(0) \leq C_5 \exp(C' T^* \langle \xi \rangle^{1/s_1} + C_6 \langle \xi \rangle^{1/s}) \leq C_5 \exp(C_7 \langle \xi \rangle^{1/s_1}).$$

Here we used that $s=\min(s_1,s_0)=s_1$, since by (5), $s_1\le 2-1/s_0(\le s_0)$. In the strictly hyperbolic case, combining (19) and (14) we have

$$\mathcal{E}(t)\lesssim\mathcal{E}(0)\le C_5\exp(C_6\langle\xi\rangle^{1/s})\le C_5\exp(C_7\langle\xi\rangle^{1/s_1}),$$

where we used that $s=\min(s_1,s_0)=s_1$, since by (6), $s_1\le s_0$. On the other hand, we see that

$$\mathcal{E}(t)\ge E_1(t)=\sum_{j=0}^{\infty}d_j\frac{r^j}{j!^{s_0}}\ge\sum_{j=0}^{\infty}|\partial_t^{j+1}v|\frac{r^j}{j!^{s_0}}\ge\sup_{j\ge0}|\partial_t^{j+1}v|\frac{r^j}{j!^{s_0}}.$$

Thus, it follows that

$$|\partial_t^{j+1}v|\le C_5R_4^j j!^{s_0}\exp(C_7\langle\xi\rangle^{1/s_1}),$$

which completes the proofs of Theorems 1.1 and 1.2.

Remark 4.3. Using (7), one can find positive constants C and R such that

$$|\partial_t^j v(0,\xi)|\le CR^j j!^s\langle\xi\rangle^j\quad\text{for any } j\in\mathbb{N} \text{ and any } \xi\in\mathbb{R}.$$

This inequality can be derived for any type of equation (even if the equation is not hyperbolic). With this rough estimate, one cannot get the thought for estimate (19), since

$$\mathcal{E}(0)\le C_5\sum_{j=0}^{\infty}(Rr\langle\xi\rangle)^j=\infty\quad\text{for sufficiently large } \xi,$$

and, indeed, only the formal Cauchy problem is solvable (see [Ko]). For this reason, in the proof of (19) we used the hyperbolicity, comparing with the solution $\mathcal{V}$ to the hyperbolic equation by Lemma 4.1.

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