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Quasi-parabolic analytic transformations of $\mathbf{C}^n$. Parabolic manifolds

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Abstract. In [Rong, F., Quasi-parabolic analytic transformations of $\mathbf{C}^n$, *J. Math. Anal. Appl.* **343** (2008), 99–109], we showed the existence of "parabolic curves" for certain quasi-parabolic analytic transformations of $\mathbf{C}^n$. Under some extra assumptions, we show the existence of "parabolic manifolds" for such transformations.

1. Introduction

In this paper, we continue to study the local dynamics of quasi-parabolic analytic transformations of $\mathbf{C}^n$. An analytic germ Φ of $\mathbf{C}^n$ at a fixed point p is said to be *quasi-parabolic* if $d\Phi_p=\mathrm{Diag}(I,\Lambda)$, where I is the identity matrix and Λ is a diagonal matrix with eigenvalues λ_j, such that $|\lambda_j|=1$ and $\lambda_j\neq 1$. Such transformations have been studied by Bracci and Molino [1] in dimension two, and by the author ([4] and [5]) in any dimensions.

As in [5], we suppose that Φ is of finite *order* ν, that Φ has a *non-degenerate characteristic direction* $[v]$, and that Φ is *dynamically-separating* in the direction $[v]$ (see Section 2 for the precise definitions of the terms just mentioned). In [5], we associated with the direction $[v]$ an $(n-1,n-1)$-matrix N, whose eigenvalues are invariants of Φ in the direction $[v]$. We divide the eigenvalues of N into two sets, counted with multiplicity, $\{\gamma_j\}_{j=1}^a$ and $\{\mu_k\}_{k=1}^b$, with $a+b=n-1$, in such a way that for some positive real number α, we have

$$\begin{aligned} &\operatorname{Re}\gamma_j>\alpha>0 \quad \text{for } j=1,\dots,a,\\ &\operatorname{Re}\mu_k<\alpha \qquad\quad \text{for } k=1,\dots,b. \end{aligned} \tag{1.1}$$

A *parabolic manifold* of dimension d for Φ at p is an injective holomorphic map $\varphi\colon\Delta\to\mathbf{C}^n$ satisfying the following properties:

(i) Δ is a simply-connected domain in $\mathbf{C}^d$ with $0 \in \partial\Delta$;

(ii) φ is continuous on $\partial\Delta$ and $\varphi(0)=p$;

(iii) $\varphi(\Delta)$ is invariant under Φ and $\Phi^k(\varphi(\zeta)) \to p$, as $k \to \infty$, for any $\zeta \in \Delta$.

Furthermore, if $[\varphi(\zeta)] \to [v] \in \mathbf{P}^{n-1}$ as $\zeta \to 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbf{C}^n \setminus \{p\}$ onto $\mathbf{P}^{n-1}$), we say that φ is *tangent to* $[v]$ at p. A parabolic manifold of dimension one is also called a *parabolic curve*.

Our main result is the following.

Theorem 1.1. *Let Φ be an analytic transformation of $\mathbf{C}^n$, with a quasi-parabolic fixed point p. Assume that Φ is of finite order ν, that $[v]$ is a non-degenerate characteristic direction for Φ, and that Φ is dynamically-separating in the direction $[v]$. Let N be the matrix associated with $[v]$ and assume that we divide the eigenvalues of N as in (1.1). Then there exist at least $\nu-1$ parabolic manifolds of dimension $a+1$ tangent to $[v]$ at p.*

In Section 2, we recall some definitions and results from [5]. We prove the main theorem in Section 3, following the approach of Hakim in [3]. In Section 4, we extend some other results of Hakim in [3] to the quasi-parabolic case.

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2. Preliminaries

In this section, we briefly recall some definitions and results for quasi-parabolic analytic transformations of $\mathbf{C}^n$. We refer the reader to [5] for details (cf. [1]).

Let Φ be an analytic transformation of $\mathbf{C}^n$, with an isolated quasi-parabolic fixed point p. In some local coordinates around p, $u=(w,z) \in \mathbf{C}^l \times \mathbf{C}^m$, $l+m=n$, we can write Φ as

$$\begin{cases} w_1 = w + a_2(u) + a_3(u) + \ldots, \\ z_1 = \Lambda z + b_2(u) + b_3(u) + \ldots, \end{cases}$$

where $a_i(u)$ and $b_i(u)$ are homogeneous polynomials of degree $i \geq 2$. Let $\Phi_i(u) = (a_i(u), b_i(u))$.

Let ν (resp. μ) be the least $|j|$ for the terms w^j in the expression for w_1 (resp. z_1). If $\nu < \infty$ and $\mu \geq \nu$, then we say that Φ is *ultra-resonant*, and that the *order* of Φ is ν. This is well defined by [5, Lemma 2.3].

Assume now that Φ is ultra-resonant of order ν. A *characteristic direction* for Φ is a vector $[v]=[v_1:...:v_n]\in\mathbf{P}^{n-1}$, with $v_i=0$ for $l<i\le n$, such that $\Phi_\nu(v)=\lambda v$ for some $\lambda\in\mathbf{C}$. If $\lambda\neq 0$, we say that $[v]$ is *non-degenerate*, otherwise it is *degenerate*.

Assume now that Φ has a non-degenerate characteristic direction $[v]$. In some suitable local coordinates $u=(x,y,z)\in\mathbf{C}\times\mathbf{C}^{l-1}\times\mathbf{C}^m$, we can assume that $[v]=[1:0:0]$. We can then write Φ as

$$\begin{cases} x_1 = x+p_\nu(x,y,0)+P(x,y,z)+O(\nu+1),\\ y_1 = y+q_\nu(x,y,0)+Q(x,y,z)+O(\nu+1),\\ z_1 = \Lambda z+R(x,y,z)+O(\nu+1),\end{cases}$$

where $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ contain terms $x^i y^j z^k$ with $2\le i+|j|+|k|\le\nu$ and $|k|\ge 1$.

Write $\Lambda=\mathrm{Diag}(\Lambda_1,...,\Lambda_r)$, where $\Lambda_i=\lambda_{(i)}I_{m_i}$ with the $\lambda_{(i)}$'s being mutually different and $m_1+...+m_r=m$. Similarly, write $z=(z_{(1)},...,z_{(r)})$ with $z_{(i)}=(z_{(i),1},...,z_{(i),m_i})$ and write $R(x,y,z)=(R_1(x,y,z),...,R_r(x,y,z))$. By [5, Lemma 2.7], we can assume, after a suitable change of coordinates, that there are no terms $x^i y^j z_{(q),s}$ in $R_p(x,y,z)$ for any $1\le p\le r$ and $q\neq p$. We say that Φ is *dynamically separating* in the characteristic direction $[v]=[1:0:0]$ if $R_p(x,y,z)$ contains no terms $x^i z_{(p),s}$ with $i<\nu-1$ for any $1\le p\le r$. This is well defined by [5, Lemma 2.10].

In [5], we obtained the following result.

Theorem 2.1. ([5], Theorem 1.5) *Let Φ be an analytic transformation of $\mathbf{C}^n$, with a quasi-parabolic fixed point p. If Φ is of finite order ν and is dynamically separating in a non-degenerate characteristic direction $[v]$, then there exist at least $\nu-1$ parabolic curves tangent to $[v]$ at p.*

3. Parabolic manifolds

Let Φ be an analytic transformation of $\mathbf{C}^n$, with a quasi-parabolic fixed point. Assume that $[v]=[1:0:...:0]\in\mathbf{P}^{n-1}$ is a non-degenerate characteristic direction for Φ and that Φ is dynamically-separating in the direction $[v]$. For simplicity, we will assume that the order of Φ is $\nu=2$. (One can easily pass from $\nu=2$ to an arbitrary ν as in [3, Section 6].)

Choose local coordinates (x,y,z) such that the parabolic curve in Theorem 2.1 is given by $\zeta=0$, where $\zeta=(y,z)$. By [5, Proposition 2.14], after a finite number of blow-ups and changes of coordinates, we can write Φ as

$$(3.1)\qquad \begin{cases} x_1 = x - x^2 + O(x^2\|\zeta\|) + O_1(x^3), \\ y_1 = (I - xA)y + O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|), \\ z_1 = (\Lambda - xB)z + O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|), \end{cases}$$

where $O_1(*) = O(*\cdot(\log x)^\mu)$ for some unspecified integer $\mu \geq 0$. (Note that we always work in a region where $\log x$ is well defined.) Moreover, we can assume, without loss of generality, that both A and B are in Jordan canonical form. We also know that the eigenvalues of A and $\Lambda^{-1}B$ are invariants associated with the non-degenerate characteristic direction $[v]$. Dividing these eigenvalues into subsets according to their real parts, we can rewrite (3.1) as

$$(3.2)\qquad \begin{cases} x_1 = x - x^2 + O(x^2\|\zeta\|) + O_1(x^3), \\ u_1 = (I_p - xA_p)u + O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|), \\ v_1 = (I_q - xA_q)v + O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|), \\ r_1 = (\Lambda_l - xB_l)r + O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|), \\ s_1 = (\Lambda_m - xB_m)s + O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|). \end{cases}$$

Set $w = (u, r)$, $t = (v, s)$, $J = \mathrm{Diag}(I_p, \Lambda_l)$, $K = \mathrm{Diag}(I_q, \Lambda_m)$, $C = \mathrm{Diag}(A_p, B_l)$ and $D = \mathrm{Diag}(A_q, B_m)$. We can then rewrite (3.2) as

$$(3.3)\qquad \begin{cases} x_1 = f(x, w, t) = x - x^2 + F(x, w, t), \\ w_1 = g(x, w, t) = (J - xC)w + G(x, w, t), \\ t_1 = h(x, w, t) = (K - xD)t + H(x, w, t), \end{cases}$$

with

$$F(x, w, t) = O(x^2\|\zeta\|) + O_1(x^3),$$
$$G(x, w, t) = O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|),$$
$$H(x, w, t) = O(x\|\zeta\|^2) + O_1(x^2\|\zeta\|).$$

If $\{\gamma_j\}_{j=1}^a$ and $\{\mu_k\}_{k=1}^b$ are the eigenvalues for $L = J^{-1}C$ and $M = K^{-1}D$, respectively, we require that $\mathrm{Re}\,\gamma_j > \alpha > 0$ for $1 \leq j \leq a$ and $\mathrm{Re}\,\mu_k < \alpha$ for $1 \leq k \leq b$.

Proposition 3.1. *Let Φ be as in (3.3). Then, for all integers $m, k \geq 2$, we can choose local coordinates such that $H(x, w, 0) = O(x\|w\|^m + x^k\|w\|)$.*

Proof. Set $E_\beta = \{\gamma \in \mathbf{N} : |a_{\beta,\gamma}(\zeta)| \not\equiv 0\}$, where $a_{\beta,\gamma}(\zeta)$ are the coefficients of the terms $x^\beta(\log x)^\gamma$ in (3.3). It is a finite set as a consequence of the construction in [2].

Write $H(x,w,0)$ as

$$H(x,w,0)=\sum_{\substack{1\le\beta\le k-1\\ \gamma\in E_\beta}} x^\beta(\log x)^\gamma a_{\beta,\gamma}(w)+O_1(x^k\|w\|).$$

We want to show that there exists a change of coordinates such that $a_{\beta,\gamma}(w)=O(\|w\|^m)$. This is done by induction on β, γ and the degree d of homogeneous terms in $a_{\beta,\gamma}(w)$. We start with the smallest β, the biggest γ in E_β and the smallest d of homogeneous terms in $a_{\beta,\gamma}(w)$.

We only consider the term $x^\beta(\log x)^\gamma Q(w)$ in the expression of s_1 in (3.2), where $Q(w)$ is a $\mathbf{C}^m$-valued homogeneous polynomial of degree d. Such a term in the expression of v_1 in (3.2) can be dealt with similarly.

We need to break down Λ_m into Jordan blocks and consider each block separately. For simplicity, we will assume without loss of generality that $\Lambda_m=\lambda I_m$.

For a monomial w^l, $l=(l_1,...,l_a)\in\mathbf{N}^a$, we define the weight of w^l as $\omega(l):=\sum_{i=1}^a il_i$. We start with monomials w^l in $Q(w)$ with the smallest weight.

Let $R(w)=(R_1(w),...,R_m(w))$, with $R_i(w)=c_iw^l$, $c_i\in\mathbf{C}$, $1\le i\le m$. Since $R(w)$ is a monomial, we have $R(Jw)=\tilde\lambda R(w)$ for some constant $\tilde\lambda\in\mathbf{C}$.

If $\tilde\lambda\ne\lambda$, the following transformation cancels the term $x^\beta(\log x)^\gamma R(w)$:

$$s=\tilde s+\frac{1}{\tilde\lambda-\lambda}x^\beta(\log x)^\gamma R(w).$$

If $\tilde\lambda=\lambda$, we consider the transformation of the form

$$s=\tilde s+x^{\beta-1}(\log x)^\gamma P(w), \tag{3.4}$$

where $P(w)=(P_1(w),...,P_m(w))$, with $P_i(w)=e_iw^l$, $e_i\in\mathbf{C}$, $1\le i\le m$. One readily checks that such a transformation cancels the term $x^\beta(\log x)^\gamma R(w)$ if and only if

(3.5)

$$P((J-xC)w)-\lambda I_mP(w)+x(B_m-(\beta-1)\lambda I_m)P(w)=xR(w)+x\widetilde R(w)+O(x^2\|w\|),$$

where $\widetilde R(w)$ is a sum of monomials with bigger weights.

The linear part of the left-hand side of (3.5) is equal to

$$x[-\lambda\langle \operatorname{grad}P_i, Lw\rangle+((B_m-(\beta-1)\lambda I_m)P(w))_i],\quad 1\le i\le m.$$

Since B_m is in Jordan canonical form, the above system is reduced to

$$x\lambda\left[-\left(\frac{\partial P_i}{\partial w_1}(Lw)_1+...+\frac{\partial P_i}{\partial w_a}(Lw)_a\right)+((\mu_{i+q}-\beta+1)P_i+\varepsilon_{i,i+1}P_{i+1})\right],\quad 1\le i\le m,$$

where $\varepsilon_{i,i+1}$ is 0 or 1 if $i<m$ and $\varepsilon_{m,m+1}=0$.

By decreasing induction on i from $i=m$ to $i=1$, we see that (3.5) is satisfied with

$$e_m=[l_1\gamma_1+\ldots+l_a\gamma_a-(\mu_{m+q}-\beta+1)]^{-1}\left(-\frac{1}{\lambda}c_m\right)$$

and

$$e_i=[l_1\gamma_1+\ldots+l_a\gamma_a-(\mu_{i+q}-\beta+1)]^{-1}\left(\varepsilon_{i,i+1}e_{i+1}-\frac{1}{\lambda}c_i\right),\quad 1\le i<m.$$

Note that $\operatorname{Re}[l_1\gamma_1+\ldots+l_a\gamma_a-(\mu_{i+q}-\beta+1)]>(d\alpha-\operatorname{Re}\mu_{i+q})+\beta-1>0$, $1\le i\le m$. So $l_1\gamma_1+\ldots+l_a\gamma_a-(\mu_{i+q}-\beta+1)\ne 0$, $1\le i\le m$.

Thus, an induction on the weight $\omega(l)$ will get rid of all terms in $x^\beta(\log x)^\gamma w^l$ for fixed β and γ. Note that the transformation (3.4) also introduces new terms in $x^\beta(\log x)^{\gamma-1}w^l$, which are taken care of by our induction on γ.

Finally we finish with an induction on d and β. $\square$

For $\tau,\sigma,\rho>0$ small enough, let

$$\Delta_{\tau,\sigma,\rho}=\{(x,w)\in\mathbf{C}\times\mathbf{C}^a:|\operatorname{Im}x|\le\tau\operatorname{Re}x,\ |x|\le\sigma \text{ and } \|w\|<\rho\}.$$

We want to show that there exists a function $t=\varphi(x,w)$, analytic in some $\Delta_{\tau,\sigma,\rho}$, such that

$$\varphi(f(x,w,\varphi(x,w)),g(x,w,\varphi(x,w)))=h(x,w,\varphi(x,w)), \tag{3.6}$$

with $\lim_{\{x\to0,w\to0\}}\varphi(x,w)=0$.

Define

$$H_1(x,w,t):=t-x^MK^{-1}x_1^{-M}t_1. \tag{3.7}$$

Then as in [5, Section 3], we have

$$H_1(x,w,t)=O(x\|\zeta\|^2)+O_1(x^2\|\zeta\|),$$

and, by Proposition 3.1,

$$H_1(x,w,0)=O(x\|w\|^m,x^k\|w\|),\quad m,k\ge2. \tag{3.8}$$

The functional equation (3.6) is equivalent to

$$x^{-M}\varphi(x,w)-K^{-1}x_1^{-M}\varphi(x_1,w_1)=x^{-M}H_1(x,w,\varphi(x,w)). \tag{3.9}$$

Define

$$T\varphi(x,w) = x^M \sum_{i=0}^{\infty} K^{-i} x_i^{-M} H_1(x_i, w_i, \varphi(x_i, w_i)). \tag{3.10}$$

Let B_0 be the Banach space of holomorphic functions φ on the set $\Delta_{\tau,\sigma,\rho} \cap \{(x,w): |x|^{-\alpha}\|w\| \le 1\}$ such that

$$\|\varphi\|_0 := \sup_{x,w} \frac{|\varphi(x,w)|}{\|w\|^m + |x|^{k-1}\|w\|} \tag{3.11}$$

is bounded, endowed with the norm $\|\varphi\|_0$. Since the spectrum of $d\Phi_p$ lies in the unit circle, all the estimates in the proof of [3, Theorem 1.6] carry over to our case. Therefore, we have that T is well defined on a closed subset B_1 of B_0, with $T\varphi \in B_1$. Furthermore, the operator T is continuous and contracting on a closed subset B_2 of B_1. Hence T has a fixed point, which is a solution to (3.9), and thus to (3.6). This proves Theorem 1.1.

4. Fatou coordinates

Let Φ be as in the previous section. We want to take a closer look at the dynamics of Φ in the parabolic manifolds provided by Theorem 1.1. For this, we assume that all the eigenvalues associated with the direction $[1:0:...:0]$ have positive real parts. Thus, we can write Φ as

$$\begin{cases} x_1 = x - x^2 + F(x,w), \\ w_1 = (J - xC)w + G(x,w). \end{cases} \tag{4.1}$$

Let $\{\gamma_j\}_{j=1}^h$ be the distinct eigenvalues of $L = J^{-1}C$, with $\alpha_j = \operatorname{Re}\gamma_j$, $1 \le j \le h$, and $\alpha_1 \ge \alpha_2 \ge ... \ge \alpha_h > \alpha > 0$. Write further $J = \operatorname{Diag}(J_1, ..., J_h)$, $C = \operatorname{Diag}(C_1, ..., C_h)$ and $L = \operatorname{Diag}(L_1, ..., L_h)$. Rewrite Φ as

$$\begin{cases} x_1 = x - x^2 + F(x,w), \\ w_1^1 = (J_1 - xC_1)w^1 + G^1(x,w), \\ \quad \vdots \\ w_1^h = (J_h - xC_h)w^h + G^h(x,w). \end{cases} \tag{4.2}$$

For $j = 1, ..., h$, let $w^{\le j} = (w^1, ..., w^j)$ and $w^{>j} = (w^{j+1}, ..., w^h)$. We have the following proposition (cf. [3, Proposition 4.1]).

Proposition 4.1. *Let Φ be as in (4.2). Then, for all integers $m, k \ge 2$, we can choose local coordinates such that $G^j(x, (0, w^{>j})) = O(x\|w^{>j}\|^m + x^k\|w^{>j}\|)$ for $1 \le j \le h$.*

Proof. We fix j and let $u=w^j$ and $v=w^{>j}$. We also rewrite (4.2) as

$$\begin{cases} x_1 = x - x^2 + F(x,w), \\ w_1^{<j} = (J_{<j} - xC_{<j})w^{<j} + G^{<j}(x,w), \\ u_1 = (J_j - xC_j)u + G^j(x,w), \\ v_1 = (J_{>j} - xC_{>j})v + G^{>j}(x,w). \end{cases} \tag{4.3}$$

Here $G^{<j}=(G^1,...,G^{j-1})$, $G^{>j}=(G^{j+1},...,G^h)$, $J_{<j}=\mathrm{Diag}(J_1,...,J_{j-1})$, $J_{>j}=\mathrm{Diag}(J_{j+1},...,J_h)$, $C_{<j}=\mathrm{Diag}(C_1,...,C_{j-1})$ and $C_{>j}=\mathrm{Diag}(C_{j+1},...,C_h)$. Set $A=J_j^{-1}C_j$ and $B=J_{>j}^{-1}C_{>j}$. Let p and q be the respective dimensions of u and v. Let $\{\beta_1,...,\beta_q\}$ be the eigenvalues of B.

Write $G^j(x,(0,v))$ as

$$G^j(x,(0,v)) = \sum_{\substack{1\le\beta\le k-1 \\ \gamma\in E_\beta}} x^\beta (\log x)^\gamma a_{\beta,\gamma}(v) + O_1(x^k\|v\|),$$

where E_β is as in Proposition 3.1.

We want to show that there exists a change of coordinates such that $a_{\beta,\gamma}(v)=O(\|v\|^m)$.

Depending on the eigenvalues of J_j, we need to divide A into smaller Jordan blocks and consider each block separately. For simplicity, we will assume without loss of generality that $J_j=\lambda I_p$.

As in Proposition 3.1, we work with monomials in $a_{\beta,\gamma}(v)$. So consider terms $x^\beta(\log x)^\gamma R(v)$, where $R(v)=cv^l$ with $c\in\mathbf{C}^p$, $l=(l_1,...,l_q)\in\mathbf{N}^q$ and $l_1+...+l_q=d<m$. We prove the proposition by an induction on β, d, $\omega(l)$ and γ. We start with the smallest $(\beta,d,\omega(l))$. As for γ, there are two different cases which we discuss below.

Since $R(v)$ is a monomial, we have $R(J_{>j}v)=\tilde\lambda R(v)$ for some constant $\tilde\lambda\in\mathbf{C}$. If $\tilde\lambda\neq\lambda$, the following transformation cancels the term $x^\beta(\log x)^\gamma R(v)$:

$$u=\tilde u + \frac{1}{\tilde\lambda-\lambda} x^\beta (\log x)^\gamma R(v).$$

If $\tilde\lambda=\lambda$, there are two different cases, depending on $l\in\mathbf{N}^q$. Let

$$L=\{l=(l_1,...,l_q)\in\mathbf{N}^q:\ l_1\beta_1+...+l_a\beta_q-(\gamma_j-\beta+1)=0\}.$$

For $l\in L$, we consider a transformation of the form

$$u=\tilde u + x^{\beta-1}(\log x)^{\gamma+1}P(v),$$

where $P(v)=ev^l$ with $e\in\mathbf{C}^p$.

One readily checks that if we choose $e=-c/(\gamma+1)$ then the above transformation cancels the term $x^\beta(\log x)^\gamma R(v)$. Note that the terms in $x^\beta(\log x)^{\gamma+1}v^l$ vanish for $l\in L$. Thus, an induction on the weight $\omega(l)$ will get rid of all terms in $x^\beta(\log x)^\gamma v^l$ for fixed β and γ. Note also that we have introduced a new term $x^\beta(\log x)^{\gamma+1}\widetilde{R}(v)$, but with $\widetilde{R}(v)$ as a sum of monomials with bigger weights. So there can only be finitely many such new terms, which are taken care of by an induction on γ, starting with the smallest γ.

For $l\notin L$, we consider a transformation of the form

$$u=\tilde{u}+x^{\beta-1}(\log x)^\gamma P(v).$$

We can solve $P(v)$ as in Proposition 3.1. And an induction on the weight $\omega(l)$ will get rid of all terms in $x^\beta(\log x)^\gamma v^l$ for fixed β and γ. Note that the above transformation also introduces new terms in $x^\beta(\log x)^{\gamma-1}v^l$, with $l\notin L$, which are taken care of by an induction on γ, starting with the biggest γ.

Finally we finish with an induction on d and β. $\square$

With the above proposition, we are now ready to extend Theorems 1.9, 1.10 and 1.11 in [3] to the quasi-parabolic case. First, we have the following result.

Theorem 4.2. *Let Φ be as in (4.1). Then one can choose local coordinates (X,W) such that Φ is conjugated to $(1/X_1=1/X+1, W_1=JW)$.*

Proof. On $\Delta_W:=\Delta_{\tau,\sigma,\rho}\cap\{(x,w):|x|^{-\alpha_j+\varepsilon}\|w^{\le j}\|\le 1 \text{ for all } j\}$, with $0<\varepsilon\ll\alpha_h$, define

$$W(x,w):=x^{-L}w+\sum_{i=0}^{\infty}J^{-i}(J^{-1}x_{i+1}^{-L}w_{i+1}-x_i^{-L}w_i)=\lim_{i\to\infty}J^{-i}x_i^{-L}w_i.$$

As in [3, Lemma 4.5], one sees that the above series converges normally on Δ_W and that $W_1=JW$. We can then proceed as in the proof of [3, Theorem 1.9]. $\square$

We now get back to the general case, with Φ as in (3.3). (We are only going to state our results for $\nu=2$. The statements for $\nu>2$ are similar.) The following is an immediate consequence of the above theorem and Theorem 1.1.

Corollary 4.3. *Let Φ be as in (3.3). One can choose local coordinates (X,W,T) such that the parabolic manifold M provided by Theorem 1.1 is given by $\{(X,W,T):T=0\}$, and that the restriction of Φ to M is conjugated to $(1/X_1=1/X+1, W_1=JW)$.*

When Φ is a global isomorphism of $\mathbf{C}^n$, which takes the form (3.3) at a fixed point p, we can consider the global attracting set of Φ at p. By Theorem 1.1, we will assume that $H(x,w,0)=0$. Define

$$\Omega_p=\big\{Z\in\mathbf{C}^n\setminus\{p\}:\lim_{i\to\infty}Z_i=p \text{ and } \lim_{i\to\infty}[Z_i]=[1:0]\big\},$$

if $\operatorname{Re}\gamma_j>\alpha>0$, $1\le j\le a$, and $\operatorname{Re}\mu_k<0$, $1\le k\le b$, or

$$\Omega_p=\big\{Z\in\mathbf{C}^n\setminus\{p\}:\lim_{i\to\infty}Z_i=p,\ \lim_{i\to\infty}[Z_i]=[1:0] \text{ and } \lim_{i\to\infty}x_i^{-\alpha}t_i=0\big\},$$

if $\operatorname{Re}\gamma_j>\alpha>0$, $1\le j\le a$, and $\operatorname{Re}\mu_k\le 0$, $1\le k\le b$. (For $\nu>2$, one needs to require instead that $\lim_{i\to\infty}x_i^{-(\nu-1)\alpha}t_i=0$.) Note that the definition of Ω_p does not depend on the α chosen.

We have the following theorem, whose proof is essentially the same as that of [3, Theorem 1.10] and [3, Theorem 1.11].

Theorem 4.4. *Let Φ be a global isomorphism of $\mathbf{C}^n$. Let p be a fixed point of Φ, where Φ takes the form* (3.3). *Then Ω_p is isomorphic to $\mathbf{C}^{a+1}$.*

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