Ark. Mat., **49** (2011), 1–16
DOI: 10.1007/s11512-009-0110-8

Density of the polynomials in Hardy and Bergman spaces of slit domains

John R. Akeroyd

Abstract. It is shown that for any t, $0<t<\infty$, there is a Jordan arc Γ with endpoints 0 and 1 such that $\Gamma\setminus\{1\}\subseteq\mathbb{D}:=\{z:|z|<1\}$ and with the property that the analytic polynomials are dense in the Bergman space $\mathbb{A}^t(\mathbb{D}\setminus\Gamma)$. It is also shown that one can go further in the Hardy space setting and find such a Γ that is in fact the graph of a continuous real-valued function on $[0,1]$, where the polynomials are dense in $H^t(\mathbb{D}\setminus\Gamma)$; improving upon a result in an earlier paper.

1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z:|z|<1\}$, let $\mathbb{T}$ denote the unit circle $\{z:|z|=1\}$ and let m denote normalized Lebesgue measure on $\mathbb{T}$. The Hardy space $H^t(\mathbb{D})$, $0<t<\infty$, is the collection of functions f that are analytic in $\mathbb{D}$ such that

$$\|f\|^t_{H^t(\mathbb{D})} := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^t \, dm(\zeta) < \infty.$$

For $1\leq t<\infty$, $\|\cdot\|_{H^t(\mathbb{D})}$ defines a norm relative to which $H^t(\mathbb{D})$ forms a Banach space. Let G be a bounded, simply connected region in the complex plane $\mathbb{C}$ and let φ be a conformal mapping from $\mathbb{D}$ one-to-one and onto G. The Hardy space $H^t(G)$ is the set of functions f that are analytic in G such that $f\circ\varphi\in H^t(\mathbb{D})$; which is independent of the choice of φ. One may (equivalently) define $H^t(G)$ to be the set of functions f that are analytic in G such that $|f|^t$ has a harmonic majorant on G. If $f\in H^t(G)$, then $|f|^t$ has a least harmonic majorant u_f on G and, for fixed z_0 in G and $1\leq t<\infty$, $f\mapsto u_f(z_0)^{1/t}$ defines a norm on $H^t(G)$, equal to $\|f\circ\varphi\|_{H^t(\mathbb{D})}$, when φ is chosen so that $\varphi(0)=z_0$. This alternative approach has the advantage that it is easily extendable to multiply connected regions. Let $H^\infty(G)$ denote the collection of bounded analytic functions in G. With G as above and $0<t<\infty$, the Bergman space $\mathbb{A}^t(G)$ is defined to be the set of functions f that are analytic in G such that

$$\|f\|^t_{\mathbb{A}^t(G)} := \int_G |f|^t \, dA < \infty,$$

where A denotes two-dimensional Lebesgue measure on $\mathbb{C}$. In the case that $1 \le t < \infty$, $\|\cdot\|_{\mathbb{A}^t(G)}$ defines a norm relative to which $\mathbb{A}^t(G)$ forms a Banach space. Since G is bounded, the collection of analytic polynomials, which we denote by $\mathcal{P}$, is a subspace of both $H^t(G)$ and $\mathbb{A}^t(G)$. In an earlier paper (cf. [1]) the author constructed a Jordan (i.e., simple) arc Γ with endpoints 0 and 1—a tortuous, self-similar fractal—such that $\Gamma\setminus\{1\}\subseteq\mathbb{D}$ and with the property that $\mathcal{P}$ is dense in the Hardy space $H^t(\mathbb{D}\setminus\Gamma)$; for any prescribed t, $0<t<\infty$. In this paper we improve upon this earlier result by showing that the arc Γ can be chosen to be the graph of a continuous real-valued function on $[0,1]$; see Corollary 3.4. We then turn to the Bergman space setting and show that there is a Jordan arc Γ with endpoints 0 and 1 such that $\Gamma\setminus\{1\}\subseteq\mathbb{D}$ and with the property that $\mathcal{P}$ is dense in $\mathbb{A}^t(\mathbb{D}\setminus\Gamma)$; see Corollary 4.2. This latter result puts to rest a rather long-standing question (circa 1937); cf. [3] and [10]. Throughout this paper we confine ourselves to the Banach space setting $1\le t<\infty$. By Jensen's inequality we then have the results for all t, $0<t<\infty$.

2. Preliminaries

Let μ be a finite, positive Borel measure compactly supported in $\mathbb{C}$. Then $\mathcal{P}\subseteq L^t(\mu)$, $1\le t<\infty$. We let $P^t(\mu)$ denote the closure of $\mathcal{P}$ in $L^t(\mu)$. A point α in $\mathbb{C}$ is called a *bounded point evaluation* for $P^t(\mu)$ if there is a positive constant c such that

$$|p(\alpha)| \le c\|p\|_{L^t(\mu)}$$

for all polynomials p. If α is such a point, then by the Hahn–Banach theorem and the Riesz representation theorem, there exists k_α in $L^s(\mu)$, $1/s+1/t=1$, such that $p(\alpha)=\int pk_\alpha \, d\mu$ for all polynomials p. For any f in $P^t(\mu)$, define $\hat{f}$ at α by:

$$\hat{f}(\alpha) = \int f k_\alpha \, d\mu.$$

If, in fact, there exist positive constants c and r such that

$$|p(z)| \le c\|p\|_{L^t(\mu)}$$

whenever $|z-\alpha|<r$ and p is a polynomial, then α is called an *analytic bounded point evaluation* for $P^t(\mu)$. The set of analytic bounded point evaluations for $P^t(\mu)$ is a bounded, open subset of $\mathbb{C}$ whose components are simply connected, and is the

largest open set to which every function in $P^t(\mu)$ has a natural analytic continuation, given by $\hat{f}$. J. Thomson has established a direct sum decomposition of $P^t(\mu)$, where the components of the set of analytic bounded point evaluations play a vital role; cf. [12], Theorem 5.8, or Theorem 2.1 below.

Theorem 2.1. (J. Thomson) *Let μ be a finite, positive Borel measure compactly supported in $\mathbb{C}$ and let $\{G_n\}_{n=1}^N$ be the components (there might be none, or countably many) of the set of analytic bounded point evaluations for $P^t(\mu)$. Then there is a Borel partition $\{\Delta_n\}_{n=0}^N$ of the support of μ such that*

$$P^t(\mu)=L^t(\mu|_{\Delta_0})\oplus\left(\bigoplus_{n=1}^N P^t(\mu|_{\Delta_n})\right),$$

where $\Delta_n\subseteq\overline{G}_n$ and $P^t(\mu|_{\Delta_n})$ is irreducible, $n\geq 1$. Moreover, for $n\geq 1$, the mapping $f\mapsto\hat{f}$ is one-to-one on $P^t(\mu|_{\Delta_n})$ and, under this mapping, the Banach algebras $P^t(\mu|_{\Delta_n})\cap L^\infty(\mu|_{\Delta_n})$ and $H^\infty(G_n)$ are algebraically and isometrically isomorphic, and weak-star homeomorphic.

A recent paper of A. Aleman, S. Richter and C. Sundberg refines our understanding of the irreducible summands $P^t(\mu|_{\Delta_n})$ in Theorem 2.1; cf. [2].

Let G be a bounded, simply connected region in $\mathbb{C}$. Since the Hardy space $H^t(G)$ is conformally invariant, it is straightforward that $H^\infty(G)$ is dense in $H^t(G)$. A result of L. I. Hedberg tells us that the same holds in the Bergman space setting; for a proof in the case that $t=2$, one may consult [11], pp. 112–114.

Theorem 2.2. (L. I. Hedberg) *Let G be a bounded, simply connected region in $\mathbb{C}$. Then $H^\infty(G)$ is dense in $\mathbb{A}^t(G)$, $1\leq t<\infty$.*

Once again, let G be a bounded, simply connected region in $\mathbb{C}$ and, for z_0 in G, define $\rho_{z_0}: C_{\mathbb{R}}(\partial G)\to\mathbb{R}$ by $\rho_{z_0}(u)=\hat{u}(z_0)$, where $\hat{u}$ now denotes the continuous, real-valued function on $\overline{G}$ that is harmonic in G and has boundary values u. By the maximum principle, ρ_{z_0} defines a (positive) bounded linear functional on $C_{\mathbb{R}}(\partial G)$; and, processing the function $u\equiv 1$, we find that ρ_{z_0} is of norm equal to 1. Hence, by the Riesz representation theorem, there is a unique positive, probability measure $\omega(\,\cdot\,,G,z_0)$ with support in ∂G such that

$$\hat{u}(z_0)=\int_{\partial G} u(\zeta)\,d\omega(\zeta,G,z_0)$$

for all u in $C_{\mathbb{R}}(\partial G)$. The measure $\omega(\,\cdot\,,G,z_0)$ is the so-called *harmonic measure* on ∂G for evaluation at z_0. By Harnack's inequality, $\omega(\,\cdot\,,G,w_0)$ is boundedly equivalent to $\omega(\,\cdot\,, G, z_0)$ for any other choice of point w_0 in G. And one may interpret the distribution of $\omega(\,\cdot\,, G, z_0)$ on ∂G in terms of Brownian motion paths: For any Borel subset B of ∂G, $\omega(B, G, z_0)$ is the probability that a Brownian motion path starting at z_0 will first exit G through a point in B; cf. [7], Appendix F.

To minimize some of the technical details involved in our constructions, we follow the general plan of [1] and initially focus on the rectilinear annular region $E{:=}\{z{:}1{<}\max(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|){<}2\}$. If Γ is any Jordan arc with endpoints 1 and 2 such that $\Gamma\setminus\{1,2\}\subseteq E\setminus\{-\frac{3}{2}\}$, then let E_Γ denote the set difference $E\setminus\Gamma$. Further, let ω_Γ denote harmonic measure on ∂E_Γ for evaluation at $-\frac{3}{2}$ and let ν_Γ denote two-dimensional Lebesgue measure restricted to E_Γ. In this paper we first produce a continuous real-valued function $f\colon[1,2]{\to}[-1,1]$, where $f(1){=}f(2){=}0$, such that $\Gamma{:=}\{x{+}if(x){:}1{\le}x{\le}2\}$ has the property that $\mathcal{P}$ is dense in the Hardy space $H^t(E_\Gamma)$; see Theorem 3.3. We then produce a Jordan arc Γ with endpoints 1 and 2 such that $\Gamma\setminus\{1,2\}\subseteq E$ and with the property that the polynomials are dense in $\mathbb{A}^t(E_\Gamma)$; see Theorem 4.1. After each of these results, we indicate how minor modifications give us the aforementioned Corollaries 3.4 and 4.2. Now, by elementary methods, E_Γ is contained in the set of analytic bounded point evaluations for both $P^t(\omega_\Gamma)$ and $P^t(\nu_\Gamma)$. And if the rational function $z{\mapsto}1/z$ is in $P^t(\omega_\Gamma)$ (abbreviated, $1/z{\in}P^t(\omega_\Gamma)$), then no point in $\{z{:}\max(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|){\le}1\}$ is an analytic bounded point evaluation for $P^t(\omega_\Gamma)$ and indeed E_Γ is precisely the set of analytic bounded point evaluations for $P^t(\omega_\Gamma)$. We can then apply Theorem 2.1 and find that $\mathcal{P}$ is dense in $H^t(E_\Gamma)$. Making use of Theorem 2.2, the same argument carries over to the Bergman space setting and we find that if $1/z{\in}P^t(\nu_\Gamma)$, then $\mathcal{P}$ is dense in $\mathbb{A}^t(E_\Gamma)$. This makes our target quite clear: Construct the various Jordan arcs Γ so that $1/z{\in}P^t(\omega_\Gamma)$ (respectively, $1/z{\in}P^t(\nu_\Gamma)$). Now, if $1/z{\in}P^t(\omega_\Gamma)$, then, in the terminology of [12], there is a sequence of "light routes" from 0 to ∞; and the same applies in the Bergman space setting. And by the characters of the measures under consideration, the portions of these light routes that are within E need to converge to Γ. This points to our strategy here, which is reminiscent of an argument of W. W. Hastings; cf. [9], or the proof of Lemma 10.7 in Chapter II of [4]. In the Hardy space setting, we find a sequence of Jordan arcs $\{\gamma_n\}_{n=1}^\infty$ and a sequence of polynomials $\{p_n\}_{n=1}^\infty$ such that:

(i) γ_n has endpoints 1 and 2 and $\gamma_n\setminus\{1,2\}\subseteq E\setminus\{-\frac{3}{2}\}$, $n{\ge}1$;

(ii) for $n{\ge}1$ and $1{\le}k{\le}n$, $\int_{\partial E_{\gamma_n}}|1/z{-}p_k|^t\,d\omega_{\gamma_n}{<}1/k$;

(iii) $\{\gamma_n\}_{n=1}^\infty$ converges uniformly to a Jordan arc Γ in $(E{\cup}\{1,2\})\setminus\{-\frac{3}{2}\}$, as $n{\to}\infty$.

From (i)–(iii) it follows that $\int_{\partial E_\Gamma}|1/z{-}p_k|^t\,d\omega_\Gamma{\le}1/k$ for $k{\ge}1$, and hence $1/z{\in}P^t(\omega_\Gamma)$. The sequence of light routes here are essentially tubular neighborhoods of the arcs γ_n. In the Bergman space setting our strategy is largely the same, except that we find the process more manageable if we express the Jordan arc Γ as the limit of a sequence of continua that are not themselves Jordan arcs.

3. The Hardy space setting

We begin with two results concerning harmonic measure. The proof of the first, which is omitted, is a straightforward consequence of Lemma 5.1 in Chapter IV of [7].

Lemma 3.1. *For b and n, $0<b\leq 1$ and n a positive integer, let* $W_{b,n}=\{z:\mathrm{Im}(z)>0\}\setminus\{z:0<\mathrm{Im}(z)\leq b$ *and* $|\mathrm{Re}(z)|=k/n$ *for some* $k=0,1,2,\ldots,n\}$. *Given b as above, for any $\varepsilon>0$, there exists a positive integer N such that*

$$\omega([-1,1],W_{b,n},2i)<\varepsilon,$$

whenever $n\geq N$.

Now, in order for a Brownian motion path starting at $2i$ to first exit $W_{b,n}$ through a point in the interval $(-1,1)$, it must successfully navigate down one of the corridors in $W_{b,n}$ that leads to $(-1,1)$; and each of these corridors has aspect ratio (length to width) equal to bn. The import of Lemma 3.1 is that one can reduce the probability of such an event to as small a positive value as one might wish by choosing n to be sufficiently large. And this result does not essentially depend upon the corridors being rectilinear in shape; see the discussion on extremal length in Chapter IV of [7]. Our next result makes use of this fact, but we first need to clarify our terms. For $0<a<1$ and any positive integer k, define $f_{a,k}$ on $[1,2]$ by $f_{a,k}(x)=a\sin(2\pi kx)$. Notice that $f_{a,k}(1)=f_{a,k}(2)=0$ and that $f_{a,k}$ has period $1/k$ and amplitude a on $[1,2]$. For any function $g\colon[1,2]\to[-1,1]$ that is continuous on $[1,2]$, with $g(1)=g(2)=0$, and for any δ, $0<\delta<1$, let $T(g,\delta)=\{x+iy:1\leq x\leq 2$ and $|y-g(x)|<\delta\}$.

Lemma 3.2. *Suppose that $g\colon[1,2]\to[-1,1]$ is continuous on $[1,2]$ and that $g(1)=g(2)=0$. If $0<a<1$, $\varepsilon>0$ and $0<\delta<a$, then, provided k is sufficiently large, $\gamma:=\{x+i(g(x)+f_{a,k}(x)):1\leq x\leq 2\}$ has the property: The probability that a Brownian motion path starting at $-\frac{3}{2}$ will reach a point in $E_\gamma\cap T(g,\delta)$ before it exits E_γ is less than ε; and hence*

$$\omega_\gamma(T(g,\delta))<\varepsilon.$$

Sketch of proof. By our hypothesis, g can be uniformly approximated on $[1,2]$ by a sequence of step functions. Thus, via a piecewise analysis, we can reduce to the case that g is constant on $[1,2]$; indeed, that $g\equiv 0$ on $[1,2]$. Now, in order for a Brownian motion path starting at $-\frac{3}{2}$ to reach a point in $E_\gamma \cap T(g,\delta)$ before it exits E_γ it must successfully navigate down one of $2k$ corridors delineated by portions of γ, each of length $a-\delta$ and of maximum width $1/k$. Hence, by extremal length estimates (cf. Chapter IV of [7]), the probability of such an event can be reduced to less than any prescribed positive value by choosing k to be sufficiently large. $\square$

Theorem 3.3. *For any t, $1\le t<\infty$, there is a continuous function $f\colon [1,2]\to [-1,1]$ such that $f(1)=f(2)=0$ and the Jordan arc $\Gamma := \{x+if(x)\colon 1\le x\le 2\}$ has the property that $\mathcal{P}$ is dense in $H^t(E_\Gamma)$.*

Proof. Define g_0 on $[1,2]$ by $g_0\equiv 0$. Notice that $F_1 := \overline{E}\setminus T\big(g_0,\frac{1}{4}\big)$ is a connected, compact subset of $\mathbb{C}$ whose complement in $\mathbb{C}$ has just one component, and that component contains 0. So, by Runge's theorem, there is a polynomial p_1 such that $|1/z-p_1(z)|^t<1$ for all z in F_1. Since $|1/z-p_1(z)|^t$ is bounded on E and harmonic measure has total mass equal to 1, we can apply Lemma 3.2 and find an integer k_1 sufficiently large so that $\gamma_1 := \{x+if_{1/2,k_1}(x) : 1\le x\le 2\}$ satisfies:

$$\int_{\partial E_{\gamma_1}} \left|\frac{1}{z}-p_1(z)\right|^t d\omega_{\gamma_1}(z) < 1.$$

Define g_1 on $[1,2]$ by $g_1\equiv f_{1/2,k_1}$. Notice that $F_2 := \overline{E}\setminus T\big(g_1,\frac{1}{8}\big)$ is a connected, compact subset of $\mathbb{C}$ whose complement in $\mathbb{C}$ has just one component, and that component contains 0. So, again by Runge's theorem, there is a polynomial p_2 such that $|1/z-p_2(z)|^t<\frac{1}{2}$ for all z in F_2. Applying Lemma 3.2 once again, we can find a positive integer k_2 sufficiently large so that $\gamma_2 := \{x+i(g_1(x)+f_{1/4,k_2}(x)) : 1\le x\le 2\}$ satisfies:

$$\int_{\partial E_{\gamma_2}} \left|\frac{1}{z}-p_2(z)\right|^t d\omega_{\gamma_2}(z) < \frac{1}{2}.$$

Moreover, the probability that a Brownian motion path starting at $-\frac{3}{2}$ will reach a point in $E_{\gamma_2}\cap T\big(g_0,\frac{1}{4}\big)$ before it exits E_{γ_2} is less than the sum of:

(i) The probability that a Brownian motion path starting at $-\frac{3}{2}$ will reach a point in $E_{\gamma_2}\cap T\big(g_0,\frac{1}{4}\big)$ before it reaches a point in $E_{\gamma_2}\cap T\big(g_1,\frac{1}{8}\big)$, and before it exits E_{γ_2}.

(ii) The probability that a Brownian motion path starting at $-\frac{3}{2}$ will reach a point in $E_{\gamma_2}\cap T\big(g_1,\frac{1}{8}\big)$ before it exits E_{γ_2}.

Now, since $T\big(g_1,\frac{1}{8}\big)$ is a tube around γ_1, the probability mentioned in (i) is clearly less than the probability that a Brownian motion path starting at $-\frac{3}{2}$ will reach a point in $E_{\gamma_2}\cap T\big(g_0,\frac{1}{4}\big)$ before it exits E_{γ_2} and before it reaches a point in γ_1; i.e., before it exits E_{γ_2} or E_{γ_1}. So this value is no bigger than our estimate on the size of $\omega_{\gamma_1}\big(T\big(g_0,\frac{1}{4}\big)\big)$, given by Lemma 3.2. And, by Lemma 3.2, the probability mentioned in (ii) can be made as small as we wish by choosing k_2 sufficiently large. Therefore, by choosing k_2 sufficiently large, we also have:

$$\int_{\partial E_{\gamma_2}} \left|\frac{1}{z}-p_1(z)\right|^t d\omega_{\gamma_2}(z)<1,$$

without any modification in our earlier choice of k_1. Let $g_2=g_1+f_{1/4,k_2}$. Since $F_3:=\overline{E}\setminus T\big(g_2,\frac{1}{16}\big)$ is a connected, compact subset of $\mathbb{C}$ whose complement in $\mathbb{C}$ has just one component, and that component contains 0, we can find a polynomial p_3 such that $|1/z-p_3(z)|^t<\frac{1}{3}$ for all z in F_3. Applying Lemma 3.2, we can find a positive integer k_3 sufficiently large so that $\gamma_3:=\{x+i(g_2(x)+f_{1/8,k_3}(x)):1\le x\le 2\}$ satisfies:

$$\int_{\partial E_{\gamma_3}} \left|\frac{1}{z}-p_3(z)\right|^t d\omega_{\gamma_3}(z)<\frac{1}{3}.$$

And, arguing as above, we find that, for a sufficiently large choice of k_3 we can also ensure that

$$\int_{\partial E_{\gamma_3}} \left|\frac{1}{z}-p_1(z)\right|^t d\omega_{\gamma_3}(z)<1$$

and that

$$\int_{\partial E_{\gamma_3}} \left|\frac{1}{z}-p_2(z)\right|^t d\omega_{\gamma_3}(z)<\frac{1}{2},$$

without making a change in our earlier choices of k_1 and k_2. Continuing this process ad infinitum we find a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of Jordan arcs that satisfies conditions (i) and (ii) set forth at the end of Section 2 of this paper. Notice that this sequence of arcs also satisfy condition (iii), since γ_N is the "graph" of g_N—the Nth partial sum of the sequence of functions $\{f_{2^{-n},k_n}\}_{n=1}^{\infty}$, whose amplitudes sum to 1. Therefore, our goal is reached. $\square$

One can modify the proof of Theorem 3.3 to produce a function $f\colon[0,1]\to[-\frac{1}{2},\frac{1}{2}]$ that is continuous on $[0,1]$, with $f(0)=f(1)=0$, such that $\Gamma:=\{x+if(x):0\le x\le 1\}$ satisfies: $\mathcal{P}$ is dense in $H^t(\Omega_\Gamma)$, where $\Omega:=\{z:\max(|\mathrm{Re}(z)|,|\mathrm{Im}(z)|)<1\}$ and $\Omega_\Gamma:=\Omega\setminus\Gamma$. As before, we base our construction on the functions $f_{a,k}(x):=a\sin(2\pi kx)$, now restricted to $[0,1]$ and with reduced amplitudes. In this context we work with a process that does not in any way depend on the function $z\mapsto 1/z$. One can do this by choosing a sequence of points $\{z_n\}_{n=1}^{\infty}$ that converges to 0 such that z_n is in the interior of $T(g_n,2^{-n-3})$—now defined over $[0,1]$—and by using Runge's theorem to find a polynomial p_n such that $|p_n(z_n)|\ge n$ and yet $|p_n(z)|^t<1$, for all z in $\overline{\Omega}\setminus T(g_n,2^{-n-3})$. The rest of the construction is similar to that in the proof of Theorem 3.3, the functions g_N being the Nth partial sums of the sequence $\{f_{2^{-n-1},k_n}\}_{n=1}^{\infty}$ and $\gamma_n := \{x+ig_n(x) : 0\le x\le 1\}$, chosen so that

$$\int_{\partial\Omega_{\gamma_n}} |p_k|^t \, d\omega_{\gamma_n} \le 1,$$

whenever $1\le k\le n$; where ω_{γ_n} now denotes harmonic measure on $\partial\Omega_{\gamma_n}$ for evaluation at $-\frac{1}{2}$. The sequence $\{\gamma_n\}_{n=1}^{\infty}$ then converges uniformly to a Jordan arc Γ, with endpoints 0 and 1, such that $\Gamma\setminus\{1\}\subseteq\Omega$. And, by our construction, there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $\|p_n\|_{L^t(\omega_\Gamma)}\le 1$ and yet $|p_n(z_n)|\ge n$, for all n. Since $\{z_n\}_{n=1}^{\infty}$ converges to 0, it follows that 0 is not an analytic bounded point evaluation for $P^t(\omega_\Gamma)$. Therefore, the set of analytic bounded point evaluations for $P^t(\omega_\Gamma)$ is precisely Ω_Γ, and so we conclude (as we did before) that $\mathcal{P}$ is dense in $H^t(\Omega_\Gamma)$. One can go a step further and replace Ω by $\mathbb{D}$ in this process. Yet, in order to be sure that $\Gamma\setminus\{1\}\subseteq\mathbb{D}$, it is necessary to modify the functions $f_{a_k,k}$ so that their amplitudes a_k are themselves functions of x in $[0,1]$. The details are a bit more cumbersome, but the general process is the same.

Corollary 3.4. *For any t, $1\le t<\infty$, there is a continuous real-valued function f on $[0,1]$ such that $f(0)=f(1)=0$, $|f(x)|<\sqrt{1-x^2}$ on $[0,1)$ and the Jordan arc $\Gamma := \{x+if(x) : 0\le x\le 1\}$ has the property that $\mathcal{P}$ is dense in $H^t(\mathbb{D}\setminus\Gamma)$.*

4. The Bergman space setting

In this section we build a Jordan arc Γ with endpoints 1 and 2 such that $\Gamma\setminus\{1,2\}\subseteq E$ and with the property that $\mathcal{P}$ is dense in the Bergman space $\mathbb{A}^t(E_\Gamma)$; for any prescribed t, $1\le t<\infty$. We cannot duplicate the result of Section 3 here since the graph of any continuous real-valued function on $[1,2]$ has zero two-dimensional Lebesgue measure. The arc we construct in this context must have positive two-dimensional Lebesgue measure in any neighborhood of any of its points; and this alone is by no means sufficient to ensure that $\mathcal{P}$ is dense in $\mathbb{A}^t(E_\Gamma)$. We begin with a description of the building blocks of our construction, which have precedent in the literature; cf. [8], pp. 136–138. First of all, we call a subset of $\mathbb{C}$ a *square* if it has the form $\{x+iy : a\le x\le b \text{ and } c\le y\le d\}$, where $0<b-a=d-c$. Let $\Sigma=\{x+iy : 0\le x, y\le 1\}$. For any nonincreasing sequence $\Lambda=\{\varepsilon_n\}_{n=1}^{\infty}$ in $(0,1]$, let $\mathcal{C}_\Lambda$ denote the Cantor-type subset of $I:=[0,1]$ obtained by deleting from I the middle open interval of length $\varepsilon_1 3^{-1}$ and, proceeding recursively, in stage n, $n\ge 2$, deleting the middle open intervals of length $\varepsilon_n 3^{-n}$ from each of the intervals remaining from the previous stage. Let I_n denote the union of the intervals remaining after stage n; then $\mathcal{C}_\Lambda=\bigcap_{n=1}^{\infty} I_n$. This process generalizes the well-known case that Λ is a constant

sequence, $\varepsilon = \varepsilon_n$ for all n, where $\mathcal{C}_\Lambda$ is easily found to have Lebesgue measure equal to $1-\varepsilon$. In general, $\{x+iy : x, y \in \mathcal{C}_\Lambda\}$ is a compact subset of Σ with two-dimensional Lebesgue measure equal to the square of the Lebesgue measure of $\mathcal{C}_\Lambda$. For $n \geq 1$, let $\Sigma_n = \{x+iy : x, y \in I_n\}$—a finite, pairwise disjoint union of squares of equal size that contains $\{x+iy : x, y \in \mathcal{C}_\Lambda\}$. The equality between $\{x+iy : x, y \in \mathcal{C}_\Lambda\}$ and $\bigcap_{n=1}^{\infty} \Sigma_n$ leads to the description of a Jordan arc γ in Σ that contains $\{x+iy : x, y \in \mathcal{C}_\Lambda\}$ and has endpoints 0 and 1; cf. [8], pp. 136–138, or continue with the development here. In Figure 1 we illustrate the squares of Σ_1 along with three connecting segments. We denote this set by Σ_1^* and call it *a first string of squares for* Σ *based on its lower side*. We could base this string of squares on any of the other three sides of Σ, which would simply amount to a rotation of Σ_1^* (as depicted) through $\frac{1}{2}\pi$, π or $\frac{3}{2}\pi$ radians about the point $\frac{1}{2}+\frac{1}{2}i$. A second string of squares for Σ (based on its lower side), denoted Σ_2^*, involves the squares of Σ_2 and is obtained from a first string by replacing its squares by rotated and contracted versions of itself, as illustrated in Figure 1. One can recursively repeat this process for all n and produce a string of squares Σ_n^* for each collection of squares Σ_n.

Associated with the first and second strings of squares mentioned above are Jordan arcs γ_1 and γ_2, respectively, as illustrated in Figure 2.

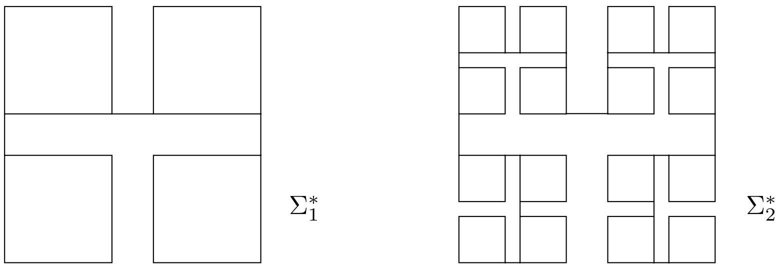

Figure 1.

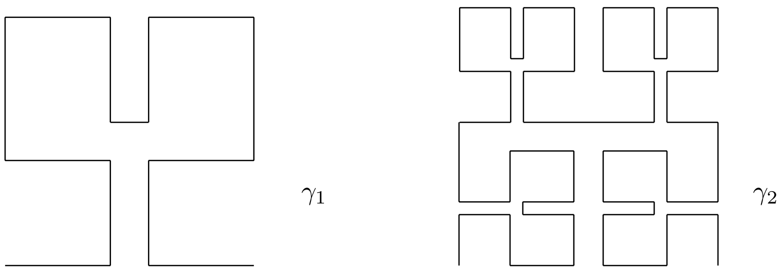

Figure 2.

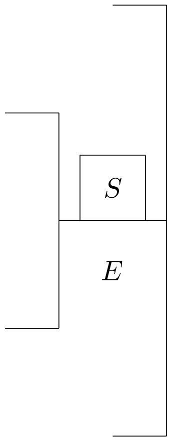

Figure 3.

In this way one generates a sequence of Jordan arcs $\{\gamma_n\}_{n=1}^{\infty}$ that converges uniformly to a Jordan arc γ in Σ, with endpoints 0 and 1, such that $\{x+iy\colon x, y\in\mathcal{C}_\Lambda\}\subseteq\gamma$. Notice that $\gamma=\bigcap_{n=1}^{\infty}\Sigma_n^*$. Countably many portions of this arc γ are segments, which, of course, have zero two-dimensional Lebesgue measure. And so γ itself is not the candidate we are looking for here. The arc that does the job for us has the form of γ, though iterated ad infinitum over all of the segments that arise. It turns out to be much easier to work with sequences of strings of squares. The proof of our main theorem proceeds along these lines.

Theorem 4.1. *For any t, $1\leq t<\infty$, there is a Jordan arc Γ with endpoints* 1 *and* 2 *such that $\Gamma\setminus\{1,2\}\subseteq E$ and with the property that $\mathcal{P}$ is dense in $\mathbb{A}^t(E_\Gamma)$.*

Proof. Let S be a square of side-length less than one, centered and resting on the interval $[1,2]$ in E; see Figure 3.

If S were enlarged to have side-length equal to 1, then $\overline{E\setminus S}$ would be a compact subset of $\mathbb{C}$ whose complement in $\mathbb{C}$ has just one component, and that component would contain 0. Thus, by Runge's theorem, we could find a polynomial p_1 such that $|1/z-p_1(z)|^t<\frac{1}{12}$ for all z in $\overline{E\setminus S}$. Therefore, we would have:

$$\int_{E\setminus S}\left|\frac{1}{z}-p_1(z)\right|^t dA(z)<1.$$

Now, since $|1/z-p_1|^t$ is integrable over E with respect to two-dimensional Lebesgue measure, $\int_{E\setminus S}|1/z-p_1(z)|^t\,dA(z)$ varies continuously with respect to the side-length

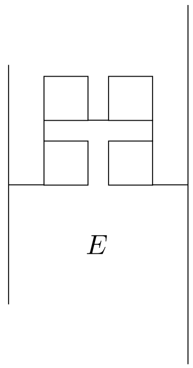

Figure 4.

of S. Hence, we can find S, centered and resting on the interval $[1,2]$, with side-length less than 1, such that

$$\int_{E\setminus S}\left|\frac{1}{z}-p_1(z)\right|^t dA(z)<1;$$

and we may assume that S has side-length at least $\frac{3}{5}$. Let $K_1=[1,2]\cup S$. Once again using the absolute continuity of the integral, we now replace S by a first string of squares S_1^* for S based on its lower side, chosen with squares sufficiently large so that

$$\int_{E\setminus S_1^*}\left|\frac{1}{z}-p_1(z)\right|^t dA(z)<1.$$

Notice that $[1,2]\setminus S$ has two components. Attaching these to S_1^*, we have a string of squares, let us call it K_1^*, that starts at 1 and ends at 2; see Figure 4.

This sets the stage for a repetition of our algorithm. On each of the exposed segments of K_1^* we now install a square, five in all, centered (between existing squares) and having side-lengths less than the lengths of the segments on which they rest; see Figure 5. Let K_2 denote the union of these five new squares along with K_1^*.

If these five new squares were increased in size so that their side-lengths were to equal the lengths of the segments on which they rest, then $\overline{E\setminus K_2}$ would be a compact set in $\mathbb{C}$ whose complement in $\mathbb{C}$ has just one component, and that component would contain 0. Therefore, by Runge's theorem, we could find a polynomial p_2 such that $|1/z-p_2(z)|^t<\frac{1}{24}$ for all z in $\overline{E\setminus K_2}$. Thus, we would have

$$\int_{E\setminus K_2}\left|\frac{1}{z}-p_2(z)\right|^t dA(z)<\frac{1}{2}.$$

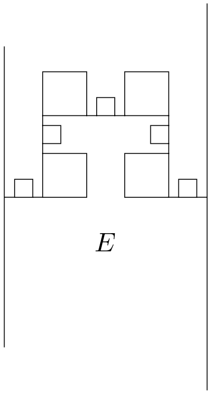

Figure 5.

And, since $|1/z-p_2|^t$ is integrable over E with respect to two-dimensional Lebesgue measure, $\int_{E\setminus K_2}|1/z-p_2(z)|^t\,dA(z)$ varies continuously with respect to the side-lengths of these five new squares. Hence, we can choose them with side-lengths less than the lengths of the segments on which they rest such that

$$\int_{E\setminus K_2}\left|\frac{1}{z}-p_2(z)\right|^t dA(z)<\frac{1}{2}.$$

Since $S_1^*\subseteq K_2$, we also have

$$\int_{E\setminus K_2}\left|\frac{1}{z}-p_1(z)\right|^t dA(z)<1.$$

And, as before, we may assume that these five new squares have side-lengths at least three-fifths the lengths of the exposed segments of K_1^* on which they rest. Proceeding recursively, we now replace S_1^* with a second string of squares for S based on its lower side, which we call S_2^* (see Figure 1); leaving the aforementioned five new squares in position. And then we replace each of these five squares by a choice of one of its first string of squares, based on the side of the square that makes contact with K_1^*. Let K_2^* denote the resulting set. Now all of these choices can be made so that

$$\int_{E\setminus K_2^*}\left|\frac{1}{z}-p_1(z)\right|^t dA(z)<1,$$

and

$$\int_{E\setminus K_2^*}\left|\frac{1}{z}-p_2(z)\right|^t dA(z)<\frac{1}{2}.$$

As before, we now install a square on each of the exposed segments of K_2^*, centered (between existing squares) and of side-lengths less than the lengths of the segments on which they rest, and proceed as in earlier iterations. We continue this process ad infinitum to obtain a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ and a sequence of strings of squares $\{K_n\}_{n=1}^{\infty}$ such that

$$\int_{E\setminus K_n}\left|\frac{1}{z}-p_k(z)\right|^t dA(z)<\frac{1}{k}, \tag{4.1}$$

whenever $1\le k\le n$. We define Γ to be the limit supremum of the sequence $\{K_n\}_{n=1}^{\infty}$, namely

$$\Gamma:=\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}K_n.$$

Then

$$E\setminus\Gamma=\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}(E\setminus K_n)$$

is the limit infimum of the sequence $\{E\setminus K_n\}_{n=1}^{\infty}$. Thus, by (4.1) and Fatou's lemma,

$$\int_{E\setminus\Gamma}\left|\frac{1}{z}-p_k(z)\right|^t dA(z)\le\frac{1}{k}$$

for $k=1,2,3,\dots$. This tells us that $1/z\in P^t(\nu_\Gamma)$. What remains to be shown is that Γ (as defined above) is a Jordan arc that lies in E, except for its endpoints 1 and 2. Once we have this, we can refer to our discussion in Section 2 to get that $\mathcal{P}$ is dense in $\mathbb{A}^t(E_\Gamma)$. To resolve this final issue concerning Γ, we first review our construction of the K_n's, starting with K_1. That we chose S to have side-length at least $\frac{3}{5}$ ensures that each of the squares in K_2 has diameter less than one-half the diameter of S. And that we chose the five additional squares of stage two to have side-lengths at least three-fifths the lengths of the exposed segments of K_1^* on which they rest, ensures that each square in K_3 has diameter less than one-half the diameter of any square in S_1^* and hence less than one-fourth the diameter of S. In general,

$$\max\{\text{diameter}(\sigma):\sigma\text{ is a square in }K_n\}\le\frac{\text{diameter}(S)}{2^{n-1}} \tag{4.2}$$

for all n. For any positive integer N, let $J_N=\bigcup_{n=N}^{\infty}K_n$. We call a square σ in some K_n a maximal square (of $\{K_n\}_{n=1}^{\infty}$) if whenever σ' is a square in some K_m, then either $\sigma'\subseteq\sigma$ or $\sigma'\cap\sigma=\varnothing$. Notice that the first square mentioned in this whole process, namely S, is a maximal square. Moreover, every maximal square rests on the interval $[1,2]$, no pair of them have any point in common and there are countably many of them: $\{\sigma_k\}_{k=1}^{\infty}$. And indeed, $[1,2]\cup\left(\bigcup_{k=1}^{\infty}\sigma_k\right)=J_1$. So J_1 looks like $[1,2]$ along with squares whose bases are the closures of the complementary intervals of some Cantor-type subset of $[1,2]$. It follows that J_1 is a continuum that is contained in $E\cup\{1,2\}$. Observe that J_2 is J_1 with four open swaths deleted; and by an open swath we mean the interior of a rectangle or a set that looks like a contracted, rotated version of $\{x+iy{:}1<x<2 \text{ and } 0<y<r\}\backslash J_1$ for some $r\geq 1$. And these swaths can be chosen with lengths less than the side-length of S. Hence, J_2 is itself a continuum contained in $E\cup\{1,2\}$. Similarly, J_3 is obtained from J_2 by deleting finitely many open swaths having lengths no greater than one-half the lengths of the earlier swaths. In general, J_N is found to be a continuum in $E\cup\{1,2\}$ that contains 1 and 2. Since $\Gamma=\bigcap_{N=1}^{\infty}J_N$, it follows that Γ itself is a continuum in $E\cup\{1,2\}$ that contains 1 and 2; cf. [13], Theorem 28.2. We now argue that every point α in $\Gamma\backslash\{1,2\}$ is a so-called *cut-point* of Γ; that is, $\Gamma\backslash\{\alpha\}$ is not connected. By our analysis above and (4.2), any point in $\Gamma\backslash\{1,2\}$ can be reached from above (and below) in $E\backslash\Gamma$ by traversing (at most) a sequence of swaths that decrease in length by one-half each time. From this it follows that any point α in $\Gamma\backslash\{1,2\}$ can be reached from above (and below) in $E\backslash\Gamma$ via a rectifiable arc. Hence, for any such α we can build a Jordan curve C_α such that $C_\alpha\subseteq E$, $1\in\text{inside}(C_\alpha)$, $2\in\text{outside}(C_\alpha)$ and $\{\alpha\}=\Gamma\cap C_\alpha$. Therefore, by the Jordan curve theorem, α is a cut-point for Γ; and indeed, $\Gamma\backslash\{1,2\}$ is the set of cut-points for Γ. Applying Theorem 28.13 of [13], we conclude that Γ is a Jordan arc that lies in E, except for its endpoints 1 and 2; which completes our proof. $\square$

The method described just after the proof of Theorem 3.3 carries over to this context to give us a Jordan arc Γ, with endpoints 0 and 1, such that $\Gamma\backslash\{1\}\subseteq\Omega$ and with the property that $\mathcal{P}$ is dense in $\mathbb{A}^t(\Omega_\Gamma)$. Let φ be a conformal mapping from $\mathbb{D}$ one-to-one and onto Ω such that $\varphi(0)=0$ and $\varphi(1)=1$; and let $\psi=\varphi^{-1}$. Then $\psi(\Gamma)$ is a Jordan arc with endpoints 0 and 1 such that $\psi(\Gamma)\backslash\{1\}\subseteq\mathbb{D}$. Let η denote two-dimensional Lebesgue measure on $\mathbb{D}\backslash\psi(\Gamma)$ and define μ on $\mathbb{D}\backslash\psi(\Gamma)$ by: $d\mu=|\varphi'|^2d\eta$. Since φ is uniformly approximable by polynomials on $\overline{\mathbb{D}}$, a change of variables argument gives us that the set of analytic bounded point evaluations for $P^t(\mu)$ is $\mathbb{D}\backslash\psi(\Gamma)$. Yet $|\varphi'|$ is bounded below by a positive constant on $\mathbb{D}$; see Theorem A.1 in Appendix A. Therefore, $\mathbb{D}\backslash\psi(\Gamma)$ is the set of analytic bounded point evaluations for $P^t(\eta)$. It now follows that $\mathcal{P}$ is dense in $\mathbb{A}^t(\mathbb{D}\backslash\psi(\Gamma))$.

Corollary 4.2. *For any t, $1\leq t<\infty$, there is a Jordan arc Γ with endpoints 0 and 1, such that $\Gamma\backslash\{1\}\subset\mathbb{D}$ and with the property that $\mathcal{P}$ is dense in $\mathbb{A}^t(\mathbb{D}\backslash\Gamma)$.*

Appendix A

We do not claim that the next (and last) result of this paper is new to the literature, yet we have no easy reference for it. We include it and a proof.

Theorem A.1. *Let G be a convex region in $\mathbb{C}$, $G\neq\mathbb{C}$, let w_0 be a point in G and let $\delta(w_0)$ denote the distance from w_0 to ∂G. If φ is a conformal mapping from $\mathbb{D}$ onto G such that $\varphi(0)=w_0$, then, for all z in $\mathbb{D}$,*

$$|\varphi'(z)| \geq \frac{\delta(w_0)}{2}.$$

Proof. Now, since G is convex, we can express G as $G=\bigcup_{n=1}^{\infty} G_n$, where $w_0\in G_n\subset G_{n+1}$ for all n and each G_n is the interior of a convex polygon. For each n, let φ_n be the conformal mapping from $\mathbb{D}$ onto G_n such that $\varphi_n(0)=w_0$ and $\psi_n'(w_0)>0$, where $\psi_n:=\varphi_n^{-1}$. Notice that $\psi_{n+1}\circ\varphi_n$ maps $\mathbb{D}$ conformally into $\mathbb{D}$ and sends 0 to 0; and hence, $0<\varphi_n'(0)\leq\varphi_{n+1}'(0)$ for all n. Therefore, by a normal families argument and a corollary to Hurwitz's theorem, $\{\varphi_n\}_{n=1}^{\infty}$ has a subsequence (indeed, $\{\varphi_n\}_{n=1}^{\infty}$) that converges uniformly on compact subsets of $\mathbb{D}$ to a conformal mapping φ from $\mathbb{D}$ onto G, where $\varphi(0)=w_0$. In view of this, we may reduce to the case that G itself is the interior of a convex polygon. Proceeding along these lines, let $[\alpha,\beta]:=\{(1-t)\alpha+t\beta:0\leq t\leq 1\}$, $\alpha\neq\beta$, be a segment contained in ∂G and let $\mathbb{H}$ be the half-plane that contains G such that $\partial\mathbb{H}=L:=\{(1-t)\alpha+t\beta:t\in\mathbb{R}\}$. Let φ be as in our hypothesis, let $\psi=\varphi^{-1}$, and let $\xi\mapsto P_{w_0}(\xi)$ denote the Poisson kernel on L for evaluation at w_0; cf. [6], p. 12. Then, by the conformal invariance of harmonic measure and the maximum principle,

$$|\psi'(\xi)| = \frac{2\pi d\omega(\xi, G, w_0)}{|d\xi|} \leq 2\pi P_{w_0}(\xi)$$

for all ξ in $[\alpha,\beta]$. Now $P_{w_0}(\xi)$ attains a maximum on L at ξ_0—the projection of w_0 on L—and that maximum value is $1/\pi|\xi_0-w_0|$. And, clearly, $\delta(w_0)\leq|\xi_0-w_0|$. Therefore, $|\varphi'(\zeta)|\geq\frac{1}{2}\delta(w_0)$ for all ζ in $\mathbb{T}$. Since G is a Smirnov domain (cf. [5], Section 10.3) it follows that $|\varphi'(z)|\geq\frac{1}{2}\delta(w_0)$ for all z in $\mathbb{D}$. $\square$

References

1. Akeroyd, J., Density of the polynomials in the Hardy space of certain slit domains, *Proc. Amer. Math. Soc.* **115** (1992), 1013–1021.
2. Aleman, A., Richter, S. and Sundberg, C., Nontangential limits in $P^t(\mu)$-spaces and the index of invariant subspaces, *Ann. of Math.* **169** (2009), 449–490.

3. Brennan, J. E., Approximation in the mean by polynomials on non-Carathéodory domains, *Ark. Mat.* **15** (1977), 117–168.
4. Conway, J. B., *The Theory of Subnormal Operators*, Math. Surveys Monogr. **36**, Amer. Math. Soc., Providence, RI, 1991.
5. Duren, P. L., *Theory of H^p Spaces*, Academic Press, New York, 1970.
6. Garnett, J. B., *Bounded Analytic Functions*, Academic Press, Orlando, FL, 1981.
7. Garnett, J. B. and Marshall, D. E., *Harmonic Measure*, Cambridge University Press, New York, 2005.
8. Gelbaum, B. R. and Olmstead, J. M. H., *Counterexamples in Analysis*, Dover, Mineola, NY, 2003.
9. Hastings, W. W., A construction of Hilbert spaces of analytic functions, *Proc. Amer. Math. Soc.* **74** (1979), 295–298.
10. Mergelyan, S. N., On the completeness of systems of analytic functions, *Uspekhi Mat. Nauk* **8** (1953), 3–63 (Russian). English transl.: *Amer. Math. Soc. Transl.* **19** (1962), 109–166.
11. Shields, A. L., Weighted shift operators and analytic function theory, in *Topics in Operator Theory*, Math. Surveys **13**, pp. 49–128, Amer. Math. Soc., Providence, RI, 1974.
12. Thomson, J. E., Approximation in the mean by polynomials, *Ann. of Math.* **133** (1991), 477–507.
13. Willard, S., *General Topology*, Addison-Wesley, Reading, MA, 1970.

John R. Akeroyd
Department of Mathematics
University of Arkansas
Fayetteville, AR 72701
U.S.A.
akeroyd@uark.edu

Received March 16, 2009
published online September 25, 2009