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Finiteness results for lattices in certain Lie groups

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This paper is dedicated to the memory of our colleague Larry Corwin.

Abstract. In this note we establish some general finiteness results concerning lattices Γ in connected Lie groups G which possess certain "density" properties (see Moskowitz, M., On the density theorems of Borel and Furstenberg, *Ark. Mat.* **16** (1978), 11–27, and Moskowitz, M., Some results on automorphisms of bounded displacement and bounded cocycles, *Monatsh. Math.* **85** (1978), 323–336). For such groups we show that Γ always has finite index in its normalizer $N_G(\Gamma)$. We then investigate analogous questions for the automorphism group $\mathrm{Aut}(G)$ proving, under appropriate conditions, that $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is discrete. Finally we show, under appropriate conditions, that the subgroup $\widetilde{\Gamma}=\{i_\gamma:\gamma\in\Gamma\}$, $i_\gamma(x)=\gamma x\gamma^{-1}$, of $\mathrm{Aut}(G)$ has finite index in $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$. We test the limits of our results with various examples and counterexamples.

1. Introduction

In this note we shall establish some general finiteness results concerning lattices Γ in certain connected Lie groups G. For all notation see the paragraph below. The Lie groups we are interested in possess "density" properties (see [10] and [11]) which we will exploit here. For these groups we shall prove that Γ always has finite index in its normalizer $N_G(\Gamma)$ (Proposition 2.1), a result which extends the classical theorem of Hurwitz that a compact Riemann surface has a finite automorphism group; here the appropriate manifolds have finite automorphism groups. In particular, our results apply to certain simply connected solvable groups G having all real roots; these are known to always contain lattices via constructions developed in [9]. In a future publication [4] we will give effective computational tools to get explicit bounds for the index of Γ in $N_G(\Gamma)$ for this class of solvable groups. Returning to the general situation, we then investigate analogous questions for the group of automorphisms $\mathrm{Aut}(G)$, proving under appropriate conditions that $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$

is discrete (Theorems 3.1 and 3.3 and their corollaries). Then under appropriate hypotheses we show that the subgroup of inner automorphisms $\widetilde{\Gamma}=\{i_\gamma:\gamma\in\Gamma\}$, with $i_\gamma(g)=\gamma g\gamma^{-1}$, has finite index in $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ (Corollary 3.8). Finally, we test the limits of our results with various examples and counterexamples.

Given a connected Lie group G with Lie algebra $\mathfrak{g}$, we denote the radical by $\mathrm{Rad}(G)$ and the center by $Z(G)$. $\mathrm{Aut}(G)$ stands for the group of C^∞ automorphisms; $\mathrm{M}(G)$ the (left) Haar measure preserving automorphisms; $\mathrm{Int}(G)$ is the subgroup of inner automorphisms $i_g(x)=gxg^{-1}$. $\mathrm{Aut}(G)$ is topologized by uniform convergence (together with the inverses) on compact sets. Since taking the differential yields a faithful smooth representation $\mathrm{Aut}(G)\to\mathrm{Aut}(\mathfrak{g})$, $\mathrm{Aut}(G)$ is a Lie group by Cartan's theorem. Because this map is injective we can consider $\mathrm{Aut}(G)$ as a subset of $\mathrm{Aut}(\mathfrak{g})$. A discrete subgroup Γ of G is a lattice if G/Γ possesses a finite regular G-invariant measure. Because the groups we are interested in contain lattices, they are unimodular [13]. $Z_G(\Gamma)$ and $N_G(\Gamma)$ denote respectively the centralizer and normalizer of Γ in G and, as above, $\widetilde{\Gamma}$ is the subgroup of $\mathrm{Int}(G)$ given by $\{i_\gamma:\gamma\in\Gamma\}$. The modulus $\Delta(\alpha)$ of an automorphism α of G is given by $\Delta(\alpha)=\mu(\alpha(S))/\mu(S)$, where μ is the left Haar measure on G and S is any set of finite positive measure. The map $\Delta\colon \mathrm{Aut}(G)\to\mathbb{R}_+^\times$ is a continuous homomorphism. The identity component of a Lie group H is indicated by H_0. For a group action $H\times X\to X$ and $Y\subseteq X$ we denote the stabilizer of Y by $\mathrm{Stab}_H(Y)$ and the orbit by $\mathcal{O}_H(Y)$.

2. Finiteness results for inner automorphisms

Let G be an arbitrary connected Lie group (or indeed a Lie group with a countable number of components) and let $B(G)$ stand for the elements in G whose conjugacy classes have compact closure. Although not obvious, this "bounded" part of G happens to be a closed subgroup. Its significance lies in the fact that $B(G)=\bigcup \mathrm{Supp}\,\mu$, where μ is an arbitrary finite, regular, $\mathrm{Int}(G)$-invariant measure on G. These facts were proved in [5], and then in more general form in [6]. In this connection an important "density" condition on G is the property $B(G)=Z(G)$. It will play a role in Proposition 2.1 below which will be the prototype of more general results along the same lines. Since our results impose hypotheses only on G and not Γ, it is not necessary to have specific knowledge of the lattice. Hence it does not matter, for example, whether the results in [9] apply to all lattices in G, or only to some. These remarks will also apply to other results in the sequel.

We remark that Proposition 2.1 is itself a considerable generalization, with the same conclusion, of a result in [1], p. 378, where G is any non-compact simple group. The result of Hurwitz applies to $\mathrm{SL}(2,\mathbb{R})$.

Proposition 2.1. *Let G be a connected Lie group and Γ be a lattice in G. Then $N_G(\Gamma)_0 \subseteq B(G)$. In particular, if $B(G)=Z(G)$ and $Z(G)$ is discrete, then $N_G(\Gamma)$ is itself discrete. Hence any lattice in such a group has finite index in its normalizer.*

Proof. Following [13], Lemma 1.6, once we know that $N_G(\Gamma)$ is discrete, finite index follows because G/Γ has finite volume, as does $G/N_G(\Gamma)$, and then we have $[N_G(\Gamma):\Gamma]=\mathrm{vol}(G/\Gamma)/\mathrm{vol}(G/N_G(\Gamma))$. Let $\{\exp(tX)\}$ be a 1-parameter subgroup of G normalizing Γ. Then $\exp(tX)\gamma\exp(-tX)=\gamma_t\in\Gamma$ for all $t\in\mathbb{R}$. For a fixed γ this is a continuous function from $\mathbb{R}\to\Gamma$ and hence is constant since Γ is discrete. Taking $t=0$ tells us that $\exp(tX)\gamma\exp(-tX)=\gamma$ for every $\gamma\in\Gamma$, so that $\exp(\mathbb{R}X)\subseteq Z_G(\Gamma)$. Now let H be a connected subgroup of G which normalizes Γ. Since all 1-parameter subgroups of H are in $Z_G(\Gamma)$ and these generate H we see that $H\subseteq Z_G(\Gamma)$.

Next we show that $Z_G(\Gamma)\subseteq B(G)$. If $g\in Z_G(\Gamma)$, then $\Gamma\subseteq Z_G(g)$ so we get a surjective map $G/\Gamma\to G/Z_G(g)$. Pushing the finite G-invariant measure on G/Γ forward gives a finite G-invariant measure on $G/Z_G(g)$ and hence by equivariance a finite $\mathrm{Int}(G)$-invariant measure on $\mathcal{O}(g)$, the conjugacy class of g. Thus $g\in B(G)$.

Hence any connected subgroup H of G which normalizes Γ is contained in $B(G)$. In particular, $N_G(\Gamma)_0\subseteq B(G)$. Since $B(G)=Z(G)$ and $Z(G)$ is discrete we see that $N_G(\Gamma)_0$ is trivial and so $N_G(\Gamma)$ is discrete. Finally because Γ is a lattice in G and $N_G(\Gamma)$ is a closed subgroup of G it follows from Lemma 1.6 of [13] that $N_G(\Gamma)/\Gamma$ has finite volume and is therefore finite. □

Corollary 2.2. *Let G be a connected semisimple Lie group without compact factors that contains a lattice Γ. Then*

$$\Gamma_0=\Gamma,\quad \Gamma_1=N_G(\Gamma_0),\quad \ldots,\quad \Gamma_i=N_G(\Gamma_{i-1}),\quad \ldots$$

is a finite increasing chain of lattices which eventually stabilizes with $\Gamma_m=N_G(\Gamma_m)$.

Thus the length m of this sequence gives us an integer-valued invariant $m(\Gamma)$ of the lattice Γ.

Proof. A semisimple Lie group of non-compact type has $B(G)=Z(G)$ (see [5]) and of course $Z(G)$ is discrete. Hence by Proposition 2.1 each Γ_i is a lattice containing the previous one. If this sequence did not stabilize, the finite index at each stage would be ≥ 2. With a fixed normalization of the Haar measure on G we would get $\mathrm{vol}(G/\Gamma_i)\leq\mathrm{vol}(G/\Gamma)/2^i$. This cannot be true because according to a result of Kazhdan–Margulis (Corollary 11.9 of [13]) there is a minimum positive volume for the fundamental domains of lattices in G. □

3. Finiteness conditions involving the full automorphism group

We now want to extend these results to the automorphism group $\mathrm{Aut}(G)$. If a connected Lie group G contains a lattice Γ and we let $\widetilde{\Gamma}=\{i_\gamma:\gamma\in\Gamma\}$ in $\mathrm{Int}(G)$ as above, we ask whether $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma):\widetilde{\Gamma}]$ is finite. If $Z(G)$ is finite this conclusion is stronger than finiteness of $[\mathrm{Stab}_{\mathrm{Int}(G)}(\Gamma):\widetilde{\Gamma}]=[N_G(\Gamma):\Gamma]$ addressed in Proposition 2.1. We shall first find conditions that guarantee that $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is discrete. The following is Proposition 1.1 of [8], whose short proof will figure in the discussion that follows.

The following "density condition" will play an important role in the sequel.

The group G has no automorphisms of bounded displacement.

The *displacement* of an automorphism α is $\{\alpha(g)g^{-1}:g\in G\}$ and *bounded displacement* means this set has compact closure. This density condition is somewhat stronger than the condition $B(G)=Z(G)$ of Proposition 2.1.

Theorem 3.1. *Let G be a connected Lie group containing a lattice Γ. Suppose that G has no non-trivial automorphisms of bounded displacement. Then* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ *is a discrete subgroup of* $\mathrm{Aut}(G)$.

Proof. Let U be a neighborhood of 1 in G such that $U\cap\Gamma=(1)$ and $F=\{\gamma_1,...,\gamma_n\}$ be a finite generating set for Γ, which exists for lattices in arbitrary connected Lie groups G (see [13], Remark 13.21, p. 210, together with remarks in [1], p. 373). A neighborhood basis of the identity I in $\mathrm{Aut}(G)$ is given by sets of the form

$$W(K,U)=\{\alpha:\alpha(g)g^{-1}\in U \text{ and } \alpha^{-1}(g)g^{-1}\in U \text{ for } g\in K\},$$

where K is compact in G and U is any neighborhood of the identity. In particular,

$$W(F,U)=\{\alpha\in\mathrm{Aut}(G):\alpha(\gamma)\gamma^{-1}\in U \text{ for all } \gamma\in F\}$$

is a neighborhood of 1 in $\mathrm{Aut}(G)$. Our sets $W(F,U)$ are open neighborhoods, and although they may not be cofinal in the neighborhood system, they will suffice for our purposes.

Let $\alpha\in W(F,U)\cap\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$. If $\alpha(\gamma_i)\gamma_i^{-1}\in U\cap\Gamma$, then since $U\cap\Gamma=(1)$ and $\alpha(\gamma_i)=\gamma_i$ for all i, it follows that $\alpha=I$ on Γ because the γ_i generate Γ. Hence the fixed point set $G_\alpha=\{g\in G:\alpha(g)=g\}$ is a closed subgroup of G containing Γ. Pushing the finite invariant measure on G/Γ forward we see that G/G_α also supports a finite invariant measure. By [6] we conclude that α has bounded displacement. Therefore $\alpha=I$ throughout G and so $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is discrete. □

As a consequence (see [11]) we get the following result.

Corollary 3.2. *If any one of the following conditions hold*

(i) $\mathrm{Rad}(G)$ *is simply connected of type E and the Levi factor of G has no compact part*;

(ii) G *is complex analytic linear*;

(iii) G *is complex analytic and* $Z(G)_0$ *is simply connected*;

then $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ *is a discrete subgroup of* $\mathrm{Aut}(G)$.

Another approach to the discreteness of $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ uses a different technology to get similar but not identical results.

Theorem 3.3. *Let* Γ *be a lattice in a connected* linear *Lie group* G *and assume that* Γ *is Zariski dense in* G. *Then any connected Lie subgroup* H *of* $\mathrm{Aut}(G)$ *which stabilizes* Γ *is trivial. In particular the identity component* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)_0$ *is trivial and* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ *is discrete. If in addition* $Z(G)$ *is discrete, then every lattice has finite index in its normalizer.*

Proof. Let α_t be a 1-parameter subgroup of $\mathrm{Aut}(G)$ which stabilizes Γ. Since Γ is discrete and the action is continuous, α_t must fix Γ as in Proposition 2.1. Now, as above, we can consider $\mathrm{Aut}(G)$ as a subset of the real algebraic group, $\mathrm{Aut}(\mathfrak{g})\subseteq\mathrm{GL}(\mathfrak{g})$. Hence the Zariski closure L of $\{\alpha_t{:}t\in\mathbb{R}\}$ in $\mathrm{Aut}(\mathfrak{g})$ must also fix Γ because fixing Γ is a Zariski-closed condition in $\mathrm{GL}(\mathfrak{g})$. Since Γ is Zariski dense in G this means that L acts as the identity on G and in particular so does α_t for all t. Now let H be any connected Lie subgroup of $\mathrm{Aut}(G)$ which stabilizes Γ. Since H is generated by its 1-parameter subgroups, H is also trivial. Finally, since $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is discrete so is $\mathrm{Stab}_{\mathrm{Int}(G)}(\Gamma)$. Combining this with the fact that now $Z(G)$ is also discrete shows that the same is true of $N_G(\Gamma)$. The proof then proceeds as in Proposition 2.1. □

Applying the density theorem of [7] we conclude the following.

Corollary 3.4. *Let* G *and* Γ *be as in Theorem* 3.3. *Then* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ *is discrete whenever* G *is a connected linear group of one of the following types*:

(i) G *is minimally almost periodic*;([1])

(ii) G *is complex*;

(iii) *the radical* $\mathrm{Rad}(G)$ *has all real eigenvalues and the Levi factor has no compact part.*

Furthermore, if $Z(G)$ *is discrete in* Γ *then* Γ *has finite index in* $N_G(\Gamma)$.

([1]) This case is due to H. Furstenberg. G is *minimally almost periodic* if all continuous finite-dimensional unitary representations are trivial.

Remark. If $Z(G)$ is trivial (as is the case for the solvable groups with real roots mentioned earlier), Theorems 3.1 and 3.3 already follow from Proposition 2.1. To see this we first show that $N_{\mathrm{Aut}(G)}(\widetilde{\Gamma})=\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$. Since $\alpha\cdot i_\gamma\cdot\alpha^{-1}=i_{\alpha(\gamma)}$, being in $N_{\mathrm{Aut}(G)}(\widetilde{\Gamma})$ just means that $i_{\alpha(\gamma)}=i_{\gamma'}$ for some $\gamma'\in\Gamma$. That is, $\alpha(\gamma)(\gamma')^{-1}\in Z(G)$. When $Z(G)$ is trivial this just says that $\alpha(\gamma)=\gamma'$, so $\alpha\in\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$. As all steps are reversible the conclusion follows. In particular, $N_{\mathrm{Int}(G)}(\widetilde{\Gamma})=\mathrm{Stab}_{\mathrm{Int}(G)}(\Gamma)$. Hence if $B(G)=Z(G)$, Proposition 2.1 tells us that $[N_G(\Gamma):\Gamma]$ is finite. Therefore so is

$$[\mathrm{Stab}_{\mathrm{Int}(G)}(\Gamma):\widetilde{\Gamma}]=[N_G(\Gamma):\Gamma].$$

Continuing our assumption that G is a connected Lie group with a lattice Γ, we now turn to the question of when $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma):\widetilde{\Gamma}]$ is finite. To do this we need the following lemma.

Lemma 3.5. *If Γ is a lattice in a connected Lie group G then $Z_G(\Gamma)=Z(G)$, and in particular $Z(\Gamma)=Z(G)\cap\Gamma$ in the following situations*:

(i) $B(G)=Z(G)$;

(ii) *G is a linear group such that Γ is Zariski-dense in G.*

Proof. In case (i), the result follows as in our proof of Proposition 2.1: if g centralizes Γ then $\Gamma\subseteq Z_G(g)$. Therefore the finite G-invariant measure on G/Γ pushes forward to a finite invariant measure on $G/Z_G(g)$, which in turn gives a finite invariant measure on the conjugacy class $\mathrm{Int}(G)\cdot g$. This class must lie in $B(G)$. Therefore $g\in Z(G)$.

In case (ii), any element centralizing Γ must be in the center of G by Zariski density. □

Lemma 3.6. *If Γ is a lattice in a connected Lie group G, then $\mathrm{Aut}(\Gamma)$ is discrete. If in addition $Z(\Gamma)=Z(G)\cap\Gamma$ then $\widetilde{\Gamma}$ is discrete in the relative topology inherited from $\mathrm{Aut}(G)$.*

Proof. Discreteness of $\mathrm{Int}(\Gamma)$ follows because Γ is finitely generated and discrete. In fact if we take $U=\{1\}$ as our neighborhood of the identity, and any finite set $F\subseteq\Gamma$, then

$$W(F)=\{\alpha\in\mathrm{Aut}(\Gamma):\alpha(\gamma)=\gamma\}$$

is a typical compact-open neighborhood of the identity operator in $\mathrm{Aut}(\Gamma)$. But Γ is finitely generated. If F is a generating set $\{\gamma_1,\dots,\gamma_n\}$ then $\alpha\in W(F)$ implies that $\alpha(\gamma_i)\gamma_i^{-1}=1$ for all i which implies that $\alpha=I$.

Consider a net $\{\gamma_\nu\}$ in Γ such that $i_{\gamma_\nu}\to I$ uniformly on compact sets $K\subseteq G$; we must show that eventually $i_{\gamma_\nu}=I$ throughout G. Take $K=F$, a finite set of generators for Γ. Since Γ is discrete we get $i_{\gamma_\nu}=I$ on F, and hence on all of Γ, for all large indices ν. That implies that $\gamma_\nu\in Z(\Gamma)$. Hence γ_ν is central in G and $i_{\gamma_\nu}=I$ on G eventually. □

Remark. The property $Z(\Gamma)=Z(G)\cap\Gamma$ holds for all types of groups we have considered so far:

(1) The groups mentioned in Corollary 3.2 have this property for various reasons, all discussed in [11].

(2) For the linear groups considered in Corollary 3.4 see [10].

We now pass from arbitrary automorphisms to the subgroup $\mathrm{M}(G)$ of automorphisms that preserve left Haar measure. This is a closed normal subgroup in $\mathrm{Aut}(G)$, and hence is a Lie subgroup since $\mathrm{M}(G)$ is the kernel of the continuous map Δ.

Proposition 3.7. *Let G be any locally compact group and Γ be a lattice in G. Then* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)\subseteq\mathrm{M}(G)$. *Hence* $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)=\mathrm{Stab}_{\mathrm{M}(G)}(\Gamma)$.

Proof. Suppose α and its inverse preserve Γ. Then $\bar\alpha(g\Gamma)=\alpha(g)\Gamma$ gives a well-defined diffeomorphism of G/Γ. Let Ω be a fundamental domain for Γ in G and $\pi\colon G\to G/\Gamma$. Then $G=\Omega\Gamma$ and so $\pi(\Omega)=G/\Gamma$. If μ is a left Haar measure on G we have $\mu(\alpha(\Omega))=\Delta(\alpha)\mu(\Omega)$. Letting $\bar A=\pi(A)$ and $\bar\mu=\pi_*(\mu)$ for sets and measures on G, we have $\bar\mu(\bar\alpha(\bar\Omega))=\Delta(\alpha)\bar\mu(\bar\Omega)$. But $\bar\alpha(\bar\Omega)=G/\Gamma=\bar\Omega$. Therefore $\bar\mu(G/\Gamma)=\Delta(\alpha)\bar\mu(G/\Gamma)$ and then $\Delta(\alpha)=1$ since $0<\bar\mu(G/\Gamma)<\infty$. □

Because G is unimodular, $\mathrm{Int}(G)$ preserves Haar measure so $\mathrm{Int}(G)\subseteq\mathrm{M}(G)$. In particular, $\mathrm{Stab}_{\mathrm{M}(G)}(\Gamma)\supseteq\widetilde\Gamma$ and when G is connected $\mathrm{Int}(G)\subseteq\mathrm{M}(G)_0$. Now assume that G is simply connected. By taking the differential on $\mathrm{Aut}(G)$, and therefore also on $\mathrm{M}(G)$ and $\mathrm{Int}(G)$, these can be regarded as subgroups of the linear group $\mathrm{GL}(\mathfrak{g})$. As was shown in [8], in this representation $\mathrm{M}(G)$ is the set of real points of an algebraic group defined over $\mathbb{R}$. Therefore $\mathrm{M}(G)_0$ has finite index in $\mathrm{M}(G)$ (see [14]).

Corollary 3.8. *Let G be a simply connected Lie group containing a lattice Γ. Suppose G satisfies either the conditions of Theorem 3.1 or 3.3. Then the stabilizer* $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma):\widetilde\Gamma]$ *is finite if* $\mathrm{M}(G)_0/\mathrm{Int}(G)$ *is compact.*

We remark that when G is a complex linear group in Corollary 3.8 we can of course take $\mathrm{M}(G)_0$ to be the identity component of the group of *holomorphic* measure-preserving automorphisms.

Proof. Since $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact so is $\mathrm{M}(G)/\mathrm{Int}(G)$ (because $M(G)$ is a real algebraic group, $M(G)_0$ has finite index in $M(G)$ by [14]). Hence $\mathrm{M}(G)/\mathrm{Int}(G)$ has finite volume since $\mathrm{Int}(G)$ is normal in $M(G)$. By pushing the measure on G/Γ forward we see that there is a finite invariant measure on $\mathrm{Int}(G)/\widetilde{\Gamma}$. Hence $\mathrm{M}(G)/\widetilde{\Gamma}$ also has a finite invariant measure. Now the closed subgroup $\mathrm{Stab}_{\mathrm{M}(G)}(\Gamma)$ of $\mathrm{M}(G)$ sits in between the two,

$$\widetilde{\Gamma} \subseteq \mathrm{Stab}_{\mathrm{M}(G)}(\Gamma) \subseteq \mathrm{M}(G).$$

By Theorem 3.1 or 3.3, $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is discrete. Hence by Proposition 3.7 it follows that $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma){:}\widetilde{\Gamma}]$ is finite. □

Next we determine when $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact, under the conditions of Corollary 3.8.

Corollary 3.9. *Let Γ be a lattice in a simply connected real Lie group G. Suppose G satisfies either the conditions of Theorem 3.1 or 3.3. Then $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact if and only if $\mathrm{M}(G)_0/\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ has finite volume and $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)/\widetilde{\Gamma}$ is finite. In particular, if the quotient space $\mathrm{M}(G)_0/\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ is known to have finite volume, then $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact if and only if $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)/\widetilde{\Gamma}$ is finite.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma) & \longrightarrow & \mathrm{M}(G)_0 \\
\uparrow & & \uparrow \\
\{i_\gamma : \gamma \in \Gamma\} = \widetilde{\Gamma} & \longrightarrow & \mathrm{Int}(G).
\end{array}$$

The subgroup $\widetilde{\Gamma} \subseteq \mathrm{Int}(G)$ is discrete by Lemma 3.6. Since $\mathrm{Int}(G)/\widetilde{\Gamma}$ supports a finite invariant measure (the push-forward of a finite invariant measure on G/Γ), $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact if and only if $\mathrm{M}(G)_0/\widetilde{\Gamma}$ has a finite invariant measure. It follows from the commutativity of the diagram above that this is equivalent to $\mathrm{M}(G)_0/\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ and $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)/\widetilde{\Gamma}$ each having finite volume. Since $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ is discrete, $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)/\widetilde{\Gamma}$ is finite. Conversely, if the latter two have finite volume so does $\mathrm{M}(G)_0/\widetilde{\Gamma}$ by [13]. This means that $\mathrm{M}(G)_0/\mathrm{Int}(G)$ also has finite volume, and then $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact because $\mathrm{Int}(G)$ is normal. □

Corollary 3.10. *Let Γ be a uniform lattice in a simply connected real Lie group G. Assume that G satisfies the conditions of Theorem 3.1 or 3.3. Then $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact if and only if $\mathrm{M}(G)_0/\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ is compact and $\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)/\widetilde{\Gamma}$ is finite.*

Proof. This follows since here $\mathrm{Int}(G)/\widetilde{\Gamma}$ is compact. □

In particular Corollary 3.9 applies when G is a simply connected solvable group of type E. In [4] we intend to conduct a detailed study of these indices for the groups and lattices that were constructed in [9]. These groups are semidirect products in which a 1-parameter group of automorphisms acts on $\mathbb{R}^n$. As we shall see in [4] when $n{=}2$, $\mathrm{M}(G)_0{=}\mathrm{Int}(G)$ and so $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is trivially compact. Hence $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma){:}\widetilde{\Gamma}]$ is finite, and in [4] we expect to get effective bounds on this index. This is no longer the case when $n{\geq}3$. Indeed then $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is $(\mathbb{R}_+^\times)^{n-2}$, so $\mathrm{M}(G)_0/\mathrm{Int}(G)$ does not have finite volume and $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma){:}\widetilde{\Gamma}]$ is infinite.

We remark that $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is also compact in any semisimple Lie group without compact factors because then $[\mathrm{Aut}(G){:}\mathrm{Int}(G)]$ is finite.

We conclude with some examples and counterexamples involving *groups of Heisenberg type*. By [3] these Lie algebras all have rational structure constants, hence the corresponding simply connected groups contain uniform lattices.

Abelian and Heisenberg cases. Suppose that $G{=}\mathbb{R}^n$, or $G{=}N_n$, the Heisenberg group of dimension $2n{+}1$, and let Γ be the usual integer lattice in G. In both these cases $\mathrm{M}(G)_0/\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma)$ does support a finite invariant measure, but $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is not compact (see, e.g., [12]). In this setting $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)/\widetilde{\Gamma}$ is always infinite, because when $G{=}\mathbb{R}^n$ we have $\mathrm{Int}(G){=}\{I\}$ and $\mathrm{M}(G)_0{=}\mathrm{SL}(n,\mathbb{R})$. Hence $\mathrm{M}(G)_0/\mathrm{Int}(G){=}\mathrm{SL}(n,\mathbb{R})$ and

$$[\mathrm{Stab}_{\mathrm{M}(G)}(\Gamma):\widetilde{\Gamma}]=\mathrm{SL}^{\pm}(n,\mathbb{Z})$$

which is infinite.

When $G{=}N_n$, we have $\mathrm{M}(G)_0{=}\mathrm{Sp}(n,\mathbb{R})\times\mathbb{R}^{2n}$ (where $\times$ stands for semidirect product), while $\mathrm{Int}(G){=}\mathbb{R}^{2n}$. Here $\mathrm{M}(G)_0/\mathrm{Int}(G){=}\mathrm{Sp}(n,\mathbb{R})$ while

$$[\mathrm{Stab}_{\mathrm{M}(G)_0}(\Gamma):\widetilde{\Gamma}]=|\mathrm{Sp}(n,\mathbb{Z})\times\mathbb{Z}^{2n}/\mathbb{Z}^{2n}|=|\mathrm{Sp}(n,\mathbb{Z})|$$

which is also infinite. Thus in both these cases, although $\mathrm{Der}_0(\mathfrak{g})/\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))$ is semisimple it is not of compact type.

Quaternionic and Cayley number analogs. Now we consider irreducible Lie algebras $\mathfrak{g}$ of Heisenberg type with center $\mathfrak{z}$ of dimension 3 or 7. This means that $\mathfrak{g}$ is either

$$\dim \mathfrak{z}=3: \quad \mathfrak{h}_n=\mathfrak{v}\oplus\mathfrak{z}=\mathbb{H}^n\oplus\mathrm{Im}(\mathbb{H}), \quad \dim \mathfrak{h}_n=4n+3,$$
$$\dim \mathfrak{z}=7: \quad \mathfrak{c}_n=\mathfrak{v}\oplus\mathfrak{z}=\mathbb{O}^n\oplus\mathrm{Im}(\mathbb{O}), \quad \dim \mathfrak{c}_n=8n+7,$$

where $\mathbb{H}$ is the set of real quaternions and $\mathbb{O}$ is the set of octonians.

Let G be the associated *simply connected* nilpotent group of Heisenberg type. $\mathrm{Der}_0(\mathfrak{g})$ denotes the derivations of $\mathfrak{g}$ of trace zero, which is the Lie algebra of $\mathrm{M}(G)$ (see [8]).

Lemma 3.11. *In these cases* $\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))=\mathrm{ad}(\mathfrak{g})$ *and* $\mathrm{Der}_0(\mathfrak{g})/\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))$ *is semisimple of compact type, where*

$$\mathrm{Der}_0(\mathfrak{g}) = \{T \in \mathrm{Der}(\mathfrak{g}) : \mathrm{tr}(T) = 0\}$$

and $\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))$ *is the nilradical.*

Proof. The nilradical $\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))$ is the largest ideal in the radical of $\mathrm{Der}_0(\mathfrak{g})$ consisting of nilpotent operators. Since $\mathfrak{g}$ is a nilpotent Lie algebra each $\mathrm{ad}\,X$ is a nilpotent derivation. Also $\mathrm{ad}(\mathfrak{g})$ is a nilpotent ideal in $\mathrm{Der}(\mathfrak{g})$ and therefore also in $\mathrm{Der}_0(\mathfrak{g})$. Hence $\mathrm{ad}(\mathfrak{g})\subseteq\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))$ as subalgebras. Now $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z}$ so $\dim \mathrm{ad}(\mathfrak{g})= \dim\mathfrak{g}-\dim\mathfrak{z}=\dim\mathfrak{v}$. On the other hand, by Theorem 5.4 of Barbano [2] (see also [8]), $\mathrm{Der}_0(\mathfrak{g})/\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))$ is not merely reductive, but in fact is compact semisimple with $\dim \mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))=\dim\mathfrak{v}$. This means that $\mathrm{Nil}(\mathrm{Der}_0(\mathfrak{g}))=\mathrm{ad}(\mathfrak{g})$. $\square$

It follows from this lemma that $\mathrm{M}(G)_0/\mathrm{Int}(G)$ is compact. (For example, the quotient $\mathrm{M}(H_n)_0/\mathrm{Int}(H_n)$ is actually the direct product $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ by [2], p. 263.) Hence by Corollary 3.10 we conclude the following result.

Corollary 3.12. *Let G be an irreducible group of Heisenberg type with center of dimension* 3 *or* 7. *Then* $[\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma):\widetilde{\Gamma}]$ *is finite for any lattice* Γ *in* G.

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