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# Ergodic complex structures on hyperkähler manifolds

by

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## 1. Introduction

### 1.1. Complex geometry and ergodic theory

For an introduction to Teichmüller theory and global Torelli theorem, please see §1.2. The basic notions of hyperkähler geometry are recalled in §2. Here we assume that a reader knows the basic definitions.

The *Teichmüller space* is defined as the space of complex structures up to isotopies: $\mathrm{Teich}:=\mathsf{Comp}/\mathrm{Diff}_0$. The *mapping class group* (also known as the *group of diffeotopies*) is the group $\Gamma:=\mathrm{Diff}/\mathrm{Diff}_0$ of connected components of the diffeomorphism group. Clearly, $\Gamma$ acts on the Teichmüller space in a natural way.

It turns out that in some important geometric situations (for the hyperkähler manifolds with $b_2>3$ and the complex tori of dimension $\geqslant 2$) the mapping group action on Teich is ergodic. This is surprising, at least to the author of the present paper, because in this case the moduli space $\mathrm{Teich}/\Gamma$ of these geometric objects is extremely pathological. In fact this quotient is so much non-Hausdorff that any two non-empty open subsets of $\mathrm{Teich}/\Gamma$ intersect (see Remark 3.12).

Complex structures with dense $\Gamma$-orbits are called *ergodic* (see Definition 1.12). From the description of the moduli in terms of homogeneous spaces and Moore's theorem on ergodic actions it follows that the set of non-ergodic complex structures on hyperkähler manifolds with $b_2>3$ and complex tori of dimension $\geqslant 2$ has measure zero (Theorem 3.9). Applying Ratner theory, we prove that the set of non-ergodic complex structures is in fact

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countable: a complex structure is non-ergodic if and only if its Picard rank is maximal (Corollary 4.12).

The density of particular families of hyperkähler manifolds in $\mathrm{Teich}/\Gamma$ has been used many times since the early 1970s. Piatetski–Shapiro and Shafarevich used density of the family of Kummer surfaces in the moduli of K3 surfaces to prove the local Torelli theorem [PS]. This theorem was generalized to a general hyperkähler manifold $M$ with $b_2 \geqslant 5$ in [AnV]. Here it was proven that any divisorial family defined by an integer class in $H^2(M, \mathbb{Z})$ is dense in $\mathrm{Teich}/\Gamma$. In [KamV] this approach was used further to study the Lagrangian fibrations on hyperkähler manifolds. Using the density argument and existence of Lagrangian fibrations it was proven that all known hyperkähler manifolds are non-hyperbolic.(1) In [MM], Markman and Mehrota show that the Hilbert space of K3 surfaces is dense in the corresponding deformation space, and prove a similar result about the generalized Kummer varieties.

Existence of ergodic complex structures leads to some interesting results about various complex-analytic quantities, such as the Kobayashi pseudometric. As a model situation, consider a function $\varphi$ on the set of equivalence classes of complex manifolds which is continuous on deformations. Since the ergodic orbit $\Gamma \cdot I$ is dense in the Teichmüller space, and $\varphi$ is constant on $\Gamma \cdot I$, this implies that $\varphi$ is constant.

In practice, such an application is hard to come by, because functions which continuously depend on the complex structure are sowewhat rare. However, there are many semicontinuous functions, and a semicontinuous function has to be constant on ergodic complex structures. Indeed, let $I, J \in \mathrm{Teich}$ be two ergodic complex structures, that is, complex structures with dense $\Gamma$-orbits, and $\varphi\colon \mathrm{Teich} \to \mathbb{R}$ be a semicontinuous (say, upper semicontinuous) $\Gamma$-invariant function. Since $I$ is a limiting point of the dense set $\Gamma \cdot J$, semicontinuity implies $\varphi(I) \geqslant \varphi(J)$. By the same reason, $\varphi(J) \geqslant \varphi(I)$, and hence $\varphi$ is constant on the set of all ergodic complex structures.

This remark can be applied to several questions of complex hyperbolicity (see §1.4).

### 1.2. Teichmüller spaces and hyperkähler geometry

We recapitulate briefly the definition of the Teichmüller space of the hyperkähler manifolds, following [V3].

*Definition* 1.1. Let $M$ be a compact complex manifold, and $\mathrm{Diff}_0(M)$ be a connected component of its diffeomorphism group (the *group of isotopies*). Denote by $\mathsf{Comp}$ the space of complex structures on $M$, equipped with the structure of a Fréchet manifold,

(1) In the present paper we generalize this further to all hyperkähler manifolds with $b_2 > 3$.

and let Teich:=$\mathsf{Comp}/\mathrm{Diff}_0(M)$ be its quotient, equipped with the quotient topology. We call it the *Teichmüller space.*

*Remark* 1.2. In many important cases, such as for Calabi–Yau manifolds [Cat], Teich is a finite-dimensional complex space; usually it is non-Hausdorff.

*Definition* 1.3. Let $\mathrm{Diff}(M)$ be the group of orientable diffeomorphisms of a complex manifold. The quotient $\mathsf{Comp}/\mathrm{Diff}=\mathrm{Teich}/\Gamma$ is called the *moduli space* of complex structures on $M$. Typically, it is very non-Hausdorff. The set $\mathsf{Comp}/\mathrm{Diff}$ corresponds bijectively to the set of isomorphism classes of complex structures.

*Definition* 1.4. A *hyperkähler structure* on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I$, $J$ and $K$, satisfying quaternionic relations

$$I\circ J=-J\circ I=K,$$

such that $g$ is Kähler for $I$, $J$ and $K$.

*Remark* 1.5. One could define a hyperkähler structure in terms of the complex geometry of its twistor space (see Definition 5.9). This was discovered in [HKLR]; see [V2] for a few historical remarks and further development of this approach.

*Remark* 1.6. A hyperkähler manifold is holomorphically symplectic: $\omega_J+i\omega_K$ is a holomorphic symplectic form on $(M,I)$. This is easily seen using a simple linear-algebraic calculation [Bes].

THEOREM 1.7. (Calabi–Yau; see [Y], [Bea] and [Bes]) *A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.*

*Remark* 1.8. The term "hyperkähler manifold" can mean many different things. In the literature, it denotes either a manifold equipped with a hyperkähler structure, or a complex manifold admitting a hyperkähler structure, or a Riemannian manifold with holonomy in $\mathrm{Sp}(n)$. In the present paper, we shall by a *hyperkähler manifold* mean a compact, complex manifold admitting a Kähler structure and a holomorphically symplectic structure. Such a complex structure is called a complex structure of hyperkähler type. We also assume tacitly that all hyperkähler manifolds are of maximal holonomy (or irreducibly holomorphically symplectic), in the sense of Definition 1.9 below.

*Definition* 1.9. A hyperkähler manifold $M$ is *of maximal holonomy* if $\pi_1(M)=0$ and $H^{2,0}(M)=\mathbb{C}$. In the literature, such manifolds are often called *irreducibly holomorphically symplectic*, or *irreducibly symplectic varieties.*

This definition is motivated by the following theorem of Bogomolov.

Theorem 1.10. ([Bog1], [Bea]) *Any hyperkähler manifold admits a finite covering which is a product of a torus and several maximal holonomy hyperkähler manifolds.*

*Remark* 1.11. Further on, all hyperkähler manifolds are assumed to be of maximal holonomy, Comp is the space of all complex structures of hyperkähler type on $M$, and Teich its quotient by $\mathrm{Diff}_0(M)$.

### 1.3. Ergodic complex structures

The main object of this paper is the following notion.

*Definition* 1.12. Let $M$ be a complex manifold, Teich be its Teichmüller space, and $I\in\mathrm{Teich}$ be a point. Consider the set $Z_I\subset\mathrm{Teich}$ of all $I'\in\mathrm{Teich}$ such that $(M,I)$ is biholomorphic to $(M,I')$ (clearly, $Z_I=\Gamma\cdot I$, where $\Gamma=\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ is a mapping group acting on Teich). A complex structure is called *ergodic* if the corresponding orbit $Z_I$ is dense in Teich.

*Remark* 1.13. The origins of this term are explained in §3.1. It is well known that almost all orbits of an ergodic action are dense (Claim 3.3). However, for a hyperkähler manifold with $b_2>3$ and a complex torus of dimension $\geqslant 2$, the mapping group action on Teich is ergodic (see Theorem 3.9 and Remark 3.10). Notice that the definition of ergodic action does not require one to fix a particular measure on the space (Remark 3.2).

In many situations, the mapping class group action on the Teichmüller space is ergodic. This implies that the non-ergodic complex structures form a set of measure zero.

Theorem 1.14. *Let $M$ be a maximal holonomy hyperkähler manifold or a compact complex torus of dimension $\geqslant 2$. Then the set* $\mathrm{Teich}_{ne}$ *of non-ergodic points has measure zero in the corresponding Teichmüller space* Teich.

*Proof.* See Theorem 3.9 below. □

*Remark* 1.15. The notion of a measure-zero subset of a manifold is independent of the choice of a smooth measure. Therefore, to state Theorem 1.14, it is not necessary to fix a particular measure on Teich.

This result follows from the ergodicity of the mapping class group action on Teich, which follows from the global Torelli theorem and the ergodicity of an arithmetic action on homogeneous spaces due to Moore [Moo]. It is not very explicit, and for a considerable period of time, no explicit examples of ergodic complex structures were known. This problem was solved by an application of a powerful theorem of Ratner (Theorem 4.2).

THEOREM 1.16. *Let $M$ be a maximal holonomy hyperkähler manifold or a compact complex torus of dimension $\geqslant 2$, and $I$ be a complex structure on $M$. Then $I$ is non-ergodic if and only if the Néron–Severi lattice of $(M, I)$ has maximal possible rank. This means that*

$$\operatorname{rk} \mathsf{NS}(M,I) = \begin{cases} b_2(M)-2, & \text{if } M \text{ is hyperkähler,} \\ (\dim_{\mathbb{C}} M)^2, & \text{if } M \text{ is a torus.} \end{cases}$$

*Proof.* See Corollary 4.12 below. □

### 1.4. Kobayashi pseudometric on hyperkähler manifolds

The ergodic properties of the mapping group action have many applications to Kobayashi hyperbolicity. For the definition of Kobayashi pseudometric, basic properties and the further reference, please see §5. For the purposes of the present paper, Kobayashi pseudometric is important because it is a complex-analytic invariant which is upper semicontinuous as a function of a complex structure [Ko], [Vo]. This suggests the following conjecture.

CONJECTURE 1.17. *Let $I, J \in \mathrm{Teich}$ be ergodic complex structures on $M$, and $d_I$ and $d_J$ be the corresponding Kobayashi pseudometrics. Then $(M, d_I)$ is isometric to $(M, d_J)$.*

*Remark* 1.18. Since the $\Gamma$-orbit of $I$ is dense in Teich, any $K \in \mathrm{Teich}$ can be obtained as a limit of $\nu_j^* I$ (see Remark 3.5), and one has $d_K \geqslant d_I$ by semicontinuity of $K$. In principle, this should give $d_J \geqslant d_I \geqslant d_J$, because both $I$ and $J$ are ergodic. To make this heuristic argument rigorous, one should make the dependency of $d_I$ and $d_J$ on diffeomorpisms $\nu_j$ explicit.

A *Kobayashi hyperbolic manifold* is a hyperbolic manifold with non-degenerate Kobayashi pseudometric. The set of Kobayashi hyperbolic complex structures is open in holomorphic families. Moreover, given a holomorphic family of Kobayashi hyperbolic manifolds, the Kobayashi pseudometric is continuous in this family [Vo]. In the presence of an ergodic complex structure, the argument used in the sketch of Conjecture 1.17 would imply that all hyperbolic complex structures on $M$ are isometric, and that the mapping class group acts by isometries. However, the isometry group of a compact metric space is always compact, and the image of the mapping class group in cohomology is usually noncompact. This can be used to prove non-hyperbolicity for manifolds admitting ergodic complex structures.

However, for hyperkähler manifolds, there exists a very simple and direct argument proving non-hyperbolicity.

The non-hyperbolicity of hyperkähler (and, more generally, Calabi–Yau) manifolds was a subject of long research, but until recently the only general result was a theorem by Campana proven in [Cam3].

A *twistor space* of a hyperkähler manifold (see Definition 5.9) is a total space of a fibration obtained from a hyperkähler rotation of a complex structure. Campana observed that the space of rational curves on $\mathrm{Tw}(M)$ transversal to the fibers is never compact (in fact it is holomorphically convex, as shown in [KalV]; see also [V4] and [DLM]). Then the limit of a sequence of rational curves in $\mathrm{Tw}(M)$ would contain an entire curve in one of the twistor fibers. This implies the Campana non-hyperbolicity theorem (Theorem 5.10): at least one of the fibers of the twistor family is not Kobayashi hyperbolic.

Another approach to non-hyperbolicity was used in [KamV]. In this paper it was shown that for all known examples of hyperkähler manifolds, the manifolds admitting a holomorphic Lagrangian fibration are dense in the moduli space. Such manifolds contain entire curves, and hence they are non-hyperbolic. However, the set of non-hyperbolic complex structures is closed in the relevant deformation space by Brody's lemma (Theorem 5.4). This is how non-hyperbolicity was proven in [KamV].

In the present paper we go by a different route, using ergodic methods and the non-hyperbolicity result of Campana. The explicit description of the set of non-ergodic complex structures (Corollary 4.12) allows one to find a twistor family with all fibers ergodic (Claim 5.11). By Campana's theorem, one of these fibers is non-hyperbolic. This gives a non-hyperbolic ergodic complex structure $I\in\mathrm{Teich}$. Then all points of the set $\Gamma\cdot I\subset\mathrm{Teich}$ are also non-hyperbolic. However, this set is dense, and hence its closure $\overline{\Gamma\cdot I}$ is the whole of Teich. Finally, we observe that the set of non-hyperbolic complex structures is closed in Teich, and therefore it contains $\overline{\Gamma\cdot I}=\mathrm{Teich}$.

## 2. Hyperkähler manifolds

In this section, we state the global Torelli theorem for hyperkähler manifolds, following [V3].

### 2.1. Bogomolov–Beauville–Fujiki form

The Bogomolov–Beauville–Fujiki form was defined in [Bog2] and [Bea], but it is easiest to describe it using the Fujiki formula, proven in [F].

THEOREM 2.1. (Fujiki) *Let $M$ be a maximal holonomy hyperkähler manifold, let $\eta\in H^2(M)$ and $n=\frac{1}{2}\dim M$. Then $\int_M \eta^{2n}=cq(\eta,\eta)^n$, where $q$ is a primitive integral quadratic form on $H^2(M,\mathbb{Z})$, and $c>0$ is a rational number.*

*Remark* 2.2. Fujiki formula (Theorem 2.1) determines the form $q$ uniquely up to a sign. For odd $n$, the sign is unambiguously determined as well. For even $n$, one needs the following explicit formula, which is due to Bogomolov and Beauville:

$$\lambda q(\eta,\eta)=\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\bar{\Omega}^{n-1}-\frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\bar{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\bar{\Omega}^{n-1}\right), \quad (2.1)$$

where $\Omega$ is the holomorphic symplectic form, and $\lambda>0$.

*Definition* 2.3. Let $q\in\operatorname{Sym}^2(H^2(M,\mathbb{Z})^*)$ be the integral form defined by Theorem 2.1 and formula (2.1). This form is called the *Bogomolov–Beauville–Fujiki form*.

## 2.2. Mapping class group

*Definition* 2.4. Let $\operatorname{Diff}(M)$ be the group of oriented diffeomorphisms of $M$, and $\operatorname{Diff}_0(M)$ be the group of isotopies, that is, the connected component of $\operatorname{Diff}(M)$. We call $\Gamma:=\operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ the *mapping class group* of $M$.

For Kähler manifolds of dimension $\geqslant 3$, the mapping class group can be computed using the following theorem by Sullivan.

THEOREM 2.5. (Sullivan; [Su]) *Let $M$ be a compact, simply connected Kähler manifold, with $\dim_{\mathbb{C}} M\geqslant 3$. Denote by $\Gamma_0$ the group of automorphisms of the algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_j(M)$. Then the natural map $\operatorname{Diff}(M)/\operatorname{Diff}_0\to\Gamma_0$ has finite kernel, and its image has finite index in $\Gamma_0$.*

*Definition* 2.6. Two groups $G$ and $G'$ are *commensurable* if there exist subgroups $G_1\subset G$ and $G_1'\subset G_1$ of finite index, and finite normal subgroups $G_2\subset G_1$ and $G_2'\subset G_1'$ such that $G_1/G_2$ is isomorphic to $G_1'/G_2'$. An *arithmetic group* is a group which is commensurable to an integer lattice in a rational Lie group.

*Remark* 2.7. Sullivan's theorem claims that the mapping class group of any Kähler manifold is arithmetic.(2)

Using the results of [V1], the group of automorphisms of the algebra $H^*(M,\mathbb{Z})$ can be determined explicitly, up to commensurability. This gives the following theorem, proven in [V3].

(2) In fact, Sullivan proved the arithmeticity of the mapping class group for any compact smooth manifold of dimension $\geqslant 5$.

Theorem 2.8. *Let $M$ be a maximal holonomy hyperkähler manifold, and $\Gamma_0$ be the group of automorphisms of $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_j(M)$. Consider the restriction map $\Gamma_0 \xrightarrow{\psi} \mathrm{GL}(H^2(M,\mathbb{Z}))$. Then $\psi$ has finite kernel, its image lies in the orthogonal group $O(H^2(M,\mathbb{Z}),q)$, and $\psi(\Gamma_0)$ has finite index in that group.*

### 2.3. Global Torelli theorem

*Remark* 2.9. Let $M$ be a hyperkähler manifold (as usual, we assume $M$ to be of maximal holonomy). Recall that in this situation Teich was defined as the set of all complex structures of hyperkähler type on $M$ (Remark 1.11). For any $J$ in the same connected component of Teich, $(M,J)$ is also a maximal holonomy hyperkähler manifold, because the Hodge numbers are constant in families. Therefore, $H^{2,0}(M,J)$ is 1-dimensional.

*Definition* 2.10. Let

$$\mathsf{Per}\colon \mathrm{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$$

map $J$ to the line $H^{2,0}(M,J)\in\mathbb{P}H^2(M,\mathbb{C})$. The map $\mathsf{Per}$ is called the *period map*.

*Remark* 2.11. The period map $\mathsf{Per}$ maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\mathsf{er} := \{l\in\mathbb{P}H^2(M,\mathbb{C}) : q(l,l)=0 \text{ and } q(l,\bar{l})>0\}.$$

It is called the *period space* of $M$. Indeed, any holomorphic symplectic form $l$ satisfies the relations $q(l,l)=0$ and $q(l,\bar{l})>0$, as follows from formula (2.1).

Proposition 2.12. *The period space $\mathbb{P}\mathsf{er}$ is identified with the quotient*

$$\frac{\mathrm{SO}(b_2-3,3)}{\mathrm{SO}(2)\times\mathrm{SO}(b_2-3,1)},$$

*which is a Grassmannian $\mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ of positive oriented 2-planes in $H^2(M,\mathbb{R})$.*

*Proof.* This statement is well known, but we shall sketch its proof to illustrate the constructions given below.

*Step* 1. Given $l\in\mathbb{P}H^2(M,\mathbb{C})$, the space generated by $\operatorname{Im} l$ and $\operatorname{Re} l$ is 2-dimensional, because $q(l,l)=0$, and $q(l,\bar{l})>0$ implies that $l\cap H^2(M,\mathbb{R})=0$.

*Step* 2. This 2-dimensional plane is positive, because

$$q(\operatorname{Re} l,\operatorname{Re} l) = q(l+\bar{l}, l+\bar{l}) = 2q(l,\bar{l})>0.$$

*Step* 3. Conversely, for any 2-dimensional positive plane $V\in H^2(M,\mathbb{R})$, the quadric

$$\{l\in V\otimes_{\mathbb{R}}\mathbb{C} : q(l,l)=0\}$$

consists of two lines; a choice of a line is determined by orientation. $\square$

*Definition* 2.13. Let $M$ be a topological space. We say that two points $x, y \in M$ are *non-separable* (denoted $x \sim y$) if for any open sets $V \ni x$ and $U \ni y$, one has $U \cap V \neq \varnothing$.

THEOREM 2.14. (Huybrechts; [H1], [H2]) *Two points $I, I' \in \mathrm{Teich}$ are non-separable if there exists a bimeromorphism $(M, I) \to (M, I')$.*

*Definition* 2.15. The space $\mathrm{Teich}_b := \mathrm{Teich}/\sim$ is called the *birational Teichmüller space* of $M$.

THEOREM 2.16. (Global Torelli theorem; [V3]) *The period map* $\mathsf{Per}\colon \mathrm{Teich}_b \to \mathbb{P}\mathsf{er}$ *is an isomorphism for each connected component of* $\mathrm{Teich}_b$.

*Definition* 2.17. Let $M$ be a hyperkähler manifold, let $\mathrm{Teich}_b$ be its birational Teichmüller space, and let $\Gamma$ be the mapping class group. The quotient $\mathrm{Teich}_b / \Gamma$ is called the *birational moduli space* of $M$. Its points are in bijective correspondence with the complex structures of hyperkähler type on $M$ up to a bimeromorphic equivalence.

*Remark* 2.18. The word "space" in this context is misleading. In fact, the quotient topology on $\mathrm{Teich}_b / \Gamma$ is extremely non-Hausdorff, e.g. every two non-empty open sets intersect (Remark 3.12).

The global Torelli theorem can be stated as a result about the birational moduli space.

THEOREM 2.19. ([V3, Theorem 7.2 and Remark 7.4]) *Let $(M, I)$ be a hyperkähler manifold, and $W$ be a connected component of its birational moduli space. Then $W$ is isomorphic to* $\mathbb{P}\mathsf{er}/\Gamma_I$, *where* $\mathbb{P}\mathsf{er} = \mathrm{SO}(b_2-3, 3)/\mathrm{SO}(2) \times \mathrm{SO}(b_2-3, 1)$ *and $\Gamma_I$ is an arithmetic group in* $\mathrm{O}(H^2(M, \mathbb{R}), q)$, *called the* monodromy group *of* $(M, I)$.

*Remark* 2.20. The monodromy group of $(M, I)$ can be also described as a subgroup of the group $O(H^2(M, \mathbb{Z}), q)$ generated by the monodromy transform maps for the Gauss–Manin local systems obtained from all deformations of $(M, I)$ over a complex base ([V3, Definition 7.1]). This is how this group was originally defined by Markman [M1], [M2].

*Remark* 2.21. Caution: usually the global Torelli theorem is understood as a theorem about Hodge structures. For K3 surfaces, the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure. For $\dim_{\mathbb{C}} M > 2$, this is false.

*Remark* 2.22. We shall freely identify $\mathbb{P}\mathsf{er}$ and $\mathrm{Teich}_b$.

*Remark* 2.23. By [M2, Proposition 5.14], the fibers of the natural projection

$$\mathsf{Per}\colon \mathrm{Teich} \longrightarrow \mathrm{Teich}_b$$

can be identified with a set of "Kähler chambers", which are open subsets of the space $H^{1,1}(M, I)$. Therefore, each fiber is countable or finite.

*Remark* 2.24. By [V3, Remark 4.28], outside of a countable union of complex divisors on $\mathrm{Teich}_b$, the map $\mathsf{Per}\colon \mathrm{Teich}\to\mathrm{Teich}_b$ is bijective.

*Remark* 2.25. We will be interested in ergodic (that is, measure-theoretic) properties of Teich and $\mathrm{Teich}_b$. By Remarks 2.24 and 2.23, the map $\mathsf{Per}$ is bijective outside of a measure-zero set. Therefore, any ergodicity result proven for Teich remains true for $\mathrm{Teich}_b$, and vice versa.

## 3. Ergodic complex structures on hyperkähler manifolds and tori

### 3.1. Ergodicity: basic definitions and results

*Definition* 3.1. Let $(M,\mu)$ be a space with measure, and $G$ be a group acting on $M$ preserving the sigma-algebra of measurable subsets, and mapping measure-zero sets to measure-zero sets. This action is *ergodic* if all $G$-invariant measurable subsets $M'\subset M$ satisfy $\mu(M')=0$ or $\mu(M\setminus M')=0$.

*Remark* 3.2. When one defines an ergodic action, it is usually assumed that the action of $G$ preserves the measure. However, this is not necessary. In fact, any manifold is equipped with a sigma-algebra of Lebesgue measurable sets and, moreover, the notion of a measure-zero subset set is independent of the choice of a Lebesgue measure. This means that one can define "ergodic action of a group" on a manifold not specifying the measure.

Claim 3.3. *Let $M$ be a manifold, $\mu$ be a Lebesgue measure, and $G$ be a group acting on $(M,\mu)$ ergodically. Then the set of points with non-dense orbits has measure zero.*

*Proof.* Consider a non-empty open subset $U\subset M$. Then $\mu(U)>0$, and so $M':=G\cdot U$ satisfies $\mu(M\setminus M')=0$. For any orbit $G\cdot x$ not intersecting $U$, $x\in M\setminus M'$. Therefore the set of such points has measure zero. $\square$

*Definition* 3.4. Let $M$ be a complex manifold, Teich be its Techmüller space, and $\Gamma$ be the mapping group acting on Teich. An *ergodic complex structure* is a complex structure with dense $\Gamma$-orbit.

*Remark* 3.5. Let $(M,I)$ be a manifold with ergodic complex structure, and $I'$ be another complex structure. Then there exists a sequence of diffeomorphisms $\nu_j$ such that $\nu_j^*(I)$ converges to $I'$ in the usual (Fréchet) topology on the space of complex structure tensors. This property is clearly equivalent to ergodicity of $I$.

We shall need the following result about ergodicity of an arithmetic group action on a homogeneous space. This result will be applied to a mapping class group (which is arithmetic by Theorem 2.8) and a period space, which is homogeneous (by Proposition 2.12).

*Definition* 3.6. Let $G$ be a Lie group, and $\Gamma \subset G$ be a discrete subgroup. Consider the pushforward of the Haar measure to $G/\Gamma$. We say that $\Gamma$ has *finite covolume* if the Haar measure of $G/\Gamma$ is finite. In this case $\Gamma$ is called a *lattice subgroup*.

*Remark* 3.7. Borel and Harish-Chandra proved that an arithmetic subgroup of a reductive group $G$ over $\mathbb{Q}$ is a lattice whenever $G$ has no non-trivial characters over $\mathbb{Q}$ (see e.g. [VGS]). In particular, all arithmetic subgroups of a semi-simple group defined over $\mathbb{Q}$ are lattices.

THEOREM 3.8. (Moore, [Moo, Theorem 7]) *Let $\Gamma$ be a lattice subgroup (such as an arithmetic subgroup) in a non-compact simple Lie group $G$ with finite center, and $H \subset G$ be a non-compact Lie subgroup. Then the left action of $\Gamma$ on $G/H$ is ergodic.*

## 3.2. Ergodic action on the Teichmüller space for hyperkähler manifolds and tori

THEOREM 3.9. *Let $\mathbb{P}\mathsf{er}$ be a component of a birational Teichmüller space of a hyperkähler manifold $M$, with $b_2(M)>3$, and $\Gamma_I$ be its monodromy group acting on $\mathbb{P}\mathsf{er}$. Consider the set $Z \subset \mathbb{P}\mathsf{er}$ of all points with non-dense orbits. Then the action of $\Gamma_I$ on $\mathbb{P}\mathsf{er}$ is ergodic, and $Z$ has measure zero in $\mathbb{P}\mathsf{er}$.*

*Proof. Step* 1. Let $G=\mathrm{SO}(b_2-3,3)$, $H=\mathrm{SO}(2)\times\mathrm{SO}(b_2-3,1)$, and $\Gamma \subset G$ be an arithmetic subgroup. Then the $\Gamma$-action on $G/H$ is ergodic, by Moore's theorem.

*Step* 2. The space $\mathbb{P}\mathsf{er}$ is identified with $G/H$ (Proposition 2.12), and the monodromy group is an arithmetic subgroup of $G$ by Theorems 2.8 and 2.19. Then $\Gamma_I$ acts on $\mathbb{P}\mathsf{er}$ ergodically, and the set of points with non-dense orbits has measure zero (Claim 3.3). □

*Remark* 3.10. As explained in Remark 2.25, the space $\mathrm{Teich}_b = \mathbb{P}\mathsf{er}$ is identified with Teich up to measure-zero subsets. Therefore, the set of non-ergodic complex structures on a hyperkähler manifold has measure zero in Teich.

A similar result is true for a compact torus. Here the Teichmüller space is the space of complex structure operators on $\mathbb{R}^{2n}$, identified with the quotient $\mathrm{SL}(2n,\mathbb{R})/\mathrm{SL}(n,\mathbb{C})$, and the mapping class group is $\mathrm{SL}(2n,\mathbb{Z})$ ([Cat]). For $n\geqslant 2$, the group $\mathrm{SL}(n,\mathbb{C})$ is non-compact. Thus, Theorem 3.8 can be applied, and we obtain the following statement.

Theorem 3.11. *Let* $W := \mathrm{SL}(2n,\mathbb{R})/\mathrm{SL}(n,\mathbb{C})$, $n \geqslant 2$, *be the Teichmüller space of an* $n$*-dimensional compact torus, equipped with an action of the mapping class group* $\Gamma = \mathrm{SL}(2n,\mathbb{Z})$. *Then the action of* $\Gamma$ *on* $W$ *is ergodic. In particular, the set of non-ergodic complex tori has measure zero in the corresponding Teichmüller space.*

*Remark* 3.12. Existence of erdogic complex structures means that the quotient $\mathrm{Teich}/\Gamma$ (considered with the quotient topology) is extremely non-Hausdorff. Indeed, any two non-empty open sets in Teich contain points in a dense orbit $\Gamma \cdot I$, and hence their images in $\mathrm{Teich}/\Gamma$ intersect. We obtain that any two open subsets in the moduli space $\mathrm{Teich}/\Gamma$ intersect.

## 4. Ratner orbit closure theorem and ergodic complex structure

### 4.1. Lie groups generated by unipotent elements

Here we state the basic facts of Ratner theory. We follow [KSS] and [Mor].

*Definition* 4.1. Let $G$ be a Lie group, and $g \in G$ be any element. We say that $g$ is *unipotent* if $g = e^h$ for a nilpotent element $h$ in its Lie algebra. A group $G$ is *generated by unipotent elements* if $G$ is multiplicatively generated by unipotent elements.

Theorem 4.2. (Ratner orbit closure theorem) *Let* $H \subset G$ *be a Lie subroup generated by unipotent elements, and* $\Gamma \subset G$ *be a lattice. Then the closure of any* $H$*-orbit in* $G/\Gamma$ *is the orbit of a closed, connected subgroup* $S \subset G$, *such that* $S \cap \Gamma \subset S$ *is a lattice in* $S$.

*Proof.* See [Mor, §1.1.15 (2)]. □

*Remark* 4.3. Theorem 4.2 is true if $H = H_0 \times H_1$, where $H_0$ is generated by unipotent elements, and $H_1$ is compact. Indeed, for each $x \in G/\Gamma$, one has $\overline{H \cdot x} = H_1 \cdot \overline{H_0 \cdot x}$. The inclusion $\overline{H \cdot x} \supset H_1 \cdot \overline{H_0 \cdot x}$ is obvious. The converse inclusion would follow if we prove that $H_1 \cdot \overline{H_0 \cdot x}$ is closed. However, the orbit of a closed set under a compact Lie group is always closed.

*Example* 4.4. Let $V$ be a real vector space with a non-degenerate bilinear symmetric form of signature $(3,k)$, $k > 0$. Also let $G := \mathrm{SO}^+(V)$ be a connected component of the isometry group, $H \subset G$ be a subgroup fixing a given positive 2-dimensional plane,

$$H \cong \mathrm{SO}^+(1,k) \times \mathrm{SO}(2),$$

and $\Gamma \subset G$ be an arithmetic lattice. Consider the quotient $\mathbb{P}\mathsf{er} := H \backslash G$. Then

(i) a point $J \in \mathbb{P}\mathsf{er}$ has closed $\Gamma$-orbit if and only if the orbit $H \cdot J$ in the quotient $G/\Gamma$ is closed;

(ii) the closure of $H\cdot J$ in $G/\Gamma$ is the orbit of a closed connected Lie group $S\supset H$:

$$\overline{H\cdot J}=S\cdot J\subset \mathbb{P}\mathsf{er}\,.$$

For arithmetic groups, the Ratner orbit closure theorem can be stated in a more precise way, as follows.

THEOREM 4.5. *Let $G$ be a real algebraic group defined over $\mathbb{Q}$ and with no non-trivial characters. Also let $W\subset G$ be a subgroup generated by unipotent elements, and $\Gamma\subset G$ be an arithmetic lattice. For a given $g\in G$, let $H$ be the smallest real algebraic $\mathbb{Q}$-subgroup of $G$ containing $g^{-1}Wg$. Then the closure of $Wg$ in $G/\Gamma$ is $Hg$.*

*Proof.* See [KSS, Proposition 3.3.7] or [Sh, Proposition 3.2]. □

## 4.2. Ratner theorem for Teichmüller spaces

In §4.4, we prove the following two elementary theorems.

THEOREM 4.6. *Let $G=\mathrm{SO}^+(3,k)$, $k\geqslant 1$, and $H\cong \mathrm{SO}^+(1,k)\times \mathrm{SO}(2)\subset G$. Then any closed connected Lie subgroup $S\subset G$ containing $H$ coincides with $G$ or with $H$.*

*Proof.* See Theorem 4.15. □

THEOREM 4.7. *Let $G=\mathrm{SL}(2n,\mathbb{R})$, $n\geqslant 2$, and $H\cong \mathrm{SL}(n,\mathbb{C})\subset G$. Then any closed connected Lie subgroup $S\subset G$ containing $H$ coincides with $G$ or with $H$.*

*Proof.* See Theorem 4.17. □

Now we can apply these theorems to characterize the ergodic and non-ergodic complex structures.

THEOREM 4.8. *Let $M$ be a hyperkähler manifold, $\mathbb{P}\mathsf{er}$ be its period space, and*

$$I\in \mathbb{P}\mathsf{er}=Gr_{++}(H^2(M,\mathbb{R}))$$

*be a point associated with a positive 2-plane $V\subset H^2(M,\mathbb{R})$. Then the $\Gamma$-orbit of $I$ is dense in $\mathbb{P}\mathsf{er}$ unless the plane $V$ is rational, that is, it satisfies $\dim_{\mathbb{Q}}(V\cap H^2(M,\mathbb{Q}))=2$.*

*Proof.* Let $\Gamma\subset G$ be the monodromy group of $M$, that is, the image of the mapping class group in $G$, where $G=\mathrm{SO}^+(H^2(M,\mathbb{R}),q)$. $\Gamma$ is an arithmetic lattice in $G$, as shown in Theorem 2.8. Since $I$ is non-ergodic, the closure $\overline{\Gamma\cdot I}$ of $\Gamma\cdot I$ is strictly smaller than $\mathbb{P}\mathsf{er}$. By Ratner's theorem, there exists a subgroup $S\subsetneq G$ containing $H$ such that $\overline{\Gamma\cdot I}=S\cdot I$, and $S\cap\Gamma$ is a lattice in $S$. Theorem 4.6 implies that $S=H$. Since $S\cap\Gamma$ is a lattice, this set is Zariski dense in $S$. By Theorem 4.5, $S=H$ is a rational subgroup of $G$. Conversely, if $H$ is rational, its image is closed in $G/\Gamma$ as follows from Theorem 4.5. □

Theorem 4.9. *Let $M$ be a compact complex torus of dimension $n\geqslant 2$,* Teich *be its Teichmüller space*, $\mathrm{Teich}=\mathrm{SL}(2n,\mathbb{R})/\mathrm{SL}(n,\mathbb{C})$, *and $I\in\mathrm{Teich}$ be a point associated with a complex structure*, $I\in\mathrm{End}(\mathbb{R}^{2n})$. *Then we have that the point $I$ is non-ergodic if and only if $H^{1,1}(M,\mathbb{R})\subset H^2(M,\mathbb{R})$ is a rational subspace.*

*Proof.* Let $\Gamma=\mathrm{SL}(2n,\mathbb{Z})$ be the mapping class group of $M$. A point $I$ is non-ergodic if its $\Gamma$-orbit in $\mathrm{Teich}=\mathrm{SL}(2n,\mathbb{R})/\mathrm{SL}(n,\mathbb{C})$ is not dense. By the Ratner orbit closure theorem, $\overline{\Gamma\cdot I}=S\cdot I$, where $S\supset\mathrm{SL}(n,\mathbb{C})$ is a connected Lie subgroup of $\mathrm{SL}(2n,\mathbb{R})$. Since an intermediate subgroup $\mathrm{SL}(2n,\mathbb{R})\supset S\supset\mathrm{SL}(n,\mathbb{C})$ is equal to either $\mathrm{SL}(2n,\mathbb{R})$ or $\mathrm{SL}(n,\mathbb{C})$, the point $I$ is non-ergodic if and only if $S=\mathrm{SL}(n,\mathbb{C})$ and the orbit $\Gamma\cdot I$ is closed. By Theorem 4.5, this happens if and only if the stabilizer $\mathsf{St}(I)\cong\mathrm{SL}(n,\mathbb{C})$ of $I$ is a rational subgroup of $\mathrm{SL}(2n,\mathbb{R})$. The centralizer $Z(\mathsf{St}(I))$ is a group $R_I\cong U(1)=\cos t+\sin t\cdot I$, and $Z(Z(\mathsf{St}(I))=\mathsf{St}(I)$. Hence rationality of $\mathsf{St}(I)$ is equivalent to rationality of $R_I$.

However, the space $H^2(M)^{R_I}$ of $R_I$-invariants is $H^{1,1}(M)$, and, conversely, $R_I$ is a subgroup of $\mathrm{SL}(2n,\mathbb{R})=\mathrm{SL}(H^1(M,\mathbb{R}))$ acting trivially on $H^{1,1}(M)$. Therefore, $R_I$ is rational if and only if $H^{1,1}(M)\subset H^2(M,\mathbb{R})$ is rational. □

We have just proven density of certain orbits of $\Gamma$ in the period space, but for geometric applications, one would need density of orbits in the Teichmüller space. This is already true for a torus, because for the torus the period space coincides with the Teichmüller space. For a hyperkähler manifold with rational curves, a similar result can be obtained directly.

Corollary 4.10. *Let $(M,I)$ be a hyperkähler manifold with Picard group of non-maximal rank. Assume that $(M,I)$ contains no rational curves. Then $I$ is an ergodic complex structure.*

*Proof.* Let $\mathrm{Teich}_0\subset\mathrm{Teich}$ be the set of all Hausdorff points in Teich. By Theorem 2.14, $\mathrm{Teich}_0$ is the set of all complex structures on $M$ admitting no non-trivial birational models. However, any birational map between complex manifolds with trivial canonical bundle must blow down some subvariety, and hence such maps do not exist when one has no rational curves. Therefore, $I\in\mathrm{Teich}_0$. Now, the period map restricted to $\mathrm{Teich}_0$ is a homeomorphism, and $\Gamma\cdot\mathsf{Per}(I)$ is dense in $\mathbb{P}\mathsf{er}$ by Theorem 4.8. Therefore, $\Gamma\cdot I$ is dense in an appropriate connected component of $\mathrm{Teich}_0$, but $\mathrm{Teich}_0$ is dense in Teich by Remark 2.24. □

Corollary 4.10 is already sufficient for many applications dealing with hyperbolicity; indeed, to prove that a manifold is non-hyperbolic, it suffices to show that it contains rational curves. However, for many applications a full strength ergodicity result is required.

THEOREM 4.11. *Let $M$ be a hyperkähler manifold, and $I$ be a complex structure of non-maximal Picard rank. Then $I$ is ergodic.*

*Proof.* See §4.3. □

COROLLARY 4.12. *Let $M$ be a hyperkähler manifold or a complex torus of complex dimension $\geqslant 2$. Then $M$ is non-ergodic if and only if its Néron–Severi lattice has maximal possible rank. In particular, there are only countably many non-ergodic complex structures.*

*Proof.* By definition, the Néron–Severi lattice is a lattice of integer (1,1)-classes in $H^2(M)$. It is easy to see that it has maximal possible rank if and only if $\mathsf{Per}(I)$ is rational (for hyperkähler manifolds). For complex tori, the argument is given in the proof of Theorem 4.9. The countability of the set of such complex structures is also well known and easy to check. □

### 4.3. Density of non-Hausdorff orbits

Fix a connected component of a Teichmüller space of hyperkähler manifold. Abusing the notation, we denote it by Teich, and denote the subgroup of the mapping class group fixing Teich by $\Gamma$.

Let $[I]\in\mathbb{P}\mathsf{er}$ be a point in the period space of $M$. The Hodge decomposition of $H^2(M)$ is determined by the periods, and we denote the corresponding (1,1)-space by $H^{1,1}([I])$. The *positive cone* $\mathrm{Pos}([I])$ is the set of all real (1,1)-classes $v\in H^{1,1}([I])$ satisfying $q(v,v)>0$. A subset $K\subset\mathrm{Pos}([I])$ is called a *Kähler chamber* if it is a Kähler cone for some $I\in\mathrm{Teich}$ satisfying $\mathsf{Per}(I)=[I]$. We have already used the following result, which is due to Markman.

PROPOSITION 4.13. *Different Kähler chambers of $[I]$ do not intersect, and* $\mathrm{Pos}([I])$ *is the closure of their union. Moreover, there is a bijective correspondence between points of $\mathsf{Per}^{-1}([I])$ in one Teichmüller component and the set of Kähler chambers of $[I]$.*

*Proof.* See [M2, Proposition 5.14]. □

Consider the set Hyp of pairs $I\in\mathrm{Teich}$ and $\omega\in\mathrm{Kah}(M,I)$, where $\mathrm{Kah}(M,I)$ denotes the Kähler cone. One should think of Hyp as of the Teichmüller space of all hyperkähler metrics on a holomorphically symplectic manifold. Let $F$ be the set of all pairs $[I]\in\mathbb{P}\mathsf{er}$ and $\omega\in\mathrm{Pos}([I])$. Consider the period map $\mathsf{Per}_h\colon\mathrm{Hyp}\to F$ mapping $(I,\omega)$ to $(\mathsf{Per}(I),\omega)$. By Proposition 4.13, $\mathsf{Per}_h$ is injective with dense image.

To prove that $\Gamma\cdot I$ is dense in Teich is the same as to show that

$$\Gamma\cdot(I,\mathrm{Kah}(M,I))$$

is dense in $\mathrm{Hyp}\subset F$ (Proposition 4.13). We consider $F$ as a homogeneous space of an appropriate Lie group. To show that $\Gamma\cdot(I, \mathrm{Kah}(M,I))$ is dense in $F$, we show that $\mathrm{Kah}(M,I)$ contains an orbit of its Lie subgroup and apply Ratner's theorem to this homogeneous space.

Our arguments are based on the following lemma.

Lemma 4.14. *Let $(M,I)$ be a hyperkähler manifold, $\omega\in\mathrm{Kah}(M,I)$ be a Kähler class, $H^{1,1}_I(M,\mathbb{Q})$ be the space of rational $(1,1)$-classes, and $l\in H^{1,1}_I(M,\mathbb{Q})^\perp$ be a $(1,1)$-class orthogonal to $H^{1,1}_I(M,\mathbb{Q})$. Then $V\cap\mathrm{Pos}(M,I)\subset\mathrm{Kah}(M,I)$, where $V=\langle\omega,l\rangle$ is a 2-dimensional space generated by $l$ and $\omega$.*

*Proof.* As follows from [H3] and [Bou] (see [AmV, Theorem 1.19] for a precise statement), $\mathrm{Kah}(M,I)$ is a subset of the positive cone given by a set of linear inequalities

$$\mathrm{Kah}(M,I)=\{\omega\in\mathrm{Pos}(M,I): q(\omega,l_j)>0\},$$

where $l_j$ is a countable set of rational $(1,1)$-classes. This means that for any $\omega\in\mathrm{Kah}(M,I)$ and any $l\in H^{1,1}_I(M,\mathbb{Q})^\perp$, the sum $l+\omega$ also belongs to $\mathrm{Kah}(M,I)$, as long as it has a positive square. □

*Proof of Theorem* 4.11. As we have already observed, to prove Theorem 4.11 it would suffice to show that $\Gamma\cdot(I,\mathrm{Kah}(M,I))$ is dense in $F$. Consider the set $F_1$ of all $([I],\eta)\in F$ such that $q(\eta,\eta)=1$. Clearly,

$$F_1=\frac{\mathrm{SO}(3,b_2-3)}{\mathrm{SO}(2)\times\mathrm{SO}(b_2-3)}.$$

Indeed, $F_1$ is identified with the set of pairs

$$\{(W,\omega): W\in\mathrm{Gr}_{++}(H^2(M,\mathbb{R})),\ \omega\in W^\perp \text{ and } q(\omega,\omega)>0\}.$$

By Lemma 4.14, for any $\omega\in\mathrm{Kah}(M,I)$ and any $l\in H^{1,1}_I(M,\mathbb{Q})^\perp$, the whole set $\mathrm{Pos}(M,I)\cap\langle\omega,l\rangle$ belongs to $\mathrm{Kah}(M,I)$. Choose $l$ in such a way that $q(l,l)<0$; since $\mathrm{Pic}(M)$ is not of maximal rank, this is always possible. Consider the group $H_0$ of oriented isometries of $V:=\langle\omega,l\rangle$; we extend its action to $H^2(M,\mathbb{R})$ by requiring $H_0$ to act trivially on $V^\perp$. By Lemma 4.14, $H_0$ preserves $\mathrm{Kah}(M,I)$. To prove density of $\Gamma\cdot(I,\mathrm{Kah}(M,I))$ in $F_1$, it would suffice to show that a $\Gamma$-orbit of the set $(I,H_0\cdot\omega)$ is dense in $F_1$. This is the same as to show that a $\Gamma$-orbit of an appropriate point in

$$\frac{\mathrm{SO}(3,b_2-3)}{H_0\cdot(\mathrm{SO}(2)\times\mathrm{SO}(b_2-3))}=\frac{\mathrm{SO}(3,b_2-3)}{\mathrm{SO}(2)\times\mathrm{SO}(1,b_2-3)}$$

is dense.

We have arrived at the situation described in Theorems 4.6 and 4.8. Here it was shown that any orbit of $\Gamma$ is either closed or dense. For this orbit to be closed, the stabilizer of a pair

$$[I]\in \mathrm{Gr}_{++}(H^2(M,\mathbb{R})) \quad \text{and} \quad V=\langle \omega, l\rangle$$

has to be rational; since $[I]$ is irrational, this is impossible. This proves Theorem 4.11. □

### 4.4. Maximal subgroups of Lie groups

THEOREM 4.15. *Let $(V,q)$ be a real vector space equipped with a non-degenerate quadratic form. Let $G=\mathrm{SO}^+(V)$ be the connected component of the group of isometries of $V$ and $W\subset V$ be a subspace with $q|_W$ non-degenerate. Consider the subgroup $H\subset G$ consisting of all isometries preserving $W\subset V$. Then any closed connected Lie subgroup $S\subsetneq G$ containing $H$ coincides with $H$.*

*Proof.* Let $\mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{s}$ be the Lie algebras of $H$, $G$ and $S$, respectively. Then $\mathfrak{h}=\mathfrak{so}(W)\oplus\mathfrak{so}(W^\perp)$. The quotient $\mathfrak{g}/\mathfrak{h}$ is identified with $\mathrm{Hom}(W,W^\perp)$, and hence it is an irreducible representation of $\mathfrak{h}$. Since $\mathfrak{s}/\mathfrak{h}$ is a proper $\mathfrak{h}$-subrepresentation of $\mathfrak{g}/\mathfrak{h}$, it is equal to 0. □

*Remark* 4.16. The proof of Theorem 4.15 is intuitively very clear: any isometry fixing $W$ is contained in $H$; if we add an isometry which moves $W$, the resulting group should contain all isometries. A similar argument works for the pair $\mathrm{SL}(n,\mathbb{C})\subset\mathrm{SL}(2n,\mathbb{R})$, if we think of $\mathrm{SL}(n,\mathbb{C})$ as of a group fixing a subspace of $(1,0)$-vectors in the complexification of $\mathbb{C}^n$.

THEOREM 4.17. *Let $W$ be a complex vector space, $G=\mathrm{SL}(W,\mathbb{R})$ be the group of its real volume-preserving automorphisms, and $H\cong\mathrm{SL}(W,\mathbb{C})\subset G$ be the group of complex volume-preserving automorphisms of $W$. Then any closed connected Lie subgroup $S\subsetneq G$ containing $H$ coincides with $H$.*

*Proof.* Let $\mathfrak{h}_\mathbb{C}$, $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{s}_\mathbb{C}$ be the complexified Lie algebras of $H$, $G$ and $S$, respectively. Consider the space $W_\mathbb{C}:=W\otimes_\mathbb{R}\mathbb{C}$, and let $W_\mathbb{C}:=W^{1,0}\oplus W^{0,1}$ be its Hodge decomposition. Then

$$\mathfrak{h}_\mathbb{C}=\mathfrak{sl}(W^{1,0})\oplus\mathfrak{sl}(W^{0,1})$$

and

$$\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}=\mathrm{Hom}(W^{1,0},W^{0,1})\oplus\mathrm{Hom}(W^{0,1},W^{1,0}).$$

Both the components $\mathrm{Hom}(W^{1,0},W^{0,1})$ and $\mathrm{Hom}(W^{0,1},W^{1,0})$ are irreducible representations of $\mathfrak{h}_\mathbb{C}$. Since $\mathfrak{s}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$ is a proper $\mathfrak{h}_\mathbb{C}$-subrepresentation of $\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$, it is equal to

$\mathrm{Hom}(W^{1,0}, W^{0,1})$ or $\mathrm{Hom}(W^{0,1}, W^{1,0})$ or 0. However, $\mathfrak{s}$ is real, and hence $\mathfrak{s}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ is fixed by the anticomplex involution exchanging $W^{1,0}$ and $W^{0,1}$. Therefore, the components $\mathrm{Hom}(W^{1,0}, W^{0,1})$ and $\mathrm{Hom}(W^{0,1}, W^{1,0})$ can only be contained in $\mathfrak{s}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ together. Since $\mathfrak{s}_{\mathbb{C}}\subset\mathfrak{g}_{\mathbb{C}}$ is a proper subalgebra, $\mathfrak{s}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ must be empty. □

## 5. Twistor spaces and Kobayashi pseudometric

### 5.1. Kobayashi pseudometric and Brody lemma

This subsection is a brief introduction to the subject. For more details, please see [L], [Vo] and [D].

*Definition* 5.1. Let $M$ be a complex manifold, $x, y\in M$ be points, and $d_P$ be the Poincaré metric on the unit disk $\Delta\subset\mathbb{C}$. Define

$$\tilde{d}(x,y) := \sup_{f:\Delta\to M} d_P(f^{-1}(x), f^{-1}(y)),$$

where the supremum is taken over all holomorphic maps $f:\Delta\to M$ from the disk $\Delta$ to $M$ such that $f(\Delta)\supset\{x,y\}$. The maximal pseudometric $d$ satisfying $d(x,y)\leqslant\tilde{d}(x,y)$ is called the *Kobayashi pseudometric*. The manifold $M$ is called *Kobayashi hyperbolic* if the Kobayashi pseudometric is non-degenerate ([Ko]).

For a compact manifold, hyperbolocity can be interpreted as non-existence of entire curves.

*Definition* 5.2. An *entire curve* in a complex manifold $M$ is the image of a non-constant holomorphic map $\mathbb{C}\to M$.

The following two theorems are fundamental in hyperbolic geometry; for details and the proofs, see again [L], [Vo] and [D]. They follow from a remarkable result on convergence of disks and entire curves on complex manifolds, called Brody's lemma [Br].

THEOREM 5.3. *Let $M$ be a compact complex manifold. Then $M$ contains an entire curve if and only if it is not Kobayashi hyperbolic.*

THEOREM 5.4. *Let $I_j$ be a sequence of non-hyperbolic complex structures on a compact manifold $M$, and $I$ be its limit. Then $(M, I)$ is also non-hyperbolic.*

The main result of this section is the following theorem.

THEOREM 5.5. *Any compact hyperkähler manifold $M$ satisfying $b_2(M)>3$ is non-hyperbolic.*

*Proof.* See §5.2. □

*Remark* 5.6. For all known examples of hyperkähler manifolds, this theorem is already known, due to Kamenova and Verbitsky [KamV].

*Remark* 5.7. To prove Theorem 5.5, it would suffice to show that there exists an ergodic complex structure $I$ which is non-hyperbolic. Indeed, in this case the orbit of $I$ is dense. This implies that any complex structure can be obtained as a limit of non-hyperbolic ones (see Remark 3.5).

### 5.2. Twistor spaces and Campana's theorem

*Definition* 5.8. Let $I, J, K, g$ be a hyperkähler structure on a manifold $M$. *Induced complex structures* on $M$ are complex structures of the form

$$S^2 \cong \{L := aI + bJ + cK : a^2 + b^2 + c^2 = 1\}.$$

*Definition* 5.9. A *twistor space* $\mathrm{Tw}(M)$ of a hyperkähler manifold is a complex manifold obtained by gluing induced complex structures into a holomorphic family over $\mathbb{C}P^1$. More formally:

*Let* $\mathrm{Tw}(M) := M \times S^2$. *Consider the complex structure* $I_m : T_m M \to T_m M$ *on* $M$ *induced by* $J \in S^2 \subset \mathbb{H}$. *Let* $I_J$ *denote the complex structure on* $S^2 = \mathbb{C}P^1$. *The operator* $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ *satisfies* $I_{\mathrm{Tw}}^2 = -\mathrm{Id}$. *It defines an almost complex structure on* $\mathrm{Tw}(M)$. *This almost complex structure is known to be integrable* ([O] *and* [Sa]; *see* [Kal] *for a modern proof*).

Rational curves on twistor spaces were studied by Campana in a series of papers ([Cam1] and [Cam2]); among the results of this study, Campana proved the following theorem.

THEOREM 5.10. ([Cam3]) *Let* $M$ *be a hyperkähler manifold, equipped with a hyperkähler structure, and* $\mathrm{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ *be its twistor space. Then there exists an entire curve in some fiber of* $\pi$.

CLAIM 5.11. *Let* $M$ *be a hyperkähler manifold,* $b_2(M) \geqslant 4$. *Then there exists a twistor family on* $M$ *which has only ergodic fibers.*

*Proof.* By Corollary 4.12, there are only countably many complex structures which are not ergodic. The space $\mathcal{T}$ of all twistor families is identified with the set of hyperkähler metrics up to a constant multiplier. Therefore, it has real dimension $\frac{1}{6} b_2(b_2-1)(b_2-2)$, as follows from Bogomolov's local Torelli theorem and the Calabi–Yau theorem (Theorem 1.7). The space of twistor families passing through a given complex structure is parameterized by the projectivization of a Kähler cone, and hence its real dimension is $b_2-3$. There is a countable number of non-ergodic complex structures. Thus the set $\mathcal{T}_0$ of twistor families passing through non-ergodic complex structures is a union of countably many $(b_2-3)$-dimensional families. For $b_2>3$, one has $\dim_{\mathbb{R}} \mathcal{T} > b_2-3$, and hence $\mathcal{T}_0$ has measure zero in $\mathcal{T}$. □

Non-hyperbolicity of a hyperkähler manifold follows immediately from this claim. Indeed, let $\pi: \mathcal{S} \to \mathbb{C}P^1$ be a twistor family with all fibers ergodic. Theorem 5.10 implies that at least one fiber of $\pi$ is non-hyperbolic. Denote this fiber by $M$. Since $M$ is ergodic, there is a dense family of manifolds biholomorphic to $M$ in the Teichmüller space Teich. Since non-hyperbolic complex structures are closed in Teich (Theorem 5.4), this implies that all points in Teich correspond to non-hyperbolic complex structures.

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## References

[AmV] Amerik, E. & Verbitsky, M., Rational curves on hyperkähler manifolds. To appear in *Int. Math. Res. Not.*

[AnV] Anan′in, S. & Verbitsky, M., Any component of moduli of polarized hyperkähler manifolds is dense in its deformation space. *J. Math. Pures Appl.*, 101 (2014), 188–197.

[Bea] Beauville, A., Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18 (1983), 755–782.

[Bes] Besse, A. L., *Einstein Manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 10. Springer, Berlin–Heidelberg, 1987.

[Bog1] Bogomolov, F. A., On the decomposition of Kähler manifolds with a trivial canonical class. *Mat. Sb.*, 93 (1974), 573–575 (Russian); English translation in *Math. USSR–Sb.*, 22 (1974), 580–583.

[Bog2] — Hamiltonian Kählerian manifolds. *Dokl. Akad. Nauk SSSR*, 243 (1978), 1101–1104 (Russian); English translation in *Soviet Math. Dokl.*, 19 (1978), 1462–1465.

[Bou] Boucksom, S., Le cône kählérien d'une variété hyperkählérienne. *C. R. Acad. Sci. Paris Sér. I Math.*, 333 (2001), 935–938.

[Br] Brody, R., Compact manifolds and hyperbolicity. *Trans. Amer. Math. Soc.*, 235 (1978), 213–219.

[Cam1] Campana, F., Algébricité et compacité dans les espaces de cycles. *C. R. Acad. Sci. Paris Sér. A-B*, 289 (1979), A195–A197.

[Cam2] — On twistor spaces of the class $\mathscr{C}$. *J. Differential Geom.*, 33 (1991), 541–549.

[Cam3] — An application of twistor theory to the nonhyperbolicity of certain compact symplectic Kähler manifolds. *J. Reine Angew. Math.*, 425 (1992), 1–7.

[Cat] Catanese, F., A superficial working guide to deformations and moduli, in *Handbook of Moduli.* Vol. I, Adv. Lect. Math., 24, pp. 161–215. Int. Press, Somerville, MA, 2013.

[D] Demailly, J.-P., Hyperbolic algebraic varieties and holomorphic differential equations. *Acta Math. Vietnam.*, 37 (2012), 441–512.

[DLM] Deschamps, G., Le Du, N. & Mourougane, C., Hessian of the metric form on twistor spaces. Preprint, 2012. arXiv:1202.0183 [math.DG].

[F] Fujiki, A., On the de Rham cohomology group of a compact Kähler symplectic manifold, in *Algebraic Geometry* (Sendai, 1985), Adv. Stud. Pure Math., 10, pp. 105–165. North-Holland, Amsterdam, 1987.

[HKLR] Hitchin, N. J., Karlhede, A., Lindström, U. & Roček, M., Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108 (1987), 535–589.

[H1] Huybrechts, D., Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135 (1999), 63–113.

[H2] — Erratum: "Compact hyper-Kähler manifolds: basic results" [H1]. *Invent. Math.*, 152 (2003), 209–212.

[H3] — The Kähler cone of a compact hyperkähler manifold. *Math. Ann.*, 326 (2003), 499–513.

[Kal] Kaledin, D., Integrability of the twistor space for a hypercomplex manifold. *Selecta Math.*, 4 (1998), 271–278.

[KalV] Kaledin, D. & Verbitsky, M., Non-Hermitian Yang–Mills connections. *Selecta Math.*, 4 (1998), 279–320.

[KamV] Kamenova, L. & Verbitsky, M., Families of Lagrangian fibrations on hyperkähler manifolds. *Adv. Math.*, 260 (2014), 401–413.

[KSS] Kleinbock, D., Shah, N. & Starkov, A., Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory, in *Handbook of Dynamical Systems*, Vol. 1A, pp. 813–930. North-Holland, Amsterdam, 2002.

[Ko] Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings.* Pure and Applied Mathematics, 2. Marcel Dekker, New York, 1970.

[L] Lang, S., *Introduction to Complex Hyperbolic Spaces.* Springer, New York, 1987.

[M1] Markman, E., Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface. *Internat. J. Math.*, 21 (2010), 169–223.

[M2] — A survey of Torelli and monodromy results for holomorphic-symplectic varieties, in *Complex and Differential Geometry*, Springer Proc. Math., 8, pp. 257–322. Springer, Berlin–Heidelberg, 2011.

[MM] Markman, E. & Mehrotra, S., Hilbert schemes of K3 surfaces are dense in moduli. Preprint, 2012. arXiv:1201.0031 [math.AG].

[Moo] Moore, C. C., Ergodicity of flows on homogeneous spaces. *Amer. J. Math.*, 88 (1966), 154–178.

[Mor] Morris, D. W., *Ratner's Theorems on Unipotent Flows.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2005.

[O] Obata, M., Affine connections on manifolds with almost complex, quaternion or Hermitian structure. *Japan J. Math.*, 26 (1956), 43–77.

[PS] Piatetski-Shapiro, I. I. & Shafarevich, I. R., Torelli's theorem for algebraic surfaces of type K3. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35 (1971), 530–572 (Russian); English translation in *Math. USSR–Izv.*, 5 (1971), 547–588.

[Sa] Salamon, S., Quaternionic Kähler manifolds. *Invent. Math.*, 67 (1982), 143–171.

[Sh] Shah, N. A., Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.*, 289 (1991), 315–334.

[Su] Sullivan, D., Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (1977), 269–331.

[V1] Verbitsky, M., Cohomology of compact hyper-Kähler manifolds and its applications. *Geom. Funct. Anal.*, 6 (1996), 601–611.

[V2] — Hypercomplex varieties. *Comm. Anal. Geom.*, 7 (1999), 355–396.

[V3] — Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162 (2013), 2929–2986.

[V4] — Rational curves and special metrics on twistor spaces. *Geom. Topol.*, 18 (2014), 897–909.

[VGS] Vinberg, E. B., Gorbatsevich, V. V. & Shvartsman, O. V., Discrete subgroups of Lie groups, in *Lie Groups and Lie Algebras*, II, Encyclopaedia Math. Sci., 21, pp. 1–123, 217–223. Springer, Berlin–Heidelberg, 2000.

[Vo] Voisin, C., On some problems of Kobayashi and Lang; algebraic approaches, in *Current Developments in Mathematics*, 2003, pp. 53–125. Int. Press, Somerville, MA, 2003.

[Y] Yau, S. T., On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, 31 (1978), 339–411.

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