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Codimension-p Paley–Wiener theorems

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Abstract. We obtain the generalized codimension-p Cauchy–Kovalevsky extension of the exponential function $e^{i\langle \underline{y},\underline{t}\rangle}$ in $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$, where $p>1$, $\underline{y},\underline{t}\in\mathbf{R}^q$, and prove the corresponding codimension-p Paley–Wiener theorems.

0. Introduction

The Clifford algebra formulation of Euclidean spaces will be adopted. Let $\mathbf{e}_1, ..., \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j+\mathbf{e}_j\mathbf{e}_i=-2\delta_{ij}$, where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ otherwise, $i,j=1,2,...,m$. Set

$$\mathbf{R}^m=\{\underline{x}=x_1\mathbf{e}_1+...+x_m\mathbf{e}_m : x_j\in\mathbf{R},\ j=1,2,...,m\},$$

and

$$\mathbf{R}_1^m=\{x=x_0+\underline{x} : x_0\in\mathbf{R},\ \underline{x}\in\mathbf{R}^m\}.$$

$\mathbf{R}^m$ and $\mathbf{R}_1^m$ are called the *homogeneous* and *inhomogeneous* Euclidean spaces, respectively.

A number of generalizations of the classical Paley–Wiener theorem (PW theorem) to higher dimensional spaces were studied [1], [3], [6], [7] and [9]. Among the literature, by imbedding $\mathbf{R}^m$ into $\mathbf{C}^m=\mathbf{R}^m\oplus i\mathbf{R}^m$, the corresponding PW theorem is obtained in [6]. By making use of Clifford algebra a direct proof of the theorem is obtained in [9]. In [7], a different type of PW theorem in $\mathbf{C}^m$ is proved by using the heat kernel. In [3], through imbedding $\mathbf{R}^m$ into $\mathbf{R}_1^m$, in the complex structure induced by the generalized Cauchy–Riemann operator, an inhomogeneous codimension-1 result is proved. This latter result is viewed as a precise analogy to the classical result in which $\mathbf{R}$ is imbedded into the complex plane $\mathbf{C}=\mathbf{R}_1^1$.

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The standard CK extension (Cauchy–Kovalevsky extension) ([1] and [2]) asserts that any real-analytic function in an open set Q of $\mathbf{R}^q$ may be extended to become a one-sided-monogenic function in an open set of $\mathbf{R}_1^q$ that contains Q. This will be regarded as the *inhomogeneous codimension-1 CK extension.* The authors of [2] further obtain the *homogeneous codimension-p CK extension* that extends any real-analytic function in an open set Q of $\mathbf{R}^q$ into a one-sided-monogenic function in an open set of the homogeneous space $\mathbf{R}^p \oplus \mathbf{R}^q$ containing the set Q. When $p=1$ this reduces to the homogeneous codimension-1 CK extension from $\mathbf{R}^q$ to $\mathbf{R}^{q+1}$. The codimension-p CK extension is made more general in [2] through incorporating a k-left-inner monogenic weight function in $\mathbf{R}^p$, and called *generalized CK extension* or *homogeneous codimension-p CK extension with a k-monogenic function.* In this paper, with homogeneous codimension-p and generalized CK extensions of $e^{i\langle \underline{y}, \underline{t}\rangle}$ from $\mathbf{R}^q$ into $\mathbf{R}^p \oplus \mathbf{R}^q$ we prove the corresponding PW theorems in $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$, $m=p+q$. We further extend the PW theorem in terms of generalized Taylor series.

In Section 1 we recall the basic notation and terminology used in the paper. This section also serves as a survey on the theory of CK type and "generalized-CK type" extensions of real-analytic functions, as well as generalized Taylor series. In Section 2 we obtain the homogeneous codimension-p generalized CK extension of $e^{i\langle \underline{y}, \underline{t}\rangle}$ in $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$, where $\underline{y}, \underline{t} \in \mathbf{R}^q$. In Section 3 we formulate and prove the corresponding codimension-p PW theorems. Furthermore, we obtain the PW theorems in terms of monogenic extensions given by generalized Taylor series. The proofs of the results in this work are based on the inhomogeneous codimension-1 PW theorem obtained in [3].

1. Preliminaries

The reader is supposed to know the basic material on Clifford algebras, Dirac operators, Cauchy–Riemann operator and so on. The basic knowledge and notation in relation to Clifford algebras are referred to [1], [2] and [4].

The *unit sphere* $\{\underline{x} \in \mathbf{R}^m : |\underline{x}| = 1\}$ is denoted by S^{m-1}. We use $B(\underline{x}, r)$ for the open ball in $\mathbf{R}^m$ centered at $\underline{x}$ with radius r, and $\overline{B}(\underline{x}, r)$ for the topological closure of $B(\underline{x}, r)$.

Let $k \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of non-negative integers. Denote by $M^+(m, k, \mathbf{C}^{(m)})$ the space of k-homogeneous monogenic polynomials in $\mathbf{R}^m$, called *k-monogenic of degree k.*

Below by saying that f is *analytic* at a certain point we mean that f can be expanded into a Taylor series in a neighborhood of the point. Let $f(\underline{x})$ be a given analytic function in $U \subset \mathbf{R}^q$. When we say that f^* is *an inhomogeneous CK extension* of f, we mean that there exist an open set U^* of $\mathbf{R}_1^q$ containing U and a function f^*, left-monogenic with respect to the inhomogeneous Dirac operator ∂_x, such that $f^*|_{x_0=0}=f$ in U. By a *homogeneous CK extension* of f to $\mathbf{R}^{q+1}$ we mean a function f^*, left-monogenic with respect to the homogeneous Dirac operator $\partial_{\underline{x}}$ in an open set U^* in $\mathbf{R}^{q+1}$ containing U, such that $f^*|_{x_0=0}=f$ in U. The existence of such CK extensions are referred to [1] and [2].

The Fourier transform of functions in $\mathbf{R}^m$ is defined by

$$\hat{f}(\underline{\xi})=\int_{\mathbf{R}^m} e^{-i\langle \underline{x},\underline{\xi}\rangle} f(\underline{x})\,d\underline{x}$$

and the inverse Fourier transform by

$$\check{g}(\underline{x})=\frac{1}{(2\pi)^m}\int_{\mathbf{R}^m} e^{i\langle \underline{x},\underline{\xi}\rangle} g(\underline{\xi})\,d\underline{\xi},$$

where $\underline{\xi}=\xi_1\mathbf{e}_1+...+\xi_m\mathbf{e}_m$.

To extend the domain of the Fourier transform to $\mathbf{R}_1^m$, we first need to extend the exponential function $e^{i\langle \underline{x},\underline{\xi}\rangle}$. Set, for $x=x_0\mathbf{e}_0+\underline{x}$,

$$e(x,\underline{\xi})=e^{i\langle \underline{x},\underline{\xi}\rangle}e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi})+e^{i\langle \underline{x},\underline{\xi}\rangle}e^{x_0|\underline{\xi}|}\chi_-(\underline{\xi}), \tag{1}$$

where

$$\chi_\pm(\underline{\xi})=\frac{1}{2}\left(1\pm i\frac{\underline{\xi}\mathbf{e}_0}{|\underline{\xi}|}\right).$$

It is easy to verify that the functions $\chi_\pm$ satisfy the properties of projections:

$$\chi_-\chi_+=\chi_+\chi_-=0, \quad \text{and} \quad \chi_\pm^2=\chi_\pm, \quad \chi_++\chi_-=1.$$

For any fixed $\underline{\xi}$, $e(x,\underline{\xi})$ is two-sided-monogenic in $x\in\mathbf{R}_1^m$. The above extension is the inhomogeneous codimension-1 CK extension of $e(\underline{x},\underline{\xi})$ to $\mathbf{R}_1^m$. Replacing $\mathbf{e}_0$ by $\mathbf{e}_{m+1}$ in (1), where $\mathbf{e}_{m+1}$ is a basis element added to the collection $\mathbf{e}_1,...,\mathbf{e}_m$, with $\mathbf{e}_{m+1}^2=-1$ and anti-commutativity with the other $\mathbf{e}_j, j=1,...,m$, one obtains the homogeneous codimension-1 CK extension of $e^{i\langle \underline{x},\underline{\xi}\rangle}$ in $\mathbf{R}^{m+1}$. Generalizations of the exponential function of these types can be first found in F. Sommen's work [5].

We will be concerned with the direct sum decomposition of the space $\mathbf{R}^m$ into $\mathbf{R}^p\oplus\mathbf{R}^q$, where $\mathbf{R}^p$ is the real-linear span of $\mathbf{e}_1,...,\mathbf{e}_p$, and $\mathbf{R}^q$ is the real-linear span of $\mathbf{e}_{p+1},...,\mathbf{e}_m$. We call $\mathbf{R}^q$ the *codimension-p space* in $\mathbf{R}^p\oplus\mathbf{R}^q$.

Codimension-p CK extension concerns the following question. Suppose we are given a function $f(\underline{y})$ that is analytic in an open subset $U\subset\mathbf{R}^q$. Then the question is: Is there a left-monogenic function $f^*(\underline{x},\underline{y})$ in some domain in $\mathbf{R}^p\oplus\mathbf{R}^q$ containing U such that $f^*|_{\underline{x}=0}=f$ in U?

The answer to this question is positive, but, when $p>1$, the solution is not unique. For instance, $f(\underline{y})$ can first have a homogeneous codimension-1 CK extension in $\mathbf{R}^{q+1}$ which is also left-monogenic in $\mathbf{R}^p\oplus\mathbf{R}^q$.

Before answering the question in full, we first need to distinguish different types of monogenic extensions. In both the homogeneous and inhomogeneous codimension-1 cases there will be only one type, viz. the standard CK type, as the extension is unique. The CK extension from $U\subset\mathbf{R}^q$ to $U^*\subset\mathbf{R}^q_1$ corresponds to the series expansion of

$$f(x_0,\underline{y})=e^{-x_0\partial_{\underline{y}}}A(\underline{y})=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x_0^j\partial_{\underline{y}}{}^jA(\underline{y}),$$

where $\partial_{\underline{y}}$ is the Dirac operator in the homogeneous Euclidean space $\mathbf{R}^q$.

The codimension-p, $p>1$, CK extension of a function $A(\underline{y})$, $\underline{y}\in U\subset\mathbf{R}^q$, being the case $k=0$ in Lemma 1 ([2]), is a modification of the series obtained from the exponential expression $e^{-x_0\partial_{\underline{y}}}A(\underline{y})$. Lemma 1 answers the question in a more general context, called a *generalized CK extension,* where an extension is related to a k-monogenic function in $M^+(p,k,\mathbf{C}^{(p)})$. The participation of the k-monogenic function makes the extension having the role of monomial functions z^k in one complex variable, which enables one to further formulate generalized Taylor series.

Lemma 1. (The Generalized CK Extension: Homogeneous Codimension-p CK extension Associated With P_k [2, p. 265]) *Let $P_k\in M^+(p,k,\mathbf{C}^{(p)})$ be given and $A_0(\underline{y})$ be a Clifford $\mathbf{C}^{(q)}$-valued analytic function in U. Then there exists a unique sequence $(A_l(\underline{y}))_{l>0}$ of analytic functions such that the series*

$$f_{P_k}(\underline{x},\underline{y})=\sum_{l=0}^{\infty}\underline{x}^lP_k(\underline{x})A_l(\underline{y}) \tag{2}$$

is convergent in an SO(p)-invariant domain $U^*\subset\mathbf{R}^m$, which is a neighborhood of U, and the sum f is left-monogenic in U^*. The functions $A_l(\underline{y})$, $l>1$, are uniquely determined by the formulas

$$P_k(\underline{x})A_{2l}(\underline{y})=\frac{(-1)^l\Gamma(k+p/2)\partial_{\underline{y}}^{2l}[P_k(\underline{x})A_0(\underline{y})]}{2^{2l}l!\Gamma(l+k+p/2)},$$

and

$$P_k(\underline{x})A_{2l+1}(\underline{y})=\frac{(-1)^l\Gamma(k+p/2)\partial_{\underline{y}}^{2l+1}[P_k(\underline{x})A_0(\underline{y})]}{2^{2l+1}l!\Gamma(l+k+p/2+1)}.$$

We call $f_{P_k}(\underline{x},\underline{y})$ the *generalized CK extension in relation to* P_k of $A_0(\underline{y})$ and $A_0(\underline{y})$ the *initial value* of $f_{P_k}(\underline{x},\underline{y})$. Denote by $\mathcal{T}_{P_k}(U)$ the space of all functions of the form (2).

From [2] we know, if $f(\underline{x},\underline{y})$, $\underline{x}=\rho\underline{\omega}\in\mathbf{R}^p$, $\underline{y}\in\mathbf{R}^q$, is left-monogenic in $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$, that $T_k(f)(\underline{\omega},\underline{y})=\lim_{\rho\to 0}\rho^{-k}P(k)f(\rho,\underline{\omega},\underline{y})$ is the generalized Taylor coefficient of f of order k, which can be decomposed in a unique way as

$$T_k(f)(\underline{\omega},\underline{y})=\sum_{\alpha\in\mathcal{I}_k}P_{k,\alpha}(\underline{\omega})T_{k,\alpha}(f)(\underline{y}),$$

where $T_{k,\alpha}(f)(\underline{y})$ are real-analytic functions.

Denote the generalized CK extension, corresponding to $P_{k,\alpha}(\underline{\omega})T_{k,\alpha}(f)(\underline{y})$, by $T_{k,\alpha}(\underline{x},\underline{y})$, then we have

$$f(\underline{x},\underline{y})=\sum_{k=0}^{\infty}\sum_{\alpha\in\mathcal{I}_k}T_{k,\alpha}(\underline{x},\underline{y}), \tag{3}$$

where

$$T_{k,\alpha}(\underline{x},\underline{y})=\sum_{l=0}^{\infty}\underline{x}^{l}P_{k,\alpha}(\underline{x})T^{(l)}_{k,\alpha}(\underline{y}).$$

The expansion (3) is called a *generalized Taylor series.*

2. The generalized CK extension of $e^{i\langle\underline{y},\underline{t}\rangle}$ in $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$

In [1] and [4], the homogeneous codimension-1 CK extension of $e^{i\langle\underline{y},\underline{t}\rangle}$ is given by

$$e(x_1\mathbf{e}_1,\underline{y},\underline{t})=e^{i\langle\underline{y},\underline{t}\rangle}\left[\cosh(x_1|\underline{t}|)+i\sinh(x_1|\underline{t}|)\mathbf{e}_1\frac{\underline{t}}{|\underline{t}|}\right].$$

It is easy to see that for all $\underline{y},\underline{t}\in\mathbf{R}^q$, $|e(x_1\mathbf{e}_1,\underline{y},\underline{t})|\le Ce^{|x_1||\underline{t}|}$, where C is a constant.

Now we study the extension for all cases $p\ge1$ and $k\ge0$, for $P_k\in M^+(p,k,\mathbf{C}^{(p)})$ given. Let $\underline{x}=r\underline{\omega}\in\mathbf{R}^p$. In Lemma 1, setting $A_0(\underline{y})=e^{i\langle\underline{y},\underline{t}\rangle}$, we obtain the generalized, homogeneous codimension-p CK extension of $e^{i\langle\underline{y},\underline{t}\rangle}$ in $\mathcal{T}_{P_k}$, denoted by $\varepsilon^p_{P_k}$, with the expression

$$\begin{aligned}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})&=\Gamma\bigl(k+\frac{p}{2}\bigr)e^{i\langle\underline{y},\underline{t}\rangle}r^k\left(\frac{r|\underline{t}|}{2}\right)^{-k-p/2+1}\\&\quad\times\left[P_k(\underline{\omega})I_{k+p/2-1}(r|\underline{t}|)+i\underline{\omega}P_k(\underline{\omega})I_{k+p/2}(r|\underline{t}|)\frac{\underline{t}}{|\underline{t}|}\right],\end{aligned} \tag{4}$$

where

$$I_v(x)=i^{-v}J_v(ix)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(v+k+1)}\bigl(\frac{x}{2}\bigr)^{v+2k}$$

is a kind of Bessel function ([8]). Note that $e(x_1\mathbf{e}_1,\underline{y},\underline{t})=\varepsilon^1_1(x_1\mathbf{e}_1,\underline{y},\underline{t})$.

Next we will estimate $\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})$.

From [8] we know that the generating function of I_n is

$$e^{u(t+1/t)/2} = \sum_{n=-\infty}^{\infty} t^n I_n(u).$$

Taking $t=1$, we get $e^u = \sum_{n=-\infty}^{\infty} I_n(u)$.

Using $I_n(-u)=I_n(u)$, we get $e^u = 2\sum_{n=1}^{\infty} I_n(u)+I_0(u)$. As $I_n(u)>0$ when $u>0$, we have

$$\sum_{n=0}^{\infty} I_n(x) \le e^x, \quad \text{when } x>0. \tag{5}$$

When $|\underline{t}|\le\Omega$, using (4) and (5) we have

$$\begin{aligned} |\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})| &\le C\left(\frac{2}{\Omega}\right)^k \left(\frac{r\Omega}{2}\right)^{-p/2+1} [I_{k+p/2-1}(r\Omega)+I_{k+p/2}(r\Omega)] \\ &\le C[I_k(r\Omega)+I_{k+1}(r\Omega)] \le Ce^{r\Omega}, \end{aligned} \tag{6}$$

where C is a constant independent of $\underline{y}$.

On the other hand, when $k=0$ and $P_k=1$ we have the generalization of $e^{i\langle \underline{y},\underline{t}\rangle}$ in $\mathcal{T}_1(\mathbf{R}^q)$,

$$\varepsilon^p_1(\underline{x},\underline{y},\underline{t}) = \Gamma\left(\frac{p}{2}\right) e^{i\langle \underline{y},\underline{t}\rangle} \left(\frac{r|\underline{t}|}{2}\right)^{-p/2+1} \left[I_{p/2-1}(r|\underline{t}|)+iI_{p/2}(r|\underline{t}|)\underline{\omega}\frac{\underline{t}}{|\underline{t}|}\right].$$

Since for any $u>0$,

$$\left(\frac{u}{2}\right)^{-v} I_v(u) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v+k+1)}\left(\frac{u}{2}\right)^{2k} \le C\sum_{k=0}^{\infty}\frac{u^{2k}}{(2k)!} \le Ce^u,$$

and

$$\left(\frac{u}{2}\right)^{-v+1} I_v(u) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v+k+1)}\left(\frac{u}{2}\right)^{2k+1} \le C\sum_{k=0}^{\infty}\frac{u^{2k+1}}{(2k+1)!} \le Ce^u.$$

We therefore have

$$|\varepsilon^p_1(\underline{x},\underline{y},\underline{t})| \le Ce^{r|\underline{t}|} \quad \text{for any } \underline{y},\underline{t}\in\mathbf{R}^q, \tag{7}$$

where C is a constant independent of $\underline{y}$. The estimate is stronger than (6) where it is under the restriction $|\underline{t}|\le\Omega$.

3. Paley–Wiener theorems in $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$

We first study the codimension-1 case. The following result is obtained in [3].

Lemma 2. (Inhomogeneous codimension-1 PW theorem) *Let F be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e}_2, ..., \mathbf{e}_{q+1}$, and $F \in L^2(\mathbf{R}^q)$. Let Ω be a positive real number. Then the following two conditions are equivalent*:

(1) *F can be left-monogenically extended to a function defined on $\mathbf{R}_1^q$, denoted by f, and there exists a constant C such that*

$$|f(y)| \le Ce^{\Omega|y|} \quad \textit{for any } y \in \mathbf{R}_1^q.$$

(2) $\operatorname{supp}(\widehat{F}) \subset \overline{B}(0, \Omega)$.

Moreover, if one of the above conditions holds, then we have

$$f(y) = \frac{1}{(2\pi)^q} \int_{\mathbf{R}^q} e(y, \underline{\xi}) \widehat{F}(\underline{\xi})\, d\underline{\xi}, \quad y \in \mathbf{R}_1^q,$$

where $e(y, \underline{\xi})$ is given in (1) *with $\mathbf{e}_0 = 1$.*

With the difference $\mathbf{e}_1^2 = -1$ in place of $\mathbf{e}_0^2 = 1$ in the above result, we now prove the following result.

Theorem 1. (Homogeneous codimension-1 PW theorem) *Let F be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e}_2, ..., \mathbf{e}_{q+1}$, and $F \in L^2(\mathbf{R}^q)$, and let Ω be a positive real number. Then the following two conditions are equivalent*:

(1) *F can be left-monogenically extended to a function defined on $\mathbf{R}^{q+1}$, denoted by f, and there exists a constant C such that*

$$|f(x_1\mathbf{e}_1, \underline{y})| \le Ce^{\Omega|x_1\mathbf{e}_1 + \underline{y}|} \quad \textit{for any } x_1 \in \mathbf{R} \textit{ and } \underline{y} \in \mathbf{R}^q.$$

(2) $\operatorname{supp}(\widehat{F}) \subset \overline{B}(0, \Omega)$.

Moreover, if one of the above conditions holds, then we have

$$f(x_1\mathbf{e}_1, \underline{y}) = \frac{1}{(2\pi)^q} \int_{\mathbf{R}^q} e(x_1\mathbf{e}_1, \underline{y}, \underline{t}) \widehat{F}(\underline{t})\, d\underline{t}, \quad x_1 \in \mathbf{R},\ \underline{y} \in \mathbf{R}^q.$$

While the proof of Lemma 2 ([3]) may be adapted step by step to give a proof of Theorem 1, we, however, prefer to show that the main part of the theorem may be concluded from Lemma 2.

Proof. (2)⇒(1) Let

$$G(x_1\mathbf{e}_1,\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q} e(x_1\mathbf{e}_1,\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}.$$

Because $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$, we have

$$G(x_1\mathbf{e}_1,\underline{y})=\frac{1}{(2\pi)^q}\int_{\overline{B}(0,\Omega)} e(x_1\mathbf{e}_1,\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}.$$

The estimate of $\varepsilon_1^1(x_1\mathbf{e}_1,\underline{y},\underline{t})$ implies

$$|G(x_1\mathbf{e}_1,\underline{y})|\le C\|\widehat{F}\|_2 e^{\Omega|x_1|}\|\chi_{\overline{B}(0,\Omega)}\|_2\le Ce^{\Omega|x_1\mathbf{e}_1+\underline{y}|}.$$

Since $F(\underline{y})$ and $G(0,\underline{y})$ agree in $\mathbf{R}^q$, both being left-monogenic in $\mathbf{R}^{q+1}$, we conclude that $f(x_1\mathbf{e}_1,\underline{y})=G(x_1\mathbf{e}_1,\underline{y})$.

(1)⇒(2) Since $f(x_1\mathbf{e}_1,\underline{y})$ is the homogeneous codimension-1 CK extension of $f(0,\underline{y})=F(\underline{y})$, and

$$|f(x_1\mathbf{e}_1,\underline{y})|\le Ce^{\Omega|x_1\mathbf{e}_1+\underline{y}|},\quad x_1\mathbf{e}_1\in\mathbf{R}^1,\ \underline{y}\in\mathbf{R}^q,$$

we have

$$\begin{aligned}|f(x_1\mathbf{e}_1,\underline{y})|&=\left|\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)|x_1|^{2l}\partial_{\underline{y}}^{2l}[A_0(\underline{y})]}{(2l)!}+\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)|x_1|^{2l}x_1\mathbf{e}_1\partial_{\underline{y}}^{2l+1}[A_0(\underline{y})]}{(2l+1)!}\right|\\&\le Ce^{\Omega|x_1\mathbf{e}_1+\underline{y}|},\end{aligned}$$

where $f(0,\underline{y})=A_0(\underline{y})$.

Since the first and the second summations are expanded, respectively, over different groups of reduced products of the basis elements of $\mathbf{R}^1\oplus\mathbf{R}^q$, we have

$$\left|\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)|x_1|^{2l}\partial_{\underline{y}}^{2l}[A_0(\underline{y})]}{(2l)!}\right|\le Ce^{\Omega|x_1\mathbf{e}_1+\underline{y}|},$$

and

$$\left|\sum_{l=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)|x_1|^{2l}x_1\mathbf{e}_1\partial_{\underline{y}}^{2l+1}[A_0(\underline{y})]}{(2l+1)!}\right|\le Ce^{\Omega|x_1\mathbf{e}_1+\underline{y}|}.$$

On the other hand, $F(\underline{y})$ has an inhomogeneous codimension-1 CK extension $f_1(x_1,\underline{y})\in\mathbf{R}_1^q$ with $f_1(0,\underline{y})=F(\underline{y})$ and

$$f_1(x_1,\underline{y})=\sum_{l=0}^{\infty}\frac{|x_1|^{2l}\partial_{\underline{y}}^{2l}[A_0(\underline{y})]}{(2l)!}-\sum_{l=0}^{\infty}\frac{|x_1|^{2l}x_1\partial_{\underline{y}}^{2l+1}[A_0(\underline{y})]}{(2l+1)!}.$$

Then the last two inequalities imply that

$$|f_1(x_1,\underline{y})| \leq \left|\sum_{l=0}^{\infty} \frac{|x_1|^{2l}\partial_{\underline{y}}^{2l}[A_0(\underline{y})]}{(2l)!}\right| + \left|\sum_{l=0}^{\infty} \frac{|x_1|^{2l}x_1\partial_{\underline{y}}^{2l+1}[A_0(\underline{y})]}{(2l+1)!}\right|$$
$$\leq Ce^{\Omega|x_1\mathbf{e}_1+\underline{y}|} = Ce^{\Omega|x_1+\underline{y}|}.$$

Invoking Lemma 2, we conclude that $\mathrm{supp}(\widehat{F}) \subset \overline{B}(0,\Omega)$. □

Below we study the homogeneous codimension-p PW theorem in $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$ (for $p>1$) for CK extensions involving an initial value function. We need the following lemma.

Lemma 3. *Let*

$$\partial x^n = \partial x_1^{n_1} ... \partial x_l^{n_l}, \quad |n| = n_1+n_2+...+n_l,$$

where $n=(n_1,...,n_l)\in\mathbf{N}^l$ *is an* l*-dimensional multi-index. Let*

$$E(\underline{x}) = \frac{\overline{x}}{|\underline{x}|^l}, \quad \underline{x}\in\mathbf{R}^l.$$

Then

$$\left|\frac{\partial^{|n|}}{\partial x^n}E(\underline{x})\right| \leq \frac{(l-1)l...(l+|n|-2)}{|\underline{x}|^{l+|n|-1}}.$$

The lemma can be easily proved by direct computation.

Lemma 4. (Technical lemma) *Assume that* $f(\underline{x},\underline{y})$ *is left-monogenic in* $\mathbf{R}^p\oplus\mathbf{R}^q$*, and*

$$|f(\underline{x},\underline{y})| \leq Ce^{\Omega|\underline{x}|}, \quad \underline{x}\in\mathbf{R}^p, \ \underline{y}\in\mathbf{R}^q,$$

where Ω *is a positive real number. Let* $f_1(x_1\mathbf{e}_1,\underline{y})$ *be the homogeneous codimension-*1 *CK extension satisfying* $f_1(0,\underline{y})=f(0,\underline{y})$*. Then for any* $\varepsilon>0$*, there exists a constant* $C_\varepsilon>0$ *such that*

$$|f_1(x_1\mathbf{e}_1,\underline{y})| \leq C_\varepsilon e^{(\Omega+\varepsilon)|x_1|}, \quad x_1\mathbf{e}_1\in\mathbf{R}^1, \ \underline{y}\in\mathbf{R}^q.$$

Proof. Since $f(\underline{x},\underline{y})$ is left-monogenic in $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$, by Cauchy's formula, we have

$$f(\underline{x},\underline{y}) = \frac{1}{\omega_{m-1}} \int_{\partial B(\underline{x}+\underline{y},\rho)} E(\underline{t}-(\underline{x}+\underline{y}))\, d\sigma(\underline{t}) f(\underline{t}),$$

where ρ is any positive real number.

Then

$$f(0,\underline{y})=\frac{1}{\omega_{m-1}}\int_{\partial B(\underline{y},\rho)}E(\underline{t}-\underline{y})\,d\sigma(\underline{t})f(\underline{t}).$$

By Lemma 3 and the condition $|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|}$, we have

$$|\partial^l_{\underline{y}}f(0,\underline{y})|\le C\frac{(m-1)m...(m+l-2)}{\rho^l}e^{\Omega\rho}. \tag{8}$$

Since f_1 is the homogeneous codimension-1 CK extension and $f(0,\underline{y})=f_1(0,\underline{y})$, the above estimate (8) implies that

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\le\sum_{l=0}^{\infty}\frac{|x_1|^l|\partial^l_{\underline{y}}[f(0,\underline{y})]|}{l!}\le\sum_{l=0}^{\infty}\frac{|x_1|^l(m-1)m...(m+l-2)}{l!\rho^l}e^{\Omega\rho}. \tag{9}$$

For any $\varepsilon>0$, set $l_0=[2\Omega(m-2)/\varepsilon]+1$. Then $l>l_0$ will imply that $(m-2)/l<\varepsilon/2\Omega$. Thus, if $l>l_0$, then $1+(m-2)/l<1+\varepsilon/2\Omega$, and from (9) we get

$$\begin{aligned}|f_1(x_1e_1,\underline{y})|&\le C\sum_{l=0}^{\infty}\frac{|x_1|^l(m-1)m...(m+l-2)}{l!\rho^l}e^{\Omega\rho}\\&\le C\sum_{l=0}^{l_0}\frac{|x_1|^l(m-1)m...(m+l-2)}{l!\rho^l}e^{\Omega\rho}+C_\varepsilon\sum_{l=l_0}^{\infty}\frac{|x_1|^l(1+\varepsilon/2\Omega)^{l-l_0}}{\rho^{m+l-1}}e^{\Omega\rho},\end{aligned}$$

where $C_\varepsilon=(m-1)(m/2)...(m+l_0-2)/l_0$.

Taking $\rho=|x_1|(1+\varepsilon/\Omega)$, we have

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\le C_\varepsilon e^{(\Omega+\varepsilon)|x_1|}.$$

The proof is complete. □

Theorem 2. (Homogeneous codimension-p PW Theorem) *Let F be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e}_{p+1},...,\mathbf{e}_{p+q}$, and $F\in L^2(\mathbf{R}^q)$, and let Ω be a positive real number. Then the following two assertions are equivalent*:

(1) *F has a homogeneous codimension-p CK extension to $\mathbf{R}^{p+q}$, denoted by f, and there exists a constant C such that*

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{y}\in\mathbf{R}^q.$$

(2) $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$.

Moreover, if one of the above conditions holds, we have

$$f(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon_1^p(\underline{x},\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{y}\in\mathbf{R}^q.$$

Proof. (2)⇒(1) Set

$$G(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon_1^p(\underline{x},\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}.$$

As $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$, we have

$$G(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\overline{B}(0,\Omega)}\varepsilon_1^p(\underline{x},\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}. \tag{10}$$

Owing to the estimate (7) of $\varepsilon_1^p(\underline{x},\underline{y},\underline{t})$, we have

$$|G(\underline{x},\underline{y})|\le C\|\widehat{F}\|_2e^{\Omega|\underline{x}|}\|\chi_{\overline{B}(0,\Omega)}\|_2\le Ce^{\Omega|\underline{x}|}\quad\text{for any } \underline{x}\in\mathbf{R}^p \text{ and } \underline{y}\in\mathbf{R}^q.$$

Since ε_1^p is the codimension-p CK extension of $e^{i\langle\underline{y},\underline{t}\rangle}$, through the integral representation (10), the function $G(\underline{x},\underline{y})$ is the codimension-p CK extension of $F(\underline{y})$. As both $G(\underline{x},\underline{y})$ and $f(\underline{x},\underline{y})$ are codimension-p CK extensions from the same function $F(\underline{y})$ on $\mathbf{R}^q$, they have to be identical on $\mathbf{R}^{p+q}$.

(1)⇒(2) Set $A_0(\underline{y})=F(\underline{y})$. Let $f_1(x_1\mathbf{e}_1,\underline{y})$ be the homogeneous codimension-1 CK extension satisfying $f_1(0,\underline{y})=A_0(\underline{y})=f(0,\underline{y})$. Lemma 4 then asserts that for any $\varepsilon>0$, $x_1\mathbf{e}_1\in\mathbf{R}^1$, $\underline{y}\in\mathbf{R}^q$,

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\le C_\varepsilon e^{(\Omega+\varepsilon)|x_1|}\le C_\varepsilon e^{(\Omega+\varepsilon)|x_1\mathbf{e}_1+\underline{y}|}.$$

By invoking Theorem 1, we get $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega+\varepsilon)$. Letting $\varepsilon\to0$, we conclude that $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$. The proof is complete. □

The following variation of Theorem 2 holds.

Theorem 3. *Let F be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e}_{p+1},\dots,\mathbf{e}_{p+q}$, and $F\in L^2(\mathbf{R}^q)$. Let Ω be a positive real number. Then the following assertions hold*:

(1) *If* $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$, *then F has a homogeneous codimension-p CK extension to $\mathbf{R}^{p+q}$, denoted by f, and there exists a constant C such that*

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}+\underline{y}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q.$$

(2) *If there exists a homogeneous codimension-p CK extension of F to $\mathbf{R}^{p+q}$, denoted by f and a constant C such that*

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}+\underline{y}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q,$$

then $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\sqrt{2}\Omega)$.

Moreover, in each of the above cases, we have

$$f(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon_1^p(\underline{x},\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q.$$

Proof. The assertion (1) is the easy part. We only indicate how to prove (2). Note that, if in Lemma 4, instead of the assumption

$$|f(\underline{x},\underline{y})|\leq Ce^{\Omega|\underline{x}|}$$

one adopts the weaker assumption

$$|f(\underline{x},\underline{y})|\leq Ce^{\Omega|\underline{x}+\underline{y}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q,$$

then the inequality (8) becomes

$$|\partial^l_{\underline{y}}f(0,\underline{y})|\leq C\frac{(m-1)m...(m+l-2)}{\rho^{m+l-1}}e^{\sqrt{2}\Omega|\rho+\underline{y}|}.$$

This will modify the inequality (9). By taking $\rho=|x_1|(1+\varepsilon/\Omega)$ in the modified inequality, we have, for any $\varepsilon>0$,

$$|f(x_1\mathbf{e}_1,\underline{y})|\leq C_\varepsilon e^{\sqrt{2}(\Omega+\varepsilon)|x_1\mathbf{e}_1+\underline{y}|},\quad x_1\mathbf{e}_1\in\mathbf{R}^1,\ \underline{y}\in\mathbf{R}^q.$$

The proof is complete by invoking Theorem 1. $\square$

Next we extend Theorem 2 to the generalized CK extension case involving a k-monogenic weight function.

Lemma 5. *Let $P_k(\underline{x})\in M^+(p,k,\mathbf{C}^{(p)})$ be given. Denote by $f_{P_k}(\underline{x},\underline{y})$ the generalized CK extension of $F(\underline{y})$ in relation to P_k, which is left-monogenic in $\mathbf{R}^m$. Assume that*

$$|f_{P_k}(\underline{x},\underline{y})|\leq Ce^{\Omega|\underline{x}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q,$$

where Ω is a positive real number. Let $f_1(x_1\mathbf{e}_1,\underline{y})$ be the homogeneous codimension-1 CK extension of the same initial value $f_1(0,\underline{y})=F(\underline{y})$. Then for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\leq C_\varepsilon e^{(\Omega+\varepsilon)|x_1|}\quad \text{for any } x_1\mathbf{e}_1\in\mathbf{R}^1 \text{ and } \underline{y}\in\mathbf{R}^q.$$

Proof. Since $f_{P_k}(\underline{x},\underline{y})$ is the generalized CK extension of $F(\underline{y})$ in relation to P_k, we have

$$f_{P_k}(\underline{x},\underline{y})=\sum_{l=0}^{\infty}\underline{x}^lP_k(\underline{x})F_l(\underline{y}),$$

where

$$F_{2l}(\underline{y})=\frac{(-1)^l\Gamma(k+p/2)\partial^{2l}_{\underline{y}}F(\underline{y})}{2^{2l}l!\Gamma(l+k+p/2)}, \tag{11}$$

and

$$F_{2l+1}(\underline{y})=\frac{(-1)^l\Gamma(k+p/2)\partial_{\underline{y}}^{2l+1}F(\underline{y})}{2^{2l+1}l!\Gamma(l+k+p/2+1)}. \tag{12}$$

On the other hand, since f_{P_k} is left-monogenic in $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$, by Cauchy's integral formula, we have

$$f_{P_k}(\underline{x},\underline{y})=\frac{1}{\omega_{m-1}}\int_{\partial B(\underline{x}+\underline{y},\rho)}E(\underline{t}-(\underline{x}+\underline{y}))\,d\sigma(\underline{t})f_{P_k}(\underline{t}),$$

where ρ is any positive real number.

Then

$$\frac{\partial^{l+k}}{\partial x_1^{l+k}}f_{P_k}(\underline{x},\underline{y})=\frac{1}{\omega_{m-1}}\frac{\partial^{l+k}}{\partial x_1^{l+k}}\int_{\partial B(\underline{x}+\underline{y},\rho)}E(\underline{t}-(\underline{x}+\underline{y}))\,d\sigma(\underline{t})f_{P_k}(\underline{t}).$$

By Lemma 3 and the assumption $|f_{P_k}(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|}$, we have

$$\left|\frac{\partial^{l+k}}{\partial x_1^{l+k}}f_{P_k}(\underline{x},\underline{y})\right|\le C\frac{(m-1)m...(m+l+k-2)}{\rho^{l+k}}e^{\Omega(|\underline{x}|+\rho)}.$$

Therefore,

$$|F_l(\underline{y})|=\lim_{|\underline{x}|\to 0}\frac{1}{l!}\left|\frac{\partial^{l+k}}{\partial x_1^{l+k}}f_{P_k}(\underline{x},\underline{y})\right|\le C\frac{1}{l!}\frac{(m-1)m...(m+l+k-2)}{\rho^{l+k}}e^{\Omega\rho}. \tag{13}$$

Using (11), (12) and the above estimate (13), we have

$$\begin{aligned}|f_1(x_1\mathbf{e}_1,\underline{y})|&\le\sum_{l=0}^{\infty}\frac{|x_1|^l|\partial_{\underline{y}}^l[F(\underline{y})]|}{l!}\\&\le\sum_{l=0}^{\infty}\Big(l+k+\frac{p}{2}\Big)^{k+p/2}\frac{|x_1|^l(m-1)m...(m+l+k-2)}{l!\rho^{l+k}}e^{\Omega\rho}.\end{aligned}$$

For any $\varepsilon>0$, set $l_0=[2\Omega(m+k-2)/\varepsilon]+1$, then $l>l_0$ implies $(m+k-2)/l<\varepsilon/2\Omega$. Thus, if $l>l_0$, then $1+(m+k-2)/l<1+\varepsilon/2\Omega$, so

$$\begin{aligned}|f_1(x_1e_1,\underline{y})|&\le C\sum_{l=0}^{\infty}\Big(l+k+\frac{p}{2}\Big)^{k+p/2}\frac{|x_1|^l(m-1)m...(m+l+k-2)}{l!\rho^{l+k}}e^{\Omega\rho}\\&\le C\sum_{l=0}^{l_0}\Big(l+k+\frac{p}{2}\Big)^{k+p/2}\frac{|x_1|^l(m-1)m...(m+l+k-2)}{l!\rho^{l+k}}e^{\Omega\rho}\\&\quad+C_\varepsilon\sum_{l=l_0}^{\infty}\Big(l+k+\frac{p}{2}\Big)^{k+p/2}\frac{|x_1|^l(1+\varepsilon/2\Omega)^{l-l_0}}{\rho^{l+k}}e^{\Omega\rho},\end{aligned}$$

where $C_\varepsilon=(m-1)m...(m+k+l_0-2)/l_0!$.

When $|x_1|>1$, taking $\rho=|x_1|(1+\varepsilon/\Omega)^2$, we have

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\le C_\varepsilon e^{(\Omega+\varepsilon)|x_1|}. \tag{14}$$

When $|x_1|<1$, taking $\rho=2$, we get that f_{P_k} is bounded. So the inequality (14) also holds. The proof is complete. □

Theorem 4. (Generalized codimension-p Paley–Wiener Theorem) *Let $P_k\in M^+(p,k;\mathbf{C}^{(p)})$ be given, F be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e}_{p+1},...,\mathbf{e}_{p+q}$, and $F\in L^2(\mathbf{R}^q)$. Let Ω be a positive real number. Then the following two assertions are equivalent*:

(1) *F has a homogeneous codimension-p generalized CK extension to $\mathbf{R}^{p+q}$, denoted by f_{P_k}, and there exists a constant C such that*

$$|f_{P_k}(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{y}\in\mathbf{R}^q.$$

(2) $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$.

Moreover, if one of the above conditions holds, then we have

$$f_{P_k}(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{y}\in\mathbf{R}^q.$$

Proof. (2)⇒(1) Set

$$G_{P_k}(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\widehat{F}(\underline{t})\,d\underline{t}.$$

Using the same method as in the first part of Theorem 2, we can easily prove this implication.

(1)⇒(2) Let $A_0(\underline{y})=F(\underline{y})$. By CK extension, using $A_0(\underline{y})$, we can construct another left-monogenic function $f_1(x_1\mathbf{e}_1,\underline{y})$ which satisfies $f_1(0,\underline{y})=A_0(\underline{y})=f(0,\underline{y})$. If we can prove that for any $\varepsilon>0$, $x_1\mathbf{e}_1\in\mathbf{R}^1$, $\underline{y}\in\mathbf{R}^q$,

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\le C_\varepsilon e^{(\Omega+\varepsilon)|x_1\mathbf{e}_1+\underline{y}|}, \tag{15}$$

by Theorem 1, we get $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega+\varepsilon)$. Letting $\varepsilon\to0$, we get $\operatorname{supp}(\widehat{F})\subset\overline{B}(0,\Omega)$.

We are thus reduced to show inequality (15).

Since

$$f_1(x_1\mathbf{e}_1,\underline{y})=\sum_{l=0}^{\infty}(x_1\mathbf{e}_1)^lA_l(\underline{y})=\sum_{l=0}^{\infty}\frac{(-1)^l(x_1\mathbf{e}_1)^l\partial^l_{\underline{y}}[A_0(\underline{y})]}{l!},$$

and

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|}\quad\text{for any } \underline{x}\in\mathbf{R}^p \text{ and } \underline{y}\in\mathbf{R}^q,$$

by Lemma 5, for any $\varepsilon>0$, we have

$$|f_1(x_1\mathbf{e}_1,\underline{y})|\le C_\varepsilon e^{(\Omega+\varepsilon)|x_1\mathbf{e}_1+\underline{y}|}\quad \text{for any } x_1\mathbf{e}_1\in\mathbf{R}^1 \text{ and } \underline{y}\in\mathbf{R}^q.$$

So inequality (15) holds. The proof is complete. $\square$

The rest of this section will deal with a PW theorem in relation to generalized Taylor series.

Lemma 6. *Assume that* $f(\underline{x},\underline{y})$ *is left-monogenic in* $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$ *with the form* (3) *and such that*

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q, \tag{16}$$

where Ω *is a positive real number. Then for all* $k\ge 0$ *and* $\alpha\in\mathcal{I}_k$*, we have*

$$|T_{k,\alpha}(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q,$$

where C *depends on* k *and* α.

Proof. Based on (3) and using the orthogonality on the sphere between $P_k(\underline{\omega})$ and $P_l(\underline{\omega})$ $(l\ne k)$, and that between $P_k(\underline{\omega})$ and $\underline{\omega}P_l(\underline{\omega})$ for any k and l, we have

$$\int_{S^{p-1}}\overline{P}_{k,\alpha}(\underline{\omega})f(|\underline{x}|\underline{\omega},\underline{y})\,dS_{\underline{\omega}}=\sum_{l=0}^{\infty}(-1)^l|\underline{x}|^{2l+k}T_{k,\alpha}^{(2l)}(\underline{y}).$$

According to the condition (16), we obtain

$$\left|\sum_{l=0}^{\infty}\underline{x}^{2l}P_k(\underline{x})T_{k,\alpha}^{(2l)}(\underline{y})\right|\le C\left|\sum_{l=0}^{\infty}(-1)^l|\underline{x}|^{2l+k}T_{k,\alpha}^{(2l)}(\underline{y})\right|\le Ce^{\Omega|\underline{x}|}.$$

Similarly, we have

$$\int_{S^{p-1}}\overline{\underline{\omega}P}_{k,\alpha}(\underline{\omega})f(|\underline{x}|\underline{\omega},\underline{y})\,dS_{\underline{\omega}}=\sum_{l=0}^{\infty}(-1)^l|\underline{x}|^{2l+1+k}T_{k,\alpha}^{(2l+1)}(\underline{y}).$$

Using the condition (16), we also obtain

$$\left|\sum_{l=0}^{\infty}\underline{x}^{2l+1}P_k(\underline{x})T_{k,\alpha}^{(2l+1)}(\underline{y})\right|\le C\left|\sum_{l=0}^{\infty}(-1)^l|\underline{x}|^{2l+1+k}T_{k,\alpha}^{(2l+1)}(\underline{y})\right|\le Ce^{\Omega|\underline{x}|}.$$

Thus we have

$$\begin{aligned}|T_{k,\alpha}(\underline{x},\underline{y})|&=\left|\sum_{l=0}^{\infty}\underline{x}^{l}P_{k,\alpha}(\underline{x})T_{k,\alpha}^{(l)}(\underline{y})\right|\\&\le\left|\sum_{l=0}^{\infty}\underline{x}^{2l}P_{k,\alpha}(\underline{x})T_{k,\alpha}^{(2l)}(\underline{y})\right|+\left|\sum_{l=0}^{\infty}\underline{x}^{2l+1}P_{k,\alpha}(\underline{x})T_{k,\alpha}^{(2l+1)}(\underline{y})\right|\\&\le Ce^{\Omega|\underline{x}|},\quad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q.\quad\square\end{aligned}$$

We will need the following lemma.

Lemma 7. ([1, p. 281]) *Assume that $f(\underline{x})$ is left-monogenic in $\mathbf{R}^m$ and that Ω is a positive number. If there exists a constant C, such that*

$$|f(\underline{x})|\le Ce^{\Omega|\underline{x}|},\quad \underline{x}=r\underline{\omega},$$

then

$$|P(k)(f)(\underline{\omega})|\le C(1+k)^m\frac{\Omega^k}{k!},$$

where $P(k)$ is the projection onto $\mathcal{M}^+(m,k,\mathbf{C}^{(m)})$.

Theorem 5. *Assume that $f(\underline{x},\underline{y})$ is left-monogenic in $\mathbf{R}^m=\mathbf{R}^p\oplus\mathbf{R}^q$ with the form* (3)*. For any $k\ge 0$ and $\alpha\in\mathcal{I}_k$, let $T_{k,\alpha}(\underline{y})=T^{(0)}_{k,\alpha}(\underline{y})$ be analytic, defined in $\mathbf{R}^q$, taking values in $\mathbf{C}^{(q)}$, the complex Clifford algebra generated by $\mathbf{e}_{p+1},\dots,\mathbf{e}_{p+q}$, and $T_{k,\alpha}(\underline{y})\in L^2(\mathbf{R}^q)$. Assume also that*

$$\Big|\sum_{k=0}^{\infty}\sum_{\alpha\in\mathcal{I}_k}P_{k,\alpha}(\underline{x})\widehat{T}_{k,\alpha}(\underline{t})\Big|\le Ce^{\Omega|\underline{x}|}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{t}\in\mathbf{R}^q, \tag{17}$$

where Ω is a positive real number. Then the following two assertions are equivalent:

(1) *There exists a constant C such that*

$$|f(\underline{x},\underline{y})|\le Ce^{\Omega|\underline{x}|}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{y}\in\mathbf{R}^q.$$

(2) $\operatorname{supp}(\widehat{T}_{k,\alpha})\subset\overline{B}(0,\Omega)$ *for any $k\ge0$ and $\alpha\in\mathcal{I}_k$.*

Moreover, if one of the above conditions holds, we have

$$f(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\sum_{k=0}^{\infty}\sum_{\alpha\in\mathcal{I}_k}\int_{\mathbf{R}^q}\varepsilon^p_{P_{k,\alpha}}(\underline{x},\underline{y},\underline{t})\widehat{T}_{k,\alpha}(\underline{t})\,d\underline{t}\quad \textit{for any } \underline{x}\in\mathbf{R}^p \textit{ and } \underline{y}\in\mathbf{R}^q.$$

Proof. (2)$\Rightarrow$(1) The first part of the proof of Theorem 2 may be closely followed. In fact, for any $k\ge0$ and $\alpha\in\mathcal{I}_k$, set

$$G_{k,\alpha}(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon^p_{P_k}(\underline{x},\underline{y},\underline{t})\widehat{T}_{k,\alpha}(\underline{t})\,d\underline{t}.$$

As $\operatorname{supp}(\widehat{T}_{k,\alpha})\subset\overline{B}(0,\Omega)$, we have

$$G_{k,\alpha}(\underline{x},\underline{y})=\frac{1}{(2\pi)^q}\int_{\overline{B}(0,\Omega)}\varepsilon^p_{P_{k,\alpha}}(\underline{x},\underline{y},\underline{t})\widehat{T}_{k,\alpha}(\underline{t})\,d\underline{t}.$$

Since $\varepsilon^p_{P_{k,\alpha}}$ is the generalized CK extension of $e^{i\langle\underline{y},\underline{t}\rangle}$ in $\mathcal{T}_{P_k}$, the generalized Taylor coefficient of $G_{k,\alpha}(\underline{x},\underline{y})$ is $T_{k,\alpha}(\underline{y})$. That is $T_{k,\alpha}(\underline{x},\underline{y})$ and $G_{k,\alpha}(\underline{x},\underline{y})$ have the same generalized Taylor coefficient $T_{k,\alpha}(\underline{y})$, so $T_{k,\alpha}(\underline{x},\underline{y})=G_{k,\alpha}(\underline{x},\underline{y})$. Thus

$$\begin{aligned} f(\underline{x},\underline{y}) &= \sum_{k=0}^{\infty}\sum_{\alpha\in\mathcal{I}_k} T_{k,\alpha}(\underline{x},\underline{y}) \\ &= \frac{1}{(2\pi)^q}\sum_{k=0}^{\infty}\sum_{\alpha\in\mathcal{I}_k}\int_{\overline{B}(0,\Omega)} \varepsilon^p_{P_{k,\alpha}}(\underline{x},\underline{y},\underline{t})\widehat{T}_{k,\alpha}(\underline{t})\,d\underline{t} \\ &= \frac{1}{(2\pi)^q}\sum_{k=0}^{\infty}\int_{\overline{B}(0,\Omega)} \varepsilon^p_k(\underline{x},\underline{y},\underline{t})r^k\left[\sum_{\alpha\in\mathcal{I}_k} P_{k,\alpha}(\underline{\omega})\widehat{T}_{k,\alpha}(\underline{t})\right]d\underline{t}. \end{aligned}$$

Using condition (17) and Lemma 7, we have

$$\left|\sum_{\alpha\in\mathcal{I}_k} P_{k,\alpha}(\underline{\omega})\widehat{T}_{k,\alpha}(t)\right| \le C(1+k)^p\frac{\Omega^k}{k!}.$$

Adding to the inequality (5), we have

$$\begin{aligned} |f(\underline{x},\underline{y})| &\le \frac{1}{(2\pi)^q}\sum_{k=0}^{\infty}\int_{\overline{B}(0,\Omega)} |\varepsilon^p_k(\underline{x},\underline{y},\underline{t})|r^k\left|\sum_{\alpha\in\mathcal{I}_k} P_{k,\alpha}(\underline{\omega})\widehat{T}_{k,\alpha}(\underline{t})\right||d\underline{t}| \\ &\le C\sum_{k=0}^{\infty}(1+k)^p\frac{r^k\Omega^k}{k!}\left(\frac{r\Omega}{2}\right)^{-k-p/2+1}\left[I_{k+p/2-1}(r\Omega)+I_{k+p/2}(r\Omega)\right] \\ &\le C\sum_{k=0}^{\infty}\frac{(1+k)^p2^k}{k!}\left(\frac{r\Omega}{2}\right)^{-p/2+1}\left[I_{k+p/2-1}(r\Omega)+I_{k+p/2}(r\Omega)\right] \\ &\le C\sum_{k=0}^{\infty}[I_k(r\Omega)+I_{k+1}(r\Omega)] \\ &\le Ce^{r\Omega} \\ &= Ce^{\Omega|\underline{x}|}. \end{aligned}$$

(1)$\Rightarrow$(2) For any $k\ge 0$, and $\alpha\in\mathcal{I}_k$, by the generalized CK extension, using $T_{k,\alpha}(\underline{y})$, we can construct a series of left-monogenic functions $T_{k,\alpha}(x_1\mathbf{e}_1,\underline{y})\in\mathbf{R}^{q+1}$ satisfying $T_{k,\alpha}(0,\underline{y})=T_{k,\alpha}(\underline{y})$. If we can prove that for any $\varepsilon>0$, $x_1\mathbf{e}_1\in\mathbf{R}^1$ and $\underline{y}\in\mathbf{R}^q$ we have

$$|T_{k,\alpha}(x_1\mathbf{e}_1,\underline{y})| \le C_\varepsilon e^{(\Omega+\varepsilon)(|x_1\mathbf{e}_1+\underline{y}|)}, \tag{18}$$

by Theorem 1, we get $\operatorname{supp}(\widehat{T}_{k,\alpha})\subset\overline{B}(0,\Omega+\varepsilon)$. Letting $\varepsilon\to 0$, we get $\operatorname{supp}(\widehat{T}_{k,\alpha})\subset\overline{B}(0,\Omega)$.

We therefore are reduced to proving inequality (18).

Since

$$|f(\underline{x},\underline{y})| \le Ce^{\Omega|\underline{x}|} \quad \text{for any } \underline{x}\in \mathbf{R}^p \text{ and } \underline{y}\in \mathbf{R}^q.$$

Combining Lemmas 5 and 6, for any $\varepsilon>0$, we have

$$|T_{k,\alpha}(x_1\mathbf{e}_1,\underline{y})| \le C_\varepsilon e^{(\Omega+\varepsilon)(|x_1\mathbf{e}_1+\underline{y}|)} \quad \text{for any } x_1\mathbf{e}_1\in \mathbf{R}^1 \text{ and } \underline{y}\in \mathbf{R}^q.$$

The proof is complete. □

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