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Existence of knotted vortex tubes in steady Euler flows

by

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1. Introduction

The motion of particles in an ideal fluid in $\mathbb{R}^3$ is described by its velocity field $u(x,t)$, which satisfies the Euler equation

$$\partial_t u+(u\cdot\nabla)u=-\nabla P, \quad \operatorname{div} u=0,$$

for some pressure function $P(x,t)$. Equivalently, the field u satisfies the equation

$$-\partial_t u + u \times \omega = \nabla B, \quad \operatorname{div} u = 0,$$

where the field $\omega := \operatorname{curl} u$ is the *vorticity* and $B := P + \frac{1}{2}|u|^2$ is the Bernoulli function. The trajectories (or integral curves) of the vorticity $\omega(x,t)$ for fixed t are usually called *vortex lines*. A solution u to the Euler equation is called *steady* when it does not depend on time.

A domain in $\mathbb{R}^3$ that is the union of vortex lines and whose boundary is an embedded torus is a (closed) *vortex tube*. The analysis of *thin* vortex tubes, in a sense to be specified below, for solutions to the Euler equation has attracted considerable attention. A long-standing problem in this direction is Lord Kelvin's conjecture [28] that knotted and linked thin vortex tubes can arise in steady solutions to the Euler equation. This conjecture was motivated by results due to Helmholtz on the time-dependent case, which hinge on the mechanism of vorticity transport, and Maxwell's observations of what he called "water twists".

Kelvin's conjecture is basically a question on the existence of knotted invariant tori in steady solutions of the Euler equation. There is a considerable body of literature devoted to the analysis of topological and geometrical structures that appear in fluid flows, which has led to significant results e.g. on particle trajectories and vortex lines [2], [12], [21], [10], [25], on the relationship between the Euler equation and the group of volume-preserving diffeomorphisms [2], [9], [19], [6], and on the connection of the helicity with the energy functional and the asymptotic linking number [3], [15], [16], [29]. However, Kelvin's conjecture remains wide open, and indeed has been included as a major open problem in topological fluid mechanics in the surveys [27] and [23].

There is strong numerical evidence of the existence of thin vortex tubes, both in the case of steady and time-dependent fluid flows. As a matter of fact, thin vortex tubes have long played a key role in the construction and numerical exploration of possible blow-up scenarios for the Euler equation, which in turn has led to rigorous results such as [7] and [8]. A particularly influential scenario in this direction is [26], which discusses how an initial condition with a certain set of linked thin vortex tubes might lead to singularity formation in finite time. As a side remark, let us point out that thin vortex tubes of complicated knot topologies have been recently constructed experimentally in the fluid mechanics laboratory at the University of Chicago [20].

Our aim in this paper is to prove that there exist steady solutions to the Euler equation in $\mathbb{R}^3$ having thin vortex tubes of any link and knot type. The steady solutions we construct are *Beltrami fields*, that is, they satisfy the equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$ for some non-zero real constant λ. As any Beltrami field satisfies the equation $\Delta u = -\lambda^2 u$ in $\mathbb{R}^3$, it is apparent that the solutions we construct are real-analytic but do not have finite energy (i.e., u is not in $L^2(\mathbb{R}^3)$). However, our construction yields solutions with optimal decay at infinity in the class of Beltrami fields, which fall off as $|u(x)| < C/|x|$. In particular, they are in $L^p(\mathbb{R}^3)$ for all $p > 3$.

The motivation to consider the class of Beltrami solutions to the Euler equation to address the existence of linked vortex tubes comes from Arnold's structure theorem [4, Theorem II.1.2]. Under mild technical assumptions, this theorem ensures that the vortex lines of a steady solution to the Euler equation whose velocity field is not everywhere collinear with its vorticity are nicely stacked in a rigid structure akin to those appearing in the study of integrable Hamiltonian systems. Heuristically, this structure should somehow restrict the way the vortex lines are arranged; partial results in this direction have been shown in [11], where it is proved that under appropriate (strong) hypotheses the vortex lines of steady solutions with non-collinear velocity and vorticity can only be of certain knot types. In contrast, using Beltrami fields we have recently managed to produce steady solutions of the Euler equation with a set of vortex lines diffeomorphic to any locally finite link [10].

Before stating the main result we need some definitions. We will say that a bounded domain of $\mathbb{R}^3$ is a (closed) *tube* if its boundary is a smoothly embedded torus. Therefore, a vortex tube is a tube whose boundary is the union of vortex lines (or, equivalently, its boundary is an invariant torus of the vorticity field). A convenient way of constructing *thin* tubes is as metric neighborhoods of curves. Indeed, if $\gamma \subset \mathbb{R}^3$ is a closed curve, we will denote by

$$\mathcal{T}_\varepsilon(\gamma) := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \gamma) < \varepsilon\} \tag{1.1}$$

the tube of core γ and thickness ε. We are interested in the case where ε is a small positive number, which corresponds to the case of thin tubes. Obviously any finite collection of disjoint tubes in $\mathbb{R}^3$ can be isotoped to a collection of thin tubes of this form.

Let us consider a finite collection of (possibly knotted and linked) disjoint thin tubes, constructed using metric neighborhoods of curves as above. Our main result in this paper is that, if the thickness of the tubes is small enough, this collection can be transformed by a C^m-small diffeomorphism into a union of vortex tubes of a Beltrami field in $\mathbb{R}^3$. Furthermore, the structure of the vortex lines inside each vortex tube is extremely rich: Firstly, the boundary of each vortex tube is far from being the only invariant torus of the flow, as the set of invariant tori has positive Lebesgue measure. Secondly, between any pair of these invariant tori there are infinitely many periodic vortex lines. Thirdly, there is a periodic vortex line which is close to the core of the initial tube and diffeomorphic to it. More precisely, we have the following statement, where (as always henceforth) all the diffeomorphisms are assumed smooth, and all curves and surfaces are smoothly embedded in $\mathbb{R}^3$. It should be emphasized that the thinness of the tubes is crucially used in the proof of the theorem.

THEOREM 1.1. *Let* $\gamma_1, ..., \gamma_N$ *be* N *pairwise disjoint (possibly knotted and linked) closed curves in* $\mathbb{R}^3$*. For small enough* ε*, one can transform the collection of pairwise disjoint thin tubes* $\mathcal{T}_\varepsilon(\gamma_1), ..., \mathcal{T}_\varepsilon(\gamma_N)$ *by a diffeomorphism* Φ *of* $\mathbb{R}^3$*, arbitrarily close to the identity in any* C^m *norm, so that* $\Phi[\mathcal{T}_\varepsilon(\gamma_1)], ..., \Phi[\mathcal{T}_\varepsilon(\gamma_N)]$ *are vortex tubes of a Beltrami field* u*, which satisfies the equation* $\operatorname{curl} u = \lambda u$ *in* $\mathbb{R}^3$ *for some non-zero constant* λ*.*

Moreover, the field u *decays at infinity as* $|D^j u(x)| < C_j/|x|$ *for all* j *and has the following properties in each vortex tube* $\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$:

(i) *In the interior of* $\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$ *there are uncountably many nested tori invariant under the Beltrami field* u*. On each of these invariant tori, the field* u *is ergodic.*

(ii) *The set of invariant tori has positive Lebesgue measure in a small neighborhood of the boundary* $\partial\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$.

(iii) *In the region bounded by any pair of these invariant tori there are infinitely many closed vortex lines, not necessarily of the same knot type as the curve* γ_i*.*

(iv) *The image of the core curve* γ_i *under the diffeomorphism* Φ *is a periodic vortex line of* u*.*

An important property of the structure of the vortex lines inside each vortex tube, as described above, is that it is *stable* in the following sense: on the one hand, it is robust under small perturbations of the field u, meaning that the trajectories of any field which is close enough to u in a sufficiently high C^k norm have the same structure. On the other hand, the boundary of each vortex tube $\partial\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$ is Lyapunov stable under the flow of the Beltrami field u. It should be noticed too that from property (iv) in Theorem 1.1 we recover the main theorem of [10] for finite links and improve it by ensuring that the solution falls off at infinity (while in [10] we had no control at all on the growth of the solution at infinity). Besides, the proof of Theorem 1.1 shows that the vortex lines $\Phi(\gamma_i)$ are elliptic trajectories, and therefore linearly stable, while the vortex lines we constructed in [10] were hyperbolic, and thus unstable.

After establishing his structure theorem, Arnold conjectured [2] that, contrary to what happens in the non-collinear case, Beltrami fields could present vortex lines of the same topological complexity as the trajectories of any divergence-free vector field. By Kolmogorov–Arnold–Moser (KAM) theory, typically these trajectories give rise to a set of invariant tori with positive measure and chaotic regions with homoclinic tangles between these tori. Theorem 1.1 is fully consistent with this picture, and proves the existence of the aforementioned positive-measure set of invariant tori.

The paper is organized as follows. In §2 we will discuss the strategy of the proof of Theorem 1.1. This section, which serves as a guide to the paper, also allows us to explain the key difficulties that appear in the proof of this result but not in that of [10], which require the introduction of new techniques and ideas, and make this paper considerably more involved. In §3 we introduce some objects associated with the geometry of a thin tube that will be used throughout the paper. In §4 we prove some estimates for the Laplacian in a thin tube with Neumann boundary conditions. These estimates are used in §5 to study harmonic fields in thin tubes. In §6 we construct Beltrami fields in thin tubes with prescribed harmonic part. These fields are analyzed further in §7, where we prove a KAM theorem for Beltrami fields in generic thin tubes. A Runge-type approximation theorem by global Beltrami fields tending to zero at infinity is presented in §8. With all these ingredients, in §9 we prove Theorem 1.1. The paper concludes with an easy application to the Navier–Stokes equation, which we present in §10.

2. Strategy of the proof and guide to the paper

To prove Theorem 1.1, our basic goal is to establish the existence of a Beltrami field u, satisfying the equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$ for some non-zero constant λ and falling off at infinity, such that the field u has a set of N invariant tori diffeomorphic to the surfaces $\{\partial\mathcal{T}_\varepsilon(\gamma_i)\}_{i=1}^N$, with $\{\gamma_i\}_{i=1}^N$ being a set of prescribed (possibly knotted and linked) closed curves. We recall that $\mathcal{T}_\varepsilon(\gamma_i)$ is a metric neighborhood of the curve γ_i of small thickness ε, as defined in equation (1.1). By deforming them a little if necessary, we may assume without loss of generality that the curves γ_i are analytic.

The basic idea behind the proof of Theorem 1.1 is carried out in three interrelated stages. Firstly, we construct a *local Beltrami field* v, which satisfies the Beltrami equation in a neighborhood of each closed tube $\overline{\mathcal{T}_\varepsilon(\gamma_i)}$ and has a set of invariant tori given by $\partial\mathcal{T}_\varepsilon(\gamma_1), ..., \partial\mathcal{T}_\varepsilon(\gamma_N)$. Secondly, we prove that these invariant tori are "*robust*", meaning that suitably small perturbations of the local Beltrami field v still have a set of invariant tori diffeomorphic to $\partial\mathcal{T}_\varepsilon(\gamma_1), ..., \partial\mathcal{T}_\varepsilon(\gamma_N)$. To conclude, we show that the local Beltrami field v can be approximated in any C^k norm by a *global Beltrami field* u, which satisfies the Beltrami equation in the whole space $\mathbb{R}^3$ and falls off at infinity in an optimal way. The robustness of the invariant tori of the local Beltrami field ensure that u has a set of vortex tubes diffeomorphic to the initial configuration of thin tubes, and that in fact this diffeomorphism can be taken close to the identity in a C^m norm.

However, the implementation of this basic idea turns out to be extremely subtle. To understand why, one can start by noticing that the robustness of the invariant tori of the local Beltrami field relies on a KAM-type argument. To apply this KAM argument, we need a small perturbation parameter and some control on the dynamics of the local Beltrami field in a neighborhood of each invariant torus, which is required in order to show that the local Beltrami field is equivalent to a Diophantine rotation on the torus and satisfies a suitable non-degeneracy condition.

To construct local Beltrami fields in a neighborhood of the tori $\partial\mathcal{T}_\varepsilon(\gamma_i)$ with controlled behavior on these surfaces, it is natural to use some variant of the Cauchy–Kovalevsky theorem for the curl operator with Cauchy data on the tori (e.g. [10, Theorem 3.1]). This would lead to a local Beltrami field defined in a small neighborhood Ω_i of each torus $\partial\mathcal{T}_\varepsilon(\gamma_i)$. Unfortunately, the Runge-type approximation theorem we prove in this paper does not allow us to approximate a local Beltrami field defined in $\Omega_1\cup...\cup\Omega_N$ by a global Beltrami field, since the complement $\mathbb{R}^3\backslash(\Omega_1\cup...\cup\Omega_N)$ has compact connected components. As is well known, this is not just a technical issue, but a fundamental obstruction in any Runge-type theorem.

Therefore, to construct the local Beltrami field v we will not consider a Cauchy problem but a boundary value problem for the curl operator in each tube. This leads to a vector field that satisfies the Beltrami equation

$$\operatorname{curl} v = \lambda v$$

in a neighborhood of the tubes (because the boundaries are analytic), which allows us to apply our Runge-type approximation theorem. In this boundary value problem one can prescribe the normal component of v at each boundary $\partial\mathcal{T}_\varepsilon(\gamma_i)$, so that by setting it to zero we can ensure that each boundary is an invariant torus of the local Beltrami field. However, we have no control on the tangential component of v on each torus, so in principle the local Beltrami field does not necessarily satisfy the dynamical conditions our KAM-type theorem requires for the invariant tori to be preserved under small perturbations.

To overcome this difficulty we resort to a careful analysis of harmonic fields in thin tubes. Indeed, we show that in the above boundary value problem for the local Beltrami field, one can also prescribe its harmonic part (that is, the L^2 projection of v into the space of harmonic fields with tangency boundary conditions), and for small λ one expects the local Beltrami field to behave essentially as if it were harmonic. Hence, we exploit the small parameter ε (that is, the thickness of the tubes) to extract detailed analytic information about harmonic fields through energy estimates, and then utilize this knowledge to compute the dynamical properties of the local Beltrami field that are required in the KAM-type theorem under the assumption that the Beltrami parameter λ is suitably small (in fact, of order ε^3). A key ingredient in the computation of these dynamical properties is a set of energy estimates for the local Beltrami field and all its derivatives that are optimal with respect to the geometry of the tubes, i.e., the parameter ε.

The local Beltrami field can now be approximated in any C^k norm by a global Beltrami field u that falls off at infinity as $C/|x|$ using a Runge-type theorem. When using the KAM argument to guarantee that u still has a set of invariant tori diffeomorphic to $\{\partial\mathcal{T}_\varepsilon(\gamma_i)\}_{i=1}^N$, one has to face the problem that the local Beltrami field is in fact a small perturbation of a field that does *not* satisfy the non-degeneracy condition (equivalent to a rotation of the disk with constant frequency), which requires a fine analysis of the terms controlled by each small parameter: the thickness ε, the Beltrami constant λ and the error in the Runge-type approximation.

We shall next present a sketch of the proof of Theorem 1.1, where we explain the different intermediate results that are needed in the demonstration and their interrelations. This short sketch is also intended to serve as a guide to the paper. For the sake of clarity, we will divide the proof into three stages.

Stage 1. Construction of the local Beltrami field. The local Beltrami field v is obtained as the unique solution to a certain boundary value problem for the Beltrami equation. Our goal is to estimate various analytic properties of this field, and for this it is natural to introduce coordinates adapted to a Frenet frame in each tube $\mathcal{T}_\varepsilon\equiv\mathcal{T}_\varepsilon(\gamma_i)$, which essentially correspond to an arc-length parametrization of the curve γ_i and to rectangular coordinates in a transverse section of the tube. Thus we consider an angular coordinate α, taking values in $\mathbb{S}^1_\ell:=\mathbb{R}/\ell\mathbb{Z}$ (with ℓ being the length of the curve γ_i), and rectangular coordinates $y=(y_1,y_2)$ taking values in the unit 2-disk $\mathbb{D}^2$. Details are given in §3.

To extract information about the local Beltrami field, a useful tool is the boundary value problem for the Laplacian on scalar functions with zero mean and zero Neumann boundary conditions in the thin tube $\mathcal{T}_\varepsilon$:

$$\Delta\psi=\varrho \text{ in } \mathcal{T}_\varepsilon, \quad \partial_\nu\psi|_{\partial\mathcal{T}_\varepsilon}=0, \quad \int_{\mathcal{T}_\varepsilon}\psi\,dx=0.$$

When written in the natural coordinates (α,y), we obtain a boundary value problem in the domain $\mathbb{S}^1_\ell\times\mathbb{D}^2$, with the coefficients of the Laplacian in these coordinates depending on the geometry of the tube strongly through its thickness ε and the curvature and torsion of the core curve γ_i.

It is clear that we have the H^k estimate

$$\|\psi\|_{H^k(\mathcal{T}_\varepsilon)} \leqslant C_{\varepsilon,k}\|\varrho\|_{H^{k-2}(\mathcal{T}_\varepsilon)}.$$

However this estimate, where the constant $C_{\varepsilon,k}$ depends on ε in an undetermined way, is far from being enough to compute the dynamical quantities for the local Beltrami field that are needed in the KAM-type theorem. Therefore, in §4 we prove energy estimates for the function ψ and its derivatives that are optimal with respect to the parameter ε. In particular, in order to be able to compute the desired dynamical quantities for the local Beltrami field later on, it is crucial to distinguish between estimates for derivatives of ψ with respect to the "slow" variable α and the "fast" variable y. The estimates for the function ψ and its derivatives that we will need are stated in Theorem 4.11.

These estimates are immediately put to work in §5 to derive a perturbative expression for the harmonic field in each thin tube $\mathcal{T}_\varepsilon$ for small ε. (Of course, the problem degenerates for $\varepsilon=0$ and the asymptotic results we prove do not correspond to a Taylor expansion.) We will use the notation h for the harmonic field in $\mathcal{T}_\varepsilon$, which is unique up to a multiplicative constant. In fact, we need to compute h up to corrections that are suitably small for small ε; as before, different powers of ε are required for the slow and fast components of the field. To first order, the harmonic field can be written in the coordinates (α, y) as

$$h = \partial_\alpha + \tau(\alpha)(y_1\partial_2 - y_2\partial_1) + \mathcal{O}(\varepsilon), \tag{2.1}$$

where τ stands for the torsion of each curve γ_i and $\mathcal{O}(\varepsilon)$ denotes some vector field whose components in these coordinates are bounded by a multiple of ε in the $C^k(\mathbb{S}^1_\ell \times \mathbb{D}^2)$ norm. By equation (2.1), the field h is an $\mathcal{O}(\varepsilon)$ perturbation of the rotation of constant frequency given by the total torsion $\int_0^\ell \tau(\alpha)\, d\alpha$. From the point of view of KAM theory, this is a very degenerate case, so it is not hard to guess that one needs to compute h (at least) up to order $\mathcal{O}(\varepsilon^3)$, as we do in Theorem 5.1. As a matter of fact, we will see later on that the non-degeneracy condition that appears in the KAM theorem is that a certain quantity, called the normal torsion, must be non-zero, and that it actually vanishes modulo terms of order $\mathcal{O}(\varepsilon^2)$, which justifies why we need good estimates for h.

In §6 we show that, for any non-zero constant λ that is small enough (say, smaller than some fixed ε-independent constant), there is a unique vector field that is tangent to the boundary of each tube $\mathcal{T}_\varepsilon$, satisfies the equation

$$\operatorname{curl} v = \lambda v \tag{2.2}$$

in the tube and whose harmonic part (i.e., the L^2 projection on the space of harmonic fields) is h. (This is, of course, different to what happens in the case of compact manifolds without boundary, where Beltrami fields are always orthogonal to harmonic fields.) We also prove estimates that measure how the field v becomes close to h in the C^k norm for small ε and λ. The key result is Theorem 6.8, which will be crucial in verifying the conditions in the KAM argument for the preservation of invariant tori.

Stage 2. Preservation of the invariant tori. In Stage 1, for small enough ε and λ, we have constructed a local Beltrami field v, which satisfies equation (2.2) in a neighborhood of $\bigcup_{i=1}^N \overline{\mathcal{T}_\varepsilon(\gamma_i)}$ and has a set of invariant tori given by $\partial\mathcal{T}_\varepsilon(\gamma_i)$. Furthermore, the estimates we have proved provide a very convenient expression for the local Beltrami field, up to terms that are of order ε^3 (when $\lambda=\varepsilon^3$) in a suitable sense.

In §7 we analyze the robustness of the invariant tori using the Poincaré map Π defined by the local Beltrami field at a transverse section of each tube. In each tube, Π is then a diffeomorphism of the disk that preserves a certain measure, while the intersection of each invariant torus with the transverse section is an invariant circle of the Poincaré map. The persistence of the invariant tori will rely on a KAM theorem for the Poincaré map (Theorem 7.6) that applies to individual invariant circles.

To apply the KAM theorem, one has to verify that the rotation number of Π on the invariant circle is Diophantine and that Π satisfies a non-degeneracy condition (namely, that its normal torsion is non-zero). Computing these dynamical quantities using the estimates for the local Beltrami field is non-trivial. On the one hand, the rotation number depends on the behavior of the trajectories for arbitrarily large times, so a delicate treatment is required in order to uniformly control the effect of the $\mathcal{O}(\varepsilon^2)$ terms in the vector field. On the other hand, the normal torsion, which is required to be non-zero, turns out to be zero modulo $\mathcal{O}(\varepsilon^2)$, since the Poincaré map is a small perturbation of a constant-frequency rotation of the disk, which is a highly degenerate case from the point of view of KAM theory.

The estimates proved at Stage 1 are tailored to permit us to overcome these difficulties, yielding expressions for the rotation number ω_Π and the normal torsion $\mathcal{N}_\Pi$ that depend on the geometry of each curve γ_i through its curvature $\varkappa$ and torsion τ (see Theorems 7.4 and 7.8):

$$\omega_\Pi = \int_0^\ell \tau(\alpha)\, d\alpha + \mathcal{O}(\varepsilon^2),$$
$$\mathcal{N}_\Pi = -\frac{5\pi\varepsilon^2}{8}\int_0^\ell \varkappa(\alpha)^2\tau(\alpha)\, d\alpha + \mathcal{O}(\varepsilon^3).$$

These expressions allow us to prove the main result in Stage 2, which is that for "generic" curves γ_i the hypotheses of the aforementioned KAM theorem are satisfied, so that the invariant tori of the local Beltrami field v are robust: if u is a divergence-free vector field in a neighborhood of the tubes that is close enough to v in a suitable sense (e.g., in a high enough C^k norm), then u also has an invariant tube diffeomorphic to each $\mathcal{T}_\varepsilon(\gamma_i)$, and moreover the corresponding diffeomorphisms can be taken close to the identity. This is proved in Theorem 7.10.

Stage 3. Approximation by a global Beltrami field. In §*8* we prove a Runge-type approximation theorem for Beltrami fields that decay at infinity (Theorem 8.3). More precisely, we will show that the local Beltrami field v, considered in the previous stages and defined in a neighborhood of the thin tubes $\mathcal{T}_\varepsilon(\gamma_i)$, can be approximated in any C^k norm by a Beltrami field u that falls off at infinity as $|D^j u(x)|<C_j/|x|$. This decay is optimal for the class of Beltrami fields.

The proof of this theorem consists of two steps. In the first step we use functional-analytic methods to approximate the field field v by an auxiliary vector field w that satisfies the elliptic equation $\Delta w=-\lambda^2 w$ in a large ball of $\mathbb{R}^3$ that contains all the tubes. In the second step, we define the approximating global Beltrami field u in terms of a truncation of a suitable series representation of the field w, ensuring that u has the desired fall-off at infinity.

Completion of the proof of the main theorem. In §9 we use all previous results to complete the proof of Theorem 1.1. Indeed, from the robustness of the invariant tori of the local Beltrami field, it immediately follows that the global Beltrami field u has a set of thin vortex tubes equivalent through a C^m-small diffeomorphism Φ to $\{\mathcal{T}_\varepsilon(\gamma_i)\}_{i=1}^N$, provided ε is small enough. More precisely, what is proved is that there is some constant ε_0, which depends only (albeit in a rather non-trivial way) on the geometry of the curves γ_i and on the allowed smallness for the diffeomorphism Φ, so that the statement of the theorem holds for any thickness $\varepsilon\leqslant\varepsilon_0$.

The remaining properties of the vortex lines stated in Theorem 1.1 are established in this section too, using results derived throughout the paper. In particular, we show that near the curve γ_i there is an elliptic periodic trajectory of the field u. As a side remark, notice that this elliptic periodic trajectory is obviously linearly stable, but is *not* granted a priori to be Lyapunov stable. Therefore, as shown by the counterexample of Anosov and Katok (cf. e.g. [13]), it is not guaranteed that there are invariant tori of the field in a neighborhood of the elliptic trajectory. A careful (and non-trivial) analysis of the dynamics near the elliptic trajectory for small ε and λ would be required to prove the existence of these tori.

Before beginning with the technical part of the paper, it is convenient to provide a short comparison between the proof of Theorem 1.1 and that of the main theorem in reference [10], where we showed that there are steady solutions to the Euler equation with a set of vortex lines diffeomorphic to any given link. In this reference, the proof was also based on the construction of a local Beltrami field with a "robust" set of periodic trajectories, which was then approximated by a global Beltrami field. However, the implementation of this basic principle is totally different.

To begin with, the robustness of periodic trajectories in [10] relies on the hyperbolic permanence theorem (which is essentially an application of the implicit function theorem) instead of a considerably more sophisticated KAM argument. To construct a local Beltrami field with prescribed periodic trajectories γ_i that are hyperbolic, as required by the permanence theorem, it is enough to prove a suitable analog of the Cauchy–Kovalevsky theorem for the curl operator: indeed, since the field is divergence-free, it is enough to assume that the field used as Cauchy datum is exponentially contracting into the curve γ_i on the Cauchy surface to obtain a local Beltrami field that has γ_i as a hyperbolic periodic trajectory. All this is in strong contrast to the case of vortex tubes, where the construction of the local solution with a robust set of prescribed invariant tori requires the analysis of the boundary value problem and the KAM argument described in Stages 1 and 2 (§§4–7).

The approximation theorem we use in this paper is also different from the one employed in [10]. The reason for this is that in [10] we had no control on the growth of the global Beltrami field at infinity, even in the case of connected links. On the contrary, the approximation theorem we prove in this paper (Theorem 8.3) yields Beltrami fields with optimal fall-off at infinity. The proof of these approximation theorems is considerably different: the old theorem is based on an iterative scheme that uses a theorem by Lax and Malgrange and works for the curl operator in any Riemannian 3-manifold, while the new one is based on different principles and takes advantage of the geometry of Euclidean 3-space to ensure that the global Beltrami field falls off at infinity.

Notice that the main theorem in [10] applies to locally finite sets of curves, while in Theorem 1.1 in this paper we can only take finite sets of curves $\gamma_1, \dots, \gamma_N$. Some comments in this regard are made in Remark 9.2.

3. Geometry of thin tubes

In this section we introduce some notation, including a coordinate system, that will be used throughout the paper to describe functions and vector fields defined on thin tubes. These tubes will be characterized in terms of the curve that sits on its core and its thickness ε, which is a parameter that will be everywhere assumed to be suitably small.

Let us start with a closed analytic non-self-intersecting curve with an arc-length parametrization $\gamma:\mathbb{S}^1_\ell\to\mathbb{R}^3$, with $\mathbb{S}^1_\ell:=\mathbb{R}/\ell\mathbb{Z}$ (throughout the paper, when the period is 2π we will simply write $\mathbb{S}^1\equiv\mathbb{S}^1_{2\pi}$). This amounts to saying that the tangent field $\dot\gamma$ has unit norm and ℓ is the length of the curve. We will abuse the notation and denote also by γ the curve parameterized by the above map (i.e., the image set $\gamma(\mathbb{S}^1_\ell)$).

Let us denote by $\mathcal{T}_\varepsilon\equiv\mathcal{T}_\varepsilon(\gamma)$ the metric neighborhood with thickness ε of the curve γ, that is,

$$\mathcal{T}_\varepsilon:=\{x\in\mathbb{R}^3:\operatorname{dist}(x,\gamma)<\varepsilon\}.$$

This is a thin tube having the curve γ as its core. It is standard that, for small ε, the boundary $\partial\mathcal{T}_\varepsilon$ is analytic. The normal bundle of the curve γ being trivial [22], one can associate with each $\alpha\in\mathbb{S}^1_\ell$ two orthogonal unit vectors $e_j(\alpha)$ in $\mathbb{R}^3$ perpendicular to the curve at the point $\gamma(\alpha)$. For convenience, we will make the assumption that the curvature of the curve γ does not vanish, which allows us to take $e_1(\alpha)$ and $e_2(\alpha)$ as the normal and binormal vectors at the point $\gamma(\alpha)$. It is well known that the assumption that the curvature does not vanish is satisfied for generic curves [5, p. 184] (roughly speaking, "generic" refers to an open and dense set, with respect to a reasonable C^k topology, in the space of smooth curves in $\mathbb{R}^3$).

Using the vector fields $e_j(\alpha)$ and denoting by $\mathbb{D}^2$ the 2-dimensional unit disk, we can introduce analytic coordinates $(\alpha,y)\in\mathbb{S}^1_\ell\times\mathbb{D}^2$ in the tube $\mathcal{T}_\varepsilon$ via the diffeomorphism

$$(\alpha,y)\longmapsto\gamma(\alpha)+\varepsilon y_1e_1(\alpha)+\varepsilon y_2e_2(\alpha).$$

In the coordinates (α,y), a short computation using the Frenet formulas shows that the Euclidean metric in the tube reads as

$$ds^2=A\,d\alpha^2+2\varepsilon^2\tau(y_2\,dy_1-y_1\,dy_2)\,d\alpha+\varepsilon^2(dy_1^2+dy_2^2), \tag{3.1}$$

where $\varkappa\equiv\varkappa(\alpha)$ and $\tau\equiv\tau(\alpha)$ respectively denote the curvature and torsion of the curve,

$$A:=(1-\varepsilon\varkappa y_1)^2+(\varepsilon\tau)^2|y|^2, \tag{3.2}$$

and $|y|$ stands for the Euclidean norm of $y=(y_1,y_2)$. As is customary, we will denote by g_{ij} and g^{ij} the components of the metric tensor and its inverse, respectively.

We will sometimes take polar coordinates $r\in(0,1)$, $\theta\in\mathbb{S}^1:=\mathbb{R}/2\pi\mathbb{Z}$ in the disk $\mathbb{D}^2$, which are defined so that

$$y_1=r\cos\theta\quad\text{and}\quad y_2=r\sin\theta.$$

The metric then reads

$$ds^2=A\,d\alpha^2-2\varepsilon^2\tau r^2\,d\theta\,d\alpha+\varepsilon^2\,dr^2+\varepsilon^2r^2\,d\theta^2, \tag{3.3}$$

where we, with a slight abuse of notation, still call A the expression of (3.2) in these coordinates, i.e.,

$$A := (1-\varepsilon\varkappa r\cos\theta)^2+(\varepsilon\tau r)^2. \tag{3.4}$$

Notice that the coordinate r is simply the distance to the curve γ. For future reference, we record that the volume measure is written in these coordinates as

$$dV := B\,d\alpha\,dy = Br\,d\alpha\,dr\,d\theta \tag{3.5}$$

up to a factor of ε^2, where $dy:=dy_1\,dy_2$ and

$$B := 1-\varepsilon\varkappa y_1 = 1-\varepsilon\varkappa r\cos\theta. \tag{3.6}$$

4. Estimates for the Neumann Laplacian in thin tubes

In this section we will derive some estimates for the Laplace equation $\Delta\psi=\varrho$ in the thin tube $\mathcal{T}_\varepsilon$ with zero Neumann boundary conditions. As we will see, to gain control on the function ψ it is convenient to attack this equation in the coordinates $(\alpha,y)\in\mathbb{S}^1_\ell\times\mathbb{D}^2$ that we introduced in §3, in terms of which the Laplace equation can be written as

$$\Delta\psi=\varrho \text{ in } \mathbb{S}^1_\ell\times\mathbb{D}^2, \quad \partial_\nu\psi=0 \text{ on } \mathbb{S}^1_\ell\times\partial\mathbb{D}^2, \tag{4.1a}$$

where the Laplacian Δ is now interpreted as an ε-dependent differential operator in the variables (α,y) or (α,r,θ). In order to ensure the existence of solutions to this equation, we suppose that $\int\varrho\,dV=0$, which allows us to uniquely determine the solution ψ by demanding that it also has zero mean:

$$\int\psi\,d\alpha\,dy=0. \tag{4.1b}$$

(Here and in what follows, we omit the domain of integration when it is the whole domain $\mathbb{S}^1_\ell\times\mathbb{D}^2$. We could have used the measure dV in the above formula too, but this choice is slightly more convenient.) For future reference, we record here the expression of Δ in the variables (α,r,θ):

$$\begin{aligned}\Delta\psi=\frac{1}{\varepsilon^2}\left(\psi_{rr}+\frac{1}{r}\psi_r+\frac{A}{r^2B^2}\psi_{\theta\theta}\right)+\frac{1}{B^2}\psi_{\alpha\alpha}+\frac{2\tau}{B^2}\psi_{\alpha\theta}+\frac{\tau'-\varepsilon r(\varkappa\tau'-\varkappa'\tau)\cos\theta}{B^3}\psi_\theta\\ +\frac{1}{\varepsilon}\left(\frac{\varkappa\sin\theta(B^2-(\varepsilon\tau r)^2)}{rB^3}\psi_\theta-\frac{\varkappa\cos\theta}{B}\psi_r\right)+\frac{\varepsilon r(\varkappa'\cos\theta-\tau\varkappa\sin\theta)}{B^3}\psi_\alpha.\end{aligned} \tag{4.2}$$

As usual, we denote partial derivatives by subscripts when there is not risk of confusion.

Given a subset $\Omega\subset\mathbb{S}^1_\ell\times\mathbb{D}^2$, we will use the notation

$$\|\psi\|_\Omega := \biggl(\int_\Omega \psi^2\, d\alpha\, dy\biggr)^{1/2}$$

for the $L^2(\Omega)$-norm of the function ψ, omitting the subscript when Ω is the whole domain $\mathbb{S}^1_\ell\times\mathbb{D}^2$. In this section we will use the notation

$$\|\psi\|^2_{\dot H^1_\varepsilon} := \|\partial_\alpha\psi\|^2+\frac{\|D_y\psi\|^2}{\varepsilon^2} \tag{4.3}$$

for a homogeneous Sobolev norm in which the derivatives associated with the "small directions" of the thin tube are weighted with an appropriate ε-dependent factor. The usual H^k norm of the function $\psi(\alpha,y)$ will be denoted by $\|\psi\|_{H^k}$.

In what follows, we will assume that ψ is a solution of the Laplace equation (4.1) with zero mean and zero Neumann boundary conditions, which ensures that

$$\int g^{ij}\partial_i\psi\partial_j\varphi\, dV = -\int \varrho\varphi\, dV \tag{4.4}$$

for any $\varphi\in H^1(\mathbb{S}^1_\ell\times\mathbb{D}^2)$. As usual, the scripts i and j range over the set of coordinates $\{\alpha,y_1,y_2\}$, and summation over repeated indices is understood. It is important to notice that, for any function φ,

$$\int g^{ij}\partial_i\varphi\partial_j\varphi\, dV = (1+\mathcal{O}(\varepsilon))\|\varphi\|^2_{\dot H^1_\varepsilon}, \tag{4.5}$$

so that the $\dot H^1_\varepsilon$ norm is essentially a more convenient way of dealing with the natural $\dot H^1$ norm associated with the metric. We will use this identity many times in this section without further comment.

The structure of this section is the following. We will start by estimating the L^2 norm of the derivatives of the function ψ with respect to the "slow" variable α (§4.1). The proof of these estimates is standard. We recall that the reason why α is called the slow variable is that it parameterizes the "large" direction of the thin tube $\mathcal{T}_\varepsilon$, as opposed to the "fast" variable y, which is obtained by rescaling the small section of the tube. Estimates for the derivatives of ψ with respect to the "fast" variable y with optimal dependence in the small parameter ε are presented in §4.2. They are more complicated to obtain, basically because one has to consider an auxiliary function in order to get rid of the contributions to ψ that only depend on the slow variable. The resulting estimates for ψ, which we will often use in forthcoming sections, will be stated in §4.3.

4.1. Estimates for derivatives with respect to the "slow" variable

In this subsection we will prove H^k estimates for the derivatives of the function ψ with respect to the "slow" variable α (cf. Proposition 4.3). We will begin with the following proposition, where we estimate the L^2 and $\dot{H}^1_\varepsilon$ norms of the function ψ. As is customary, throughout this article we will use the letter C to denote ε-independent constants that may vary from line to line.

PROPOSITION 4.1. *The function ψ satisfies the estimate*

$$\|\psi\|+\|\psi\|_{\dot{H}^1_\varepsilon} \leqslant C\|\varrho\|.$$

Proof. By the expression of the metric in the coordinates (α, y), it is clear that for small enough ε one has

$$\int g^{ij}\partial_i\psi\,\partial_j\psi\,dV \geqslant (1+\mathcal{O}(\varepsilon))\int\left(\psi_\alpha^2+\frac{|D_y\psi|^2}{\varepsilon^2}\right)d\alpha\,dy \geqslant \frac{1}{2}\int(\psi_\alpha^2+|D_y\psi|^2)\,d\alpha\,dy.$$

The rightmost term in the inequality is bounded from below by $\frac{1}{2}\mu_1\|\psi\|^2$, where μ_1 stands for the first non-zero Neumann eigenvalue of the flat solid torus:

$$\mu_1 := \inf\left\{\int(\varphi_\alpha^2+|D_y\varphi|^2)\,d\alpha\,dy : \varphi\in C^\infty(\mathbb{S}^1_\ell\times\mathbb{D}^2), \int\varphi\,d\alpha\,dy=0 \text{ and } \|\varphi\|=1\right\}>0.$$

Hence we infer that

$$\|\psi\| \leqslant C\|\psi\|_{\dot{H}^1_\varepsilon} \tag{4.6}$$

for some $C>0$.

To conclude, we can now use the weak formulation of the equation (4.4) with $\varphi=\psi$ and the Cauchy–Schwarz inequality to derive that

$$\|\psi\|^2_{\dot{H}^1_\varepsilon} \leqslant (1+\mathcal{O}(\varepsilon))\int g^{ij}\partial_i\psi\,\partial_j\psi\,dV \leqslant (1+\mathcal{O}(\varepsilon))\|\varrho\|\,\|\psi\| \leqslant C\|\varrho\|\,\|\psi\|_{\dot{H}^1_\varepsilon}.$$

The proposition then follows from this inequality and the estimate (4.6). ☐

In the following lemma we record an elementary inequality that will be of use several times in this section.

LEMMA 4.2. *Let $D^l_{\alpha,y}$ denote the tensor of l-th order derivatives with respect to the variables (α, y). For any well-behaved (e.g., smooth) functions φ, $\widetilde{\varphi}$, χ and $\widetilde{\chi}$ on $\mathbb{S}^1_\ell\times\mathbb{D}^2$, one has*

$$\int|D^l_{\alpha,y}(Bg^{ij})\partial_i\varphi\,\partial_j\widetilde{\varphi}\,\chi\widetilde{\chi}|\,d\alpha\,dy \leqslant \varepsilon C_l\left(\int g^{ij}\partial_i\varphi\,\partial_j\varphi\,\chi\,dV\right)^{1/2}\left(\int g^{ij}\partial_i\widetilde{\varphi}\,\partial_j\widetilde{\varphi}\,\widetilde{\chi}\,dV\right)^{1/2}.$$

In particular,

$$\int|D^l_{\alpha,y}(Bg^{ij})\partial_i\varphi\,\partial_j\widetilde{\varphi}|\,d\alpha\,dy \leqslant C_l\varepsilon\|\varphi\|_{\dot{H}^1_\varepsilon}\|\widetilde{\varphi}\|_{\dot{H}^1_\varepsilon}.$$

Proof. This is an immediate consequence of the expression for the metric and the function B in the coordinates (α, y) (see equation (3.1)) and the Cauchy–Schwarz inequality. □

We are now ready to prove the estimates for the derivatives of ψ with respect to the slow variable α that we will need in this paper.

PROPOSITION 4.3. *The k-th partial derivative of the function ψ with respect to the angle α satisfies*

$$\|\partial_\alpha^{k+1}\psi\|_{\dot{H}^1_\varepsilon} \leqslant C_k \|\varrho\|_{H^k},$$

where k is any non-negative integer.

Proof. Let us take $\varphi = \psi_{\alpha\alpha}$ in equation (4.4) and integrate by parts to get

$$\int g^{ij}\partial_i\psi_\alpha\,\partial_j\psi_\alpha\,dV = \int \varrho\psi_{\alpha\alpha}\,dV - \int \partial_\alpha(Bg^{ij})\,\partial_i\psi\,\partial_j\psi_\alpha\,d\alpha\,dy.$$

The left-hand side is bounded from below by $(1+\mathcal{O}(\varepsilon))\|\psi_\alpha\|^2_{\dot{H}^1_\varepsilon}$, while from the definition of the norm $\dot{H}^1_\varepsilon$ and Lemma 4.2 it stems that

$$\left|\int \varrho\psi_{\alpha\alpha}\,dV\right| \leqslant (1+\mathcal{O}(\varepsilon))\|\varrho\|\,\|\psi_\alpha\|_{\dot{H}^1_\varepsilon},$$

$$\left|\int \partial_\alpha(Bg^{ij})\,\partial_i\psi\,\partial_j\psi_\alpha\,d\alpha\,dy\right| \leqslant C\varepsilon\|\psi\|_{\dot{H}^1_\varepsilon}\|\psi_\alpha\|_{\dot{H}^1_\varepsilon}.$$

Using Proposition 4.1 to estimate $\|\psi\|_{\dot{H}^1_\varepsilon}$ in terms of $\|\varrho\|$, we infer that

$$\|\psi_\alpha\|^2_{\dot{H}^1_\varepsilon} \leqslant C\|\varrho\|\,\|\psi_\alpha\|_{\dot{H}^1_\varepsilon},$$

which readily implies the desired bound for $k=0$. When k is a positive integer, the proof is totally analogous and can be obtained by induction on k using $\varphi = \partial_\alpha^{2k+2}\psi$, the only difference being that one needs to estimate the term $\int \varrho\varphi\,dV$ as

$$\left|\int \varrho\partial_\alpha^{2k+2}\psi\,dV\right| \leqslant C\|\varrho\|_{H^k}\sum_{j=1}^{k+1}\|\partial_\alpha^j\psi\|_{\dot{H}^1_\varepsilon} \leqslant C\|\varrho\|^2_{H^k} + C\|\varrho\|_{H^k}\|\partial_\alpha^{k+1}\psi\|_{\dot{H}^1_\varepsilon}$$

by the induction hypothesis. □

4.2. Estimates for the "fast" variables

To estimate the derivates with respect to the "fast" variable y in an optimal way, it is crucial to ensure that the terms that only depend on α are not considered when estimating the norms of the function. A convenient way of doing this is by considering the auxiliary function

$$\bar{\psi}(\alpha, y) := \psi(\alpha, y) - \frac{1}{\pi} \int_{\mathbb{D}^2} \psi(\alpha, y')\, dy', \tag{4.7}$$

which is obtained from ψ by subtracting its average in the fast variable. It should be emphasized that, if we were not to subtract this average, the estimates we would obtain would not be strong enough for our needs in later sections. The key estimate in this subsection is Theorem 4.8.

An immediate observation is that, of course

$$D_y^j \partial_\alpha^k \psi = D_y^j \partial_\alpha^k \bar{\psi} \tag{4.8}$$

whenever the number j of derivatives we take with respect to y is greater than zero, and that $\bar{\psi}$ has zero mean:

$$\int \bar{\psi}\, d\alpha\, dy = 0.$$

Moreover, the function $\bar{\psi}$ satisfies the equation

$$\Delta \bar{\psi} = \bar{\varrho} \ \text{ in } \mathbb{S}^1_\ell \times \mathbb{D}^2, \quad \partial_\nu \bar{\psi} = 0 \ \text{ on } \mathbb{S}^1_\ell \times \partial\mathbb{D}^2,$$

with

$$\bar{\varrho}(\alpha, y) := \varrho(\alpha, y) - \frac{1}{\pi} \Delta \int_{\mathbb{D}^2} \psi(\alpha, y')\, dy'.$$

The estimates for ψ we derive in the previous section guarantee that the norm of $\bar{\varrho}$ can be bounded by a multiple of the norm of the initial source term ϱ.

Proposition 4.4. *The H^k norm of $\bar{\varrho}$ is bounded by*

$$\|\bar{\varrho}\|_{H^k} \leqslant C_k \|\varrho\|_{H^k}.$$

Proof. Observe that the action of the Laplacian (which we compute in the coordinates (α, y)) on the function $\psi(\alpha, y')$ is

$$\Delta \psi(\alpha, y') = (1 + \mathcal{O}(\varepsilon)) \partial_\alpha^2 \psi(\alpha, y') + \mathcal{O}(\varepsilon) \partial_\alpha \psi(\alpha, y'),$$

where $\mathcal{O}(\varepsilon^n)$ here stands for an ε-dependent quantity $Q(\alpha, y)$ bounded as

$$\|Q\|_{H^k} \leqslant C_k \varepsilon^n$$

for all k. Therefore, one finds that

$$\begin{aligned}\|D_y^j\partial_\alpha^k\bar\varrho\| &= \left\|D_y^j\partial_\alpha^k\varrho-\frac{1}{\pi}D_y^j\partial_\alpha^k\Delta\int_{\mathbb{D}^2}\psi(\alpha,y')\,dy'\right\|\\ &= \left\|D_y^j\partial_\alpha^k\varrho-\frac{1}{\pi}\int_{\mathbb{D}^2}\partial_\alpha^{k+2}\psi(\alpha,y')\,dy'+\sum_{l=1}^{k+2}\int_{\mathbb{D}^2}\mathcal{O}(\varepsilon)\partial_\alpha^l\psi(\alpha,y')\,dy'\right\|\\ &\leqslant \|D_y^j\partial_\alpha^k\varrho\|+(1+C\varepsilon)\|\partial_\alpha^{k+2}\psi\|+C\varepsilon\sum_{l=1}^{k+1}\|\partial_\alpha^l\psi\|\\ &\leqslant C\|\varrho\|_{H^k},\end{aligned}$$

where in the last step we have used Propositions 4.1 and 4.3. The claim then follows. □

To derive energy estimates for ψ, we will use that one obviously has

$$\int g^{ij}\partial_i\bar\psi\,\partial_j\varphi\,dV=-\int\bar\varrho\varphi\,dV \tag{4.9}$$

for all $\varphi\in H^1(\mathbb{S}^1_\ell\times\mathbb{D}^2)$. Our first result will be an estimate for the L^2 norm of $\bar\psi$ and $D_y\bar\psi$. While we can readily derive a bound for these quantities using Proposition 4.1, the estimates we prove here are much sharper for small ε. This will be crucial in the derivation of optimal estimates for ψ.

PROPOSITION 4.5. *The function $\bar\psi$ satisfies the H^1 estimates*

$$\|\bar\psi\|+\|D_y\bar\psi\|\leqslant C\varepsilon^2\|\varrho\| \quad \textit{and} \quad \|\partial_\alpha\bar\psi\|\leqslant C\varepsilon\|\varrho\|.$$

Proof. Choosing $\varphi=\bar\psi$ in equation (4.9) and in view of the expression of the coefficients of the metric (3.1), one immediately obtains that

$$\|\bar\psi\|^2_{\dot H^1_\varepsilon}\leqslant(1+\mathcal{O}(\varepsilon))\|\bar\varrho\|\,\|\bar\psi\|. \tag{4.10}$$

Since $\bar\psi(\alpha,\cdot)$ has zero mean in the disk $\mathbb{D}^2$ for any fixed α, by Poincaré's inequality there is a positive constant C, independent of ε and α (namely, the first non-zero Neumann eigenvalue of the disk), such that

$$\int_{\mathbb{D}^2}|D_y\bar\psi(\alpha,y)|^2\,dy\geqslant C\int_{\mathbb{D}^2}\bar\psi(\alpha,y)^2\,dy.$$

Integrating this inequality in α, the H^1 norm of $\bar\psi$ can be estimated as

$$\|\bar\psi\|^2_{\dot H^1_\varepsilon}\geqslant\frac{1}{\varepsilon^2}\int|D_y\bar\psi(\alpha,y)|^2\,dy\,d\alpha\geqslant\frac{C}{\varepsilon^2}\|\bar\psi\|^2.$$

Together with equation (4.10), this yields $\|\bar\psi\|\leqslant C\varepsilon^2\|\bar\varrho\|$ and $\|\bar\psi\|_{\dot H^1_\varepsilon}\leqslant C\varepsilon\|\bar\varrho\|$, so the claim follows directly from Proposition 4.4. □

It is particularly easy to derive preliminary estimates (which will be instrumental in the proof of Proposition 4.7) for the derivatives of $\bar{\psi}$ with respect to α due to Proposition 4.3. Notice that these bounds will be substantially improved later on.

Proposition 4.6. *The derivative of $\bar{\psi}$ with respect to α satisfies*

$$\|D_y \partial_\alpha^{k+1} \bar{\psi}\| \leqslant C\varepsilon \|\varrho\|_{H^k} \quad \text{and} \quad \|\partial_\alpha^{k+2} \bar{\psi}\| \leqslant C_k \|\varrho\|_{H^k}$$

for any non-negative integer k.

Proof. The claim is an immediate consequence of the definition of $\bar{\psi}$ (cf. equation (4.7)) and Proposition 4.3, since obviously the $L^2(\mathbb{S}^1_\ell \times \mathbb{D}^2)$ norm of the function

$$\partial_\alpha^j \int_{\mathbb{D}^2} \psi(\alpha, y')\, dy'$$

is bounded from above by $C\|\partial_\alpha^j \psi\|$, which was in turn estimated in the aforementioned proposition. ☐

We are ready to show that the second derivatives of $\bar{\psi}$ with respect to the fast variable y are bounded by a factor of order ε^2. The proof of these estimates makes essential use of Propositions 4.5 and 4.6.

Proposition 4.7. *The second derivatives of the function $\bar{\psi}$ with respect to the fast coordinates are bounded by*

$$\|D_y^2 \bar{\psi}\| \leqslant C\varepsilon^2 \|\varrho\|.$$

Proof. We will denote by $\mathbb{D}^2_R$ the 2-dimensional disk of radius R, R being a fixed real smaller than 1. Let us start by proving interior estimates. For this, we will denote by ∂_a the derivative along a y-direction (that is, ∂_1 or ∂_2), and consider a smooth function $\chi(|y|)$ equal to 1 for $|y|<R$ and equal to zero in a neighborhood of $\partial\mathbb{D}^2$. Taking $\varphi = \partial_a(\chi^2 \partial_a \bar{\psi})$ in equation (4.9) (here and in what follows we will *not* sum over the index a) and integrating by parts, one readily obtains

$$I^2 := \int g^{ij} \partial_i \partial_a \bar{\psi}\, \partial_j \partial_a \bar{\psi}\, \chi^2\, dV = I_1 - I_2 - I_3, \tag{4.11}$$

where

$$I_1 := \int \bar{\varrho} \partial_a(\chi^2 \partial_a \bar{\psi})\, dV,$$

$$I_2 := \int \partial_a(B g^{ij})\, \partial_i \bar{\psi}\, \partial_j(\chi^2 \partial_a \bar{\psi})\, d\alpha\, dy,$$

$$I_3 := \int \partial_a \bar{\psi}\, g^{ij} \partial_j(\chi^2)\, \partial_i \partial_a \bar{\psi}\, dV.$$

Let us estimate these integrals. The first one can easily be controlled using the Cauchy–Schwarz inequality and Propositions 4.4 and 4.5:

$$\begin{aligned}|I_1| &\leqslant \left|\int \bar\varrho\chi^2\partial_a^2\bar\psi\, dV\right| + \left|\int \bar\varrho\partial_a(\chi^2)\partial_a\bar\psi\, dV\right| \\ &\leqslant C\|\bar\varrho\|(\|\chi\partial_a^2\bar\psi\|+\|\partial_a\bar\psi\|) \leqslant C\varepsilon\|\varrho\| I + C\varepsilon^2\|\varrho\|^2.\end{aligned}$$

The integral I_2 can be controlled using an analogous argument, Lemma 4.2 and Jensen's inequality. This leads to the estimate

$$\begin{aligned}|I_2| &\leqslant \left|\int \partial_a(Bg^{ij})\chi^2\partial_i\bar\psi\,\partial_j\partial_a\bar\psi\, d\alpha\, dy\right| + \left|\int \partial_a(Bg^{ij})\partial_i\bar\psi\,\partial_j(\chi^2)\,\partial_a\bar\psi\, d\alpha\, dy\right| \\ &\leqslant C\varepsilon\|\bar\psi\|_{\dot H^1_\varepsilon} I + C\|\partial_a\bar\psi\|\left(\int (\partial_a(Bg^{ij})\partial_i\bar\psi\,\partial_j\chi)^2\, d\alpha\, dy\right)^{1/2} \\ &\leqslant C\varepsilon^2\|\varrho\| I + C\varepsilon^2\|\varrho\|\int |\partial_a(Bg^{ij})\partial_i\bar\psi\,\partial_j\chi|\, d\alpha\, dy \\ &\leqslant C\varepsilon^2\|\varrho\| I + C\varepsilon^3\|\varrho\|\,\|\bar\psi\|_{\dot H^1_\varepsilon}\|\chi\|_{\dot H^1_\varepsilon} \\ &\leqslant C\varepsilon^2\|\varrho\| I + C\varepsilon^3\|\varrho\|^2,\end{aligned}$$

where in the fifth line we have used that $\|\chi\|_{\dot H^1_\varepsilon} = \|D_y\chi\|/\varepsilon = C/\varepsilon$. A similar argument shows that

$$|I_3| \leqslant C\|\partial_a\bar\psi\|\,\|\chi\|_{\dot H^1_\varepsilon} I \leqslant C\varepsilon\|\varrho\| I.$$

Feeding these bounds into equation (4.11), we obtain

$$I^2 \leqslant C\varepsilon\|\varrho\| I + C\varepsilon^2\|\varrho\|^2,$$

so that $I \leqslant C\varepsilon\|\varrho\|$. From the definition of the function χ and the identity (4.5) it then follows that

$$\|D_y^2\bar\psi\|_{\mathbb{S}^1_\ell\times\mathbb{D}^2_R} \leqslant C\varepsilon^2\|\varrho\|, \tag{4.12}$$

which is the desired interior estimate.

To prove the estimates up to the boundary, we begin by showing that the $\dot H^1_\varepsilon$ norm of

$$\partial_\theta\bar\psi \equiv y_1\partial_2\bar\psi - y_2\partial_1\bar\psi$$

is bounded in terms of $\|\varrho\|$. For this, we will find it convenient to take a smooth function $\chi(|y|)$, equal to 1 for $|y|>R$ and vanishing in a neighborhood of the origin, and use polar coordinates throughout without further notice. If we now take $\varphi = \chi^2\bar\psi_{\theta\theta}$ in equation (4.4) and integrate by parts, we readily find that

$$\begin{aligned}&\int g^{ij}\partial_i\bar\psi_\theta\,\partial_j\bar\psi_\theta\,\chi^2\, dV \\ &\quad = \int \bar\varrho\bar\psi_{\theta\theta}\chi^2\, dV - \int \partial_\theta(Bg^{ij})\,\partial_i\bar\psi\,\partial_j(\chi^2\bar\psi_\theta)\, d\alpha\, dy - \int g^{ij}\partial_i\bar\psi_\theta\,\bar\psi_\theta\partial_j(\chi^2)\, dV,\end{aligned}$$

where the indices i and j now range over the set $\{r,\theta,\alpha\}$. Notice that the reason we are now using polar coordinates is that ∂_θ does not commute with the derivatives with respect to (y_1,y_2). Arguing as above, one finds that

$$\int g^{ij}\partial_i\bar{\psi}_\theta\,\partial_j\bar{\psi}_\theta\,\chi^2\,dV\leqslant C\varepsilon\|\varrho\|,$$

which ensures that

$$\|\bar{\psi}_{\theta\theta}\|_{\mathbb{S}^1_\ell\times\mathbb{A}_R}+\|\bar{\psi}_{r\theta}\|_{\mathbb{S}^1_\ell\times\mathbb{A}_R}\leqslant C\varepsilon^2\|\varrho\|,\tag{4.13}$$

where $\mathbb{A}_R:=\mathbb{D}^2\backslash\overline{\mathbb{D}^2_R}$ is the annulus of inner radius R.

To estimate the derivative $\bar{\psi}_{rr}$, it now suffices to isolate this quantity in the equation $\Delta\bar{\psi}=\bar{\varrho}$. From the expression of the Laplacian in these coordinates (4.2) it is apparent that for $r>R$ one can write $\bar{\psi}_{rr}$ as

$$\varepsilon^{-2}\bar{\psi}_{rr}-\bar{\varrho}=\mathcal{O}(\varepsilon^{-2})(|\bar{\psi}_{\theta\theta}|+|\bar{\psi}_r|)+\mathcal{O}(\varepsilon^{-1})\bar{\psi}_\theta+\mathcal{O}(1)(|\bar{\psi}_{\alpha\theta}|+|\bar{\psi}_{\alpha\alpha}|+|\bar{\psi}_\alpha|).$$

From Propositions 4.5–4.7 and the estimates (4.13), it then follows that

$$\|\bar{\psi}_{rr}\|_{\mathbb{S}^1_\ell\times\mathbb{A}_R}\leqslant C\varepsilon^2\|\varrho\|.$$

This yields the desired boundary estimates, completing the proof of the proposition. □

The results we have established so far show that the derivatives of $\bar{\psi}$ can be bounded in terms of the source ϱ as

$$\|\bar{\psi}\|+\|D_y\bar{\psi}\|+\|D_y^2\bar{\psi}\|\leqslant C\varepsilon^2\|\varrho\|,\tag{4.14a}$$

$$\|\partial_\alpha\bar{\psi}\|+\|D_y\partial_\alpha\bar{\psi}\|\leqslant C\varepsilon\|\varrho\|,\tag{4.14b}$$

$$\|\partial_\alpha^2\bar{\psi}\|\leqslant C\|\varrho\|.\tag{4.14c}$$

However, having in mind the applications in the forthcoming sections, we would rather have estimates where the right-hand side always has a factor of ε^2.

In the following theorem we show that this can be achieved by replacing the L^2 norm of ϱ by its H^1 norm (thus using estimates that are weaker in terms of the gain of derivatives), and provide a generalization for higher derivatives. It is worth emphasizing that both the estimates (4.14) and those in the following theorem are optimal with respect to its dependence on the small parameter ε, as can be checked easily.

THEOREM 4.8. *For any non-negative integers j and k we have the bound*

$$\|D_y^j\partial_\alpha^k\bar{\psi}\|\leqslant C_{jk}\varepsilon^2\|\varrho\|_{H^{J+k}},$$

where $J:=\max\{j-2,0\}$.

Proof. The proof proceeds by induction on $J+k$, that is, on the number of derivatives in the right-hand side of the inequality. The case $J+k=0$ follows from Propositions 4.5 and 4.7. To avoid cumbersome notation that might obscure the argument, we will sketch the procedure for the case $J+k=1$, where one has to estimate $\partial_\alpha\bar{\psi}$ and $D_y\partial_\alpha\bar{\psi}$ (improving the bounds in Propositions 4.5 and 4.6), $D_y^2\partial_\alpha\bar{\psi}$ and $D_y^3\bar{\psi}$. Once this case has been worked out in detail, it is straightforward to prove the general result using an induction argument.

Let us begin by estimating the quantities having derivatives with respect to α, that is, $\partial_\alpha\bar{\psi}$, $D_y\partial_\alpha\bar{\psi}$ and $D_y^2\partial_\alpha\bar{\psi}$. An easy computation using the expression of the Laplacian (4.2) shows that the commutator

$$\varrho := \Delta(\partial_\alpha\bar{\psi}) - \partial_\alpha(\Delta\bar{\psi})$$

can be written as

$$\varrho = \mathcal{O}(\varepsilon^{-1})D_y^2\bar{\psi} + \mathcal{O}(\varepsilon)\bar{\psi}_{\alpha\alpha} + \mathcal{O}(1)D_y\partial_\alpha\bar{\psi} + \mathcal{O}(\varepsilon^{-1})D_y\bar{\psi} + \mathcal{O}(\varepsilon)\bar{\psi}_\alpha.$$

Now from the H^2 estimates proved in Propositions 4.5–4.7, one obtains that the L^2 norm of the commutator is bounded by

$$\|\varrho\| \leqslant C\varepsilon\|\varrho\|.$$

The function $\partial_\alpha\bar{\psi}$ obviously satisfies the zero-mean condition

$$\int_{\mathbb{D}^2} \partial_\alpha\bar{\psi}(\alpha, y')\, dy' = 0, \tag{4.15a}$$

the boundary condition

$$\partial_\nu(\partial_\alpha\bar{\psi}) = 0 \tag{4.15b}$$

on $\mathbb{S}^1_\ell \times \partial\mathbb{D}^2$, and the equation

$$\Delta(\partial_\alpha\bar{\psi}) = \partial_\alpha\bar{\varrho} + \varrho. \tag{4.15c}$$

As the L^2 norm of the right-hand side is bounded by $C\|\varrho\|_{H^1}$, an immediate application of Propositions 4.5 and 4.7 (applied to the boundary problem (4.15) rather than to (4.1)) shows that

$$\|\bar{\psi}_\alpha\| + \|D_y\bar{\psi}_\alpha\| + \|D_y^2\bar{\psi}_\alpha\| \leqslant C\varepsilon^2\|\varrho\|_{H^1}.$$

To estimate $D_y^3\bar{\psi}$ we will argue essentially as in the proof of Proposition 4.7. One starts by proving interior estimates, which are obtained by feeding the test function

$$\varphi := \partial_a^2(\chi^2\partial_a^2\bar{\psi})$$

into the identity (4.9). As before, $\chi(y)$ is a smooth function which vanishes in a neighborhood of $\partial\mathbb{D}^2$ and is identically equal to 1 in a disk of radius R and the script a (which is not summed) denotes any y direction. In order to get boundary estimates, one can start by noticing that the same argument as we have used above to control derivatives with respect to α also works for $\partial_\theta\bar{\psi}\equiv y_1\partial_2\bar{\psi}-y_2\partial_1\bar{\psi}$. Indeed, the L^2 norm of the commutator

$$\tilde{\varrho}:=\Delta(\partial_\theta\bar{\psi})-\partial_\theta(\Delta\bar{\psi})$$

is also bounded by $C\varepsilon\|\varrho\|$ as a consequence of Propositions 4.5–4.7, and besides $\partial_\theta\bar{\psi}$ satisfies the boundary value problem

$$\Delta(\partial_\theta\bar{\psi})=\partial_\theta\bar{\varrho}+\tilde{\varrho} \text{ in } \mathbb{S}^1_\ell\times\mathbb{D}^2, \quad \partial_\nu\partial_\theta\bar{\psi}=0 \text{ on } \mathbb{S}^1_\ell\times\partial\mathbb{D}^2, \quad \int_{\mathbb{D}^2}\partial_\theta\bar{\psi}(\alpha,y')\,dy'=0.$$

It then follows that the L^2 norm of the derivatives $D^2_y\partial_\theta\bar{\psi}$ are bounded by $C\varepsilon^2\|\varrho\|_{H^1}$. Hence one can now differentiate the equation $\Delta\bar{\psi}=\bar{\varrho}$ with respect to r and isolate $\partial^3_r\bar{\psi}$ to show that the norm $\|\partial^3_r\bar{\psi}\|_{\mathbb{S}^1_\ell\times\mathbb{A}_R}$ in a neighborhood of the boundary is bounded by $C\varepsilon^2\|\varrho\|_{H^1}$, as claimed.

This proves the induction hypotheses for $J+k=1$. For higher values of $J+k$, the idea is exactly the same. Derivatives with respect to α (or θ) are dealt with by taking derivatives directly in the equation and invoking the induction hypotheses. To estimate the highest derivative with respect to r, one combines interior estimates with the test function

$$\varphi=\partial_a^{J+k+1}(\chi^2\partial_a^{J+k+1}\bar{\psi})$$

with the direct isolation of $\partial_r^{J+k+2}\bar{\psi}$ in the equation $\partial_r^{J+k}(\Delta\bar{\psi}-\bar{\varrho})=0$. □

4.3. Pointwise estimates

Taking into account that $D_y\bar{\psi}=D_y\psi$, we can combine the results in the previous two subsections to obtain H^k estimates for ψ in which the derivatives with respect to the fast and slow variables are controlled in terms of different (and optimal) powers of ε. To begin with, we can put together Propositions 4.3 and 4.6 and Theorem 4.8 to arrive at the following bounds.

THEOREM 4.9. *The functions ψ and $\bar{\psi}$ satisfy the H^k estimate*

$$\|\psi\|_{H^{k+2}}\leqslant C_k\|\varrho\|_{H^k} \quad \textit{and} \quad \|\bar{\psi}\|_{H^k}\leqslant C_k\varepsilon^2\|\varrho\|_{H^k}$$

for any non-negative integer k.

Remark 4.10. It is not hard to prove that these estimates are optimal with respect to the dependence on ε. To have a rough idea of why this is true, the reader might want to consider the problem

$$\psi_{\alpha\alpha}+\frac{\Delta_y\psi}{\varepsilon^2}=\varrho \text{ in } \mathbb{S}^1_\ell\times\mathbb{D}^2, \quad \partial_\nu\psi=0 \text{ on } \mathbb{S}^1_\ell\times\partial\mathbb{D}^2, \quad \int\psi\,d\alpha\,dy=0, \tag{4.16}$$

which can be understood as a simplified version of the problem (4.1). Here and in what follows, $\Delta_y\psi:=\partial_1^2\psi+\partial_2^2\psi$ stands for the standard Laplacian in the variables y.

For future reference, it is convenient to invoke the Sobolev embedding theorem and record the following estimate for the C^k norm of the function ψ.

THEOREM 4.11. *The function ψ satisfies the C^k estimates*

$$\|\psi\|_{C^k}\leqslant C_k\|\varrho\|_{H^k} \quad \textit{and} \quad \|D_y\psi\|_{C^k}\leqslant C_k\varepsilon^2\|\varrho\|_{H^{k+3}}$$

for any non-negative integer k.

Proof. This follows immediately from Theorem 4.9 upon noticing that $D_y\bar{\psi}=D_y\psi$ and using the Sobolev inequality $\|\varphi\|_{C^k}\leqslant C\|\varphi\|_{H^{k+2}}$. □

Remark 4.12. It is clear that the same bounds we have proved for the function ψ hold true if we assume that ψ solves the model problem (4.16) instead of $\Delta\psi=\varrho$. In particular, for future reference we record here that this function also satisfies the estimates

$$\|\psi\|_{H^{k+2}}+\frac{\|D_y^2\psi\|_{H^k}}{\varepsilon^2}\leqslant C_k\|\varrho\|_{H^k} \quad \text{and} \quad \|D_y\psi\|_{H^k}\leqslant C_k\varepsilon^2\|\varrho\|_{H^k}. \tag{4.17}$$

5. Harmonic fields in thin tubes

In this section we use the estimates proved in §4 to compute the harmonic field in a thin tube up to terms that are suitably bounded for small ε.

We recall that a vector field h in the tube $\mathcal{T}_\varepsilon$ is *harmonic* if it is divergence-free, irrotational, and tangent to the boundary. The vector space of all harmonic fields in the tube will be denoted by

$$\mathcal{H}(\mathcal{T}_\varepsilon):=\{h\in C^\infty(\Omega,\mathbb{R}^3):\operatorname{div}h=0,\operatorname{curl}h=0 \text{ and } h\cdot\nu=0\}. \tag{5.1}$$

It is standard that the space $\mathcal{H}(\mathcal{T}_\varepsilon)$ is 1-dimensional, as it is isomorphic to the first cohomology group of the tube with real coefficients.

Let us consider the vector field in $\mathbb{S}^1_\ell\times\mathbb{D}^2$ defined by

$$h_0:=B^{-2}(\partial_\alpha+\tau\partial_\theta). \tag{5.2}$$

It can be readily checked that h_0 is irrotational (with respect to the metric (3.3)) and tangent to the boundary.

By the Hodge decomposition [14], there is a function ψ such that

$$h := h_0 + \nabla\psi \tag{5.3}$$

is harmonic. Clearly this is the only harmonic vector field in $\mathbb{S}^1_\ell \times \mathbb{D}^2$ up to a multiplicative constant. By $\nabla\psi$ we are denoting the gradient of the function ψ with respect to the metric (3.3), that is,

$$\nabla\psi = \frac{\psi_\alpha + \tau\psi_\theta}{B^2}\partial_\alpha + \frac{\psi_r}{\varepsilon^2}\partial_r + \frac{A\psi_\theta + \varepsilon^2 r^2 \tau\psi_\alpha}{(\varepsilon r B)^2}\partial_\theta. \tag{5.4}$$

The fact that the field h is divergence-free implies that ψ solves the Neumann boundary value problem

$$\Delta\psi = \varrho \ \text{ in } \mathbb{S}^1_\ell \times \mathbb{D}^2, \quad \partial_\nu\psi = 0 \ \text{ on } \mathbb{S}^1_\ell \times \partial\mathbb{D}^2,$$

where

$$\varrho := \varepsilon B^{-3} r(\tau\varkappa \sin\theta - \varkappa' \cos\theta) \tag{5.5}$$

is minus the divergence of h_0 (and, as such, satisfies $\int \varrho\, dV = 0$). We will also assume that $\int \psi\, d\alpha\, dy = 0$ in order to determine ψ uniquely.

In the following two sections we will need some estimates for the harmonic field h (or, equivalently, for the function ψ) that depend on the particular form of the source term ϱ. These estimates are obtained in the following theorem, where we calculate, up to some controllable error, some derivatives of the function ψ that we will use later on. To simplify the notation, we will write $\mathcal{O}(\varepsilon^n)$ for any function χ satisfying the bound $\|\chi\|_{C^k} \leqslant C_k \varepsilon^n$ for all k (and also for numbers whose absolute value is smaller than $C\varepsilon^n$, but the meaning should be clear from the context).

THEOREM 5.1. *Consider the functions*

$$\varphi_0 := \tfrac{1}{8}\varepsilon^3 (r^3 - 3r)(\tau\varkappa \sin\theta - \varkappa' \cos\theta), \tag{5.6}$$

$$\varphi_1 := \tfrac{13}{96}\varepsilon^4 (r^4 - 2r^2)(\tau\varkappa^2 \sin 2\theta - \varkappa\varkappa' \cos 2\theta), \tag{5.7}$$

which are obviously of order $\mathcal{O}(\varepsilon^3)$ and $\mathcal{O}(\varepsilon^4)$, respectively. Then ψ is related to these functions through the estimates

$$\begin{aligned} \psi &= \mathcal{O}(\varepsilon^2), \\ D_y\psi &= D_y\varphi_0 + \mathcal{O}(\varepsilon^4), \\ \partial_\theta\psi &= \partial_\theta\varphi_0 + \partial_\theta\varphi_1 + \mathcal{O}(\varepsilon^5). \end{aligned}$$

Proof. Let $\varrho_0 := \varepsilon r(\tau\varkappa \sin\theta - \varkappa' \cos\theta)$. It is easy to see that the function $\varphi_0(\alpha, y)$ is the only solution to the problem

$$\Delta_y \varphi_0 = \varepsilon^2 \varrho_0 \;\text{ in } \mathbb{S}^1_\ell \times \mathbb{D}^2, \quad \partial_r \varphi_0|_{r=1} = 0, \quad \int_{\mathbb{D}^2} \varphi_0\, dy = 0,$$

for any value of the angle α.

Consider now the function $\psi_1 := \psi - \varphi_0$, which obviously has zero normal derivative on $\mathbb{S}^1_\ell \times \partial\mathbb{D}^2$ and has zero mean because so do ψ and φ_0. The Laplacian of ψ_1 is given by

$$\Delta\psi_1 = \varrho - \Delta\varphi_0 = (\varrho - \varrho_0) + \left(\frac{\Delta_y \varphi_0}{\varepsilon^2} - \Delta\varphi_0\right).$$

Using equation (5.5), the first term in brackets can easily be shown to be

$$\varrho - \varrho_0 = 3\varepsilon^2 \varkappa r^2 \cos\theta\, (\tau\varkappa \sin\theta - \varkappa' \cos\theta) + \mathcal{O}(\varepsilon^3),$$

while the second can easily be dealt with using the formula (4.2) for the Laplacian:

$$\frac{\Delta_y \varphi_0}{\varepsilon^2} - \Delta\varphi_0 = \frac{\varkappa}{\varepsilon}\left(\cos\theta\, \partial_r \varphi_0 - \frac{\sin\theta\, \partial_\theta \varphi_0}{r}\right) + \mathcal{O}(\varepsilon^3).$$

Using the definition of φ_0, this yields

$$\Delta\psi_1 = \varrho_1 + \varrho_2 + \mathcal{O}(\varepsilon^3),$$

with

$$\varrho_1 := \tfrac{13}{8}\varepsilon^2 \varkappa r^2 (\varkappa\tau \sin 2\theta - \varkappa' \cos 2\theta) \quad \text{and} \quad \varrho_2 := \tfrac{1}{8}\varepsilon^2 (3 - 14r^2)\varkappa\varkappa'$$

of order $\mathcal{O}(\varepsilon^2)$, so the estimates we proved in Theorem 4.11 then ensure that

$$\psi_1 = \mathcal{O}(\varepsilon^2) \quad \text{and} \quad D_y \psi_1 = \mathcal{O}(\varepsilon^4).$$

This shows that $\psi = \mathcal{O}(\varepsilon^2)$ and $D_y\psi = D_y\varphi_0 + \mathcal{O}(\varepsilon^4)$.

To calculate $D_y\psi$ up to $\mathcal{O}(\varepsilon^5)$ we will consider two auxiliary functions φ_1 and φ_2. The function $\varphi_1(\alpha, y)$ is the solution to the problem

$$\Delta_y \varphi_1 = \varepsilon^2 \varrho_1 \;\text{ in } \mathbb{S}^1_\ell \times \mathbb{D}^2, \quad \partial_r \varphi_1|_{r=1} = 0, \quad \int_{\mathbb{D}^2} \varphi_1\, dy = 0.$$

The existence and uniqueness of the solution are standard given that the right-hand side satisfies

$$\int_{\mathbb{D}^2} \varrho_1\, dy = 0$$

for all α. In fact, the solution can be computed in closed form using separation of variables, which readily yields the formula for φ_1 given in the statement (equation (5.7)).

The function φ_2 is the solution to the problem

$$\partial_\alpha^2\varphi_2+\frac{\Delta_y\varphi_2}{\varepsilon^2}=\varrho_2 \text{ in } \mathbb{S}^1_\ell\times\mathbb{D}^2, \quad \partial_r\varphi_2|_{r=1}=0, \quad \int\varphi_2\,d\alpha\,dy=0.$$

Again, the existence and uniqueness of solutions is standard because the right-hand side satisfies the zero-mean condition

$$\int\varrho_2\,d\alpha\,dy=0.$$

(The reason why we are including derivatives with respect to α in the definition of φ_2 but not in that of φ_1 is that the source term ϱ_2 has zero mean when averaged with respect to α and y but not when averaged in y only.) Obviously φ_2 does not depend on θ.

We can now estimate $\psi_2:=\psi_1-\varphi_1-\varphi_2$ using the same argument as we used with ψ_1. We start by noticing that, by construction, $\partial_r\psi_2=0$ when $r=1$ and the integral

$$\int_{\mathbb{S}^1_\ell\times\mathbb{D}^2}\psi_2\,d\alpha\,dy$$

is zero. The Laplacian of ψ_2 is

$$\begin{aligned}\Delta\psi_2&=\varrho_1+\varrho_2-\Delta\varphi_1-\Delta\varphi_2+\mathcal{O}(\varepsilon^3)\\&=\left(\frac{\Delta_y\varphi_1}{\varepsilon^2}-\Delta\varphi_1\right)+\left(\partial_\alpha^2\varphi_2+\frac{\Delta_y\varphi_2}{\varepsilon^2}-\Delta\varphi_2\right)+\mathcal{O}(\varepsilon^3)=\mathcal{O}(\varepsilon^3).\end{aligned}$$

To pass to the third line we have used the expression of the Laplacian in polar coordinates (equation (4.2)) and of φ_1, as well as the estimates for φ_2 that stem from Theorem 4.11 and Remark 4.12. Another application of Theorem 4.11 shows that $D_y\psi_2=\mathcal{O}(\varepsilon^5)$. Since $\partial_\theta\varphi_2=0$ and

$$\psi=\varphi_0+\varphi_1+\varphi_2+\psi_2,$$

the theorem follows. □

6. Beltrami fields with prescribed harmonic part

Our goal in this section is to construct a Beltrami field v in the thin tube $\mathcal{T}_\varepsilon$, tangent to the boundary, whose harmonic part is a fixed harmonic field h. We will be particularly interested in the way the Beltrami field v is related to the harmonic field h as the parameter λ tends to zero, since in the next section it will be crucial to have good estimates for this relation in order to compute some dynamical quantities of the field v.

This section is divided into three parts. In §6.1 we prove an existence result for a boundary value problem for the curl operator in tubes in which we can prescribe the harmonic part of the solution (Corollary 6.3). In §6.2 we will provide estimates for an auxiliary vector equation with constant coefficients (Proposition 6.6), which are used in §6.3 to prove the desired estimates, optimal in ε, for the boundary value problem under consideration (Theorem 6.8).

6.1. An existence result for the curl operator

Let us begin by making precise what we understand by the harmonic part of a vector field w in the thin tube $\mathcal{T}_\varepsilon$ that is tangent to the boundary. We will define its *harmonic part* to be the vector field

$$\mathcal{P}w := \frac{h}{\|h\|^2_{L^2(\mathcal{T}_\varepsilon)}} \int_{\mathcal{T}_\varepsilon} h\cdot w\, dx,$$

that is, its projection to the space of harmonic vector fields $\mathcal{H}(\mathcal{T}_\varepsilon)$, as introduced in equation (5.1). In this subsection, we will denote by

$$\|v\|^2_{H^k(\mathcal{T}_\varepsilon)} := \sum_{j=0}^{k} \int_{\mathcal{T}_\varepsilon} |D^j v|^2\, dx$$

the usual H^k norm of a vector field in the tube $\mathcal{T}_\varepsilon \subset \mathbb{R}^3$ and write $L^2(\mathcal{T}_\varepsilon)$ for $H^0(\mathcal{T}_\varepsilon)$. Throughout, we will use the notation C_ε for positive constants, possibly not uniformly bounded in the small parameter ε, that may vary from line to line.

In the following proposition we present the basic existence result that we will use to show the existence of Beltrami fields with prescribed harmonic part. The result is probably known to some experts but we have not found it in the literature. The proof relies on a duality argument for a suitable energy functional and the Fredholm alternative theorem.

PROPOSITION 6.1. *Let f be an L^2 vector field in $\mathcal{T}_\varepsilon$ which is divergence-free. There is a countable subset of the real line without accumulation points such that, if the constant λ does not belong to it, then the equation*

$$\operatorname{curl} w - \lambda w = f, \quad \operatorname{div} w = 0, \tag{6.1}$$

has a unique H^1 solution w that is tangent to the boundary $\partial\mathcal{T}_\varepsilon$ and has zero harmonic part.

Proof. Let us consider the Hilbert spaces of vector fields

$$\mathcal{F} := \{F \in L^2(\mathcal{T}_\varepsilon, \mathbb{R}^3) : \operatorname{div} F = 0\},$$
$$\mathcal{W} := \{w \in H^1(\mathcal{T}_\varepsilon, \mathbb{R}^3) : w \text{ is tangent to } \partial\mathcal{T}_\varepsilon\},$$

where the tangency condition is to be understood in terms of traces. It is well known [14] that, because of the tangency condition imposed on $\mathcal{W}$, the H^1 norm is equivalent to the norm

$$\|w\|_{\mathcal{W}}^2 := \|\operatorname{curl} w\|_{L^2(\mathcal{T}_\varepsilon)}^2 + \|\operatorname{div} w\|_{L^2(\mathcal{T}_\varepsilon)}^2 + \|\mathcal{P}w\|_{L^2(\mathcal{T}_\varepsilon)}^2$$

in the sense that, for all $w \in \mathcal{W}$,

$$\frac{\|w\|_{\mathcal{W}}}{C_\varepsilon} \leqslant \|w\|_{H^1(\mathcal{T}_\varepsilon)} \leqslant C_\varepsilon \|w\|_{\mathcal{W}}.$$

Consider the scalar product $E: \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ associated with the norm $\|\cdot\|_{\mathcal{W}}$, given by

$$E[w, u] := \int_{\mathcal{T}_\varepsilon} (\operatorname{curl} w \cdot \operatorname{curl} u + \operatorname{div} w \, \operatorname{div} u + \mathcal{P}w \cdot \mathcal{P}u)\, dx.$$

By the Riesz representation theorem, for any L^2 vector field F there is a unique $w_F \in \mathcal{W}$ such that

$$E[w_F, u] = \int_{\mathcal{T}_\varepsilon} F \cdot \operatorname{curl} u \, dx \tag{6.2}$$

for all $u \in \mathcal{W}$.

We claim that, for any $F \in \mathcal{F}$, condition (6.2) is equivalent to demanding that w_F be an H^1 solution to the equation

$$\operatorname{curl} w_F = F, \quad \operatorname{div} w_F = 0, \tag{6.3}$$

in the tube $\mathcal{T}_\varepsilon$, that is tangent to the boundary and has zero harmonic part. One side of the implication is obvious, so we only need to prove that if w_F satisfies condition (6.2) with $F \in \mathcal{F}$, then it solves equation (6.3).

For this, let us take suitable choices of the field u in equation (6.2). Letting $u = \mathcal{P}w_F$ be the harmonic part of w_F, we obtain that $\mathcal{P}w_F$ is zero. To see that $\operatorname{div} w_F$ is zero too, it suffices to take $u = \nabla\psi$, with ψ being the solution to the boundary value problem

$$\Delta\psi = \operatorname{div} w_F \ \text{ in } \mathcal{T}_\varepsilon, \quad \partial_\nu \psi|_{\partial\mathcal{T}_\varepsilon} = 0, \quad \int_{\mathcal{T}_\varepsilon} \psi \, dx = 0.$$

Notice that, in order to show that the solution exists and $\nabla\psi$ belongs to the space $\mathcal{W}$, we need to use that w_F is in H^1 and is tangent to the boundary.

Since $\mathcal{P}w_F$ and $\operatorname{div} w_F$ are zero, condition (6.2) can then be written as

$$\int_{\mathcal{T}_\varepsilon} \operatorname{curl} G\cdot u\, dx+\int_{\partial\mathcal{T}_\varepsilon}(G\times\nu)\cdot u\, d\sigma=0 \tag{6.4}$$

for all $u\in\mathcal{W}$, where

$$G:=\operatorname{curl} w_F-F$$

and $d\sigma$ denotes the induced surface measure on the boundary. Letting u vary over the space of smooth vector fields of compact support in $\mathcal{T}_\varepsilon$, we immediately infer that G must be irrotational. Now that we know that the first term in (6.4) is always zero, we can take an arbitrary field u tangent to the boundary to derive that $G\times\nu$ must be zero, so that G is divergence-free, irrotational and orthogonal to the boundary. By the Hodge decomposition theorem, this ensures that the field G is identically zero. Hence we have proved that w_F is the only solution to the problem (6.3). Notice that, as an immediate consequence of the equivalence of the H^1 norm to $\|\cdot\|_{\mathcal{W}}$ on $\mathcal{W}$,

$$\|w_F\|_{H^1(\mathcal{T}_\varepsilon)}\leqslant C_\varepsilon\|F\|_{L^2(\mathcal{T}_\varepsilon)} \tag{6.5}$$

for some constant that depends on ε but not on F.

Let us consider the operator K mapping a field $F\in\mathcal{F}$ to its solution w_F, which we regard as a linear map $K:\mathcal{F}\to\mathcal{F}$ whose image lies in $\mathcal{W}$. It is standard in view of the estimate (6.5) that the operator K is compact, so by the Fredholm alternative the equation

$$Kw-\frac{1}{\lambda}w=\tilde{f} \tag{6.6}$$

has a unique solution for each $\tilde{f}\in\mathcal{F}$ and any constant $1/\lambda$ that does not belong to the spectrum of the adjoint operator K^*. This spectrum is a bounded countable set that only accumulates at zero.

Let w be the only solution to equation (6.6) with $\tilde{f}=-Kf/\lambda$, for an arbitrary field f in $\mathcal{F}$. Since $w=K(\lambda w+f)$ belongs to the image of K, from the above discussion it stems that the field w is tangent to the boundary, divergence-free and has zero harmonic part, and by the definition of K satisfies $\operatorname{curl} w=\lambda w+f$. The proposition then follows. □

Remark 6.2. The proof of the proposition shows that the values of λ for which there is not a unique solution to the equation (6.1) are given by the reciprocal of the eigenvalues of the compact operator K^* that we defined in the proof.

As a direct application of the previous proposition, we derive the following corollary, which gives the existence result for Beltrami fields with prescribed harmonic part that we will use later on. Of course, it is apparent that this corollary is totally different from the results that one has on compact manifolds without boundary, where the Beltrami equation has no solutions but for λ belonging to a countable set and these solutions (i.e., the eigenfunctions of curl) are necessarily orthogonal to harmonic fields.

COROLLARY 6.3. *For any constant λ not belonging to a certain ε-dependent countable set without accumulation points, there is a unique solution to the equation*

$$\operatorname{curl} v = \lambda v, \quad \operatorname{div} v = 0,$$

in the tube $\mathcal{T}_\varepsilon$ that is tangent to the boundary and whose harmonic part is $\mathcal{P}v = h$.

Proof. It follows immediately from Proposition 6.1 by taking $v = h + w$, with w being the only solution to equation (6.1) with $f = \lambda h$. □

6.2. Estimates for an equation with constant coefficients

In this section we will prove estimates for a curl-type equation with constant coefficients on the domain $\mathbb{S}^1_\ell \times \mathbb{D}^2$. This equation with constant coefficients is closely related to the Beltrami equation on the thin tube $\mathcal{T}_\varepsilon$ and will be used subsequently to estimate the difference between the Beltrami field v and its harmonic part h. As before, the natural coordinates on the domain $\mathbb{S}^1_\ell \times \mathbb{D}^2$ will be denoted by (α, y), and we will consider polar coordinates (α, r, θ) where convenient.

Vector fields on $\mathbb{S}^1_\ell \times \mathbb{D}^2$ are regarded as functions $W : \mathbb{S}^1_\ell \times \mathbb{D}^2 \to \mathbb{R}^3$, whose components are denoted by

$$W = (W_\alpha, W_1, W_2).$$

We will sometimes write $W_y := (W_1, W_2)$. On these vector fields we will consider the action of the differential operators

$$\begin{aligned} \operatorname{Div} W &:= \partial_\alpha W_\alpha + \partial_1 W_1 + \partial_2 W_2, \\ \operatorname{Curl}_\varepsilon W &:= \left(\partial_1 W_2 - \partial_2 W_1, \frac{\partial_2 W_\alpha}{\varepsilon^2} - \partial_\alpha W_2, \partial_\alpha W_1 - \frac{\partial_1 W_\alpha}{\varepsilon^2} \right), \end{aligned}$$

and we will say that a vector field W is tangent to the boundary if $y \cdot W_y = 0$ (in the sense of traces) on the torus $|y| = 1$. Obviously, Div and $\operatorname{Curl}_\varepsilon$ are related to the standard divergence and curl operators through a rescaling, but for our purposes these forms of the operators are more convenient.

We will also consider the functional $\mathcal{Q}$ that maps each field W to the real number

$$\mathcal{Q}[W] := \int W_\alpha \, d\alpha \, dy.$$

This is obviously related to the projection onto the space of "harmonic fields" associated with the operators Div and $\mathrm{Curl}_\varepsilon$, which is spanned by the constant field $(1,0,0)$, so when $\mathcal{Q}[W]=0$ we will simply say that the harmonic part of $\mathcal{Q}[v]$ is zero. All integrals are taken over $\mathbb{S}^1_\ell\times\mathbb{D}^2$ unless otherwise stated. To control the behavior of the vector fields, in addition to the usual H^k norms we will consider the norms

$$\|W\|^2_{H^k_\varepsilon} := \|W_\alpha\|^2_{H^k} + \varepsilon^2\|W_y\|^2_{H^k}. \tag{6.7}$$

These norms should not be confused with the norm $\|\cdot\|_{\dot H^1_\varepsilon}$ that we considered for scalar functions in §4. We will simply write $\|\cdot\|_\varepsilon$ for the corresponding L^2_ε norm ($k=0$), and reserve the notation $\|\cdot\|$ for the usual L^2 norm, where factors of ε do not appear.

In the following proposition we compute the lowest eigenvalue of a self-adjoint operator associated with $\mathrm{Curl}_\varepsilon$ to show how the L^2_ε norm of a vector field W can be controlled using that of $\mathrm{Curl}_\varepsilon W$.

Proposition 6.4. *Let W be an H^1 vector field on $\mathbb{S}^1_\ell\times\mathbb{D}^2$ with* $\mathrm{Div}\, W=0$ *that is tangent to the boundary and has zero harmonic part. Then*

$$\|\mathrm{Curl}_\varepsilon W\|_\varepsilon \geqslant \frac{C\|W\|_\varepsilon}{\varepsilon}.$$

Proof. It is an easy consequence of [31] that $\mathrm{Curl}_\varepsilon$ defines an unbounded self-adjoint operator on the Hilbert space

$$\mathcal{H} := \{W\in L^2(\mathbb{S}^1_\ell\times\mathbb{D}^2,\mathbb{R}^3) : \mathrm{Div}\, W=0,\ \mathcal{Q}[W]=0 \text{ and } W \text{ is tangent to the boundary}\},$$

endowed with the scalar product associated with the norm $\|\cdot\|_\varepsilon$. The domain of this operator consists of the H^1 vector fields in $\mathcal{H}$ such that $\mathrm{Curl}_\varepsilon W$ is also in $\mathcal{H}$. Hence, to prove the proposition it is enough to see that the eigenvalues of this operator, in absolute value, satisfy $|\mu|>C/\varepsilon$.

Exploiting the symmetry of the equations (that is, rotation of the angles α and θ), it is not hard to see that the eigenvalue equation

$$\mathrm{Curl}_\varepsilon W = \mu W,$$

with W divergence-free, tangent to the boundary and with zero harmonic part, can be solved in closed form. Indeed, the symmetry ensures that the eigenfunctions can be chosen of the form (here, and in similar formulas, i denotes the imaginary unit)

$$W = e^{in\alpha+im\theta}(v_1(r)e_\alpha + v_2(r)e_r + v_3(r)e_\theta),$$

where n and m are integers, $v_j(r)$ are functions of the radial variable and the unit vectors e_α, e_r and e_θ are defined in the obvious way:

$$e_\alpha := (1,0,0), \quad e_r := (0,\cos\theta,\sin\theta) \quad \text{and} \quad e_\theta := (0,-\sin\theta,\cos\theta).$$

Using this expression, a tedious but straightforward computation shows that the eigenvalues with the smallest absolute value are $\pm j_1^{(1)}/\varepsilon$, where $j_1^{(1)}$ denotes the first positive zero of the Bessel function J_1. □

In the proof of the main result of this subsection we will need the following identity.

LEMMA 6.5. *Let W be a vector field tangent to the boundary of $\mathbb{S}^1_\ell \times \mathbb{D}^2$. Then*

$$\varepsilon^2\|\partial_\alpha W_y\|^2 + \|D_y W_y\|^2 + \|\partial_\alpha W_\alpha\|^2 + \varepsilon^{-2}\|D_y W_\alpha\|^2 = \|\operatorname{Div} W\|^2 + \|\operatorname{Curl}_\varepsilon W\|_\varepsilon^2 - \int_{\mathbb{S}^1_\ell\times\partial\mathbb{D}^2} |W_y|^2\, d\alpha\, d\theta,$$

where we are writing $d\alpha\, d\theta$ for the induced surface measure on $\mathbb{S}^1_\ell \times \partial\mathbb{D}^2$.

Proof. One can easily check that

$$\|\operatorname{Div} W\|^2 + \|\operatorname{Curl}_\varepsilon W\|_\varepsilon^2 = \|\partial_\alpha W_\alpha\|^2 + \sum_{i=1}^2 \left(\varepsilon^2\|\partial_\alpha W_i\|^2 + \frac{\|\partial_i W_\alpha\|^2}{\varepsilon^2}\right) + \sum_{i,j=1}^2 \|\partial_i W_j\|^2 + S,$$

where

$$S := \sum_{i,j\in\{\alpha,1,2\}} \int (\partial_i W_i\, \partial_j W_j - \partial_j W_i\, \partial_i W_j)\, d\alpha\, dy.$$

We can now integrate by parts to write

$$\begin{aligned} S &= -\sum_{i,j=1}^2 \int_{\mathbb{S}^1_\ell\times\partial\mathbb{D}^2} (y_j W_i \partial_i W_j + y_j W_\alpha \partial_\alpha W_j)\, d\alpha\, d\theta \\ &= \sum_{i,j=1}^2 \int_{\mathbb{S}^1_\ell\times\partial\mathbb{D}^2} W_j \partial_i (y_j W_i)\, d\alpha\, d\theta = \sum_{i=1}^2 \int_{\mathbb{S}^1_\ell\times\partial\mathbb{D}^2} W_i^2\, d\alpha\, d\theta. \end{aligned}$$

To pass to the second identity we have used that $y\cdot W_y$ is zero on the boundary, so the second summand in the first integrand vanishes, and taken advantage of the fact that W is tangent to the boundary to integrate by parts a second time. □

In the following proposition, which is the main result in this subsection, we prove an estimate for the operators Div and $\operatorname{Curl}_\varepsilon$ that is "optimal" with respect to the small parameter ε. In the proof of this version of the inequalities we will also derive another one in which the right-hand side has one derivative less than the left-hand side, as is customary. However, the estimate we will need later on is the former, which is why it is the one that appears in the statement. It is not hard to check that the dependence on ε of both estimates is sharp.

PROPOSITION 6.6. *Let W be any vector field in $\mathbb{S}^1_\ell \times \mathbb{D}^2$ that is tangent to the boundary and satisfies the equation*

$$\operatorname{Curl}_\varepsilon W = F, \quad \operatorname{Div} W = \varrho,$$

for a scalar function ϱ and a vector field F. Then

$$\begin{aligned}\|W_\alpha\|_{H^k} + \|D_y W_\alpha\|_{H^k} &\leqslant C_k(\varepsilon \|F\|_{H^k_\varepsilon} + \|\varrho\|_{H^k} + |\mathcal{Q}[W]|),\\ \|\partial_\alpha W_\alpha\|_{H^k} &\leqslant C_k(\|F\|_{H^k_\varepsilon} + \|\varrho\|_{H^k}),\\ \|W_y\|_{H^k} + \varepsilon\|\partial_\alpha W_y\|_{H^k} + \|D_y W_y\|_{H^k} &\leqslant C_k(\|F\|_{H^k_\varepsilon} + \|\varrho\|_{H^k} + |\mathcal{Q}[W]|)\end{aligned}$$

for constants that depend on k but not on ε.

Proof. We can write the field W as the sum of three fields:

$$W = V + \left(\partial_\alpha \psi, \frac{\partial_1 \psi}{\varepsilon^2}, \frac{\partial_2 \psi}{\varepsilon^2}\right) + \left(\frac{\mathcal{Q}[W]}{|\mathbb{S}^1_\ell \times \mathbb{D}^2|}, 0, 0\right). \tag{6.8}$$

The scalar function ψ that appears in the second vector field is defined as the only solution to the Neumann boundary value problem

$$\partial_\alpha^2 \psi + \frac{\Delta_y \psi}{\varepsilon^2} = \varrho \text{ in } \mathbb{S}^1_\ell \times \mathbb{D}^2, \quad \partial_\nu \psi|_{\mathbb{S}^1_\ell \times \partial\mathbb{D}^2} = 0, \quad \int_{\mathbb{S}^1_\ell \times \mathbb{D}^2} \psi \, d\alpha \, dy = 0, \tag{6.9}$$

and it should be noticed that the third vector field (which corresponds to the harmonic part of W) is constant. As a consequence of these definitions and the properties of W, the field V is tangent to the boundary, has zero harmonic part ($\mathcal{Q}[V]=0$) and satisfies the equation

$$\operatorname{Curl}_\varepsilon V = F, \quad \operatorname{Div} V = 0. \tag{6.10}$$

The H^k estimates stated in Remark 4.12 (equation (4.17)), applied to the boundary problem (6.9), provide suitable control of the second field that appears in equation (6.8) (that is, the "gradient" part) and its derivatives, as they show that

$$\begin{aligned}\|\partial_\alpha \psi\|_{H^k} + \|\partial_\alpha^2 \psi\|_{H^k} + \frac{\|D_y \partial_\alpha \psi\|_{H^k}}{\varepsilon} &\leqslant C\|\varrho\|_{H^k},\\ \frac{\|D_y \psi\|_{H^k}}{\varepsilon^2} + \frac{\|D_y \partial_\alpha \psi\|_{H^k}}{\varepsilon} + \frac{\|D_y^2 \psi\|_{H^k}}{\varepsilon^2} &\leqslant C\|\varrho\|_{H^k}.\end{aligned}$$

The third field in equation (6.8) is trivial to control as it is constant. Therefore, to prove the proposition it is enough to derive suitable estimates for the field V.

Hence, our goal is to show that the vector field V satisfies

$$\|V_\alpha\|_{H^k}+\|D_yV_\alpha\|_{H^k}\leqslant C\varepsilon\|F\|_{H^k_\varepsilon}, \tag{6.11a}$$

$$\|\partial_\alpha V_\alpha\|_{H^k}\leqslant C\|F\|_{H^k_\varepsilon}, \tag{6.11b}$$

$$\|V_y\|_{H^k}+\varepsilon\|\partial_\alpha V_y\|_{H^k}+\|D_yV_y\|_{H^k}\leqslant C\|F\|_{H^k_\varepsilon}. \tag{6.11c}$$

For this we start by noticing that, when applied to equation (6.10), the L^2 estimate proved in Proposition 6.4 yields $\|V\|_\varepsilon\leqslant C\varepsilon\|F\|_\varepsilon$, or equivalently

$$\|V_\alpha\|\leqslant C\varepsilon\|F\|_\varepsilon \quad\text{and}\quad \|V_y\|\leqslant C\|F\|_\varepsilon. \tag{6.12}$$

To prove the inequalities (6.11), we start by using estimates for boundary traces and interpolation to control the term

$$S:=\int_{\mathbb{S}^1_\ell\times\partial\mathbb{D}^2}|V_y|^2=\|V_y\|^2_{L^2(\mathbb{S}^1_\ell\times\partial\mathbb{D}^2)}\leqslant C\|V_y\|^2_{L^2(\mathbb{S}^1_\ell)\times H^{1/2}(\mathbb{D}^2)}\leqslant C\|V_y\|(\|V_y\|+\|D_yV_y\|)$$

that appears in Lemma 6.5. Together with the L^2 estimate (6.12), we can then apply Lemma 6.5 to the field V to infer that

$$\|\partial_\alpha V_\alpha\|\leqslant C\|F\|_\varepsilon,\quad \|D_yV_\alpha\|\leqslant C\varepsilon\|F\|_\varepsilon,$$
$$\|\partial_\alpha V_y\|\leqslant \frac{C}{\varepsilon}\|F\|_\varepsilon,\quad \|D_yV_y\|\leqslant C\|F\|_\varepsilon.$$

This proves the estimate (6.11) for $k=0$.

The derivation of higher-order estimates from these inequalities is standard. To show the basic ideas, let us sketch the proof of the $k=1$ estimates. Interior estimates are obtained by considering the identity shown in Lemma 6.5 for the vector field $\partial_i(\chi V)$, where $i=1,2$ and χ is a smooth function, compactly supported in $\mathbb{S}^1_\ell\times\mathbb{D}^2$ and equal to one in the region $|y|<\frac{1}{2}$. Besides, it is easy to estimate derivatives of V with respect to α because the field $\partial^j_\alpha V$ is still tangent to the boundary, divergence-free, has zero harmonic part and satisfies the equation

$$\operatorname{Curl}_\varepsilon(\partial^j_\alpha V)=\partial^j_\alpha F.$$

Analogous properties can also be shown for the globally defined field

$$\partial_\theta V+(0,V_2,-V_1),$$

with the angular derivative $\partial_\theta:=y_1\partial_2-y_2\partial_1$, so the same argument can be applied for this field. Since the second summand is obviously controlled by the previous estimates, this yields good H^1 estimates for $\partial_\theta V$. To get estimates for the field V up to the boundary, it now suffices to control the radial derivative ∂_rV in the region $|y|>\frac{1}{2}$, and this can be readily done by writing the equation (6.10) in polar coordinates, taking its derivative with respect to r and isolating the terms having two radial derivatives. This process can be readily iterated to get the desired H^k estimates. The details, which can easily be filled in using these comments, are omitted. □

6.3. Estimates for Beltrami fields in thin tubes

In this subsection we establish some estimates for Beltrami fields defined on the tube $\mathcal{T}_\varepsilon$. The point of these estimates will be to control the difference between a Beltrami field v and its harmonic part h in terms of the Beltrami parameter λ, while ensuring that the dependence on ε of these estimates is optimal. As we did in §4 (but not §6.1), we will identify $\mathcal{T}_\varepsilon$ with $\mathbb{S}^1_\ell\times\mathbb{D}^2$ through the coordinates (α,y).

Given a vector field v in the tube, we will consider its components (v_α,v_1,v_2), defined through the relation

$$v=v_\alpha\partial_\alpha+v_1\partial_1+v_2\partial_2.$$

We will use these coordinates to define the norms H^k_ε, just as in equation (6.7), of vector fields in $\mathcal{T}_\varepsilon$, using the obvious formula

$$\|v\|^2_{H^k_\varepsilon}:=\|v_\alpha\|^2_{H^k}+\varepsilon^2\|v_y\|^2_{H^k}.$$

Here the Sobolev norms in the right-hand side denote the usual scalar norms and we are using the notation $v_y:=(v_1,v_2)$. It should be noticed that these norms reflect to some extent the effect of the metric on the way vectors are measured; in particular, the L^2 norm defined by the above formula is obviously equivalent through constants that do not depend on ε to the perhaps more appealing expression

$$\int g(v,v)\,dV,$$

where g denotes the expression of the Euclidean metric in these coordinates and dV is the normalized volume (cf. equations (3.1) and (3.5)).

We will begin with the following result, whose proof hinges on the analogous estimates proved for an equation with constant coefficients that we established in Proposition 6.6.

Proposition 6.7. *Suppose that the vector field w is tangent to the boundary, has zero harmonic part ($\mathcal{P}w=0$) and satisfies the equation*

$$\operatorname{curl} w=f,\quad \operatorname{div} w=0, \tag{6.13}$$

in $\mathcal{T}_\varepsilon$. Then

$$\|w\|_{H^k_\varepsilon}\leqslant C_k\varepsilon\|f\|_{H^k_\varepsilon}.$$

Proof. In the notation of §6.2, let us consider the vector field $W:\mathbb{S}^1_\ell\times\mathbb{D}^2\to\mathbb{R}^3$ given by

$$W:=(Aw_\alpha+\varepsilon^2\tau(y_2w_1-y_1w_2),w_1+\tau y_2w_\alpha,w_2-\tau y_1w_\alpha),$$

where we recall that A is the function defined in equation (3.2). It can be checked that W is given by the components (in the coordinates (α, y)) of the 1-form dual to w up to a rescaling of the y components by a factor of ε^{-2}. It is worth noticing that, as the field w is tangent to the boundary of the tube $\mathcal{T}_\varepsilon$, a simple computation using the definition of W shows that the field W is tangent to the boundary $\mathbb{S}^1_\ell \times \partial\mathbb{D}^2$, that is, $y \cdot W_y = 0$ on $|y| = 1$. Moreover, the components of the field w on the tube can be recovered from those of W through the relations

$$w_\alpha = \frac{W_\alpha + \varepsilon^2 \tau (y_1 W_2 - y_2 W_1)}{B^2}, \tag{6.14a}$$

$$w_1 = W_1 - \frac{\tau y_2 [W_\alpha + \varepsilon^2 \tau (y_1 W_2 - y_2 W_1)]}{B^2}, \tag{6.14b}$$

$$w_2 = W_2 + \frac{\tau y_1 [W_\alpha + \varepsilon^2 \tau (y_1 W_2 - y_2 W_1)]}{B^2}, \tag{6.14c}$$

where the function B was defined in equation (3.6).

Using the notation

$$B_1 := \frac{B}{B^2 + \varepsilon^2 \tau^2 y_2^2} \quad \text{and} \quad B_2 := \frac{B}{B^2 + \varepsilon^2 \tau^2 y_1^2},$$

we will also consider the field $F: \mathbb{S}^1_\ell \times \mathbb{D}^2 \to \mathbb{R}^3$ and the function $\varrho_W: \mathbb{S}^1_\ell \times \mathbb{D}^2 \to \mathbb{R}$ defined by

$$F := \left(\frac{A f_\alpha + \varepsilon^2 \tau (y_2 f_1 - y_1 f_2)}{B}, \frac{f_1 + \tau y_2 f_\alpha}{B_1}, \frac{f_2 - \tau y_1 f_\alpha}{B_2} \right),$$

$$\varrho_W := \partial_\alpha \left[W_\alpha \left(1 - \frac{1}{B} \right) \right] + \partial_1 \left[W_1 \left(1 - \frac{1}{B_1} \right) \right] + \partial_2 \left[W_2 \left(1 - \frac{1}{B_2} \right) \right].$$

Notice that, by the definition of the functions A and B, we trivially have

$$\|F\|_{H^k_\varepsilon} \leqslant C \|f\|_{H^k_\varepsilon}, \tag{6.15}$$

$$\|\varrho_W\|_{H^k} \leqslant C\varepsilon (\|W\|_{H^k} + \|\partial_\alpha W_\alpha\|_{H^k} + \|D_y W_y\|_{H^k}). \tag{6.16}$$

A straightforward computation shows that equation (6.13) can be written in terms of the field W as

$$\operatorname{Curl}_\varepsilon W = F, \quad \operatorname{Div} W = \varrho_W. \tag{6.17}$$

To derive estimates for W, we start by analyzing $\mathcal{Q}[W]$. To this end, we recall (cf. e.g. [31]) that if $d\sigma$ denotes the induced surface measure on the disk

$$\{(\alpha, y) : \alpha = \alpha_0\} \subset \mathcal{T}_\varepsilon,$$

the fact that the field w has zero harmonic part ($\mathcal{P}w=0$) is equivalent to the assertion that

$$0=\int_{\{\alpha=\alpha_0\}} g(w,\nu)\,d\sigma=\int_{\mathbb{D}^2} Bw_\alpha|_{\alpha=\alpha_0}\,dy$$

for each angle α_0. Since $A-B$ is of order ε, using the above equality one can estimate

$$\begin{aligned}|\mathcal{Q}[W]|&=\left|\int W_\alpha\,d\alpha\,dy\right|\\&=\left|\int (Aw_\alpha+\varepsilon^2\tau(y_2w_1-y_1w_2))\,d\alpha\,dy\right|\\&=\left|\int ((A-B)w_\alpha+\varepsilon^2\tau(y_2w_1-y_1w_2))\,d\alpha\,dy\right|\\&\leqslant C\varepsilon\|w_\alpha\|+C\varepsilon^2\|w_y\|\\&\leqslant C\varepsilon\|W_\alpha\|+C\varepsilon^2\|W_y\|,\end{aligned}\tag{6.18}$$

where to derive the last inequality we have used the expression of the components of w in terms of those of W given in equation (6.14).

We are now ready to derive some estimates for the vector field W. Since W is tangent to $\mathbb{S}^1_\ell\times\partial\mathbb{D}^2$, we can apply Proposition 6.6 to equation (6.17) to obtain

$$\|W_\alpha\|_{H^k}\leqslant C(\varepsilon\|F\|_{H^k_\varepsilon}+\|\varrho_W\|_{H^k}+|\mathcal{Q}[W]|),\tag{6.19a}$$

$$\|\partial_\alpha W_\alpha\|_{H^k}\leqslant C(\|F\|_{H^k_\varepsilon}+\|\varrho_W\|_{H^k}),\tag{6.19b}$$

$$\|W_y\|_{H^k}+\|D_yW_y\|_{H^k}\leqslant C(\|F\|_{H^k_\varepsilon}+\|\varrho_W\|_{H^k}+|\mathcal{Q}[W]|).\tag{6.19c}$$

Going back to equation (6.16), this yields

$$\|\varrho_W\|_{H^k}\leqslant C\varepsilon(\|F\|_{H^k_\varepsilon}+\|W_\alpha\|_{H^k}+\|W_y\|_{H^k}).$$

Plugging this inequality and the bound (6.18) for $\mathcal{Q}[W]$ into equation (6.19), we immediately get

$$\begin{aligned}\|W_\alpha\|_{H^k}&\leqslant C\varepsilon(\|F\|_{H^k_\varepsilon}+\|W_\alpha\|_{H^k}+\|W_y\|_{H^k}),\\\|W_y\|_{H^k}&\leqslant C(\|F\|_{H^k_\varepsilon}+\varepsilon\|W_\alpha\|_{H^k}+\varepsilon\|W_y\|_{H^k}).\end{aligned}$$

From these estimates one readily infers that, for small ε,

$$\|W_\alpha\|_{H^k}+\varepsilon\|W_y\|_{H^k}\leqslant C\varepsilon\|F\|_{H^k_\varepsilon}.$$

In view of the formulas (6.14), $\|w\|_{H^k_\varepsilon}\leqslant C\|W\|_{H^k_\varepsilon}$, so the proposition follows from the bound (6.15) for the H^k_ε norm of F. □

We are now ready to prove the main result in this section, which estimates the difference between a Beltrami field in the tube and its harmonic part in terms of the Beltrami parameter λ.

THEOREM 6.8. *Let λ be any non-zero real constant that is smaller in absolute value than some fixed positive constant Λ. For small enough ε, the problem*

$$\operatorname{curl} v = \lambda v$$

has a unique solution in the tube $\mathcal{T}_\varepsilon$ that is tangent to the boundary and whose harmonic part $\mathcal{P}v$ is the harmonic field h. Moreover, the difference between the field v and the harmonic field is bounded pointwise by

$$\|v_\alpha - h_\alpha\|_{C^k(\mathbb{S}^1_\ell \times \mathbb{D}^2)} + \varepsilon \|v_y - h_y\|_{C^k(\mathbb{S}^1_\ell \times \mathbb{D}^2)} \leqslant C_{k,\Lambda}\varepsilon|\lambda|,$$

where the constant only depends on k and Λ.

Proof. Let us write $w := v - h$, so that the field w is tangent to the boundary, has zero harmonic part and satisfies the equation

$$\operatorname{curl} w = \lambda w + \lambda h, \quad \operatorname{div} w = 0. \tag{6.20}$$

We proved in Proposition 6.1 and Remark 6.2 that this equation has a unique solution if and only if $1/\lambda$ is not an eigenvalue of the operator K^* introduced in the proof of the aforementioned proposition. As λ is real, this is equivalent to $1/\lambda$ being eigenvalue of the operator K, which means that there is a divergence-free L^2 field u such that

$$Ku = \frac{u}{\lambda}.$$

Suppose that $1/\lambda$ is an eigenvalue of K. By the properties of K proved in Proposition 6.1, this means that the non-zero vector field Ku is tangent to the boundary, has zero harmonic part and satisfies

$$\operatorname{curl}(Ku) = \lambda Ku, \quad \operatorname{div}(Ku) = 0.$$

Applying the estimate proved in Proposition 6.7 to this equation, we get

$$\|Ku\|_{L^2_\varepsilon} \leqslant C\varepsilon|\lambda| \, \|Ku\|_{L^2_\varepsilon},$$

which means that

$$|\lambda| > \frac{C}{\varepsilon}$$

for some positive constant that does not depend on ε.

Therefore the problem (6.20) has a unique solution for all $|\lambda|\leqslant\Lambda$, provided ε is small enough. Moreover, when applied to equation (6.20), Proposition 6.7 ensures that

$$\|w\|_{H^k_\varepsilon} \leqslant C_k\varepsilon|\lambda|(\|w\|_{H^k_\varepsilon}+\|h\|_{H^k_\varepsilon}),$$

which yields

$$\|w\|_{H^k_\varepsilon} \leqslant C_k\varepsilon|\lambda|\,\|h\|_{H^k_\varepsilon}$$

provided that ε is smaller than some constant of the form C/Λ. Since the H^k_ε norm of the harmonic field is bounded uniformly in ε as

$$\|h\|_{H^k_\varepsilon} \leqslant C_k$$

due to equation (5.3) and the estimates in Theorem 5.1, we obtain

$$\|w\|_{H^k_\varepsilon} \leqslant C_k\varepsilon|\lambda|.$$

C^s pointwise estimates for w are immediately obtained from this bound by taking $k=s+2$ and using Sobolev embeddings. □

7. A KAM theorem for Beltrami fields with small $\boldsymbol{\lambda}$

Let us consider the harmonic field h in the tube $\mathcal{T}_\varepsilon$, as introduced in equation (5.3). As before, we will assume that the thickness ε is small. By Theorem 6.8, for any λ smaller in absolute value than some fixed ε-independent constant, there is a unique solution v to the Beltrami equation

$$\operatorname{curl} v = \lambda v$$

in $\mathcal{T}_\varepsilon$ that is tangent to the boundary and whose harmonic part is h. We are interested in the case where the Beltrami constant λ is suitably small. *For simplicity of notation, throughout this section we will take* $\lambda:=\varepsilon^3$ (*although we could have taken any non-zero constant* $\lambda=\mathcal{O}(\varepsilon^3)$) *and refer to the vector field* v *corresponding to this choice of* λ *as the local Beltrami field.*

Our objective in this section is to study some fine dynamical properties of the local Beltrami field v. More precisely, we will show that for small values of ε and "most" core curves γ, the boundary of the tube $\mathcal{T}_\varepsilon$ is an invariant torus of the field v that is preserved (i.e., there is a small perturbation of $\partial\mathcal{T}_\varepsilon$ that is still invariant) under suitably small perturbations of v. For this, we will see that the key point is the analysis of the harmonic field h, which is close to the local Beltrami field v as a consequence of the estimates proved in Theorem 6.8 and the fact that the Beltrami parameter $\lambda=\varepsilon^3$ is small. At this point, it is worth emphasizing that, to some extent, the proofs of the dynamical properties that we study in this section ultimately depend on the estimates derived in §4. In particular, if the dependence on ε of these estimates were worse, we would not be able to check the non-degeneracy conditions of the KAM theorem we prove in this section.

This section is divided into four parts. In §7.1 we consider the trajectories of the local Beltrami field, after a rescaling of the field (which does not alter their geometric structure), and calculate these trajectories perturbatively in a suitable range of time (Proposition 7.1). In §7.2 and §7.3 we use these perturbative expressions to compute the rotation number and normal torsion of the Poincaré map of the local Beltrami field (Theorems 7.4 and 7.8), which are the quantities that control the stability of the torus in the KAM theorem that we establish in §7.4 (Theorem 7.10).

7.1. Trajectories of the local Beltrami field

In this subsection we aim to compute the trajectories of the local Beltrami field v perturbatively in the small parameter ε. From Theorem 6.8 and the fact that the Beltrami parameter is $\lambda=\varepsilon^3$, it follows that the local Beltrami field v is close to the harmonic field in the sense that

$$\|v_\alpha - h_\alpha\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)} < C_k\varepsilon^4 \quad \text{and} \quad \|v_y - h_y\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)} < C_k\varepsilon^3. \tag{7.1}$$

Since the harmonic field can be written as $h=h_0+\nabla\psi$ (with h_0 and $\nabla\psi$ respectively given by (5.2) and (5.4)), from the estimates for ψ proved in Theorem 5.1 we infer that

$$v_\alpha = B^{-2}(1+\psi_\alpha+\tau\psi_\theta)+\mathcal{O}(\varepsilon^4) = 1+\mathcal{O}(\varepsilon).$$

In particular, as v_α does not vanish for small ε, we can consider the analytic vector field

$$X := \frac{v}{v_\alpha}. \tag{7.2}$$

We will use this vector field to study the geometric structure of the trajectories of the local Beltrami field, since both fields have the same unparameterized trajectories and the vector field X presents certain computational advantages, as we shall see in the next subsection. Before we go on, and identifying the tube $\mathcal{T}_\varepsilon$ with $\mathbb{S}^1_\ell\times\mathbb{D}^2$ through the coordinates (α, y), let us note that the field X is well defined in a small neighborhood of $\mathbb{S}^1_\ell\times\overline{\mathbb{D}^2}$ because so is the local Beltrami field v.

The trajectories of X are given by the parametrization $(\alpha(s), r(s), \theta(s))$, where these functions satisfy the system of ordinary differential equations (ODEs)

$$\dot{\alpha} = 1, \tag{7.3a}$$

$$\dot{r} = \frac{B^2 \psi_r}{\varepsilon^2(1+\psi_\alpha+\tau\psi_\theta)} + \mathcal{O}(\varepsilon^3), \tag{7.3b}$$

$$\dot{\theta} = \frac{\tau+(\varepsilon r)^{-2} A\psi_\theta+\tau\psi_\alpha}{1+\psi_\alpha+\tau\psi_\theta} + \mathcal{O}(\varepsilon^3). \tag{7.3c}$$

These equations can be read off from the definition of the field X and its connection with the harmonic field (equation (7.1)) and formulas (5.3) and (5.4) for the harmonic field and the gradient of ψ. The point of the trajectory not only depends on the "flow parameter" s, but also on the initial conditions $(\alpha_0, r_0, \theta_0)$ at $s=0$. Without loss of generality, in this section we will always take $\alpha_0=0$, and make the dependence of the trajectory on (r_0, θ_0) explicit by writing

$$(\alpha(s; r_0, \theta_0), r(s; r_0, \theta_0), \theta(s; r_0, \theta_0))$$

when appropriate.

Throughout, it will be convenient to denote by θ not only the angular coordinate in $\mathbb{S}^1$, but also its lift to the real line. It should be noticed that the formulas we will give below are actually valid for the lifted coordinate too, which will be of use in §7.2.

In the following lemma we will compute the trajectory of the field X at time $s\in[0,\ell]$ up to a controllable error. We will assume that r_0 is bounded away from 0 so that the trajectory cannot reach the coordinate singularity $\{r=0\}$ (i.e. $\{(\alpha, r, \theta): r=0\}$, by abuse of notation) at any time $s\in[0,\ell]$. This is convenient in view of the terms $1/r^2$ that appear in the equations and is not a restriction for the applications that we have in mind, as we will be only concerned with initial conditions near the invariant torus $\{r=1\}$. For simplicity, in this lemma we will abuse the notation and denote by $\mathcal{O}(\varepsilon^j)$ a quantity $Q(r_0, \theta_0, s)$ that is uniformly bounded as

$$|\partial_{r_0}^k \partial_{\theta_0}^l Q(r_0, \theta_0, s)| < C_{kl}\varepsilon^j$$

for r_0 in any fixed compact set of the interval $(0,1]$ (which is the domain where polar coordinates define a diffeomorphism), $\theta_0\in\mathbb{S}^1$ and $s\in[0,\ell]$, provided ε is small enough.

Proposition 7.1. *Consider the solution to the system (7.3) with initial condition $(0, r_0, \theta_0)$ and $r_0>0$. At time $s\in[0,\ell]$, this solution is given by*

$$\begin{aligned} \alpha(s; r_0, \theta_0) &= s, \\ r(s; r_0, \theta_0) &= r_0 + \mathcal{O}(\varepsilon), \\ \theta(s; r_0, \theta_0) &= \theta^{(0)}(s) + \varepsilon\theta^{(1)}(s) + \varepsilon^2\theta^{(2)}(s) + \mathcal{O}(\varepsilon^3), \end{aligned}$$

where each quantity $\theta^{(j)}(s) \equiv \theta^{(j)}(s; r_0, \theta_0)$ *is of order* $\mathcal{O}(1)$ *and given by*

$$\theta^{(0)}(s) := \theta_0 + \int_0^s \tau(\bar{s})\, d\bar{s},$$

$$\theta^{(1)}(s) := \frac{r_0^2 - 3}{8r_0}[\varkappa(s) \sin\theta^{(0)}(s) - \varkappa(0) \sin\theta_0],$$

$$\begin{aligned}\theta^{(2)}(s) := {} & \frac{12 - 5r_0^2}{32} \int_0^s \varkappa(\bar{s})^2 \tau(\bar{s})\, d\bar{s} + \frac{3(r_0^4 + 2r_0^2 - 3)\varkappa(s)\varkappa(0)}{64r_0^2} \cos\theta_0 \, \sin\theta^{(0)}(s) \\ & - \frac{(3 - r_0^2)^2 \varkappa(s)\varkappa(0)}{64r_0^2} \sin\theta_0 \cos\theta^{(0)}(s) + \frac{(27 - 50r_0^2 + 25r_0^4)\varkappa(s)^2}{384r_0^2} \sin 2\theta^{(0)}(s) \\ & + \frac{(27 + 14r_0^2 - 31r_0^4)\varkappa(0)^2}{384r_0^2} \sin 2\theta_0.\end{aligned}$$

Proof. Converting the ODEs (7.3) into integral equations, one has, for $s \in [0, \ell]$,

$$\alpha(s; r_0, \theta_0) = s, \tag{7.4a}$$

$$r(s; r_0, \theta_0) = r_0 + \int_0^s \frac{B^2 \psi_r}{\varepsilon^2(1 + \psi_\alpha + \tau\psi_\theta)}\, d\bar{s} + \mathcal{O}(\varepsilon^3), \tag{7.4b}$$

$$\theta(s; r_0, \theta_0) = \theta_0 + \int_0^s \frac{\tau + (\varepsilon r)^{-2} A\psi_\theta + \tau\psi_\alpha}{1 + \psi_\alpha + \tau\psi_\theta}\, d\bar{s} + \mathcal{O}(\varepsilon^3). \tag{7.4c}$$

In these equations all the functions under the integral signs are evaluated along the trajectories, i.e., at the point

$$\alpha = \bar{s}, \quad r = r(\bar{s}; r_0, \theta_0), \quad \theta = \theta(\bar{s}; r_0, \theta_0). \tag{7.5}$$

Let us solve the equations perturbatively. We start by noticing that, as a consequence of the bounds for ψ derived in Theorem 5.1 (and its connection with the functions φ_0 and φ_1 introduced in this theorem), the integrands can be expanded in ε as

$$\frac{\tau + (\varepsilon r)^{-2} A\psi_\theta + \tau\psi_\alpha}{1 + \psi_\alpha + \tau\psi_\theta} = \left(\tau + \frac{A\psi_\theta}{\varepsilon^2 r^2} + \tau\psi_\alpha\right)(1 - \psi_\alpha + \mathcal{O}(\varepsilon^3)) = \tau + \frac{A\psi_\theta}{\varepsilon^2 r^2} + \mathcal{O}(\varepsilon^3), \tag{7.6}$$

$$\frac{B^2\psi_r}{\varepsilon^2(1 + \psi_\alpha + \tau\psi_\theta)} = \frac{\partial_r \varphi_0}{\varepsilon^2} + \mathcal{O}(\varepsilon^2). \tag{7.7}$$

Since $r_0 > 0$ and $s \in [0, \ell]$ (which allows us to control the effect of the denominator $1/r^2$ for small enough ε), we immediately infer from equations (7.4b) and (7.4c) that

$$r(s; r_0, \theta_0) = r_0 + \mathcal{O}(\varepsilon) \quad \text{and} \quad \theta(s; r_0, \theta_0) = \theta^{(0)}(s) + \mathcal{O}(\varepsilon). \tag{7.8}$$

Of course, we define each function $\theta^{(j)}(s)$ as in the statement of the proposition.

Now that we have the zeroth-order expression of the trajectories, we will next compute them up to second-order corrections. For convenience we will use the notation

$$\mathcal{R}_0(r):=\tfrac{1}{2}(r^3-3r) \quad \text{and} \quad \mathcal{R}_1(r):=\tfrac{13}{96}(r^4-2r^2)$$

for the dependence of φ_0 and φ_1 on r, respectively. We start with the analysis of the radial coordinate of the trajectories. Using again equation (7.7) and the zeroth-order estimates for the trajectories (7.8), we derive that

$$\begin{aligned} r(s;r_0,\theta_0) &= r_0+\int_0^s\left[\frac{\partial_r\varphi_0(\bar{s},r_0+\mathcal{O}(\varepsilon),\theta^{(0)}(\bar{s})+\mathcal{O}(\varepsilon))}{\varepsilon^2}+\mathcal{O}(\varepsilon^2)\right]d\bar{s}+\mathcal{O}(\varepsilon^3)\\ &= r_0+\int_0^s\frac{\partial_r\varphi_0(\bar{s},r_0,\theta^{(0)}(\bar{s}))}{\varepsilon^2}\,d\bar{s}+\mathcal{O}(\varepsilon^2)\\ &= r_0+\varepsilon\mathcal{R}_0'(r_0)\int_0^s[\tau(\bar{s})\varkappa(\bar{s})\sin\theta^{(0)}(\bar{s})-\varkappa'(\bar{s})\cos\theta^{(0)}(\bar{s})]\,d\bar{s}+\mathcal{O}(\varepsilon^2)\\ &= r_0-\varepsilon\mathcal{R}_0'(r_0)\int_0^s\frac{d}{d\bar{s}}(\varkappa(\bar{s})\cos\theta^{(0)}(\bar{s}))\,d\bar{s}+\mathcal{O}(\varepsilon^2)\\ &= r_0+\varepsilon r^{(1)}(s)+\mathcal{O}(\varepsilon^2), \end{aligned} \tag{7.9}$$

where

$$r^{(1)}(s):=\tfrac{3}{8}(1-r_0^2)[\varkappa(s)\cos\theta^{(0)}(s)-\varkappa(0)\cos\theta_0].$$

To pass to the second line we have used the mean-value theorem and the obvious C^k bound $\varphi_0=\mathcal{O}(\varepsilon^3)$, and to complete the calculation we have just plugged in the formulas for φ_0 (equation (5.6)) and the definition of $\theta^{(0)}(s)$. In a totally analogous manner we can compute $\theta(s;r_0,\theta_0)$, up to $\mathcal{O}(\varepsilon^2)$,

$$\begin{aligned} \theta(s;r_0,\theta_0) &= \theta^{(0)}(s)+\int_0^s\left[\frac{A\psi_\theta}{(\varepsilon r)^2}+\mathcal{O}(\varepsilon^3)\right]\Bigg|_{(s,r_0+\mathcal{O}(\varepsilon),\theta^{(0)}(s)+\mathcal{O}(\varepsilon))}d\bar{s}+\mathcal{O}(\varepsilon^3)\\ &= \theta^{(0)}(s)+\int_0^s\frac{\partial_\theta\varphi_0(\bar{s},r_0,\theta^{(0)}(s))}{(\varepsilon r_0)^2}\,d\bar{s}+\mathcal{O}(\varepsilon^2)\\ &= \theta^{(0)}(s)+\frac{\varepsilon\mathcal{R}_0(r_0)}{r_0^2}\int_0^s[\tau(\bar{s})\varkappa(\bar{s})\cos\theta^{(0)}(\bar{s})+\varkappa'(\bar{s})\sin\theta^{(0)}(\bar{s})]\,d\bar{s}\\ &= \theta^{(0)}(s)+\varepsilon\theta^{(1)}(s)+\mathcal{O}(\varepsilon^2). \end{aligned} \tag{7.10}$$

To complete the proof we need to calculate $\theta(s;r_0,\theta_0)$ up to $\mathcal{O}(\varepsilon^3)$. The procedure is as above but the computations are more tedious. We start by noticing that the term $A\psi_\theta$ that appears in the integrand (7.6) can be written as

$$A\psi_\theta=\partial_\theta\varphi_0+(\partial_\theta\varphi_1-2\varepsilon\varkappa r\cos\theta\,\partial_\theta\varphi_0)+\mathcal{O}(\varepsilon^5), \tag{7.11}$$

where we have used the estimates we established in Theorem 5.1 (recall that φ_1 was introduced in equation (5.7)). Notice that the first summand is of order $\mathcal{O}(\varepsilon^3)$, while the term in brackets is of order $\mathcal{O}(\varepsilon^4)$.

We can now use the integral equation (7.4c) for the trajectories, the expansion (7.6) for the integrand (together with (7.11)), and the expression for the trajectories up to second-order terms (equations (7.9) and (7.10)) to get

$$\theta(s;r_0,\theta_0)=\theta^{(0)}(s)+\int_0^s \frac{\partial_\theta\varphi_0}{\varepsilon^2 r^2}\bigg|_{(\bar{s},r_0+\varepsilon r^{(1)}+\mathcal{O}(\varepsilon^2),\theta^{(0)}+\varepsilon\theta^{(1)}+\mathcal{O}(\varepsilon^2))} d\bar{s} \\ +\int_0^s \frac{\partial_\theta\varphi_1-2\varepsilon\varkappa r\cos\theta\,\partial_\theta\varphi_0}{\varepsilon^2 r^2}\bigg|_{(\bar{s},r_0+\varepsilon r^{(1)}+\mathcal{O}(\varepsilon^2),\theta^{(0)}+\varepsilon\theta^{(1)}+\mathcal{O}(\varepsilon^2))} d\bar{s}+\mathcal{O}(\varepsilon^3).$$

For simplicity of notation, we omit the argument $\bar{s}$ when there is no risk of confusion. An elementary Taylor expansion and the mean-value theorem show that the first integral (let us call it I_1) is given by

$$\begin{aligned} I_1&=\int_0^s \left[\frac{\partial_\theta\varphi_0}{\varepsilon^2 r^2}+\partial_r\left(\frac{\partial_\theta\varphi_0}{\varepsilon r^2}\right)r^{(1)}(\bar{s})+\frac{\partial_\theta^2\varphi_0}{\varepsilon r^2}\theta^{(1)}(\bar{s})\right]\bigg|_{(\bar{s},r_0,\theta^{(0)})} d\bar{s}+\mathcal{O}(\varepsilon^3)\\ &=\varepsilon\theta^{(1)}(s)+\int_0^s \left[\partial_r\left(\frac{\partial_\theta\varphi_0}{\varepsilon r^2}\right)r^{(1)}(\bar{s})+\frac{\partial_\theta^2\varphi_0}{\varepsilon r^2}\theta^{(1)}(\bar{s})\right]\bigg|_{(\bar{s},r_0,\theta^{(0)})} d\bar{s}+\mathcal{O}(\varepsilon^3). \end{aligned}$$

The second integral, which we call I_2, can be immediately simplified using the mean-value theorem, finding that

$$I_2=\int_0^s \frac{\partial_\theta\varphi_1-2\varepsilon\varkappa r\cos\theta\,\partial_\theta\varphi_0}{\varepsilon^2 r^2}\bigg|_{(\bar{s},r_0,\theta^{(0)})} d\bar{s}+\mathcal{O}(\varepsilon^3).$$

The integrals I_1 and I_2 can be computed in closed form after replacing the functions φ_0 and φ_1 by their expressions, given in equations (5.6) and (5.7). For example,

$$I_2=J_1+J_2+\mathcal{O}(\varepsilon^3),$$

where

$$\begin{aligned} J_1&:=\int_0^s \frac{\partial_\theta\varphi_1}{\varepsilon^2 r^2}\bigg|_{(\bar{s},r_0,\theta^{(0)})} d\bar{s}\\ &=\frac{\varepsilon^2\mathcal{R}_1(r_0)}{r_0^2}\int_0^s [\tau(\bar{s})\varkappa(\bar{s})^2\cos 2\theta^{(0)}(s)+\varkappa(\bar{s})\varkappa'(\bar{s})\sin 2\theta^{(0)}(\bar{s})]\,d\bar{s}\\ &=\frac{\varepsilon^2\mathcal{R}_1(r_0)}{2r_0^2}\int_0^s \frac{d}{d\bar{s}}(\varkappa(\bar{s})^2\sin 2\theta^{(0)}(\bar{s}))\,d\bar{s}\\ &=\frac{13\varepsilon^2(r_0^2-2)}{96}[\varkappa(s)^2\sin 2\theta^{(0)}(s)-\varkappa(0)^2\sin 2\theta_0] \end{aligned}$$

and

$$\begin{aligned}J_2 &:= -2\int_0^s \frac{\varkappa\cos\theta\,\partial_\theta\varphi_0}{\varepsilon r}\bigg|_{(\bar s, r_0,\theta^{(0)})} d\bar s\\ &= -\frac{2\varepsilon^2\mathcal{R}_0(r_0)}{r_0}\int_0^s \varkappa\cos\theta^{(0)}[\tau\varkappa\cos\theta^{(0)}+\varkappa'\sin\theta^{(0)}]\,d\bar s\\ &= -\frac{\varepsilon^2\mathcal{R}_0(r_0)}{r_0}\int_0^s\left[\tau\varkappa^2+\frac12\frac{d}{d\bar s}(\varkappa^2\sin 2\theta^{(0)})\right]d\bar s\\ &= \frac{\varepsilon^2(3-r_0^2)}{16}\left[2\int_0^s\tau\varkappa^2\,d\bar s+\varkappa(s)^2\sin 2\theta^{(0)}(s)-\varkappa(0)^2\sin 2\theta_0\right].\end{aligned}$$

The other terms can be dealt with using analogous arguments, arriving at the formula for $\theta^{(2)}(s)$ that appears in the statement. □

To conclude this subsection, we will show that the trajectories of the field X on the invariant torus $r=1$ satisfy certain functional equation up to some controllable errors. The reason why we need to consider this way of describing trajectories on the invariant torus is that, in order to compute the rotation number later on, we will need to understand the trajectories on the torus for arbitrarily large times. The expression for the trajectories we obtained in Proposition 7.1 is not well suited for this purpose, while the functional equation below turns out to be much more convenient. In order to describe the errors that appear in the functional equation, in the following proposition we will use the notation $s\mathcal{O}(\varepsilon^n)$ for any quantity $Q(\theta_0,s)$ that is bounded as

$$|\partial^j_{\theta_0}Q(\theta_0,s)|\leqslant C_j(1+|s|)\varepsilon^n \quad\text{and}\quad |\partial_s\partial^j_{\theta_0}Q(\theta_0,s)|\leqslant C_j\varepsilon^n$$

for non-negative integer j (we could have considered higher derivatives with respect to s too, but we will not need this feature).

Proposition 7.2. *Consider the trajectories of the system of ODEs (7.3) with initial condition $(\alpha_0,r_0,\theta_0)=(0,1,\theta_0)$. The function $\theta(s)\equiv\theta(s;1,\theta_0)$ satisfies the approximate functional equation*

$$\theta(s)=\theta_0+\int_0^s\tau(\bar s)\,d\bar s-\frac{\varepsilon}{4}[\varkappa(s)\sin\theta(s)-\varkappa(0)\sin\theta_0]+s\mathcal{O}(\varepsilon^2).$$

Proof. The starting point is the differential equations for the trajectories (7.3) with initial radius $r_0=1$. It is obvious that the radial component of the trajectory is

$$r(s;1,\theta_0)=1,$$

which simply shows that the set $\{r=1\}$ is an invariant torus. Therefore, equation (7.3c) for $\theta(s)$ becomes

$$\dot\theta(s)=\frac{\tau+\varepsilon^{-2}A\psi_\theta+\tau\psi_\alpha}{1+\psi_\alpha+\tau\psi_\theta}+\mathcal{O}(\varepsilon^3)=\tau+\frac{\varepsilon^{-2}A\psi_\theta-\tau^2\psi_\theta}{1+\psi_\alpha+\tau\psi_\theta}+\mathcal{O}(\varepsilon^3), \tag{7.12}$$

where the functions in the right-hand side are evaluated along the trajectories $(s,1,\theta(s))$. Expanding the fraction that appears in the second identity using the estimates for ψ proved in Theorem 5.1 we therefore arrive at

$$\dot{\theta}(s)=\tau(s)+\frac{\partial_\theta\varphi_0(s,1,\theta(s))}{\varepsilon^2}+\mathcal{O}(\varepsilon^2). \tag{7.13}$$

Notice that the fraction is of order $\mathcal{O}(\varepsilon)$. Converting (7.12) into an integral equation, an immediate consequence of this estimate is that

$$\theta(s)=\theta^{(0)}(s)+\int_0^s\frac{\partial_\theta\varphi_0(\bar{s},1,\theta(\bar{s}))}{\varepsilon^2}d\bar{s}+s\mathcal{O}(\varepsilon^2).$$

This integral can be evaluated, modulo $s\mathcal{O}(\varepsilon^2)$, using the formula (7.13) for the derivative $\dot{\theta}$, the expression for φ_0, and integration by parts, we obtain

$$\begin{aligned}\int_0^s\frac{\partial_\theta\varphi_0}{\varepsilon^2}d\bar{s}&=-\frac{\varepsilon}{4}\int_0^s[\varkappa\tau\cos\theta+\varkappa'\sin\theta]\,d\bar{s}=-\frac{\varepsilon}{4}\int_0^s[\varkappa(\dot{\theta}+\mathcal{O}(\varepsilon))\cos\theta+\varkappa'\sin\theta]\,d\bar{s}\\&=-\frac{\varepsilon}{4}\int_0^s\frac{d}{d\bar{s}}(\varkappa\sin\theta)\,d\bar{s}+s\mathcal{O}(\varepsilon^2)=-\frac{\varepsilon}{4}[\varkappa(s)\sin\theta(s)-\varkappa(0)\sin\theta_0]+s\mathcal{O}(\varepsilon^2).\end{aligned}$$

Here, of course, all the integrands are evaluated at the point $(\bar{s},1,\theta(\bar{s}))$. □

7.2. Rotation number of the Poincaré map of the local Beltrami field

We will denote by ϕ_s the time-s flow of the field X, which maps each point (α_0,r_0,θ_0) to the trajectory of the ODEs (7.3) at time s that has the latter values as initial conditions. Since the field X, introduced in equation (7.2), is tangent to the boundary of the domain $\mathbb{S}^1_\ell\times\mathbb{D}^2$, it is standard that the flow ϕ_s is a well-defined diffeomorphism of $\mathbb{S}^1_\ell\times\mathbb{D}^2$ for all values of s.

Let us now consider the Poincaré map of the field X, which is the tool we will use to analyze the dynamical properties of the flow (and which coincides with that of the local Beltrami field v). For this, we start by considering the section $\{\alpha=\alpha_0\}$, which is clearly transverse to the vector field X. The Poincaré map of this section, $\Pi_{\alpha_0}:\overline{\mathbb{D}^2}\to\overline{\mathbb{D}^2}$, sends each point $(r_0,\theta_0)\in\overline{\mathbb{D}^2}$ to the first point at which the trajectory $\phi_s(\alpha_0,r_0,\theta_0)$ intersects the section $\{\alpha=\alpha_0\}$ (with $s>0$). The reason why we are considering the field X is that it is *isochronous* in the sense that this first return point is given by the time-ℓ flow of X, that is,

$$\Pi_{\alpha_0}(r_0,\theta_0)=\phi_\ell(\alpha_0,r_0,\theta_0). \tag{7.14}$$

We will omit the subscript when $\alpha_0=0$, and use Cartesian coordinates y in the disk when convenient.

One should notice that, since we are assuming that the curve γ is analytic, the boundary of the tube $\mathcal{T}_\varepsilon$ is also an analytic surface, so it is standard [24] that the field v is analytic in a neighborhood of the closure $\overline{\mathcal{T}_\varepsilon}$. This ensures that the Poincaré map is also a well-defined analytic map in a neighborhood of the closed disk $\overline{\mathbb{D}^2}$.

In the following proposition we will show that the Poincaré map of the Beltrami field preserves a measure on the disk. For later convenience, we will state this result in terms of the associated 2-form rather than the measure.

PROPOSITION 7.3. *The Poincaré map* Π *preserves the positive measure on the disk corresponding to the* 2*-form*

$$\Lambda := G_\Lambda(r,\theta) r\, dr \wedge d\theta$$

on the disk $\mathbb{D}^2$, *with*

$$G_\Lambda(r,\theta) := Bv_\alpha|_{\alpha=0} = 1+\varepsilon\varkappa(0) r\cos\theta + \mathcal{O}(\varepsilon^2). \tag{7.15}$$

Proof. That the function $G_\Lambda(y) := Bv_\alpha|_{\alpha=0}$ has indeed the form given by the right-hand side of (7.15) is an immediate consequence of the estimates for the function ψ proved in Theorem 5.1 and equation (7.1). Given a Borel set $\mathcal{B}\subset\overline{\mathbb{D}^2}$ and a small positive δ, let us denote by

$$\mu(\mathcal{B}) := \int_{\mathcal{B}} \Lambda$$

its area and let

$$\mathcal{B}_\delta := (-\delta,\delta)\times\mathcal{B}$$

be a small thickening of the set $\{0\}\times\mathcal{B}$ in the closed domain $\mathbb{S}^1_\ell\times\overline{\mathbb{D}^2}$.

Since the divergence of v is zero, from the definition of X it stems that its flow preserves the volume

$$d\widetilde{V} := v_\alpha\, dV.$$

Clearly the $\widetilde{V}$-volume of the set $\mathcal{B}_\delta$ is

$$\begin{aligned}\widetilde{V}(\mathcal{B}_\delta) &:= \int_{\mathcal{B}_\delta} d\widetilde{V} = \int_{-\delta}^{\delta}\int_{\mathcal{B}} Bv_\alpha\, dy\, d\alpha \\ &= 2\delta(1+\mathcal{O}(\delta)) \int_{\mathcal{B}} Bv_\alpha|_{\alpha=0}\, dy = 2\delta\mu(\mathcal{B}) + \mathcal{O}(\delta^2).\end{aligned} \tag{7.16}$$

Let us now observe that the image of the set $\mathcal{B}_\delta$ under the time-ℓ flow ϕ_ℓ is given by

$$\phi_\ell(\mathcal{B}_\delta) = \bigcup_{-\delta<\alpha<\delta} \{\alpha\}\times\Pi_\alpha(\mathcal{B}). \tag{7.17}$$

By the continuous dependence of the flow on the initial conditions and equation (7.14), the Poincaré maps corresponding to different values of the angle α satisfy

$$\|\Pi_\alpha - \Pi\|_{C^0(\mathbb{D}^2)} \leqslant C\delta$$

for $|\alpha| \leqslant \delta$, so we can use the decomposition (7.17) to show that the $\widetilde{V}$-volume of $\phi_\ell(\mathcal{B}_\delta)$ is

$$\begin{aligned}\widetilde{V}(\phi_\ell(\mathcal{B}_\delta)) &= \int_{-\delta}^{\delta}\int_{\Pi_\alpha(\mathcal{B})} Bv_\alpha\,dy\,d\alpha \\ &= \int_{-\delta}^{\delta}\biggl(\int_{\Pi(\mathcal{B})} Bv_\alpha|_{\alpha=0}\,dy + \mathcal{O}(\delta)\biggr)\,d\alpha = 2\delta\mu(\Pi(\mathcal{B})) + \mathcal{O}(\delta^2).\end{aligned} \tag{7.18}$$

Equating the $\widetilde{V}$-volumes of $\mathcal{B}_\delta$ and $\phi_\ell(\mathcal{B}_\delta)$, given by equations (7.16) and (7.18), and considering small values of δ, we then obtain that

$$\mu(\mathcal{B}) = \mu(\Pi(\mathcal{B})),$$

as claimed. □

Since the local Beltrami field v is tangent to the boundary of the domain $\mathbb{S}^1_\ell \times \mathbb{D}^2$, the image of $\partial\mathbb{D}^2$ under the Poincaré map Π is also contained in $\partial\mathbb{D}^2$. Hence, the restriction of Π to $\partial\mathbb{D}^2$ defines an analytic diffeomorphism of the circle, which will be denoted by

$$\Pi|_{\partial\mathbb{D}^2} : \partial\mathbb{D}^2 \longrightarrow \partial\mathbb{D}^2.$$

Using the coordinate θ to identify the circle $\partial\mathbb{D}^2$ with $\mathbb{R}/2\pi\mathbb{Z}$, the latter circle diffeomorphism can be naturally lifted to a diffeomorphism of the real line that we will denote by $\overline{\Pi}: \mathbb{R} \to \mathbb{R}$. As is well known, a basic tool in the study of circle diffeomorphisms is the *rotation number* (or *frequency*) of the map, which is defined as

$$\omega_\Pi := \lim_{n\to\infty} \frac{\overline{\Pi}^n(\theta_0) - \theta_0}{n}. \tag{7.19}$$

Here $\overline{\Pi}^n$ denotes the nth iterate of $\overline{\Pi}$ and θ_0 is any real number. Since the Poincaré map of a flow is homotopic to the identity, it is standard that the above limit exists and is independent of the choice of θ_0 [30].

We shall next compute the rotation number of the circle diffeomorphism $\Pi|_{\partial\mathbb{D}^2}$ using the functional equation satisfied (up to controllable errors) by the trajectories of the field X on the invariant torus. The reason is that, in order to compute the rotation number to order $\mathcal{O}(\varepsilon^2)$, we need to iterate the Poincaré map an arbitrarily large number of times, which requires fine control of the growth of the errors for large times.

The following theorem asserts that the rotation number is given by the total torsion not only modulo $\mathcal{O}(\varepsilon)$, as can be shown without relying on the functional equation, but also modulo $\mathcal{O}(\varepsilon^2)$. The fact that the $\mathcal{O}(\varepsilon)$ correction is zero will be important later on.

THEOREM 7.4. *The rotation number of the circle diffeomorphism* $\Pi|_{\partial\mathbb{D}^2}$ *is*

$$\omega_\Pi = \int_0^\ell \tau(\alpha)\,d\alpha + \mathcal{O}(\varepsilon^2).$$

Proof. By the definition of the flow, equation (7.19) simply asserts that

$$\omega_\Pi := \lim_{n\to\infty} \frac{\theta(n\ell)-\theta_0}{n},$$

where $\theta(n\ell)\equiv\theta(n\ell;1,\theta_0)$ denotes the angular component of the trajectory solving the system (7.3) with initial condition $(0,1,\theta_0)$, evaluated at time $n\ell$. Since the curvature $\varkappa(\alpha)$ and torsion $\tau(\alpha)$ are ℓ-periodic, Proposition 7.2 then ensures that

$$\omega_\Pi = \lim_{n\to\infty} \frac{1}{n}\left(\int_0^{n\ell} \tau\,ds - \frac{\varepsilon\varkappa(0)}{4}[\sin\theta(n\ell)-\sin\theta_0] + n\ell\mathcal{O}(\varepsilon^2)\right) = \int_0^\ell \tau\,ds + \mathcal{O}(\varepsilon^2). \qquad \square$$

7.3. The non-degeneracy condition for the Poincaré map

In this subsection we will compute a quantity associated with the Poincaré map Π (sometimes called the normal torsion of the map) that was introduced to analyze the stability of individual invariant tori of symplectic diffeomorphisms [18], [17]. As we shall see, the assumption that the normal torsion is non-zero plays a role that is analogous to the twist condition in the classical theorem by Arnold and Moser on perturbations of integrable symplectic maps. As the name can be misleading, it is worth emphasizing that, in principle, the normal torsion has nothing to do with the torsion of a curve.

Let us begin by introducing some notation. We will consider a domain $\mathcal{D}$ in the plane that contains the closed unit disk $\overline{\mathbb{D}^2}$ and a map $\widehat{\Pi}\colon\mathcal{D}\to\mathbb{R}^2$. (Eventually, we will be interested in taking as $\widehat{\Pi}$ the Poincaré map Π introduced in the previous subsection.) A closed curve $\Gamma\subset\mathcal{D}$ is *invariant* if its image $\widehat{\Pi}(\Gamma)$ is contained in Γ. If Γ is an invariant curve of $\widehat{\Pi}$, one says that $\widehat{\Pi}|_\Gamma$ is *conjugate to a rotation* of frequency ω through the diffeomorphism $\Theta\colon\mathbb{S}^1\to\Gamma$ if

$$\Theta^{-1}\circ\widehat{\Pi}|_\Gamma\circ\Theta(\vartheta) = \vartheta+\omega$$

for all ϑ in $\mathbb{S}^1$. When $\Gamma=\partial\mathbb{D}^2$, we will abuse the notation and also denote by Θ the diffeomorphism $\mathbb{S}^1\to\mathbb{S}^1$ corresponding to the angular component of the above diffeomorphism $\mathbb{S}^1\to\partial\mathbb{D}^2$. (Therefore, in the case of $\partial\mathbb{D}^2$ the above diffeomorphism will read as $y=(\cos\Theta(\vartheta),\sin\Theta(\vartheta))$ in Cartesian coordinates and $(r,\theta)=(1,\Theta(\vartheta))$ in polar coordinates.) From the context it will be clear which interpretation of Θ must be considered in each case.

Definition 7.5. Let $\widehat{\Pi}:\overline{\mathbb{D}^2}\to\overline{\mathbb{D}^2}$ be a diffeomorphism of the disk that preserves the measure defined by the 2-form $G(r,\theta)r\,dr\wedge d\theta$. We will denote the radial and angular components of $\widehat{\Pi}$ by $(\widehat{\Pi}_r,\widehat{\Pi}_\theta)$, respectively. Assume that $\widehat{\Pi}|_{\partial\mathbb{D}^2}$ is a diffeomorphism of the circle $\partial\mathbb{D}^2\to\partial\mathbb{D}^2$ that is conjugate to a rotation of frequency ω through a diffeomorphism Θ, which we regard here as a map $\mathbb{S}^1\to\mathbb{S}^1$. The *normal torsion* of the map $\widehat{\Pi}$ on the invariant circle $\partial\mathbb{D}^2$ is the real number

$$\mathcal{N}_{\widehat{\Pi}}:=\int_0^{2\pi}\frac{1}{\Theta'(\vartheta+\omega)\Theta'(\vartheta)}\frac{\partial_r\widehat{\Pi}_\theta(1,\Theta(\vartheta))}{G(1,\Theta(\vartheta))}\,d\vartheta.$$

The reason why we consider the above quantity is that it appears in a non-degeneracy condition of a theorem by González-Enríquez and de la Llave [17]. In fact, the result [17, Theorem 1] is much more general, and we will only need a concrete application that we state next in a form that is particularly well suited for our purposes. The normal torsion was also considered for the same purpose by Herman in [18] when $G(r,\theta)=1$ and $\Theta(\vartheta)=\vartheta$. Before stating the theorem, let us recall that a number ω is *Diophantine* if there exist a positive constant C and $\nu>1$ such that

$$\left|\frac{\omega}{2\pi}-\frac{p}{k}\right|\geqslant\frac{C}{k^{1+\nu}} \tag{7.20}$$

for any integers p and k with $k\geqslant1$.

THEOREM 7.6. (González-Enríquez and de la Llave [17]) *Consider a small neighborhood $\mathcal{D}$ of the closed unit disk $\overline{\mathbb{D}^2}$ in $\mathbb{R}^2$. Take an analytic map $\widehat{\Pi}:\mathcal{D}\to\mathbb{R}^2$ that is a diffeomorphism onto its image preserving the measure $G(r,\theta)r\,dr\,d\theta$, with G analytic in $\mathcal{D}$. Suppose that the following two conditions hold*:

(i) *The circle $\partial\mathbb{D}^2$ is invariant, and $\widehat{\Pi}|_{\partial\mathbb{D}^2}$ is conjugate through an analytic diffeomorphism $\Theta:\mathbb{S}^1\to\partial\mathbb{D}^2$ to a rotation whose frequency ω satisfies a Diophantine condition.*

(ii) *The normal torsion $\mathcal{N}_{\widehat{\Pi}}$ of the map $\widehat{\Pi}$ on the invariant circle $\partial\mathbb{D}^2$ is non-zero.*

Then for each $\delta>0$ and positive integer m there are $\delta'>0$ and an integer k such that, if an analytic map $\widetilde{\Pi}:\mathcal{D}\to\mathbb{R}^2$ preserving the same measure $G(r,\theta)r\,dr\,d\theta$ satisfies

$$\|\widehat{\Pi}-\widetilde{\Pi}\|_{C^k(\mathcal{D})}<\delta',$$

one can transform the circle $\partial\mathbb{D}^2$ by a diffeomorphism Ψ of $\mathbb{R}^2$ so that $\Psi(\partial\mathbb{D}^2)$ is an invariant curve for the map $\widetilde{\Pi}$. Moreover, the map $\widetilde{\Pi}|_{\Psi(\partial\mathbb{D}^2)}$ is also conjugate to a rotation of frequency ω and the difference $\Psi-\mathrm{id}$ can be assumed to be supported in a small neighborhood of $\partial\mathbb{D}^2$ and to satisfy

$$\|\Psi-\mathrm{id}\|_{C^m}<\delta.$$

Proof. The statement is simply a rewording of [17, Theorem 47], in the particular case of planar maps and omitting some quantitative estimates that will not be needed in the rest of the paper. The only point that requires more elaboration is to check what the degeneracy condition looks like in the situation we are considering in this section.

For the benefit of the reader, let us give some details about how the statement is derived from [17, Theorem 47], borrowing some notation from this reference without further mention. The map $\widehat{\Pi}$ is obviously symplectic with respect to the analytic 2-form $\Lambda := G(r,\theta) r\, dr \wedge d\theta$. This 2-form is obviously exact, since $\Lambda = d\beta$ with the smooth 1-form on $\mathcal{D}$,

$$\beta := \left(\int_0^r G_\Lambda(\bar{r},\theta) \bar{r}\, d\bar{r} \right) d\theta.$$

Moreover, the 1-forms $\widehat{\Pi}^* \beta - \beta$ and $\widetilde{\Pi}^* \beta - \beta$ are exact because

$$\int_{\partial \mathbb{D}^2} (\widehat{\Pi}^* \beta - \beta) = \int_{\mathbb{D}^2} (d\widehat{\Pi}^* \beta - d\beta) = \int (\widehat{\Pi}^* \Lambda - \Lambda) = 0,$$

and analogously for $\widetilde{\Pi}$.

As before, let us now regard Θ as a diffeomorphism of $\mathbb{S}^1$. Consider the embedding $K: \mathbb{S}^1 \to \mathcal{D}$ given by

$$K(\vartheta) := (\cos \Theta(\vartheta), \sin \Theta(\vartheta)),$$

which is analytic by hypothesis.

Therefore, the only hypothesis of [17, Theorem 47] that is not immediate is the non-degeneracy condition. Let us take Cartesian components (y_1, y_2) in $\mathcal{D}$ and call $(\widehat{\Pi}_1, \widehat{\Pi}_2)$ the Cartesian components of the map $\widehat{\Pi}$. In the aforementioned reference, the condition is that the average of the function

$$S(\vartheta) := P(\vartheta + \omega) \cdot [D\widehat{\Pi}(\vartheta) J(\vartheta)^{-1} P(\vartheta)]$$

be non-zero, where the dot denotes the Euclidean scalar product. Here

$$P(\vartheta) := \Theta'(\vartheta)^{-1}(-\sin \Theta(\vartheta), \cos \Theta(\vartheta)),$$

$$(D\widehat{\Pi})_{ij}(\vartheta) := \frac{\partial \widehat{\Pi}_i}{\partial y_j}(K(\vartheta))$$

is the Jacobian matrix of $\widehat{\Pi}$ and

$$J(\vartheta) := G(K(\vartheta)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Expressing the Cartesian components in polar coordinates,

$$\widehat{\Pi}_1 = \widehat{\Pi}_r \cos \widehat{\Pi}_\theta \quad \text{and} \quad \widehat{\Pi}_2 = \widehat{\Pi}_r \sin \widehat{\Pi}_\theta,$$

we immediately obtain that

$$S(\vartheta)=\frac{\partial_r\widehat{\Pi}_r\sin(\widehat{\Pi}_\theta-\Theta(\vartheta+\omega))+\widehat{\Pi}_r\partial_r\widehat{\Pi}_\theta\cos(\widehat{\Pi}_\theta-\Theta(\vartheta+\omega))}{G\Theta'(\vartheta)\Theta'(\vartheta+\omega)},$$

where the functions whose argument has not been specified are evaluated at $K(\vartheta)$. Since $\widehat{\Pi}_r(K(\vartheta))=1$ because the circle $\partial\mathbb{D}^2$ is invariant and, by hypothesis, $\widehat{\Pi}|_{\partial\mathbb{D}^2}$ is conjugate to the rotation of frequency ω through the diffeomorphism Θ (i.e.,

$$\widehat{\Pi}_\theta(K(\vartheta))=\Theta(\vartheta+\omega)$$

for all ϑ), we get

$$S(\vartheta)=\frac{1}{\Theta'(\vartheta+\omega)\Theta'(\vartheta)}\frac{\partial_r\widehat{\Pi}_\theta(K(\vartheta))}{G(K(\vartheta))}.$$

In view of the way we defined the normal torsion (cf. Definition 7.5), the statement then follows immediately from [17, Theorem 47] after realizing that

$$\widehat{\Pi}(K(\vartheta))=K(\vartheta+\omega).$$ □

In order to calculate the normal torsion of the Poincaré map Π, let us begin by computing the diffeomorphism that conjugates $\Pi|_{\partial\mathbb{D}^2}$ to a rotation when its rotation number satisfies a Diophantine condition.

Proposition 7.7. *Suppose that the rotation number ω_Π of the Poincaré map of the local Beltrami field is Diophantine. Then the circle diffeomorphism $\Pi|_{\partial\mathbb{D}^2}$ is conjugate to a rotation of frequency ω_Π through an analytic diffeomorphism $\Theta:\mathbb{S}^1\to\partial\mathbb{D}^2$ that satisfies*

$$\Theta(\vartheta)=\vartheta-\tfrac{1}{4}\varepsilon\varkappa(0)\sin\vartheta+\mathcal{O}(\varepsilon^2).$$

Proof. Since the rotation number ω_Π satisfies a Diophantine condition, the map $\Pi|_{\partial\mathbb{D}^2}$ is conjugate to a rotation of frequency ω_Π through an analytic diffeomorphism Θ [30, Theorem 1.3]. This diffeomorphism can be understood as a change of coordinates $\theta_0=\Theta(\vartheta)$.

Let us now compute the diffeomorphism Θ. Writing $\theta_0=\Theta(\vartheta)$, the fact that $\widehat{\Pi}|_{\partial\mathbb{D}^2}$ is conjugate to a rotation of frequency ω_Π through Θ means that the trajectory $\theta(\ell;1,\Theta(\vartheta))$ at time ℓ corresponds to $\Theta(\vartheta+\omega_\Pi)$, for any choice of ϑ. Using the equation for the trajectory proved in Proposition 7.2 at time ℓ, this means that Θ must satisfy the equation

$$\Theta(\vartheta+\omega_\Pi)=\Theta(\vartheta)+\omega_\Pi+\tfrac{1}{4}\varepsilon\varkappa(0)[\sin\Theta(\vartheta)-\sin\Theta(\vartheta+\omega_\Pi)]+\mathcal{O}(\varepsilon^2). \tag{7.21}$$

Here we have used the expression for ω_Π proved in Theorem 7.4 (which allows us to replace $\int_0^\ell\tau\,d\bar{s}=\omega_\Pi+\mathcal{O}(\varepsilon^2)$) and that the function $\varkappa$ is ℓ-periodic.

To analyze this equation, let us write the $\mathcal{O}(\varepsilon^2)$ term in the right-hand side of (7.21) as $R(\vartheta)$. Let us identify $\partial\mathbb{D}^2$ with $\mathbb{S}^1=\mathbb{R}/2\pi\mathbb{Z}$ through the angular coordinate θ_0, so that Θ is regarded as a diffeomorphism $\mathbb{S}^1\to\mathbb{S}^1$. With a slight abuse of notation, let us still denote by Θ its lift $\mathbb{R}\to\mathbb{R}$. It is standard that this lift can be written as

$$\Theta(\vartheta)=\vartheta+H(\vartheta), \tag{7.22}$$

where $H(\vartheta)$ is an analytic 2π-periodic function.

Let us consider the 2π-periodic function

$$F(\vartheta):=H(\vartheta)+\tfrac{1}{4}\varepsilon\varkappa(0)\sin\Theta(\vartheta). \tag{7.23}$$

Equation (7.21) then reads as

$$F(\vartheta+\omega_\Pi)=F(\vartheta)+R(\vartheta), \tag{7.24}$$

with $R=\mathcal{O}(\varepsilon^2)$. Consider the Fourier series of the functions F and R,

$$F(\vartheta)=\sum_{k=-\infty}^{\infty}\widehat{F}_k e^{ik\vartheta} \quad\text{and}\quad R(\vartheta)=\sum_{k=-\infty}^{\infty}\widehat{R}_k e^{ik\vartheta}.$$

Equation (7.24) then asserts that for any non-zero integer k the Fourier coefficients of F and R are related through the identity

$$\widehat{F}_k=\frac{\widehat{R}_k}{e^{ik\omega_\Pi}-1}. \tag{7.25}$$

We can obviously take $\widehat{F}_0=0$; moreover, $\widehat{R}_0=0$ because it is a necessary condition for the existence of the diffeomorphism Θ.

Since ω_Π satisfies the Diophantine condition (7.20), for large integer values of k we have the elementary inequality

$$|e^{ik\omega_\Pi}-1|>C|k|^{-\nu},$$

so that from equation (7.25) the H^m norm of F can be estimated by

$$\|F\|_{H^m}\leqslant C\bigg(\sum_{k=-\infty}^{\infty}(1+k^2)^m|k|^{2\nu}|\widehat{R}_k|^2\bigg)^{1/2}\leqslant C\|R\|_{H^{m+\nu}}\leqslant C_m\varepsilon^2$$

for any non-negative integer m. To derive the last inequality, which shows that $F=\mathcal{O}(\varepsilon^2)$, we have used that $R=\mathcal{O}(\varepsilon^2)$. In view of equations (7.22) and (7.23), this ensures that

$$\Theta(\vartheta)=\vartheta-\tfrac{1}{4}\varepsilon\varkappa(0)\sin\Theta(\vartheta)+\mathcal{O}(\varepsilon^2).$$

In turn, this readily leads to the expression for the diffeomorphism Θ provided in the statement. □

We are ready to provide a closed formula for the normal torsion of the Poincaré map, up to terms of order $\mathcal{O}(\varepsilon^3)$. The leading term only depends on the geometry of the curve (through its curvature and torsion) and, as is to be expected, not on the section of $\mathbb{S}^1_\ell \times \mathbb{D}^2$ we used to define the Poincaré map.

THEOREM 7.8. *Suppose that the rotation number ω_Π satisfies a Diophantine condition. Then the normal torsion of the Poincaré map of the local Beltrami field v on the invariant circle $\partial\mathbb{D}^2$ is*

$$\mathcal{N}_\Pi = -\frac{5\pi\varepsilon^2}{8}\int_0^\ell \varkappa(\alpha)^2\tau(\alpha)\,d\alpha + \mathcal{O}(\varepsilon^3).$$

Proof. By definition, the normal torsion is

$$\mathcal{N}_\Pi := \int_0^{2\pi} \frac{1}{\Theta'(\vartheta+\omega_\Pi)\Theta'(\vartheta)}\frac{\partial_r\Pi_\theta(1,\Theta(\vartheta))}{G_\Lambda(1,\Theta(\vartheta))}\,d\vartheta,$$

where $\Pi_\theta(r,\theta)$ is the angular component of the Poincaré map, $\Theta(\vartheta)$ is the diffeomorphism defined in Proposition 7.7 (considered as a map $\mathbb{S}^1 \to \mathbb{S}^1$) and the function $G_\Lambda(r,\theta)$ is given by equation (7.15).

We have already computed all the terms we need to evaluate the integrand up to an $\mathcal{O}(\varepsilon^3)$ error. Indeed, from Propositions 7.1, 7.3 and 7.7 and Theorem 7.4 it stems that, setting $T := \int_0^\ell \tau(\alpha)\,d\alpha$,

$$\begin{aligned}
\omega_\Pi &= T + \mathcal{O}(\varepsilon^2),\\
\Theta'(\vartheta) &= 1 - \tfrac{1}{4}\varepsilon\varkappa(0)\cos\vartheta + \mathcal{O}(\varepsilon^2),\\
\Theta'(\vartheta+\omega_\Pi) &= 1 - \tfrac{1}{4}\varepsilon\varkappa(0)\cos(\vartheta+T) + \mathcal{O}(\varepsilon^2),\\
G_\Lambda(1,\Theta(\vartheta)) &= 1 + \varepsilon\varkappa(0)\cos\vartheta + \mathcal{O}(\varepsilon^2)\\
\frac{\partial\Pi^\theta}{\partial r}(1,\Theta(\vartheta)) &= \varepsilon\frac{\partial\theta^{(1)}}{\partial r_0}\left(\ell;1,\vartheta - \frac{\varepsilon\varkappa(0)}{4}\sin\vartheta\right) + \varepsilon^2\frac{\partial\theta^{(2)}}{\partial r_0}(\ell;1,\vartheta) + \mathcal{O}(\varepsilon^3),
\end{aligned}$$

the functions $\theta^{(j)}(s;r_0,\theta_0)$ being those in Proposition 7.1. Plugging these expressions into the integral for the normal torsion and using trigonometric identities, one arrives at the expression

$$\begin{aligned}
\mathcal{N}_\Pi &= \int_0^{2\pi}\bigg[\frac{\varepsilon\varkappa(0)}{2}(\sin(\vartheta+T)-\sin\vartheta) - \frac{5\varepsilon^2}{16}\int_0^{2\pi}\varkappa(\alpha)^2\tau(\alpha)\,d\alpha\\
&\qquad + \frac{5\varkappa^2\varepsilon^2}{48}\sin T\cos(2\vartheta+T) + \mathcal{O}(\varepsilon^3)\bigg]d\vartheta\\
&= -\frac{5\pi\varepsilon^2}{8}\int_0^{2\pi}\varkappa^2\tau\,d\alpha + \mathcal{O}(\varepsilon^3),
\end{aligned}$$

as claimed. ∎

7.4. A KAM theorem for generic tubes

In this subsection we will use the previous results to prove a theorem on the preservation of invariant tori for divergence-free vector fields that are close to the local Beltrami field v for small enough ε. As we shall see, the hypotheses of the KAM theorem will hold true as long as the core curve γ of the tube satisfies certain generic geometric conditions. Details on the validity of these conditions are given below.

Since the local Beltrami field v is analytic in a neighborhood of the closure of the tube, which we identify with $\mathbb{S}^1_\ell \times \mathbb{D}^2$ via the coordinates (α, y), we may assume that v is defined in some domain $\mathbb{S}^1_\ell \times \mathcal{D}$, with $\mathcal{D}$ being a neighborhood of the closed unit disk in the plane. To measure the smallness of a field, we will use the norm $\|u\|_{C^k(\mathbb{S}^1_\ell \times \mathcal{D})}$, which we define in terms of its components in the coordinates (α, y) as

$$\|u\|_{C^k(\mathbb{S}^1_\ell \times \mathcal{D})} := \|u_\alpha\|_{C^k(\mathbb{S}^1_\ell \times \mathcal{D})} + \|u_y\|_{C^k(\mathbb{S}^1_\ell \times \mathcal{D})}.$$

We will sometimes find it convenient to refer to a domain bounded by an invariant torus as an *invariant tube* of the field. We say that a field u is *orbitally conjugate to a rotation* of frequency ω on an invariant torus Σ if there are global coordinates $(\widetilde{\alpha}, \tilde{\theta}) \colon \Sigma \to \mathbb{S}^1 \times \mathbb{S}^1$ in which the vector field is linear with frequency ω up to a multiplicative factor, that is,

$$u|_\Sigma = F(\widetilde{\alpha}, \tilde{\theta})(\partial_{\widetilde{\alpha}} + \omega \partial_{\tilde{\theta}}),$$

where $F \colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ is a non-vanishing function.

In the following lemma we will show that, for a generic core curve γ, the rotation number of the Poincaré map of the local Beltrami field is Diophantine and its normal torsion is non-zero. To make precise what we understand by "generic", we will say that a certain property holds for a *C^m-dense set of closed analytic curves* if, given any closed analytic curve γ in $\mathbb{R}^3$, one can deform it by a diffeomorphism Φ of $\mathbb{R}^3$, with $\|\Phi - \mathrm{id}\|_{C^m(\mathbb{R}^3)}$ as small as one wishes, so that the curve $\Phi(\gamma)$ has the desired property. Notice that this does not imply that the property holds for an open set of curves.

LEMMA 7.9. *Let m be any positive integer. The set of closed analytic curves for which the Poincaré map of the local Beltrami field v has a Diophantine rotation number ω_Π and non-zero normal torsion $\mathcal{N}_\Pi$ on the invariant circle $\partial\mathbb{D}^2$ is C^m-dense.*

Proof. The result is not hard to prove using the expressions for the rotation number and the normal torsion derived in Theorems 7.4 and 7.8,

$$\omega_\Pi = \int_0^\ell \tau \, d\alpha + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \mathcal{N}_\Pi = -\frac{5\pi\varepsilon^2}{8} \int_0^\ell \varkappa^2 \tau \, d\alpha + \mathcal{O}(\varepsilon^3).$$

A way of making things precise is the following. We will consider deformations of the curve γ, labeled by a parameter δ. More concretely, let us denote by $(T(\alpha), N(\alpha), B(\alpha))$ the Frenet trihedron of the curve γ at the point $\gamma(\alpha)$ (one should not mistake the binormal vector $B(\alpha)$ for the function B that we introduced in equation (3.6), which will not be used in this proof). Let $F(\alpha)$ be an analytic ℓ-periodic function and consider the family of curves $\Gamma(\alpha,\delta)$ in $\mathbb{R}^3$ given by

$$\Gamma(\alpha,\delta) := \gamma(\alpha) + \delta F(\alpha) B(\alpha).$$

If δ is close to zero, $\gamma_\delta \equiv \Gamma(\cdot,\delta)$ is a closed analytic curve. Notice that, for $\delta \neq 0$, α is no longer an arc-length parametrization of γ_δ but, due to the properties of the binormal field,

$$|\Gamma'| = 1 + \mathcal{O}(\delta^2).$$

Here and in what follows, we denote by a prime ($'$) the derivatives with respect to α. In particular, the length of γ_δ is $\ell + \mathcal{O}(\delta^2)$. We will label the geometric quantities associated with the curve γ_δ with a subscript δ (e.g., $\varkappa_\delta$ and τ_δ for its curvature and torsion).

The results we have presented in this section carry over immediately when one does not only consider the tube $\mathcal{T}_\varepsilon$ associated with the curve γ, but the family of tubes $\mathcal{T}_\varepsilon(\gamma_\delta)$. The dependence of the various quantities on the small parameter δ is smooth and can be controlled easily. A tedious but straightforward computation using the well-known formulas for the curvature and torsion of a parameterized curve shows that

$$\begin{aligned}
\varkappa_\delta^2 &= \varkappa^2 + 2\delta\varkappa(2F'\tau + F\tau') + \mathcal{O}(\delta^2),\\
\tau_\delta &= \tau - \delta\left[\varkappa F' + \frac{d}{d\alpha}\left(\frac{F'' - \tau^2 F}{\varkappa}\right)\right] + \mathcal{O}(\delta^2).
\end{aligned}$$

Since the length of γ_δ differs from ℓ by an $\mathcal{O}(\delta^2)$ term, one readily finds that the rotation number of the Poincaré map Π_δ associated with the harmonic field of the tube $\mathcal{T}_\varepsilon(\gamma_\delta)$ is

$$\omega_{\Pi_\delta} = \int_0^\ell \tau_\delta(\alpha)\, d\alpha + \mathcal{O}(\varepsilon^2 + \delta^2) = \omega_\Pi + \delta \int_0^\ell \varkappa'(\alpha) F(\alpha)\, d\alpha + \mathcal{O}(\varepsilon^2 + \delta^2).$$

Similarly, the dependence of the normal torsion on δ can be shown to be

$$\mathcal{N}_{\Pi_\delta} = \mathcal{N}_\Pi - \frac{5\pi\delta\varepsilon^2}{8} \int_0^\ell (2\varkappa''' + 3\varkappa^2\varkappa' - 6\varkappa\tau\tau' - 6\tau^2\varkappa') F\, d\alpha + \mathcal{O}(\delta^2 + \varepsilon^3).$$

We recall that, although the Poincaré map Π_δ depends smoothly on the parameter δ, the terms $\mathcal{O}(\varepsilon^2 + \delta^2)$ and $\mathcal{O}(\delta^2 + \varepsilon^3)$ are continuous, but possibly not differentiable, functions of δ.

Perturbing the curve γ a little if necessary to ensure that the functions $\varkappa'$ and

$$2\varkappa'''+3\varkappa^2\varkappa'-6\varkappa\tau\tau'-6\tau^2\varkappa'$$

are not identically zero, we deduce that the function F can be chosen so that the integrals

$$\int_0^\ell \varkappa' F\, d\alpha \quad \text{and} \quad \int_0^\ell (2\varkappa'''+3\varkappa^2\varkappa'-6\varkappa\tau\tau'-6\tau^2\varkappa')F\, d\alpha$$

are non-zero. For small enough ε this ensures that, as δ takes values in a small enough interval $(-\delta_0,\delta_0)$, the values taken by the continuous function ω_{Π_δ} (resp. $\mathcal{N}_{\Pi_\delta}$) cover an interval centered at ω_Π (resp. $\mathcal{N}_\Pi$) of radius $C\delta_0$ (resp. $C\varepsilon^2\delta_0$). Since Diophantine numbers have full Lebesgue measure, this immediately implies that one can choose an arbitrarily small δ such that the rotation number ω_{Π_δ} satisfies a Diophantine condition and the normal torsion $\mathcal{N}_{\Pi_\delta}$ is non-zero. □

We can now show that, for a generic core curve γ and small enough ε, a suitably small perturbation of the local Beltrami field still has an invariant torus that is close to the original one.

THEOREM 7.10. *For any positive integer m there is a C^m-dense set of closed analytic curves γ with the following* KAM-*type property: for any $\delta>0$, there is another positive integer k and some $\delta'>0$ such that any analytic divergence-free vector field u whose difference with the local Beltrami field v of the tube $\mathcal{T}_\varepsilon\equiv\mathcal{T}_\varepsilon(\gamma)$ is bounded by*

$$\|u-v\|_{C^k(\mathbb{S}^1_\ell\times\mathcal{D})}<\delta' \tag{7.26}$$

possesses an invariant tube. Furthermore, one can find a diffeomorphism Ψ of $\mathbb{R}^3$, with $\|\Psi-\mathrm{id}\|_{C^m(\mathbb{R}^3)}<\delta$ and $\Psi-\mathrm{id}$ supported in a small neighborhood of the torus $\partial\mathcal{T}_\varepsilon$, such that $\Psi(\mathcal{T}_\varepsilon)$ is an invariant tube of u, and u is orbitally conjugate on the invariant torus $\partial\Psi(\mathcal{T}_\varepsilon)$ to a Diophantine rotation.

Proof. By Lemma 7.9, we can deform the curve that lies at the core of the tube $\mathcal{T}_\varepsilon$ by a diffeomorphism of $\mathbb{R}^3$, arbitrarily close to the identity in the C^m norm, so that, if we consider the local Beltrami field (which we still denote by v) associated with the deformed tube, its rotation number ω_Π satisfies a Diophantine condition and its normal torsion $\mathcal{N}_\Pi$ is non-zero. As before, we will identify this deformed tube with the domain $\mathbb{S}^1_\ell\times\mathbb{D}^2$ through adapted coordinates (α,y).

Consider the Poincaré map Π of the local Beltrami field v, which can be safely considered as a diffeomorphism Π from $\mathcal{D}$ onto its image, with $\mathcal{D}$ being a neighborhood of the closed unit disk $\overline{\mathbb{D}^2}$. The Poincaré map of the field u, also defined on the section

$\{\alpha=0\}$, is another diffeomorphism of $\mathcal{D}$ onto its image that we will denote by $\widetilde{\Pi}$. The vector fields u and v being close by (7.26), it is apparent that

$$\|\Pi-\widetilde{\Pi}\|_{C^k(\mathcal{D})}<C\delta' \tag{7.27}$$

as the Poincaré map is simply obtained by integrating the associated vector field along a trajectory between two consecutive intersections with the section $\{\alpha=0\}$. To avoid cumbersome notation related to the intersection of the domains of auxiliary maps, wherever appropriate we will assume that the Poincaré maps are defined in a domain slightly larger than $\mathcal{D}$ without further mention. Obviously there is no loss of generality in this assumption.

We have seen in Proposition 7.3 that the Poincaré map Π preserves the 2-form $\Lambda=G_\Lambda(r,\theta)r\,dr\wedge d\theta$, where

$$G_\Lambda(r,\theta):=Bv_\alpha|_{\alpha=0}=1+\mathcal{O}(\varepsilon),$$

and v_α denotes the α-component of the local Beltrami field v. Mimicking the proof of this proposition, we immediately obtain that the Poincaré map $\widetilde{\Pi}$ of the divergence-free field u preserves the 2-form $\tilde{\Lambda}:=G_{\tilde{\Lambda}}(r,\theta)r\,dr\wedge d\theta$, with

$$G_{\tilde{\Lambda}}(r,\theta):=Bu_\alpha|_{\alpha=0}.$$

Notice that u_α does not vanish in $\{0\}\times\mathcal{D}$ because the difference $|u_\alpha-v_\alpha|$ is small and the α-component of the local Beltrami field is close to 1.

Our next goal is to relate the above invariant 2-forms to apply Theorem 7.6. More concretely, we will show that there is a C^m-small diffeomorphism Φ such that

$$\Lambda=\Phi^*\tilde{\Lambda}, \tag{7.28}$$

where Φ^* is the pullback of the diffeomorphism. This will be done using Moser's trick. We start by noticing that the difference between these 2-forms is obviously exact, as

$$\Lambda-\tilde{\Lambda}=d\Gamma$$

with the 1-form Γ given by

$$\Gamma:=\left(\int_0^r[G_\Lambda(\bar{r},\theta)-G_{\tilde{\Lambda}}(\bar{r},\theta)]\bar{r}\,d\bar{r}\right)d\theta.$$

Although we are making computations in polar coordinates, it is readily seen that all the objects we are considering are well defined also at the origin, and therefore determine smooth forms in the whole disk $\mathcal{D}$.

Consider the non-autonomous vector field of class C^∞

$$Z_s := \frac{\int_0^r [G_\Lambda(\bar r,\theta) - G_{\tilde\Lambda}(\bar r,\theta)]\bar r\, d\bar r}{r[(1-s)G_\Lambda(r,\theta) + sG_{\tilde\Lambda}(r,\theta)]}\partial_r,$$

where s will be the flow parameter. It should be noticed that, by the assumptions on the vector fields, the denominator behaves as $r+\mathcal{O}(r^2)$ while the numerator is of order $\mathcal{O}(r^2)$. The field Z_s satisfies the C^m bound

$$\|Z_s\|_{C^m(\mathcal{D})} < C\delta' \tag{7.29}$$

for all $s\in[0,1]$ as a consequence of the estimate (7.26).

The time-s flow of the non-autonomous field Z_s, which will be denoted by ϕ_s, is given by the solution to the initial value problem

$$\frac{\partial}{\partial s}\phi_s x = Z_s(\phi_s x), \quad \phi_0 x = x.$$

Consider the s-dependent 2-form

$$\Lambda_s := (1-s)\Lambda + s\tilde\Lambda.$$

A simple computation shows that

$$\frac{\partial}{\partial s}(\phi_s^*\Lambda_s) = \phi_s^*\left(\frac{\partial}{\partial s}\Lambda_s + L_{Z_s}\Lambda_s\right) = \phi_s^*(\tilde\Lambda - \Lambda + d i_{Z_s}\Lambda_s) = 0, \tag{7.30}$$

where L_{Z_s} denotes the Lie derivative along Z_s and the last equality follows immediately from the definition of Λ_s and the fact that the interior product of Z_s with Λ_s is

$$i_{Z_s}\Lambda_s = \Gamma.$$

Thus, if we set $\Phi:=\phi_1$, we obtain equation (7.28) from (7.30) and the definition of Λ_s. Moreover,

$$\|\Phi - \mathrm{id}\|_{C^m(\mathcal{D})} < C\delta' \tag{7.31}$$

because Φ is the time-1 flow of the vector field Z_s, whose C^m norm is controlled by equation (7.29).

Let us now consider the map

$$\widehat\Pi := \Phi^{-1}\circ\widetilde\Pi\circ\Phi,$$

which is a diffeomorphism from a neighborhood of $\overline{\mathbb{D}^2}$ (which we still take as $\mathcal{D}$) onto its image. We shall next relate the new map $\widehat\Pi$ to the Poincaré map Π. By the definition of

Φ and the relation between the invariant 2-forms (7.28), the map $\widehat{\Pi}$ preserves the same 2-form Λ as the Poincaré map Π, and $\widehat{\Pi}$ is close to Π by (7.31):

$$\|\Pi-\widehat{\Pi}\|_{C^m(\mathcal{D})} < C\delta'. \tag{7.32}$$

We are now ready to apply Theorem 7.6 with the maps Π and $\widehat{\Pi}$. Indeed, the previous arguments and the way we have deformed the curve γ ensure that the following statements hold true:

(i) The rotation number ω_Π satisfies a Diophantine condition, so the map $\Pi|_{\partial\mathbb{D}^2}$ is analytically conjugate to a rotation of frequency ω_Π (cf. Proposition 7.7).

(ii) The normal torsion $\mathcal{N}_\Pi$ is non-zero.

(iii) The C^k norm of $\Pi-\widehat{\Pi}$ is at most $C\delta'$ by equation (7.32), with δ' arbitrarily small.

Hence Theorem 7.6 ensures that, for any given integer m, if the integer k is large enough and δ' is sufficiently small there is a diffeomorphism $\hat{\Psi}$ of $\mathcal{D}$ with $\hat{\Psi}-\mathrm{id}$ arbitrarily small in the $C^m(\mathcal{D})$ norm and supported in a neighborhood of $\partial\mathbb{D}^2$ such that the curve $\hat{\Psi}(\partial\mathbb{D}^2)$ is invariant under the map $\widehat{\Pi}$. Furthermore, the restriction of $\widehat{\Pi}$ to this invariant curve, $\hat{\Pi}|_{\hat{\Psi}(\partial\mathbb{D}^2)}$, also has rotation number ω_Π.

The definition of $\widehat{\Pi}$ then ensures that the curve $\Phi\circ\hat{\Psi}(\partial\mathbb{D}^2)$ is invariant under the Poincaré map $\widetilde{\Pi}$ of the field u. (As a side remark, notice that this invariant curve is close to $\partial\mathbb{D}^2$ but, in principle, is not contained in the closure of $\mathbb{D}^2$, which is the reason why we are considering a slightly larger disk $\mathcal{D}$ throughout the proof.) It is standard that this is equivalent to saying that there is an invariant torus of u whose intersection with the disk $\{\alpha=0\}$ is precisely the aforementioned curve. Since the $C^m(\mathcal{D})$ norm of the diffeomorphism $\Phi\circ\hat{\Psi}$ is arbitrarily small by the properties of $\hat{\Psi}$ and equation (7.31), one can take a diffeomorphism $\bar{\Psi}$ of $\mathbb{S}^1_\ell\times\mathcal{D}$ such that $\bar{\Psi}(\mathbb{S}^1_\ell\times\partial\mathbb{D}^2)$ is an invariant torus of the field u and $\bar{\Psi}-\mathrm{id}$ is small in the C^m norm and is supported in a neighborhood of $\mathbb{S}^1_\ell\times\mathbb{D}^2$. Indeed, this diffeomorphism can be defined as follows. Take the solution to the system of ODEs

$$\frac{dr}{d\alpha}=\frac{u_r(\alpha,r,\theta)}{u_\alpha(\alpha,r,\theta)} \quad \text{and} \quad \frac{d\theta}{d\alpha}=\frac{u_\theta(\alpha,r,\theta)}{u_\alpha(\alpha,r,\theta)},$$

with the trajectory (r,θ) parametrized by the angle α and depending on the initial conditions (r_0,θ_0). Consider the function φ_α mapping the initial conditions (r_0,θ_0) to its time-α flow $(r(\alpha;r_0,\theta_0),\theta(\alpha;r_0,\theta_0))$. Then it is easy to check that the diffeomorphism $\bar{\Psi}$ is given, in polar coordinates, by

$$\bar{\Psi}(\alpha,r,\theta):=(\alpha,\varphi_\alpha\circ\Phi\circ\hat{\Psi}(r,\theta)).$$

Actually, this formula simply asserts that the intersection of the invariant torus of the field u with each section $\{\alpha=\alpha_0\}$, understood as a curve in the disk $\mathcal{D}$, is the image under φ_{α_0} of the invariant curve $\Phi\circ\hat{\Psi}(\partial\mathbb{D}^2)$ at $\{\alpha=0\}$. Clearly the formula for $\bar{\Psi}$ and the estimates for the maps Φ and $\hat{\Psi}$ imply that

$$\|\bar{\Psi}-\mathrm{id}\|_{C^m(\mathbb{S}^1_\ell\times\mathcal{D})}\leqslant C\|\Phi\circ\hat{\Psi}-\mathrm{id}\|_{C^m(\mathcal{D})}$$

can be made arbitrarily small.

To complete the proof of the theorem, it suffices to recall that the map $\widehat{\Pi}|_{\hat{\Psi}(\partial\mathbb{D}^2)}$ has rotation number ω_Π. This implies that the Poincaré map $\widetilde{\Pi}$ of the vector field u, restricted to the invariant curve $\Phi\circ\hat{\Psi}(\partial\mathbb{D}^2)$, also has rotation number ω_Π, which trivially implies that the vector field u itself is orbitally conjugate to a rotation of frequency ω_Π on the invariant torus $\bar{\Psi}(\mathbb{S}^1_\ell\times\partial\mathbb{D}^2)$. The existence of the diffeomorphism Ψ of $\mathbb{R}^3$ that appears in the statement of the theorem is then immediate. □

Remark 7.11. Ultimately, Theorem 7.10 is a result on the preservation of invariant tori for small perturbations of the harmonic field h, rather than of the local Beltrami field v. Indeed, if we had considered a Beltrami field with small but otherwise arbitrary parameter λ (possibly 0), we would have found the same expressions for the rotation number and normal torsion of its Poincaré map, the only change being that the error terms would be $\mathcal{O}(\varepsilon^2+|\lambda|)$ and $\mathcal{O}(\varepsilon^3+|\lambda|)$, respectively.

We will conclude this section with a result on the persistence of another invariant set: we will show that, for an open and dense set of core curves γ and small enough ε, any divergence-free vector field that is a small perturbation of the local Beltrami field v has an elliptic periodic trajectory close to the core curve γ. We recall that a periodic trajectory is *elliptic* if the non-trivial eigenvalues of the associated monodromy matrix all have unit modulus but are different from 1.

PROPOSITION 7.12. *Suppose that the total torsion of the curve γ satisfies*

$$\int_0^\ell \tau(\alpha)\,d\alpha\neq n\pi$$

for all integers n. Then for any $\delta>0$, there exists some $\delta'>0$ such that any divergence-free vector field u in the tube $\mathcal{T}_\varepsilon$ which is close to the local Beltrami field v in the sense that

$$\|u-v\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<\delta'$$

also has an elliptic periodic trajectory diffeomorphic to the curve $\{y=0\}$. Moreover, the corresponding diffeomorphism Ψ is bounded by

$$\|\Psi-\mathrm{id}\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<\delta$$

and is different from the identity only in a small neighborhood of the curve $\{y=0\}$.

Proof. From the expressions (5.2)–(5.4) for the harmonic field h, the estimates for the function ψ proved in Theorem 5.1 and the connection between the local Beltrami field and h (see equation (7.1)), we infer that the difference $v-\tilde{h}_0$ can be estimated as

$$\|v-\tilde{h}_0\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<C_k\varepsilon,$$

with the vector field $\tilde{h}_0$ defined as

$$\tilde{h}_0:=\partial_\alpha+\tau(\alpha)(y_1\partial_2-y_2\partial_1).$$

It is clear that $(\alpha(s),y(s))=(s,0)$ is an ℓ-periodic trajectory of the field $\tilde{h}_0$. Setting

$$T:=\int_0^\ell\tau(\alpha)\,d\alpha,$$

an easy computation shows that the monodromy matrix of this trajectory for the field $\tilde{h}_0$ is

$$\begin{pmatrix}1&0&0\\0&\cos T&-\sin T\\0&\sin T&\cos T\end{pmatrix}.$$

The non-trivial eigenvalues of this matrix are $e^{\pm iT}$, and hence different from ±1 by the hypotheses of the theorem, thus showing that $\{y=0\}$ is an elliptic trajectory of $\tilde{h}_0$.

It is then standard that any divergence-free field u that is close enough to $\tilde{h}_0$ (say, $\|u-\tilde{h}_0\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<\delta_1$)) has an elliptic periodic trajectory given by the image of $\{y=0\}$ under a diffeomorphism Ψ with $\|\Psi-\mathrm{id}\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<\delta$. There is no loss of generality in assuming that $\Psi-\mathrm{id}$ is supported in a small neighborhood of $\{y=0\}$. Since

$$\|u-\tilde{h}_0\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<\|u-v\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}+\|v-\tilde{h}_0\|_{C^k(\mathbb{S}^1_\ell\times\mathbb{D}^2)}<\delta'+C_k\varepsilon,$$

the theorem then follows by taking ε and δ' small enough. □

8. Approximation by Beltrami fields with decay

In this section we prove a result that allows us to approximate a field v that satisfies the Beltrami equation

$$\operatorname{curl} v=\lambda v$$

on a neighborhood of a compact set S, by a global Beltrami field u, which satisfies

$$\operatorname{curl} u=\lambda u$$

in the whole space $\mathbb{R}^3$ and falls off at infinity as $1/|x|$. Throughout we will assume that the complement $\mathbb{R}^3\backslash S$ is a connected set. It is not hard to see that this condition is necessary.

It will be more convenient for us to work with an auxiliary elliptic equation instead of considering the Beltrami equation directly. To this end, let us denote by

$$G(x) := \frac{\cos\lambda|x|}{4\pi|x|}$$

the Green function of the operator $\Delta+\lambda^2$ in $\mathbb{R}^3$, which satisfies the distributional equation

$$\Delta G+\lambda^2 G=-\delta_0,$$

with the Dirac measure δ_0 supported at 0. We will use the notation $\mathbb{B}_R$ for the ball in $\mathbb{R}^3$ centered at the origin and of radius R.

The following lemma, which shows how to "sweep" the singularities of the Green function, will be used in the demonstration of the global approximation theorem. Its proof is based on a duality argument and the Hahn–Banach theorem.

LEMMA 8.1. *Take $R>0$ and consider a domain $U\subset\mathbb{R}^3\backslash\mathbb{B}_{2R}$ and a compact set $S\subset\mathbb{B}_R$ whose complement $\mathbb{R}^3\backslash S$ is connected. Let us consider the vector field*

$$v(x) := \sum_{m=1}^{M} \varrho_m G(x-x_m),$$

where $\{x_m\}_{m=1}^M$ is a finite set of points in $\mathbb{B}_R\backslash S$ and $\varrho_m\in\mathbb{R}^3$ are constant vectors. Then, for any $\delta>0$, there is a finite set of points $\{z_j\}_{j=1}^J$ in the domain U and constant vectors $c_j\in\mathbb{R}^3$ such that the finite linear combination

$$w(x) := \sum_{j=1}^{J} c_j G(x-z_j) \tag{8.1}$$

approximates the field v uniformly in S as

$$\|v-w\|_{C^0(S)}<\delta.$$

Proof. Consider the space $\mathcal{U}$ of all vector fields that are linear combinations of the form (8.1), where the points z_j belong to the set U and the coefficients $c_j\in\mathbb{R}^3$ are constant vectors. Restricting these fields to the set S, $\mathcal{U}$ can be regarded as a subspace of the Banach space $C^0(S,\mathbb{R}^3)$ of continuous vector fields on S.

By the Riesz–Markov theorem, the dual of $C^0(S,\mathbb{R}^3)$ is the space $\mathcal{M}(S,\mathbb{R}^3)$ of the finite vector-valued Borel measures on $\mathbb{R}^3$ whose support is contained in the set S. Let us take any measure $\mu\in\mathcal{M}(S,\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} f\cdot d\mu=0$ for all $f\in\mathcal{U}$. Let us now define a field $F\in L^1_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3)$ as

$$F(x):=\int_{\mathbb{R}^3} G(x-\tilde{x})\,d\mu(\tilde{x}),$$

so that F satisfies the equation

$$\Delta F+\lambda^2 F=-\mu.$$

Notice that F is identically zero on the open set U by the definition of the measure μ, that $\mathbb{R}^3\backslash S$ is connected and that F satisfies the elliptic equation

$$\Delta F+\lambda^2 F=0$$

in $\mathbb{R}^3\backslash S$. Hence the unique continuation theorem ensures that the field F vanishes on the complement of S. It then follows that the measure μ also annihilates any field of the form $\varrho_m G(x-x_m)$ because, as the points x_m do not belong to S,

$$0=F(x_m)\cdot\varrho_m=\int_{\mathbb{R}^3} G(x-x_m)\,\varrho_m\cdot d\mu(x).$$

Therefore

$$\int_{\mathbb{R}^3} v\cdot d\mu=0,$$

which implies that v can be uniformly approximated on S by elements of the subspace $\mathcal{U}$ as a consequence of the Hahn–Banach theorem. The lemma then follows. □

As an intermediate step before proving the global approximation result for the Beltrami equation, we will establish the following proposition on the approximation of solutions to the elliptic equation $\Delta v=-\lambda^2 v$ by solutions defined in a large ball. Throughout, we will say that a differential equation holds in a closed set if it holds in a neighborhood of this set.

PROPOSITION 8.2. *Let v be a vector field which satisfies the equation*

$$\Delta v=-\lambda^2 v \tag{8.2}$$

in a compact subset S of $\mathbb{R}^3$. Assume that its complement $\mathbb{R}^3\backslash S$ is connected and that S is contained in the ball $\mathbb{B}_R$. Then for any $\delta>0$ and any positive integer k there is a vector field w satisfying the equation

$$\Delta w=-\lambda^2 w$$

in $\mathbb{B}_R$ that approximates the field v in S as

$$\|v-w\|_{C^k(S)}<\delta. \tag{8.3}$$

Here δ is any fixed positive constant.

Proof. By hypothesis, there is an open subset $\Omega\supset S$ such that the field v satisfies the equation (8.2) in Ω. We may assume that Ω is contained in the ball $\mathbb{B}_R$. Let us take a smooth function $\chi:\mathbb{R}^3\to\mathbb{R}$ equal to 1 in a closed set $S'\subset\Omega$, whose interior contains S, and which is identically zero outside Ω. Defining a smooth extension $\tilde{v}$ of the field v to $\mathbb{R}^n$ by setting $\tilde{v}:=\chi v$, we obviously have

$$\tilde{v}(x)=\int_{\mathbb{R}^3} G(x-\tilde{x})\varrho(\tilde{x})\,d\tilde{x} \tag{8.4}$$

with $\varrho:=-\Delta\tilde{v}-\lambda^2\tilde{v}$.

The vector field ϱ is necessarily supported in $\Omega\backslash S'$. Therefore, an easy continuity argument ensures that one can approximate the integral (8.4) in the compact set S' by a finite Riemann sum of the form

$$\hat{v}(x):=\sum_{m=1}^{M}\varrho_m G(x-x_m)$$

so that, for any constant $\delta'>0$,

$$\|\tilde{v}-\hat{v}\|_{C^0(S')}<\delta'.$$

Here ϱ_m are constant vectors in $\mathbb{R}^3$ and x_m are points that lie in $\Omega\backslash S'$.

Let us take a domain $U\subset\mathbb{R}^3\backslash\mathbb{B}_{2R}$. Lemma 8.1 asserts that there is a vector field of the form

$$w(x):=\sum_{j=1}^{J}c_j G(x-z_j)$$

such that

$$\|\hat{v}-w\|_{C^0(S')}<\delta',$$

where $\{z_j\}_{j=1}^J$ is a finite set of points in U and $c_j\in\mathbb{R}^3$ are constant vectors. Therefore,

$$\|v-w\|_{C^0(S')}<2\delta'. \tag{8.5}$$

To complete the proof of the proposition, notice that the field v satisfies

$$\Delta v+\lambda^2 v=0$$

in the set S' (whose interior contains S) and w satisfies the same equation in the ball $\mathbb{B}_{2R}$. By standard elliptic estimates, it follows that the C^0 approximation (8.5) can be promoted to the C^k bound

$$\|v-w\|_{C^k(S)}<C_k\delta'.$$

Choosing δ' small enough, the result follows. □

We are now ready to prove the global approximation theorem for the Beltrami equation with solutions that decay at infinity. To construct these solutions, we will truncate a suitable series representation for the fields in a large ball obtained using Proposition 8.2 and act on them using a convenient differential operator.

THEOREM 8.3. *Let v be a vector field that satisfies the Beltrami equation*

$$\operatorname{curl} v = \lambda v$$

in a compact set $S \subset \mathbb{R}^3$, where λ is a non-zero constant and the complement $\mathbb{R}^3 \backslash S$ is connected. Then there is a global Beltrami field u, satisfying the equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$, which falls off at infinity as $|D^j u(x)| < C_j/|x|$ and approximates the field v in the C^k norm as

$$\|u - v\|_{C^k(S)} < \delta.$$

Here δ is any positive constant.

Proof. Let us assume that the compact set S is contained in the ball $\mathbb{B}_{R/2}$. As the Beltrami field v satisfies the equation

$$\Delta v + \lambda^2 v = 0$$

in S, by Proposition 8.2 there is a field w satisfying

$$\Delta w + \lambda^2 w = 0 \tag{8.6}$$

in the ball $\mathbb{B}_R$ and such that

$$\|v - w\|_{C^{k+2}(S)} < \delta'. \tag{8.7}$$

Let us take spherical coordinates (r, θ, φ) in the ball $\mathbb{B}_R$. Writing the field w as a series of spherical harmonics and using the equation (8.6) we immediately obtain that w can be written as a series

$$w = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} j_l(\lambda r) Y_{lm}(\theta, \varphi).$$

Here j_l is the spherical Bessel function, Y_{lm} are the spherical harmonics and $c_{lm} \in \mathbb{R}^3$ are constant vectors. Therefore, given any $\delta' > 0$ there is an integer L such that the finite sum

$$\hat{u} := \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} j_l(\lambda r) Y_{lm}(\theta, \varphi)$$

approximates the field w in L^2 sense,

$$\|\hat{u}-w\|_{L^2(\mathbb{B}_R)} < \delta'. \tag{8.8}$$

By the properties of spherical Bessel functions, the vector field $\hat{u}$ satisfies the equation

$$\Delta\hat{u}+\lambda^2\hat{u}=0 \tag{8.9}$$

in $\mathbb{R}^3$ and falls off at infinity as $|D^j\hat{u}(x)|<C_j/|x|$.

In view of equations (8.6) and (8.9), standard elliptic estimates allow us to pass from the L^2 bound (8.8) to the C^{k+2} estimate

$$\|\hat{u}-w\|_{C^{k+2}(\mathbb{B}_{R/2})} < C_k\delta'.$$

From this inequality and the bound (8.7) we infer

$$\|\hat{u}-v\|_{C^{k+2}(S)} < C\delta'. \tag{8.10}$$

Let us now set

$$u := \frac{\operatorname{curl}\operatorname{curl}\hat{u}+\lambda\operatorname{curl}\hat{u}}{2\lambda^2}.$$

A simple computation shows that the vector field thus defined satisfies the Beltrami equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$ and falls off as $|D^j u(x)|<C_j/|x|$ by the properties of $\hat{u}$. Moreover,

$$\begin{aligned}\|u-v\|_{C^k(S)} &= \left\|\frac{\operatorname{curl}\operatorname{curl}\hat{u}+\lambda\operatorname{curl}\hat{u}}{2\lambda^2}-v\right\|_{C^k(S)} = \left\|\frac{(\operatorname{curl}+\lambda)\operatorname{curl}(\hat{u}-v)}{2\lambda^2}\right\|_{C^k(S)}\\ &\leqslant C\|\hat{u}-v\|_{C^{k+2}(S)} < C\delta',\end{aligned}$$

as we wanted to prove. □

Remark 8.4. The fall-off at infinity of the global Beltrami field u is obtained from the truncation of the explicit series representation for the auxiliary field w. This is the reason why Theorem 8.3 does not work in arbitrary open Riemannian 3-manifolds, unlike the approximation theorem we used in [10]. Notice that the latter theorem does not yield any control at infinity whatsoever for the global Beltrami fields.

9. Proof of the main theorem

We are now ready to give the proof of Theorem 1.1. Let us begin by considering one of the curves (say, γ_1) in the statement of the theorem. By perturbing this curve with a C^m-small diffeomorphism if necessary, we can assume that γ_1 is an analytic curve whose curvature $\varkappa$ does not vanish anywhere [5, p. 184], and that its total torsion satisfies

$$\int_0^\ell \tau(\alpha)\, d\alpha \neq \pi k$$

for all integers k. As before, ℓ denotes the length of the curve γ_1. It is worth emphasizing that, as these conditions are open, they are obviously preserved if we deform the curve γ_1 with a diffeomorphism that is close enough to the identity in the C^m norm ($m\geqslant 3$).

Let us consider the tube $\mathcal{T}_\varepsilon(\gamma_1)$ of core curve γ_1 and thickness ε, and the corresponding harmonic field h, given by the expression (5.3) in the coordinates (α, y) adapted to the tube. Throughout we will assume that ε is small enough. By Theorem 6.8, we can consider the associated local Beltrami field v_1, which is given by the only solution to the Beltrami equation with parameter $\lambda=\varepsilon^3$,

$$\operatorname{curl} v_1 = \varepsilon^3 v_1,$$

in the tube $\mathcal{T}_\varepsilon(\gamma_1)$, that is tangent to the boundary and whose harmonic part is the field h. (Notice that in this section v_1 will *not* stand for the component of a vector field in the direction of the coordinate y_1.) Since the boundary of the tube is analytic, it is well known [24] that the local Beltrami field v_1 is analytic in the closure of a small neighborhood Ω_1 of $\overline{\mathcal{T}_\varepsilon(\gamma_1)}$.

Since the total torsion of the curve is not an integral multiple of π, Theorem 7.10 and Proposition 7.12 ensure that, given any $\delta>0$, we can deform the curve γ_1 by a diffeomorphism of $\mathbb{R}^3$ arbitrarily close to the identity in the C^m norm so that any divergence-free vector field u in Ω_1 with

$$\|u - v_1\|_{C^k(\Omega_1)} < \delta'$$

has

- an invariant tube given by $\Psi_1[\mathcal{T}_\varepsilon(\gamma_1)]$;
- an elliptic periodic trajectory given by $\tilde{\Psi}_1(\gamma_1)$.

Moreover, on the invariant torus $\partial\Psi_1[\mathcal{T}_\varepsilon(\gamma_1)]$, the field u is orbitally conjugate to a Diophantine rotation, and therefore ergodic. Here Ψ_1 and $\tilde{\Psi}_1$ are diffeomorphisms of $\mathbb{R}^3$ with

$$\|\Psi_1 - \mathrm{id}\|_{C^m(\mathbb{R}^3)} + \|\tilde{\Psi}_1 - \mathrm{id}\|_{C^m(\mathbb{R}^3)} < \delta$$

and such that the differences $\Psi_1 - \mathrm{id}$ and $\tilde{\Psi}_1 - \mathrm{id}$ are supported on small neighborhoods U_1, $\widetilde{U}_1$ of $\partial\mathcal{T}_\varepsilon(\gamma_1)$ and γ_1, respectively. The constants k and δ' depend on m, δ and on the geometry of the curve γ_1.

We can apply the same argument for each curve γ_i $(1 \leqslant i \leqslant N)$, thereby obtaining (for small enough ε) a collection of local Beltrami fields v_i satisfying the equation

$$\operatorname{curl} v_i = \varepsilon^3 v_i$$

in the closure of a neighborhood Ω_i of the closed tube $\overline{\mathcal{T}_\varepsilon(\gamma_i)}$ and such that any divergence-free vector field u in Ω_i with $\|u - v_i\|_{C^k(\Omega_i)} < \delta'$ has an invariant torus, where the field u is ergodic, and an elliptic periodic trajectory. Furthermore, they are respectively given by $\partial\Psi_i[\mathcal{T}_\varepsilon(\gamma_i)]$ and $\tilde{\Psi}_i(\gamma_i)$, where Ψ_i and $\tilde{\Psi}_i$ are C^m-small diffeomorphisms of $\mathbb{R}^3$ with $\Psi_i - \mathrm{id}$ and $\tilde{\Psi}_i - \mathrm{id}$ supported in small neighborhoods U_i, $\widetilde{U}_i$ of $\partial\mathcal{T}_\varepsilon(\gamma_i)$ and γ_i, in each case. We may assume that the complement

$$\mathbb{R}^3 \backslash (\Omega_1 \cup \ldots \cup \Omega_N)$$

is connected and that the sets $\bar{\Omega}_i$ are pairwise disjoint.

Let us define a vector field v in $\bar{\Omega}_1 \cup \ldots \cup \bar{\Omega}_N$ by setting it equal to the local Beltrami field v_i in each set $\bar{\Omega}_i$. By Theorem 8.3, there is a Beltrami field u, which satisfies the equation

$$\operatorname{curl} u = \varepsilon^3 u$$

in $\mathbb{R}^3$, that falls off at infinity as $|D^j u(x)| < C_j/|x|$ and approximates the field v as

$$\|u - v\|_{C^k(\Omega_1 \cup \ldots \cup \Omega_N)} < \delta'.$$

Therefore, if we define the diffeomorphism Φ of $\mathbb{R}^3$ as

$$\Phi(x) := \begin{cases} \Psi_i(x), & \text{if } x \in U_i, \\ \tilde{\Psi}_i(x), & \text{if } x \in \widetilde{U}_i, \\ x, & \text{otherwise,} \end{cases}$$

it follows that, for each i, $\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$ is a vortex tube of the Beltrami field u, and that $\Phi(\gamma_i)$ is an elliptic periodic trajectory. Besides, the Beltrami field u is ergodic (and orbitally conjugate to a Diophantine rotation) on each invariant torus $\partial\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$.

The field u being orbitally conjugate to a Diophantine rotation on each invariant torus $\partial\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$, it follows [13] that this invariant torus is accumulated by a Cantor-like set of invariant tori with positive Lebesgue measure. On these invariant tori, the field is also orbitally conjugate to Diophantine rotations. The corresponding set of Diophantine frequencies is Cantor-like because the normal torsion $\mathcal{N}_\Pi$ is non-zero. Therefore, if we consider the trajectories of the field u between two of these invariant tori (lying on the same vortex tube $\Phi[\mathcal{T}_\varepsilon(\gamma_i)]$), Angenent's dichotomy [1] asserts that either there is a horseshoe-type invariant set between them or there are invariant tori where the field is conjugate to a rotation of rational frequency. In both cases, there are infinitely many periodic trajectories between these tori. The theorem then follows.

Remark 9.1. It is worth giving some additional details about what we understand by a thin tube. What we have proved is that for any set of smooth periodic curves $\gamma_1, ..., \gamma_N$, $\delta>0$ and $m\in\mathbb{N}$, there is a positive constant ε_0, depending on δ, m and the geometry of the curves, such that the statement of Theorem 1.1 holds true for any thickness $\varepsilon<\varepsilon_0$, the diffeomorphism Φ that maps the tubes $\mathcal{T}_\varepsilon(\gamma_i)$ into vortex tubes being bounded by $\|\Phi-\mathrm{id}\|_{C^m(\mathbb{R}^3)}<\delta$. It should be noticed that both Theorem 1.1 and its proof hold verbatim if one takes tubes $\mathcal{T}_{\varepsilon_1}(\gamma_1), ..., \mathcal{T}_{\varepsilon_N}(\gamma_N)$ of different thickness, as long as $\varepsilon_1, ..., \varepsilon_N$ are small enough (that is, smaller than the above constant ε_0).

Remark 9.2. Contrary to what happened in the main theorem of [10], the proof of Theorem 1.1 does not work in general (even if we drop the requirement that the field u decays at infinity, replacing Theorem 8.3 by [10, Theorem 3.6]) if we substitute the finite set of curves $\{\gamma_i\}_{i=1}^N$ by an infinite set $\{\gamma_i\}_{i=1}^\infty$ that is locally finite. The reason is that the "maximal thickness" $\varepsilon_0(\gamma_i)$ associated with each curve γ_i individually does not need to be bounded away from zero: since all the vector fields v_i, defined in a neighborhood of the tube $\overline{\mathcal{T}_{\varepsilon_i}(\gamma_i)}$ (whose thickness can vary from tube to tube), must satisfy the Beltrami equation

$$\operatorname{curl} v_i = \lambda v_i$$

with the *same* constant λ for all i. Since this λ must be of order $\mathcal{O}[\varepsilon_0(\gamma_i)^3]$ for all i, it is clear that the construction breaks down if the infimum of the positive quantities $\{\varepsilon_0(\gamma_i)\}_{i=1}^\infty$ is 0. On the other hand, if this infimum is positive, we can apply the approximation theorem in [10, Theorem 3.6] to construct a global Beltrami field with all the properties listed in the statement of Theorem 1.1 with the exception that its growth at infinity is not controlled.

10. A comment about the Navier–Stokes equation

To conclude, we will present an easy application of Theorem 1.1 to the existence of (time-dependent) solutions to the Navier–Stokes equation that have a prescribed set of stationary (possibly knotted and linked) vortex tubes.

For this, let us take the global Beltrami field u that we considered in §9. That is, u satisfies the equation

$$\operatorname{curl} u = \varepsilon^3 u$$

in $\mathbb{R}^3$, falls off at infinity as $C/|x|$ and has a set of thin invariant tubes given by $\{\Phi[\mathcal{T}_\varepsilon(\gamma_i)]\}_{i=1}^N$, where the diffeomorphism Φ is close to the identity and ε is a small constant. Then the analytic time-dependent field

$$w(x,t) := u(x) e^{-\nu\varepsilon^6 t}$$

is a solution of the Navier–Stokes equation

$$\partial_t w + (w\cdot\nabla) w = \nu\Delta w - \nabla P, \quad \operatorname{div} w = 0,$$

in $\mathbb{R}^3$ with pressure $P(x,t) = c - \frac{1}{2}|w(x,t)|^2$. As the vortex lines of w (which are the trajectories of the vorticity $\operatorname{curl} w(x,t)$ for fixed t) coincide with those of u at all times, up to a reparametrization, it follows that w is a solution to the Navier–Stokes equation with the desired properties.

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