

A weighted estimate for the square function on the unit ball in \mathbb{C}^n

Stefanie Petermichl and Brett D. Wick

Abstract. We show that the Luzin area integral or the square function on the unit ball of \mathbb{C}^n , regarded as an operator in the weighted space $L^2(w)$ has a linear bound in terms of the invariant A_2 characteristic of the weight. We show a dimension-free estimate for the “area-integral” associated with the weighted $L^2(w)$ norm of the square function. We prove the equivalence of the classical and the invariant A_2 classes.

1. Introduction

Weighted inequalities for singular integral operators appear naturally in many areas of analysis. The theory of weights is very well understood for the “real analysis” case. In a fundamental paper of Hunt, Muckenhoupt, and Wheeden, [4], it is shown that the so-called Calderón–Zygmund operators in harmonic analysis are bounded on weighted L^p spaces if and only if the weight satisfies the A_p condition. Once this characterization was known, it then became of interest to determine exactly *how* the norms of the operators from harmonic analysis are bounded in terms of the characteristic of the weight. One seeks the smallest power $r=r(p)$ so that $\|Tf\|_{L^p(\omega)} \leq CQ_p(w)^r \|f\|_{L^p(\omega)}$, with C an absolute constant. The best possible power for $p=2$ usually is conjectured to be 1. Though this is only known for very specific operators. The first successful estimate is for a dyadic analog of the square function, see [3] and the martingale transforms in [16] shortly after. As for classical Calderón–Zygmund operators, the best bound is only known for specific operators with certain invariance properties, such as the Hilbert and Beurling transform and the Riesz transforms. See [8], [9] and [10]. Such estimates have applications in partial differential equations, see [1] and [10], and have received considerable attention.

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Much of the theory of harmonic analysis can be extended to the boundary of the unit ball \mathbb{B} since it is a domain of homogeneous type. Many of the notions make sense on the boundary of the unit ball, but some care is needed. In studying “complex analysis” questions, the choice of metric plays a distinguished role, and in this context one should use the non-isotropic metric. This provides some additional difficulties since the “balls” in this metric are actually elliptical, and this provides some difference in the geometry. So, it makes sense to ask about the boundedness properties of singular integral operators whose natural domain are functions on the boundary of the unit ball, \mathbb{S} , for example the Cauchy transform. In particular the paper [6] demonstrates that the Cauchy transform is bounded on $L^p(w)$ if and only if $w \in A_p$. One can then inquire about other operators from harmonic analysis, such as the square function

A major motivation for this note was the paper [3]. In this paper, the authors determined the exact dependence of the norm of the square function (Luzin’s area integral, g-functions, etc.) on $L^2(\mathbb{T}; w)$ in terms of the invariant characteristic of the weight. We extend the work in [3] to the case of the unit ball and its boundary \mathbb{S} . It is interesting to note that our proof has an underlying dyadic idea – similar to all other proofs of optimal bounds in weighted spaces. This is so, even though the unit sphere itself lacks a simple dyadic structure. One manages to utilize the dyadic model in one real variable to obtain a result on the unit sphere, despite the differences in geometry.

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1.1. Definitions

Let \mathbb{B} denote the unit ball in \mathbb{C}^n , i.e. $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ and let $\mathbb{S} = \{z \in \mathbb{C}^n : |z| = 1\}$. We write $z = (z_1, \dots, z_n)$ with $z_k = x_k + iy_k$ and recall that

$$\partial_k = \frac{\partial}{\partial z_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) \quad \text{and} \quad \bar{\partial}_k = \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right).$$

The *Bergman kernel* on the unit ball is given by

$$K(z, w) = \frac{n!}{\pi^n (1 - \langle z, w \rangle)^{n+1}}.$$

Define

$$g_{ij}(z) = \partial_i \bar{\partial}_j \log K(z, z) = \frac{n+1}{(1-|z|^2)^2} [(1-|z|^2)\delta_{ij} + \bar{z}_i z_j]$$

The *Bergman metric* on \mathbb{B} is

$$\beta^2(z, \xi) = \sum_{i,j=1}^n g_{ij}(z) \xi_i \bar{\xi}_j.$$

So the volume element associated to the Bergman metric is

$$dg(z) = K(z, z) dV(z) = \frac{n!}{\pi^n} \frac{dV(z)}{(1-|z|^2)^{n+1}} = \frac{d\nu(z)}{(1-|z|^2)^{n+1}}.$$

Here V is Lebesgue measure on the ball and ν is normalized Lebesgue measure on \mathbb{B} .

Let g^{ij} be the inverse of the matrix g_{ij} , so the *Laplace–Beltrami operator* on the ball is given by

$$\tilde{\Delta} = 4 \sum_{i,j=1}^n g^{ij} \partial_j \bar{\partial}_i = 4 \frac{(1-|z|^2)}{n+1} \sum_{i,j=1}^n [\delta_{ij} - \bar{z}_i z_j] \partial_j \bar{\partial}_i.$$

It also has a radial form, given by

$$\tilde{\Delta} f(z) = \frac{(1-r^2)}{n+1} \left[(1-r^2) f''(r) + \frac{2n-r^2-1}{r} f'(r) \right]$$

if $f(z) = f(|z|)$ for $f \in C^2(\mathbb{B})$. The *invariant gradient* of a $C^1(\mathbb{B})$ -function is the vector field given by

$$\tilde{\nabla} u = 2 \sum_{i,j=1}^n g^{ij} (\bar{\partial}_i u \partial_j + \partial_j u \bar{\partial}_i).$$

The *Poisson kernel* for $\tilde{\Delta}$ is given by $P(z, \zeta) = (1-|z|^2)^n / |1-\langle z, \zeta \rangle|^{2n}$ and the *Green’s function* for $\tilde{\Delta}$ is given by

$$G(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt.$$

For $n=1$ this of course becomes $\log(1/|z|)$. Let $f \in L^2(\mathbb{S})$, define the *Poisson–Szegő integral*

$$\tilde{f}(z) = \int_{\mathbb{S}} P(z, \zeta) f(\zeta) d\sigma(\zeta),$$

where σ is normalized surface measure. For any $\zeta \in \mathbb{S}$ we define the *Korányi admissible approach region* (the analogue of the non-tangential approach region or cone in the unit disc) with aperture $a > 0$ as

$$\Gamma_a(\zeta) = \{z \in \mathbb{B} : |1 - \langle z, \zeta \rangle| < a(1 - |z|^2)\}.$$

Note that we must assume $a > \frac{1}{2}$ otherwise $\Gamma_a(\zeta)$ is empty. It is well known that \tilde{f} converges to f admissibly almost everywhere. The generalized Luzin area integral or square function on the ball with respect to $\Gamma_a(\xi)$ is

$$S_a(f)(\zeta) = \left(\int_{\Gamma_a(\zeta)} |\tilde{\nabla} \tilde{f}(z)|^2 dg(z) \right)^{1/2}.$$

We will be concerned with weighted L^2 spaces on the ball. We say that a positive $w \in L^1_{loc}(\mathbb{S})$ is in the class A_p if

$$Q_p(w) = \sup_B \left(\frac{1}{\sigma(B)} \int_B w(\zeta) d\sigma(\zeta) \right) \left(\frac{1}{\sigma(B)} \int_B w^{1/(1-p)}(\zeta) d\sigma(\zeta) \right)^{p-1} < \infty,$$

where the supremum runs over all non-isotropic balls B on \mathbb{S} . The case of A_2 is especially interesting for us, and a weight is in this class if

$$Q_2(w) = \sup_B \left(\frac{1}{\sigma(B)} \int_B w(\zeta) d\sigma(\zeta) \right) \left(\frac{1}{\sigma(B)} \int_B w^{-1}(\zeta) d\sigma(\zeta) \right) < \infty,$$

and so $Q_2(w) = Q_2(w^{-1})$.

Here, we will be more concerned with the invariant A_2 class, denoted by \tilde{A}_2 . A weight w is in \tilde{A}_2 if and only if

$$\tilde{Q}_2(w) = \sup_{z \in \mathbb{B}} \tilde{w}(z) \widetilde{w^{-1}}(z) < \infty.$$

The quantity $\tilde{Q}_2(w)$ is invariant under Möbius transforms and is therefore more suited for questions on the ball. We will see later that these two classes are the same, i.e. that $A_2 = \tilde{A}_2$ and that

$$Q_2(w) \lesssim \tilde{Q}_2(w) \lesssim Q_2(w)^2,$$

where the implied constants depend upon the dimension n . Here and throughout the paper $A \lesssim B$ means $A \leq CB$ for some absolute constant C .

Let $\mathcal{L}^2(w)$ denote the space of measurable functions in the ball that are square integrable with respect to the measure $\tilde{w}(z)G(z) dg(z)$. Also let $L^2(w)$ denote the

space of measurable functions that are square integrable with respect to the measure $w(\xi) d\sigma(\xi)$. We define the operator

$$\tilde{\nabla}: L^2(w) \longrightarrow \mathcal{L}^2(w)$$

by sending the function f to $\tilde{\nabla} \tilde{f}$, where \tilde{f} is the Poisson–Szegő extension. It is an easy calculation that

$$\|S_a f\|_{L^2(w)} \leq c(n) \|\tilde{\nabla} \tilde{f}\|_{\mathcal{L}^2(w)}.$$

For more information on some of these concepts the reader can consult [2], [5], [12], [13], [14] or [15].

1.2. Main results

The main result in this paper is the following:

Theorem 1.1. *Under the assumptions above, we have*

$$\|S_a(f)\|_{L^2(w)} \lesssim \tilde{Q}_2(w) \|f\|_{L^2(w)},$$

where the implied constant may depend upon the dimension n . Moreover,

$$\|\tilde{\nabla}(\tilde{f})\|_{\mathcal{L}^2(w)} \lesssim \tilde{Q}_2(w) \|f\|_{L^2(w)},$$

where we have no dependence on the dimension. Moreover, the dependence upon $\tilde{Q}_2(w)$ is sharp for both inequalities.

This theorem is proved by showing that the following string of inequalities holds:

$$(1.1) \quad \|S_a(f)\|_{L^2(w)} \stackrel{(1)}{\leq} c(n) \|\tilde{\nabla} \tilde{f}\|_{\mathcal{L}^2(w)} \stackrel{(2)}{\leq} c \tilde{Q}_2 \|f\|_{L^2(w)}.$$

Inequality (1) is shown by changing the order of integration, while inequality (2) is demonstrated by Bellman function techniques.

The other result demonstrated in this note is the equivalence between the weight classes A_2 and \tilde{A}_2 . This is a notable result since the corresponding fact fails in \mathbb{R}^n .

Theorem 1.2. *With the notation above, we have*

$$Q_2(w) \lesssim \tilde{Q}_2(w) \lesssim Q_2(w)^2.$$

In particular the classes A_2 and \tilde{A}_2 define the same class of weights on the unit sphere \mathbb{S} .

This result is demonstrated in the last section.

2. The “area-integral” estimate

We prove part (2) of inequality (1.1) using Bellman functions.

2.1. The Bellman function and its properties

Let us consider the function

$$B(X, x, w, v) = \left(1 + \frac{1}{Q}\right)X - \frac{x^2}{Qw} - \frac{Q^2x^2}{Q^2w + (4Q^2 + 1)w - w^2v - 4Q^2/v}$$

on the domain

$$\mathcal{O} = \{(X, x, w, v) > 0 : x^2 < Xw \text{ and } 1 \leq vw \leq Q\}.$$

Note that it is convenient to think of $B = Q^{-1}B_1 + B_2$, where

$$B_1 = X - \frac{x^2}{w} \quad \text{and} \quad B_2 = X - \frac{Q^2x^2}{Q^2w + (4Q^2 + 1)w - w^2v - 4Q^2/v}.$$

On \mathcal{O} , it enjoys the following properties:

$$(2.1) \quad 0 \leq B \leq 2X,$$

$$(2.2) \quad -d^2B \geq C \frac{1}{Q^2}v(dx)^2.$$

It is of course very hard to guess such a function. It was taken from [3] where a careful analysis was done to come up with this expression. The properties stated above are a direct calculation, we refer the reader to [3] for details and briefly sketch here how to get the estimates. The upper estimate on B for (2.1) is obvious. For the positivity of B , we split B into B_1 and B_2 and note that $X - x^2/w \geq 0$ and hence it suffices to show that

$$\frac{Q^2x^2}{Q^2w + (4Q^2 + 1)w - w^2v - 4Q^2/v} \leq \frac{x^2}{w}.$$

To see the latter, note that it is equivalent to $w^2v^2 + 4Q^2 \leq (4Q^2 + 1)vw$ which in turn is equivalent to $(vw - 2Q)^2 \leq (2Q - 1)^2vw$. The last inequality is true for $1 \leq vw \leq Q$. To establish (2.2), observe that $-d^2B \geq (2/Q)vx^2(dx/x - dw/w)^2$. Also notice that $B_2(X, x, w, v) = B_1(X, x, w')$, where $w' = w + ((4Q^2 + 1)w - w^2v - 4Q^2/v)/Q^2$. Then use the chain rule to get the estimate $-d^2B_2 \gtrsim (1/Q^2)vx^2(dw/w)^2$. Combining the estimates on the Hessians of B_1 and B_2 gives $-d^2B \gtrsim (1/Q^2)v(dx)^2$.

2.2. A dimension-free Littlewood–Paley formula

On the unit disc one has a Littlewood–Paley formula saying that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\bar{h}(e^{i\theta}) d\theta = \frac{1}{\pi} \int_{\mathbb{D}} \langle \nabla \tilde{f}(z), \nabla \tilde{h}(z) \rangle \log \frac{1}{|z|} dA(z),$$

as long as $f(0)h(0)=0$. In [17] a Littlewood–Paley formula for the unit ball, using the Bergman metric was derived:

$$\int_{\mathbb{S}} f(\xi)\bar{h}(\xi) d\sigma(\xi) = C_n \int_{\mathbb{B}} \langle \tilde{\nabla} \tilde{f}(z), \tilde{\nabla} \tilde{h}(z) \rangle G(z) dg(z),$$

where $f, h \in L^2(\mathbb{S})$ with $f(0)h(0)=0$. Here, the author was mostly concerned with showing that there be a finite constant C_n . This result is not immediate, recall that the ball, equipped with the Bergman metric is not a compact manifold as the metric blows up on \mathbb{S} .

Lemma 2.1. *With $u \in C^2(\mathbb{B}) \cap C(\overline{\mathbb{B}})$ and under the assumptions above, we have the dimension-free formula*

$$\int_{\mathbb{B}} \tilde{\Delta} u(z) G(z) dg(z) = \int_{\mathbb{S}} u(\xi) d\sigma(\xi) - u(0),$$

here σ is normalized surface measure.

The constants in this lemma can be seen by testing Green’s formula for the unit ball \mathbb{B} from [17] using the radial function $f(z) = f(|z|) = |z|^2$.

2.3. The main inequality

Lemma 2.2. *Given $\mathbf{x} \in \mathbb{R}^k$ and two smooth functions $B(\mathbf{x})$ and $\tilde{\mathbf{v}}(z) = (\tilde{v}_1(z), \dots, \tilde{v}_k(z))$, where the \tilde{v}_s are Poisson extensions, we have the following formula for the invariant Laplacian for $b(z) = B(\tilde{\mathbf{v}}(z))$:*

$$\tilde{\Delta} b(z) = 4 \sum_{i,j=1}^n g^{ij} (d^2 B(\tilde{\mathbf{v}}(z)) \bar{\partial}_i \tilde{\mathbf{v}}(z), \overline{\partial_j \tilde{\mathbf{v}}(z)})$$

If $-d^2 B(\mathbf{x}) \geq F(\mathbf{x})(dx_s)^2$ then we have the estimate $-\tilde{\Delta} b(z) \geq F(\tilde{\mathbf{v}}(z)) |\tilde{\nabla} \tilde{v}_s(z)|^2$. Here $(dx_s)^2$ is the operator represented by the matrix with an entry 1 in the position corresponding to the second derivative in the s^{th} variable and 0 entries everywhere else.

Proof. The proof is certainly a direct computation, using harmonicity of the entries of \tilde{v} . \square

To arrive at the main inequality we let $X = \widetilde{f^2/w}(z)$, $x = \tilde{f}(z)$, $w = \tilde{w}(z)$ and $v = \widetilde{w^{-1}}(z)$. Here, we would like to stress the order of the operations, for example $\widetilde{w^{-1}}$ and \tilde{w}^{-1} have different meaning. We substitute these quantities into our Bellman function $B(X, x, w, v)$, letting

$$b(z) := B\left(\frac{\widetilde{f^2}}{w}(z), \tilde{f}(z), \tilde{w}(z), \widetilde{w^{-1}}(z)\right).$$

Notice that with our choice of $Q = \tilde{Q}(w)$ and choices for variables all entries are in our domain \mathcal{O} .

Using the above Lemma 2.2 and the concavity property (2.2) of the Bellman function B we have

$$-\tilde{\Delta}b(z) \geq C \frac{1}{Q^2} \widetilde{w^{-1}} |\tilde{\nabla} \tilde{f}|^2.$$

Using the lower bound of the size condition of the Bellman function (2.1) we also see that $b(z) \geq 0$ and consequently $\int_{\mathbb{S}} b \, d\sigma \geq 0$. The upper estimate in (2.1) gives $b(0) \leq \widetilde{f^2/w}(0) = \int_{\mathbb{S}} (f^2/w) \, d\sigma$. So we have

$$\begin{aligned} \int_{\mathbb{S}} \frac{f^2}{w} \, d\sigma &\geq b(0) - \int_{\mathbb{S}} b \, d\sigma = \int_{\mathbb{B}} -\tilde{\Delta}b(z) G(z) \, dg(z) \\ &\geq C \frac{1}{Q^2} \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \widetilde{w^{-1}}(z) G(z) \, dg(z). \end{aligned}$$

Here we applied Green’s formula to b . This proves the second inequality. The estimate is valid for any weight $w \in A_2$. Replacing w by w^{-1} and recalling that $w^{-1} \in A_2$, we arrive at the desired result.

3. The square function estimate

We now turn to (1) of inequality (1.1). Let $\mathcal{L}^2(w)$ denote the space of measurable functions on the unit ball \mathbb{B} that are square integrable with respect to $\tilde{w}(z)G(z) \, dg(z)$. We have

$$\|S_a(f)\|_{L^2(w)} \leq c(n) \|\tilde{\nabla} \tilde{f}\|_{\mathcal{L}^2(w)}.$$

This follows from the following observation, $(1/\sigma(Q)) \int_Q w(\eta) d\sigma(\eta) \leq c(n)\tilde{w}(z)$. Indeed, use the observation that for any non-isotropic ball $Q=Q(\xi, \delta)$ of radius δ and center $\xi \in \mathbb{S}$ there exists a $z_\xi \in \mathbb{B}$ such that $\sigma(Q) = (1 - |z_\xi|^2)^n$ with $\xi = z_\xi/|z_\xi|$. Additionally, by the triangle inequality for the non-isotropic metric (recall the triangle inequality also holds in \mathbb{B}), for any $\eta \in Q(\xi, \delta)$,

$$|1 - \langle z_\xi, \eta \rangle|^{1/2} \leq |1 - \langle \xi, \eta \rangle|^{1/2} + |1 - \langle z_\xi, \xi \rangle|^{1/2} \leq \delta + \sigma(Q)^{1/2n}.$$

Then for any $Q=Q(\xi, \delta)$,

$$\begin{aligned} \frac{1}{\sigma(Q)} \int_Q w(\eta) d\sigma(\eta) &= \frac{1}{\sigma(Q)} \int_Q \frac{(1 - |z_\xi|^2)^n}{|1 - \langle z_\xi, \eta \rangle|^{2n}} \frac{|1 - \langle z_\xi, \eta \rangle|^{2n}}{(1 - |z_\xi|^2)^n} w(\eta) d\sigma(\eta) \\ &= \frac{1}{\sigma(Q)} \int_Q \mathcal{P}_{z_\xi}(\eta) \frac{|1 - \langle z_\xi, \eta \rangle|^{2n}}{(1 - |z_\xi|^2)^n} w(\eta) d\sigma(\eta) \\ &\leq \frac{(\delta + \sigma(Q)^{1/2n})^{4n}}{\sigma(Q)^2} \int_Q \mathcal{P}_{z_\xi}(\eta) w(\eta) d\sigma(\eta) \\ &= c_1(n)\tilde{w}(z_\xi). \end{aligned}$$

Since $c_l(n)\delta^{2n} \leq \sigma(Q) \leq c_u(n)\delta^{2n}$ (there is only a comparison, due to the fact that the “balls” in this metric are elliptical) then notice that

$$c_1(n) \leq \frac{(1 + c_u^{1/2n})^{4n}}{c_l(n)^2}.$$

With this observation we now estimate $\|S_a(f)\|_{L^2(w)}$,

$$\begin{aligned} \int_{\mathbb{S}} S_a^2(f)(\zeta) w(\zeta) d\sigma(\zeta) &= \int_{\mathbb{S}} \int_{\Gamma_a(\zeta)} |\tilde{\nabla} f(z)|^2 dg(z) w(\zeta) d\sigma(\zeta) \\ &= \int_{\mathbb{B}} |\tilde{\nabla} f(z)|^2 \int_{\mathbb{S}} \mathbf{1}_{\Gamma_a(\zeta)}(z) w(\zeta) d\sigma(\zeta) dg(z). \end{aligned}$$

For fixed $z \in \mathbb{B}$, let

$$E(z) := \left\{ \zeta \in \mathbb{S} : \left| 1 - \left\langle \frac{z}{|z|}, \zeta \right\rangle \right|^{1/2} < (a^{1/2} + 1)(1 - |z|^2)^{1/2} \right\}.$$

Then one sees by the triangle inequality for the non-isotropic metric that for any $z \in \Gamma_a(\zeta)$ we have $\zeta \in E(z)$. Note that $E(z)$ is a non-isotropic ball with center $z/|z|$ and radius $(a^{1/2} + 1)(1 - |z|^2)^{1/2}$. Continuing our estimate we have,

$$\begin{aligned}
 \int_{\mathbb{S}} S_a^2(f)(\zeta)w(\zeta) d\sigma(\zeta) &= \int_{\mathbb{S}} \int_{\Gamma_a(\zeta)} |\tilde{\nabla} \tilde{f}(z)|^2 dg(z)w(\zeta) d\sigma(\zeta) \\
 &= \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \int_{\mathbb{S}} \mathbf{1}_{\Gamma_a(\zeta)}(z)w(\zeta) d\sigma(\zeta) dg(z) \\
 &\leq \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \left(\frac{1}{\sigma(E(z))} \int_{E(z)} w(\zeta) d\sigma(\zeta) \right) \sigma(E(z)) dg(z) \\
 &\leq c_1(n) \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \tilde{w}(z) \sigma(E(z)) dg(z) \\
 &\leq c_1(n)c_u(n)(a^{1/2}+1)^{2n} \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \tilde{w}(z)(1-|z|^2)^n dg(z).
 \end{aligned}$$

We now use the inequality $(1-|z|^2)^n \leq 4n^2 G(z)/(n+1)$ in the last integral from above, and continue the estimate:

$$\begin{aligned}
 \int_{\mathbb{S}} S_a^2(f)(\zeta)w(\zeta) d\sigma(\zeta) &\leq c_1(n)c_u(n)(a^{1/2}+1)^{2n} \frac{4n^2}{n+1} \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \tilde{w}(z)G(z) dg(z) \\
 &= c(n) \int_{\mathbb{B}} |\tilde{\nabla} \tilde{f}(z)|^2 \tilde{w}(z)G(z) dg(z).
 \end{aligned}$$

This proves inequality (1) in (1.1).

4. Sharpness of the linear dependence on $\tilde{Q}_2(\omega)$.

To see that our estimate is sharp, we supply a family of examples. We utilize power weights adapted to the non-isotropic metric on the sphere. Let $\omega_\alpha(\xi) = |1-\langle \xi, \eta_0 \rangle|^\alpha$ for some fixed $\eta_0 \in \mathbb{S}$. Choose $f_\alpha(\xi) = |1-\langle \xi, \eta_0 \rangle|^{-\alpha}$. One can observe that $\omega_\alpha \in \dot{A}_2$ if and only if $|\alpha| < n-1$. To see this, one integrates using the following formula (valid for $n \geq 2$) from [15]:

$$\int_{\mathbb{S}} g(\langle \xi, \eta \rangle) d\sigma(\xi) = \frac{n-1}{\pi} \int_{\mathbb{T}} \int_0^1 (1-r^2)^{n-2} g(re^{i\theta}) r dr d\theta$$

with $g(z) = |1-z|^\alpha$. The largest contribution of the integrand in our case appears near η_0 . Recall that $\tilde{Q}_2(\omega_\alpha) = \sup_z \tilde{\omega}_\alpha(z) \tilde{\omega}_\alpha^{-1}(z)$. The supremum here is attained for z on the ray from 0 to η_0 . One sees that for $\alpha \rightarrow n-1$ we have $\tilde{Q}_2(\omega_\alpha) \sim (n-1-\alpha)^{-1}$. Similarly one calculates that $\|f_\alpha\|_{L^2(\omega_\alpha)} \sim (n-1-\alpha)^{-1/2}$. It remains to see that $\|\tilde{\nabla} \tilde{f}_\alpha\|_{L^2(w)} \gtrsim (n-1-\alpha)^{-3/2}$ and $\|S_a f_\alpha\|_{L^2(\omega_\alpha)} \gtrsim (n-1-\alpha)^{-3/2}$. Letting $\alpha \rightarrow n-1$ then establishes sharpness. To see this, one computes the invariant gradient of the Poisson extension of the function f_α directly. One then establishes the weighted norm of the area integral similar to [3], just more computationally involved.

5. The comparison of classical and invariant A_2

In this section we establish the fact that the classes A_2 and \tilde{A}_2 are in fact the same. This is shown by demonstrating that

$$Q_2(w) \lesssim \tilde{Q}_2(w) \lesssim Q_2^2(w),$$

where the implied constants only depend upon the dimension.

We begin with the following fact. First, some notation. Let

$$f_Q = \frac{1}{\sigma(Q)} \int_Q f(\xi) d\sigma(\xi) \quad \text{and} \quad w(Q) = \int_Q w(\xi) d\sigma(\xi).$$

Now, observe that by Cauchy–Schwarz’s inequality

$$f_Q^2 w(Q) \leq Q_2(w)(f^2 w)(Q).$$

If $P \subset Q$, and applying the above to $\mathbf{1}_{Q \setminus P}$ we get

$$(5.1) \quad \left(1 - \frac{\sigma(P)}{\sigma(Q)}\right)^2 \leq Q_2(w)(w(Q) - w(P)).$$

Upon rearrangement we arrive at

$$(5.2) \quad w(P) \leq \left(1 - \frac{(1 - \sigma(P)/\sigma(Q))^2}{Q_2(w)}\right) w(Q).$$

Given any non-isotropic ball Q on the sphere, choose $z_Q \in \mathbb{B}$ so that the center of Q is $z_Q/|z_Q|$ and its volume is $(1 - |z_Q|)^n$ (hence its radius is $\sim(1 - |z_Q|)^{1/2}$). There is a one-to-one correspondence between Q ’s and z_Q ’s. The value $\tilde{w}(z_Q)$ will approximately correspond to $w(Q)$.

We estimate

$$\begin{aligned} \tilde{w}(z_Q) \widetilde{w^{-1}}(z_Q) &\geq \left(\int_Q P_{z_Q} w d\sigma\right) \left(\int_Q P_{z_Q} w^{-1} d\sigma\right) \\ &\gtrsim \left(\frac{(1 - |z_Q|^2)^n}{(1 - |z_Q|)^{2n}}\right)^2 \left(\int_Q w d\sigma\right) \left(\int_Q w^{-1} d\sigma\right) \\ &\gtrsim \frac{1}{(1 - |z_Q|)^{2n}} \left(\int_Q w d\sigma\right) \left(\int_Q w^{-1} d\sigma\right) \\ &\gtrsim Q_2(w). \end{aligned}$$

Hence $Q_2(w) \lesssim \tilde{Q}_2(w)$.

We now turn to the other inequality. We have to estimate $\tilde{w}(z) \widetilde{w^{-1}}(z) \lesssim Q_2^2(w)$ with implied constant independent of w and Q . Using the Möbius invariance of \tilde{Q}_2

it suffices to assume that $z=(r, 0, \dots, 0)$ with $0 < r < 1$ arbitrary. Now fix r and let Q be the non-isotropic ball with center $(1, 0, \dots, 0)$ and $\sigma(Q)=(1-r)^n$.

To obtain the upper estimate, we exhaust the sphere by enlarging Q . Let us denote by cQ the ball with the same center and c^n -fold volume (hence $c^{1/2}$ -fold radius). Then

$$(5.3) \quad \int_{\mathbb{S}} P_z w \, d\sigma = \int_Q P_z w \, d\sigma + \sum_{k=1}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} P_z w \, d\sigma$$

Observe that for all $\xi \in \mathbb{S}$,

$$P_z(\xi) \leq \frac{(1+r)^n}{(1-r)^n}.$$

Moreover, if $\xi \in 2^k Q \setminus 2^{k-1} Q$ we have the better estimate

$$P_z(\xi) \lesssim 2^{-2nk} \frac{(1+r)^n}{(1-r)^n}.$$

So we get

$$\tilde{w}(z) \lesssim \frac{(1+r)^n}{(1-r)^n} \sum_{k=0}^{\infty} 2^{-2nk} w(2^k Q)$$

and similarly for $\widetilde{w^{-1}}$. We start to estimate the product

$$\begin{aligned} \tilde{w}(z) \widetilde{w^{-1}}(z) &\lesssim \frac{(1+r)^{2n}}{(1-r)^{2n}} \sum_{k=0}^{\infty} 2^{-2nk} w(2^k Q) \sum_{l=0}^{\infty} 2^{-2nl} w^{-1}(2^l Q) \\ &\leq \frac{(1+r)^{2n}}{(1-r)^{2n}} \sum_{k=0}^{\infty} 2^{-2nk} w(2^k Q) \sum_{l=0}^k 2^{-2nl} w^{-1}(2^l Q) \\ &\quad + \frac{(1+r)^{2n}}{(1-r)^{2n}} \sum_{k=0}^{\infty} 2^{-2nk} w(2^k Q) \sum_{l=k}^{\infty} 2^{-2nl} w^{-1}(2^l Q). \end{aligned}$$

The latter sums are equivalent (with roles of w and w^{-1} switched) so we estimate the second sum only. Iterating the estimate (5.2) we estimate for $l \geq k$,

$$w(2^k Q) \leq \left(1 - \frac{1-4^{-n}}{Q_2(w)}\right)^{l-k} w(2^l Q),$$

and using that $\sigma(2^k Q) = 2^{nk}(1-r)^n$ we get

$$\begin{aligned} \frac{(1+r)^{2n}}{(1-r)^{2n}} \sum_{k=0}^{\infty} 2^{-2nk} w(2^k Q) \sum_{l=k}^{\infty} 2^{-2nl} w^{-1}(2^l Q) &\lesssim Q_2(w) \sum_{k=0}^{\infty} 2^{-2nk} \sum_{s=0}^{\infty} \left(1 - \frac{1-4^{-n}}{Q_2(w)}\right)^s \\ &\lesssim Q_2^2(w). \end{aligned}$$

6. Concluding remarks

The result in [6] establishes that the Cauchy transform on $L^p(w)$ is bounded if and only if the weight is in A_p . It is however not known how the norm of the Cauchy transform grows with respect to the characteristic of the weight. Thus, it is an interesting task to understand the dependence of the norm of the Cauchy transform on weighted L^2 spaces of the unit sphere by means of Bellman functions. On the unit disc, using Bellman function techniques, a very simple proof of the boundedness of the Cauchy transform on weighted L^2 spaces is given, see [7] for continuity and [11] for the sharp result in terms of the invariant characteristic. In the case of the disc the sharp weighted bound for the square function was a first step and a tool to obtain the sharp bound for the Cauchy transform, which was a major motivation for this note.

As for the Cauchy transform C , the goal is to provide a linear and sharp estimate of the form $\|Cf\|_{L^2(\omega)} \lesssim \tilde{Q}_2(\omega)\|f\|_{L^2(\omega)}$, where the implied constant is independent of f , ω and the dimension n . The claim is that the invariant characteristic of the weight is the correct one, so that the dependence is both linear and no dependence on the dimension occurs. Our dimension-free estimate on the “area integral” illustrates that this guess was a good one. The missing ingredient for the estimate for the Cauchy transform is the following formula: $|\tilde{\nabla} Cf| \sim |\tilde{\nabla} \tilde{f}|$, which holds in one dimension, but is not true in several variables. The estimate for the square function avoids this deficiency.

Conjecture 6.1. Let $w \in A_2$ and C denote the Cauchy transform in \mathbb{C}^n . Then, is it true that

$$\|C(f)\|_{L^2(w)} \lesssim \tilde{Q}_2(w)\|f\|_{L^2(w)},$$

where the implied constant does not depend upon the dimension?

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Stefanie Petermichl
Department of Mathematics
University of Texas at Austin
Austin, TX 78712
U.S.A.
stefanie@math.utexas.edu

Brett D. Wick
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240-0001
U.S.A.
brett.d.wick@vanderbilt.edu

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