Acta Math., 210 (2013), 261–318
DOI: 10.1007/s11511-013-0095-9

Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$

by

Rupert L. Frank
Princeton University
Princeton, NJ, U.S.A.

Enno Lenzmann
University of Copenhagen
Copenhagen, Denmark

1. Introduction

Fractionalpowers of the Laplacian arise in a numerous variety of equations in mathematical physics and related fields; see, e.g., [1], [9], [14], [17], [20], [24], [28], [34] and the references therein. Here, a central role within these models is often played by so-called *ground state solutions*, or simply *ground states*. By this, we mean non-trivial, non-negative and radial functions $Q=Q(|x|)\geqslant 0$ that vanish at infinity and satisfy (in the distributional sense) an equation of the form

$$(-\Delta)^s Q+F(Q)=0 \quad \text{in } \mathbb{R}^d. \tag{1.1}$$

As usual, the fractional Laplacian $(-\Delta)^s$ with $0<s<1$ is defined via its multiplier $|\xi|^{2s}$ in Fourier space, whereas $F(Q)$ denotes some given non-linearity. In most examples of interest, the existence of ground states $Q=Q(|x|)\geqslant 0$ follows from variational arguments, applied to a suitable minimization problem whose Euler–Lagrange equation is given by (1.1). Moreover, based on this variational approach, it is natural in these cases to require that a ground state is also a minimizer for some related variational problem in addition to just being a non-negative and radial solution of (1.1). Indeed, we will make use of this (strengthened) notion of a ground state in this paper further below.

In striking contrast to the question of existence, it seems fair to say that extremely little is known about uniqueness of ground states $Q=Q(|x|)\geqslant 0$ for problems like (1.1), except for the "classical" limiting case with $s=1$, where standard ordinary differential equation (ODE) methods are applicable. Indeed, to the best of the authors' knowledge, the only examples for which uniqueness of ground states for (1.1) has been proven are:

- Ground state solitary waves for the Benjamin–Ono equation in $d=1$ dimension; see [5].
- Optimizers for fractional Sobolev inequalities in $d\geqslant 1$ dimensions; see [13] and [23].

In fact, in both cases the unique ground states are known in closed form. However, the uniqueness proof of both results hinges on a very specific feature of each problem: In the first case, the proof is intimately linked to complex analysis and special identities exhibited by the (completely integrable) Benjamin–Ono equation; whereas, in the second case, the conformal symmetry of Sobolev inequalities plays a key role in the uniqueness proof. In particular, the specific arguments developed in [1], [5], [13] and [23] are apparently of no use in a more general setting. Hence, we see that a satisfactory understanding of uniqueness for ground states of problems like (1.1) is largely missing. Clearly, the main analytical obstruction is that shooting arguments and other ODE techniques (which are essential in the classical case $s=1$; see, e.g., [21], [30] and [31]) are not applicable to the non-local operator $(-\Delta)^s$ when $0<s<1$.

In the present paper, we address the question of uniqueness for a general class of the form (1.1) in $d=1$ space dimension. More precisely, we prove uniqueness of ground states $Q\in H^s(\mathbb{R})$ for the non-linear model problem

$$(-\Delta)^s Q+Q-Q^{\alpha+1}=0 \quad \text{in } \mathbb{R}. \tag{1.2}$$

Here we assume that $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$ holds, where the critical exponent $\alpha_{\max}(s)$ is defined as

$$\alpha_{\max}(s):=\begin{cases} 4s/(1-2s) & \text{for } 0<s<\frac{1}{2}, \\ \infty & \text{for } \frac{1}{2}\leqslant s<1. \end{cases} \tag{1.3}$$

Technically speaking, the condition that α be strictly less than $\alpha_{\max}(s)$, which is vacuous if $s\geqslant\frac{1}{2}$, ensures that the non-linearity in equation (1.2) is *H^s-subcritical*. In fact, it turns out that this condition on α is necessary to have existence of ground states for (1.2), since (by so-called Pohozaev identities) it is easy to see that (1.2) does not admit any non-trivial solutions in $H^s(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ when $\alpha\geqslant\alpha_{\max}(s)$ holds.

Apart from being a natural model case for equation (1.1) in one space dimension, we remark that equation (1.2) and its solutions provide *solitary wave solutions* of three fundamental non-linear dispersive model equations in $d=1$ dimension. Namely, the generalized Benjamin–Ono equation and Benjamin–Bona–Mahony equation, as well as the fractional non-linear Schrödinger equation given by

$$u_t+((-\Delta)^s u)_x+u^\alpha u_x=0, \tag{gBO}$$

$$u_t+u_x+((-\Delta)^s u)_t+u^\alpha u_x=0, \tag{gBBM}$$

$$iu_t-(-\Delta)^s u+|u|^\alpha u=0. \tag{fNLS}$$

Note that in (gBO) and (gBBM) we assume that $\alpha\in\mathbb{N}$ is an integer and that $u=u(t,x)$ is real-valued.([1]) Suppose now that $Q=Q(|x|)\geqslant 0$ solves (1.2). Then it is elementary to see that the functions

$$\begin{aligned}
u_c(t,x)&=(c(\alpha+1))^{1/\alpha}Q(c^{1/2s}(x-(1+c)t)),\\
u_c(t,x)&=(c(\alpha+1))^{1/\alpha}Q\Big(\Big(\frac{c}{1+c}\Big)^{1/2s}(x-(1+c)t)\Big),\\
u_\omega(t,x)&=e^{i\omega t}\omega^{1/\alpha}Q(\omega^{1/2s}x)
\end{aligned}$$

provide solitary wave solutions of (gBO), (gBBM) and (fNLS), respectively. In the first two (water wave) examples, the parameter $c>0$ corresponds to the traveling speed of the wave to the right; whereas the parameter $\omega>0$ plays the role of an oscillation frequency of the solitary wave for (fNLS). There is numerous literature on the evolutions problems mentioned above; see, e.g., [3], [7], [20], [25] and [34] for results on solitary waves for (gBO), (gBBM) and (fNLS).

In all these cases, the uniqueness and the so-called non-degeneracy (see below) of the ground states $Q=Q(|x|)\geqslant 0$ are of fundamental importance in the stability and blowup analysis for the corresponding solitary waves $u_c(t,x)$ and $u_\omega(t,x)$ above. So far, except for the special case $s=\frac{1}{2}$ and $\alpha=1$ in [5] and a perturbative result for s close to 1 in [20], no rigorous results have been derived in this direction, and hence these properties of $Q=Q(|x|)$ have been imposed in terms of assumptions, partly supported by numerical evidence, and they have been left as main open problems; see, e.g., the recent paper by Kenig–Martel–Robbiano [20]. In particular, the rigorous understanding of blowup phenomena close to $Q=Q(|x|)$ in the L^2-*critical* setting, i.e., when $\alpha=4s$ holds in (gBO), (gBBM) and (fNLS), has been hindered so far by the absence of any essential result for the non-local equation (1.2) in this important case. Here, as a particularly intriguing feature of the fractional Laplacian $(-\Delta)^s$, the slow algebraic decay of $Q=Q(|x|)$ (see Proposition 1.1 below) is expected to lead to strong corrections to blowup rates that follow from a scaling analysis.

In Theorems 2.3 and 2.4 below, we prove uniqueness and so-called non-degeneracy of ground states for equation (1.2) in the full range $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$. In particular, these main results can be viewed as the sine qua non for future work on solitary waves and blowup for dispersive non-linear partial differential equations (PDEs) with ground state solitary wave profiles $Q=Q(x)$ that satisfy (1.2).

Before we formulate the main results of this paper, let us first recall some facts about existence, regularity and spatial decay of ground state solutions of equation (1.2).

([1]) We could extend to complex-valued u and non-integer α, by replacing $u^\alpha u_x$ with $|u|^\alpha u_x$. Indeed, such models are also of interest in the PDE literature; see, e.g., [20].

Indeed, by following the seminal approach of M. Weinstein in [33] and [34], we notice that problem (1.2) has indeed non-trivial solutions $Q\in H^s(\mathbb{R})$, which are optimizers of the Gagliardo–Nirenberg-type inequality

$$\int_{\mathbb{R}} |u|^{\alpha+2}\,dx \leqslant C_{\alpha,s}\left(\int_{\mathbb{R}} |(-\Delta)^{s/2}u|^2\,dx\right)^{\alpha/4s}\left(\int_{\mathbb{R}} |u|^2\,dx\right)^{\alpha(2s-1)/4s+1}, \tag{1.4}$$

in one space dimension. Here $C_{\alpha,s}>0$ denotes the optimal constant depending on α and s. Equivalently, this claim follows from considering the 'Weinstein' functional

$$J^{s,\alpha}(u) := \left(\int_{\mathbb{R}} |u|^{\alpha+2}\,dx\right)^{-1}\left(\int_{\mathbb{R}} |(-\Delta)^{s/2}u|^2\,dx\right)^{\alpha/4s}\left(\int_{\mathbb{R}} |u|^2\,dx\right)^{\alpha(2s-1)/4s+1}, \tag{1.5}$$

defined for $u\in H^s(\mathbb{R})$ with $u\not\equiv 0$. Clearly, every minimizer $Q\in H^s(\mathbb{R})$ for $J^{s,\alpha}$ optimizes the interpolation estimate (1.4) and vice versa. In addition, any such non-negative $Q\in H^s(\mathbb{R})$ is found to satisfy equation (1.2) after some suitable rescaling $Q(x)\mapsto aQ(bx)$ with some positive constants $a>0$ and $b>0$.

In summary, we have the following existence result and fundamental properties of solutions to (1.2), which we can infer from the literature.

PROPOSITION 1.1. *Let $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$. Then the following holds*:

(i) (Existence) *There exists a solution $Q\in H^s(\mathbb{R})$ of (1.2) such that $Q=Q(|x|)>0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function $Q\in H^s(\mathbb{R})$ is a minimizer for $J^{s,\alpha}$.*

(ii) (Symmetry and monotonicity) *If $Q\in H^s(\mathbb{R})$ with $Q\geqslant 0$ and $Q\not\equiv 0$ solves (1.2), then there exists $x_0\in\mathbb{R}$ such that $Q(\cdot-x_0)$ is even, positive and strictly decreasing in $|x-x_0|$.*

(iii) (Regularity and decay) *If $Q\in H^s(\mathbb{R})$ solves (1.2), then $Q\in H^{2s+1}(\mathbb{R})$. Moreover, we have the decay estimate*

$$|Q(x)|+|xQ'(x)| \leqslant \frac{C}{1+|x|^{1+2s}}$$

for all $x\in\mathbb{R}$ and some constant $C>0$.

Remarks. (1) As for the proof of part (i), we can refer to Weinstein's paper [34] where concentration/compactness-type arguments are used to show existence of minimizers for $\frac{1}{2}\leqslant s<1$. But the method can be applied to the range $0<s<\frac{1}{2}$ as well; see also [4]. Moreover, by *strict rearrangement inequalities* for $\int_{\mathbb{R}} |(-\Delta)^{s/2}u|^2\,dx$ when $0<s<1$ (see, e.g., [16]), we can deduce that any minimizer $Q\in H^s(\mathbb{R})$ for $J^{s,\alpha}$ must be equal (apart from translation and phase) to its symmetric-decreasing rearrangement $Q^*=Q^*(|x|)$. See also §2 and §5 below.

(2) To derive part (ii), we can directly adapt the moving plane method recently developed in [26], combined with some properties of the integral kernel for $((-\Delta)^s+1)^{-1}$ on $\mathbb{R}$. For more details, we refer to Appendix B.

(3) The regularity proof of part (iii) is worked out in Appendix B. Moreover, it is easy to see that in fact $Q\in H^k(\mathbb{R})$ for all $k\geqslant 1$, if $Q=Q(x)$ is positive or if the exponent $\alpha\geqslant 1$ is a positive integer; see also [22] for an analyticity result in this case. See [20] and the references given there for the spatial decay estimate stated above.

(4) For positive solutions $Q=Q(x)$ of (1.2), we also have the lower bound

$$Q(x)\geqslant c(1+|x|)^{-1-2s}$$

with some constant $c>0$.

2. Main results

We now formulate the main results of this paper about ground state solutions to (1.2) that we define as follows.

Definition 2.1. Let $Q\in H^s(\mathbb{R})$ be an even and positive solution of (1.2). If

$$J^{s,\alpha}(Q)=\inf_{u\in H^s(\mathbb{R})\setminus\{0\}} J^{s,\alpha}(u),$$

then we say that $Q\in H^s(\mathbb{R})$ is a *ground state solution* of equation (1.2).

Remark 2.2. In fact, there are several constrained variational problems that are equivalent to the unconstrained problem of minimizing $J^{s,\alpha}$ on $H^s(\mathbb{R})\setminus\{0\}$.

For example, in the so-called L^2-subcritical case when $0<\alpha<4s$, the constrained minimization problem, with parameter $N>0$,

$$E(N)=\inf\left\{\frac{1}{2}\int_{\mathbb{R}}|(-\Delta)^{s/2}u|^2\,dx-\frac{1}{\alpha+2}\int_{\mathbb{R}}|u|^{\alpha+2}\,dx : u\in H^s(\mathbb{R}) \text{ and } \int_{\mathbb{R}}|u|^2\,dx=N\right\}$$

is attained at $u\in H^s(\mathbb{R})$ if and only if $u=e^{i\vartheta}\lambda^{1/\alpha}Q(\lambda^{1/2s}(\cdot+y))$ with some $\vartheta\in\mathbb{R}$, $y\in\mathbb{R}$ and $\lambda>0$ chosen to ensure that $\int_{\mathbb{R}}|u|^2\,dx=N$ holds. Here $Q\in H^s(\mathbb{R})$ is a ground state solution of (1.2) in the sense of Definition 2.1 above.

Our first main result establishes the so-called *non-degeneracy* of the linearization associated with positive solutions Q of (1.2) that are *local* minimizers for $J^{s,\alpha}$; and thus our result holds in particular when Q is a ground state solution. As already mentioned, the non-degeneracy of the linearization around ground states plays a fundamental role for the stability and blowup analysis for solitary waves for related time-dependent equations

such as the generalized (gBO) and (gBBM) equations; see, e.g., [3], [7], [20], [25] and [34], where the non-degeneracy of L_+ is imposed in terms of a spectral assumption, or proven for s close to 1 by perturbation arguments.

We have the following general non-degeneracy result.

THEOREM 2.3. (Non-degeneracy) *Let* $0<s<1$ *and* $0<\alpha<\alpha_{\max}(s)$. *Suppose that* $Q\in H^s(\mathbb{R})$ *is a positive solution of* (1.2) *and consider the linearized operator*

$$L_+ = (-\Delta)^s + 1 - (\alpha+1)Q^\alpha$$

acting on $L^2(\mathbb{R})$. *Then the following conclusion holds*: *If* Q *is a local minimizer for* $J^{s,\alpha}$, *then* L_+ *is non-degenerate*, *i.e.*, *its kernel satisfies*

$$\ker L_+ = \operatorname{span}\{Q'\}.$$

In particular, *any ground state solution* $Q=Q(|x|)>0$ *of equation* (1.2) *has a non-degenerate linearized operator* L_+.

Remarks. (1) In fact, we will prove the non-degeneracy of L_+ under the (weaker) second-order condition

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} J^{s,\alpha}(Q+\varepsilon\eta) \geqslant 0 \quad \text{for all } \eta\in C_0^\infty(\mathbb{R}),$$

which clearly holds true when $Q\in H^s(\mathbb{R})$ is a local minimizer for $J^{s,\alpha}$.

(2) A fundamental application of Theorem 2.3 arises in the stability and blowup analysis of solitary waves for related time-dependent equations; notably in terms of a *coercivity estimate,* which readily follows from the non-degeneracy of L_+. More precisely, for suitable 2-dimensional subspaces $M\subset L^2(\mathbb{R})$, we can derive the lower bound

$$\langle \eta, L_+\eta\rangle \geqslant \delta\|\eta\|_{H^s}^2 \quad \text{for } \eta\perp M,$$

where $\delta>0$ is some positive constant independent of η. For example, by using the result of Theorem 2.3, it is to easy see that $M=\operatorname{span}\{\phi, Q'\}$ is a suitable choice, where $\phi=\phi(x)$ denotes the first eigenfunction of L_+ acting on $L^2(\mathbb{R})$.

Let us briefly comment on the proof of Theorem 2.3. The essential idea of the proof is to find a suitable substitute for Sturm–Liouville theory in order to estimate the number of sign changes for the *second eigenfunction*(s) for "fractional" Schrödinger operators of the form

$$H = (-\Delta)^s + V$$

in $d=1$ space dimension. In fact, it turns out that a key step in the proof of Theorem 2.3 follows from an argument in [11] developed for the classical ODE case when $s=1$ holds, *provided* we know that any (even) second eigenfunction of L_+ can change its sign only once on the positive real line $\{x>0\}$. Obviously, the crux of that matter is that $(-\Delta)^s$ is a non-local operator when $0<s<1$; and hence estimating the number of zeros for eigenfunctions of $H=(-\Delta)^s+V$ requires new arguments and insights, which substitute classical ODE techniques.

Let us briefly explain how we tackle this difficulty. First, we recall the known fact that $(-\Delta)^s$ can be regarded as a Dirichlet–Neumann operator for a suitable elliptic problem on the upper half-plane $\mathbb{R}^2_+=\{(x,y)\in\mathbb{R}^2:y>0\}$; see, e.g., the recent work by Caffarelli–Silvestre in [9] and also Graham–Zworski in [19] and Maz′ya in [29, Theorem 7.1.1.1] for this observation in the context of geometry and the theory of function spaces, respectively. Using now this extension to the upper half-plane $\mathbb{R}^2_+$, we derive—as a technical key result—a variational characterization of the eigenfunctions (and eigenvalues) for fractional Schrödinger operators $H=(-\Delta)^s+V$ in terms of the Dirichlet-type functional

$$A(u)=\iint_{\mathbb{R}^2_+}|\nabla u(x,y)|^2y^{1-2s}\,dx\,dy+\int_{\mathbb{R}}V(x)|u(x,0)|^2\,dx,$$

which is defined for a suitable class of functions $u=u(x,y)$ on the upper half-plane $\mathbb{R}^2_+$, where $u(x,0)$ denotes the trace of $u(x,y)$ on the boundary $\partial\mathbb{R}^2_+=\mathbb{R}\times\{0\}$. Moreover, for the variational problem based on the functional $A(u)$, we establish a nodal domain bound à la Courant. From such estimates we can finally deduce a sharp upper bound on the number of sign changes for any second eigenfunction of the non-local operator $H=(-\Delta)^s+V$ acting on $L^2(\mathbb{R})$, as needed in the proof of Theorem 2.3. Furthermore, this estimate for the second eigenfunctions of H involving general powers of $(-\Delta)^s$ and a broad (Kato-type) class of potentials V on $\mathbb{R}$ can be viewed as a generalization and extension of the inspiring work by Bañuelos–Kulczycki in [6], which studies eigenfunctions for $\sqrt{-\Delta}$ on the unit interval in $\mathbb{R}$. We believe that our techniques allow one to extend the results of [6] from $\sqrt{-\Delta}$ to the case of $(-\Delta)^s$ with arbitrary $0<s<1$, which was left as an open problem in [6].

We now turn to the second main result of this paper, which proves global uniqueness of ground state solutions. As a consequence, we also obtain uniqueness of optimizers for the interpolation inequality (1.4) up to scaling and translations.

THEOREM 2.4. (Uniqueness) *Let $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$. Then the ground state solution $Q=Q(|x|)>0$ of equation (1.2) is unique.*

Furthermore, every optimizer $v\in H^s(\mathbb{R})$ for the Gagliardo–Nirenberg inequality (1.4) is of the form $v=\beta Q(\gamma(\cdot+y))$ with some $\beta\in\mathbb{C}$, $\beta\neq0$, $\gamma>0$ and $y\in\mathbb{R}$.

Remarks. (1) Under the assumption that $Q=Q(|x|)>0$ minimizes $J^{s,\alpha}$, we remark that Theorem 2.4 generalizes the striking result by Amick and Toland in [5] about uniqueness of positive solutions $Q=Q(|x|)>0$ that satisfy

$$(-\Delta)^{1/2}Q+Q-Q^2=0 \quad \text{in } \mathbb{R}. \tag{2.1}$$

In fact, it was proven in [5] that (apart from translations) the function

$$Q(x)=\frac{2}{1+x^2}$$

is the only positive solution of (2.1) in $H^{1/2}(\mathbb{R})$. However, the remarkably elegant approach taken in [5] makes essential use of complex analysis (e.g., harmonic conjugates and Cauchy–Riemann equations) in combination with very specific identities derived from (2.1). In particular, it appears to be a hopeless enterprise to try to generalize the arguments in [5] to different powers of the fractional Laplacians $(-\Delta)^s$ with $s\neq\frac{1}{2}$ or non-quadratic non-linearities $f(Q)=Q^{\alpha+1}$ with $\alpha\neq 1$.

(2) The uniqueness result of Theorem 2.4 gives an affirmative answer to an open question recently posed by Kenig–Martel–Robbiano in [20].

(3) The uniqueness of optimizers for the interpolation inequality (1.4) follows directly from the ground state uniqueness and the strict rearrangement inequalities in [16]; see also §5 below.

Let us briefly explain the strategy behind the proof of the ground state uniqueness result of Theorem 2.4. First, we fix $0<s_0<1$ and $0<\alpha<\alpha_{\max}(s_0)$ and suppose that $Q_0=Q_0(|x|)>0$ is a ground state solution to (1.2) with $s=s_0$. By the non-degeneracy result of Theorem 2.3, the associated linearized operator L_+ is invertible on $L^2_{\text{even}}(\mathbb{R})\perp\ker L_+$. Hence, by using an implicit function argument, we can construct around $(Q_0,1)$ a locally unique branch of solutions (Q_s,λ_s) (in some suitable Banach space) which satisfy

$$(-\Delta)^s Q_s+\lambda_s Q_s-|Q_s|^\alpha Q_s=0, \tag{2.2}$$

where $s\in[s_0,s_0+\delta)$ and $\delta>0$ being sufficiently small. Here the function λ_s is introduced to ensure that the conservation law[(2)]

$$\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx=\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx$$

holds along the branch (Q_s,λ_s). Furthermore, we show that positivity is preserved along the branch, i.e., we have $Q_s=Q_s(|x|)>0$ for all $s\in[s_0,s_0+\delta)$ due to $Q_0=Q_0(|x|)>0$

(2) Equivalently, we could keep $\lambda_s\equiv 1$ at the expense of varying $\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx$. However, we realized that using λ_s is more convenient when we derive a-priori bounds for Q_s.

initially. Note that, although we start from a ground state solution for $s=s_0$, it *cannot* be inferred that Q_s (up to a rescaling) is also a ground state solution; i.e., a global minimizer for $J^{s,\alpha}$. Therefore, the global continuation of the branch (Q_s,λ_s) to $s=1$, say, is far from obvious.

However, as an essential step in the uniqueness proof, we show that the branch (Q_s,λ_s) can be *indeed* continued for all $s\in[s_0,1)$. This global continuation will be based on the non-degeneracy result of Theorem 2.3 in combination with the a-priori bounds

$$\int_{\mathbb{R}} |(-\Delta)^{s/2}Q_s|^2\,dx\sim 1, \quad \int_{\mathbb{R}} |Q_s|^2\,dx\sim 1 \quad \text{and} \quad \lambda_s\sim 1.$$

Here it turns out that establishing the upper bound $\int_{\mathbb{R}} |Q_s|^2\,dx\lesssim 1$ is the most delicate step and thus it requires a careful analysis of the problem. In addition to a-priori regularity bounds, the strict positivity and monotonicity of $Q_s=Q_s(|x|)>0$ also enters in a significant way, since it allows us to derive the uniform decay estimate $Q_s(|x|)\lesssim |x|^{-1}$ for $|x|\gtrsim 1$. The latter fact then guarantees relative compactness of $\{Q_s\}_s$ in certain L^p-norms.

Once we have established that (Q_s,λ_s) can be extended to $s=1$, we conclude that $Q_s\to Q_*$ (in some suitable sense) and $\lambda_s\to\lambda_*$ as $s\to 1$, where $Q_*=Q_*(|x|)>0$ and $\lambda_*>0$ satisfy

$$-\Delta Q_*+\lambda_* Q_*-Q_*^{\alpha+1}=0.$$

For this limiting equation, it is well known (by standard ODE techniques) that uniqueness of even and positive solutions $Q_*=Q_*(|x|)>0$ holds true. Furthermore, by Pohozaev identities and the conservation law for $\int_{\mathbb{R}} |Q_s|^{\alpha+2}\,dx$ and the fact that Q_0 is a ground state, we deduce that the limit $\lambda_*=\lambda_*(s_0,\alpha)$ only depends on s_0 and α. Hence, we can conclude that two different branches (Q_s,λ_s) and $(\widetilde{Q}_s,\tilde{\lambda}_s)$ (both starting from a ground state with $s=s_0$) must converge to the same limit (Q_*,λ_*). By using the known non-degeneracy for the linearization around (Q_*,λ_*), we can infer that the branches (Q_s,λ_s) and $(\widetilde{Q}_s,\tilde{\lambda}_s)$ must intersect for some $s\in[s_0,1)$ in contradiction to the local uniqueness of branches. This fact establishes uniqueness of ground states for all $0<s_0<1$ and all $0<\alpha<\alpha_{\max}(s_0)$, as stated in Theorem 2.4.

Finally, we mention that the second part of Theorem 2.4 follows from the fact that every optimizer for (1.4) must be equal to its symmetric-decreasing rearrangement modulo scaling and translation. The proof of this will be mainly based on strict rearrangement inequalities for $(-\Delta)^s$.

Extension of the main results to higher dimensions

With regard to possible extensions to higher dimensions, we remark that most of the arguments presented here can be easily generalized to $d\geqslant 2$ dimensions. However, as the only notable exception, the proof of the oscillation estimate for the second eigenfunction of L_+ (see Theorem 3.4 below) hinges on the fact that L_+ acts on functions in $d=1$ dimension. How to obtain a similar oscillation estimate for radial eigenfunctions of L_+ in $d\geqslant 2$ dimensions remains the chief open problem. If this could be solved, the analogous non-degeneracy result of Theorem 2.3 would readily follow for $d\geqslant 2$ dimensions. Moreover, the uniqueness proof of Theorem 2.4 would allow for a straightforward adaptation to $d\geqslant 2$ dimensions, since we have uniqueness and non-degeneracy of positive radial solutions $Q=Q(|x|)>0$ in $H^1(\mathbb{R}^d)$ for the limiting equation

$$-\Delta Q+Q-Q^{\alpha+1}=0 \quad \text{in } \mathbb{R}^d,$$

where $0<\alpha<\infty$ for $d=2$, and $0<\alpha<4/(d-2)$ for $d\geqslant 3$; see, e.g., [21].

Plan of the paper

We organize this paper as follows. In §3 we establish (as a technical key fact) a variational principle for fractional Schrödinger operators $H=(-\Delta)^s+V$ in terms of a *local energy functional*. As a main consequence, we obtain a sharp bound on the number of sign changes for any second eigenfunction of H. Then, in §4, we prove Theorem 2.3. Here we will make essential use of the main result from §3. Finally, in §5, we establish the uniqueness of ground states as stated in Theorem 2.4. The proof will be based on the non-degeneracy result of Theorem 2.3, combined with an elaborate global continuation argument. The appendices contain several technical results and proofs needed in this paper.

Notation

Throughout this paper, we employ standard notation for L^p-spaces and Sobolev spaces $H^s(\mathbb{R})$ of order $s\in\mathbb{R}$. We use $\langle f,g\rangle=\int_{\mathbb{R}}\bar{f}g\,dx$ to denote the inner product on $L^2(\mathbb{R})$. (In fact, we will mostly deal with real-valued functions and hence complex conjugation is redundant.) Furthermore, we make the usual abuse of notation by writing both $f=f(x)$ and $f=f(|x|)$ whenever $f:\mathbb{R}\to\mathbb{R}$ is an even function. The (open) positive real axis will be denoted by $\mathbb{R}_+=(0,\infty)$. Also, we use the standard notation $X\lesssim Y$ to denote $X\leqslant CY$ for some constant $C>0$ that only depends on some fixed quantities. We also write $X\sim Y$ if $X\lesssim Y\lesssim X$. Sometimes we write $X\lesssim_{a,b,\dots} Y$ to underline that C depends on the fixed quantities $a,b,\dots$.

Acknowledgments

R. F. acknowledges support from NSF grant PHY-0652854. E. L. was supported by a Steno fellowship from the Danish science research council, and he also gratefully acknowledges partial support from NSF grant DMS-0702492.

3. An oscillation estimate for $H=(-\Delta)^s+V$

This section serves as a preliminary discussion for §4 below, where we prove the non-degeneracy result of Theorem 2.3. More precisely, the present section deals with "fractional" Schrödinger operators

$$H=(-\Delta)^s+V$$

acting on $L^2(\mathbb{R})$. As our key technical result in this section, we prove a sharp bound on the number of sign changes for the *second eigenfunction*(s) of the non-local operator H, which we formulate in Theorem 3.4 below. The proof will be based on a variational characterization of the eigenvalues for H in terms of a *local energy functional* and associated nodal domain bound à la Courant; see Corollary 3.8 and Theorem 3.9 below. As mentioned in §2 above, the result derived below can be regarded as a generalization of [6], where properties of eigenfunctions of $\sqrt{-\Delta}$ on the unit interval in $\mathbb{R}$ are studied.

Let us first introduce a suitable class of potentials V for the fractional Schrödinger operators discussed here. In many respects (e.g., perturbation theory and properties of eigenfunctions), the following *"Kato class"* (denoted by K_s) is a natural choice.

Definition 3.1. Let $0<s<1$. We say that the potential $V\in K_s$ if and only if $V:\mathbb{R}\to\mathbb{R}$ is measurable and satisfies, where $E>0$ is a positive number,

$$\lim_{E\to\infty}\big\|((-\Delta)^s+E)^{-1}|V|\big\|_{L^\infty\to L^\infty}=0.$$

Remarks. (1) If $V\in K_s$, then $H=(-\Delta)^s+V$ defines a unique self-adjoint operator on $L^2(\mathbb{R})$ with form domain $H^s(\mathbb{R})$, and the corresponding heat kernel e^{-tH} maps $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})\cap C^0(\mathbb{R})$ for any $t>0$. In particular, any L^2-eigenfunction of H is continuous and bounded. See also [10] for equivalent definitions of K_s and further background material.

(2) If $V\in K_s$, then V is relatively form-bounded (with relative bound less than 1) with respect to $(-\Delta)^s$. That is, for every $0<\varepsilon<1$, there is a constant $C_\varepsilon>0$ such that

$$\langle\psi,|V|\psi\rangle\leqslant\varepsilon\langle\psi,(-\Delta)^s\psi\rangle+C_\varepsilon\langle\psi,\psi\rangle$$

for all $\psi\in H^s(\mathbb{R})$. In fact, the latter condition is also sufficient for V to be in K_s, provided that C_ε depends on ε in some explicit way.

(3) In terms of L^p-spaces, we can derive the following useful criterion for real-valued V to be in K_s:

- if $0<s\leqslant\frac{1}{2}$ and $V\in L^p(\mathbb{R})$ for some $p>1/2s$, then $V\in K_s$;
- if $\frac{1}{2}<s<1$ and $V\in L^p(\mathbb{R})$ for some $p\geqslant 1$, then $V\in K_s$.

See Lemma C.1 for further details on this sufficient condition.

Let us now assume that $V\in K_s$ holds. Suppose that ψ is an L^2-eigenfunction of $H=(-\Delta)^s+V$. Then, by the previous remark, we have that ψ is a continuous and bounded function on $\mathbb{R}$. Note also that we can always assume that ψ is real-valued, since $H=(-\Delta)^s+V$ is a real operator (i.e., it preserves real and imaginary parts). In particular, we can define what it means that $\psi(x)$ changes its sign N times.

Definition 3.2. Let $\psi\in C^0(\mathbb{R})$ be real-valued and let $N\geqslant 1$ be an integer. We say that $\psi(x)$ changes its sign N times if there exist points $x_1<\ldots<x_{N+1}$ such that $\psi(x_j)\neq 0$ for $j=1,\ldots,N+1$ and $\operatorname{sign}(\psi(x_j))=-\operatorname{sign}(\psi(x_{j+1}))$ for $j=1,\ldots,N$.

Remark 3.3. For $\psi\in C^0(\mathbb{R})$ we can define the *nodal domains* of $\psi(x)$ as the connected components of the open set $\{x\in\mathbb{R}:\psi(x)\neq 0\}$. If $\psi(x)$ cannot vanish to second order, then clearly the maximal number of sign changes of $\psi(x)$ equals $K-1$, where K is the number of nodal domains of ψ. But, in what follows, we prefer to work with the weaker notion of sign changes of $\psi(x)$.

We are now ready to state the following main result of this section.

THEOREM 3.4. (An oscillation estimate for H) *Let $0<s<1$, $V\in K_s$, and consider $H=(-\Delta)^s+V$ acting on $L^2(\mathbb{R})$. Suppose that $\lambda_2<\inf\sigma_{\mathrm{ess}}(H)$ is the second eigenvalue of H and let $\psi_2\in H^s(\mathbb{R})\cap C^0(\mathbb{R})$ be a corresponding real-valued eigenfunction. Then $\psi_2=\psi_2(x)$ changes its sign at most twice on $\mathbb{R}$.*

In particular, if $\psi_2=\psi_2(|x|)$ is an even eigenfunction, then $\psi_2(|x|)$ changes its sign exactly once on the positive axis $\{x>0\}$.

Remarks. (1) The reader who is mainly interested in applying this technical result may fast forward to §4 at first reading.

(2) By Perron–Frobenius arguments (see Appendix C), the first eigenfunction $\psi_1=\psi(x)>0$ of H is always strictly positive. Hence, by the self-adjointness of H, we easily see that ψ_2 changes its sign at least once in order to satisfy the orthogonality condition $\langle\psi_1,\psi_2\rangle=0$.

The proof of Theorem 3.4 will be given at the end of this section. But first we have to establish some auxiliary facts in the following subsections. In particular, we derive the key variational principle of eigenvalues of H in terms of a *local energy functional*, which we formulate in Corollary 3.8 below.

3.1. Extension to $\mathbb{R}^2_+$ and a sharp trace inequality

We recall the known fact that the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^d$ can be expressed as the Dirichlet-to-Neumann operator for a suitable *local* problem on the upper half-space $\mathbb{R}^{d+1}_+=\{(x,y):x\in\mathbb{R}^d \text{ and } y>0\}$. See the recent work by Caffarelli–Silvestre [9] for this fact. We also refer to the work of Graham–Zworski [19], where this observation occurred in a geometric context; see [12] for a comparison and extension of [9] and [19]. Our trace inequalities can also be viewed as sharp versions of certain bounds in [29, Theorem 7.1.1.1].

We consider $d=1$ space dimension in the sequel. Let $0<s<1$ be given and set $a=1-2s$. For a measurable function $f:\mathbb{R}\to\mathbb{R}$, we (first formally) define its extension $\mathcal{E}_a f:\mathbb{R}^2_+\to\mathbb{R}$ as

$$(\mathcal{E}_a f)(x,y) := \int_{\mathbb{R}} \frac{1}{y} P_a\left(\frac{x-x'}{y}\right) f(x')\,dx', \tag{3.1}$$

where the convolution kernel $P_a:\mathbb{R}\to\mathbb{R}$ is given by

$$P_a(x) := \frac{\Gamma\left(\frac{1}{2}(2-a)\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(1-a)\right)} \frac{1}{(1+x^2)^{(2-a)/2}}. \tag{3.2}$$

Under suitable assumptions on f it is known (see, e.g., [9]) that $w=\mathcal{E}_a f$ solves the boundary value problem

$$\begin{cases} \operatorname{div}(y^a \nabla w) = 0, & \text{in } \mathbb{R}^2_+, \\ w = f, & \text{on } \partial\mathbb{R}^2_+ = \mathbb{R}\times\{0\}. \end{cases} \tag{3.3}$$

Here the boundary condition $w=f$ is understood in some suitable sense, which will be formulated below. If f is sufficiently regular, then we also have that

$$\lim_{y\to 0} y^a \partial_y w(\cdot,y) = -c_a(-\Delta)^s f, \tag{3.4}$$

where $c_a>0$ is some explicit constant; see Proposition 3.5 below.

To give a precise meaning to the statements mentioned above, we first recall the definition of the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R})$ as the completion of $C_0^\infty(\mathbb{R})$ with respect to the quadratic form $\|(-\Delta)^{s/2} f\|^2$. It follows from Hardy's inequality that this completion is a space of functions when $0<s<\frac{1}{2}$. On the other hand, if $\frac{1}{2}\leqslant s\leqslant 1$, this completion is not a space of functions, but rather a space of equivalence classes of functions differing by an additive constant. (To see this phenomenom for $s=1$, consider a smoothened version of the sequence $f_n(x)=(1-|x|/n)_+$. Similar examples can be constructed for any $\frac{1}{2}\leqslant s<1$.) For simplicity, we shall write elements of $\dot{H}^s(\mathbb{R})$ still as functions, but with the understanding that for $s\geqslant\frac{1}{2}$ equalities are understood modulo constants.

Next, for $-1<a<1$ given, we introduce the weighted homogeneous Sobolev space $\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$ as the completion of $C_0^\infty(\overline{\mathbb{R}^2_+})$ with respect to the quadratic form

$$\|u\|^2_{\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)} := \iint_{\mathbb{R}^2_+} |\nabla u|^2 y^a \, dx\, dy. \tag{3.5}$$

Similarly as before, this completion is a space of functions for $0<a<1$ and a space of equivalence classes modulo constants for $-1<a\leqslant 0$. (These facts are known, but they are also consequences of our analysis below.) We note that, if $a=1-2s$, then $0<a<1$ if and only if $0<s<\frac{1}{2}$. Moreover, by scaling, one sees that $\int_{\mathbb{R}} y^{-1} P_a(x/y)\, dx$ is a constant independent of y (indeed, it is 1, as we shall see below). Hence, if f is an equivalence class modulo constants, then so is $\mathcal{E}_a f$. We have the following basic result.

PROPOSITION 3.5. *Let $0<s<1$, $f\in\dot{H}^s(\mathbb{R})$ and set $a=1-2s$. Then $\mathcal{E}_a f\in\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$ and we have that*

$$\iint_{\mathbb{R}^2_+} |\nabla \mathcal{E}_a f|^2 y^a \, dx\, dy = c_a \|(-\Delta)^{s/2} f\|_2^2, \tag{3.6}$$

where

$$c_a := 2^a \frac{\Gamma\big(\frac{1}{2}(1+a)\big)}{\Gamma\big(\frac{1}{2}(1-a)\big)}. \tag{3.7}$$

Moreover, the function $w=\mathcal{E}_a f$ is a weak solution to the partial differential equation

$$\operatorname{div}(y^a \nabla w) = 0 \quad \text{in } \mathbb{R}^2_+. \tag{3.8}$$

Finally,

$$w(\cdot,\varepsilon)\to f \ \text{ in } \dot{H}^s(\mathbb{R}) \quad \text{and} \quad -\frac{\varepsilon^a}{c_a}\frac{\partial w}{\partial y}(\cdot,\varepsilon)\to (-\Delta)^s f \ \text{ in } \dot{H}^{-s}(\mathbb{R}),$$

both as $\varepsilon\to 0$.

Proof. We begin by writing

$$\|(-\Delta)^{s/2} f\|_2^2 = \int_{\mathbb{R}} \overline{(-\Delta)^s f(x)} f(x)\, dx,$$

where the right-hand side should be understood as the duality pairing between $\dot{H}^{-s}$ and $\dot{H}^s$. Our goal now is to express both functions (which are strictly speaking distributions) on the right-hand side as boundary values of functions defined on the upper half-plane $\mathbb{R}^2_+$. We put $w=\mathcal{E}_a f$ and claim that

$$w(\cdot,\varepsilon)\to f \ \text{ in } \dot{H}^s(\mathbb{R}) \quad \text{and} \quad -\frac{\varepsilon^a}{c_a}\frac{\partial w}{\partial y}(\cdot,\varepsilon)\to (-\Delta)^s f \ \text{ in } \dot{H}^{-s}(\mathbb{R}), \tag{3.9}$$

both as $\varepsilon\to 0$.

These properties are easily seen in Fourier space. Indeed, using the computation [2, equation (11.4.44)] of the Fourier transform of P_a, we see that

$$(\mathcal{E}_a f)(x,y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) m_a(|\xi| y) e^{i\xi\cdot x}\, d\xi, \tag{3.10}$$

where

$$m_a(r) := \frac{2}{\Gamma\left(\frac{1}{2}(1-a)\right)} \left(\frac{r}{2}\right)^{(1-a)/2} K_{(1-a)/2}(r)$$

and $K_{(1-a)/2}$ is a Bessel function of the third kind. From standard properties of these functions (see, e.g., [2] again) we know that $m_a(0)=1$ and $0<m_a(r)\leqslant A_a$ for all $r\geqslant 0$. This means that

$$m_a(|\xi|\varepsilon)\to 1 \ \text{ as } \varepsilon\to 0 \quad \text{and} \quad 0<m_a(|\xi|\varepsilon)\leqslant A_a,$$

and hence

$$\int_{\mathbb{R}} |\xi|^{2s} \left|m_a(|\xi|\varepsilon)-1\right|^2 |\hat{f}(\xi)|^2\, d\xi \to 0 \quad \text{as } \varepsilon\to 0,$$

by dominated convergence. This proves the first relation in (3.9). In order to prove the second one, we note that

$$\frac{\partial w}{\partial y}(x,\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)|\xi| m_a'(|\xi|\varepsilon) e^{i\xi\cdot x}\, d\xi.$$

and that, again by properties of Bessel functions,

$$\lim_{r\to 0} r^a m_a'(r) = -c_a$$

and $0<-r^a m_a(r)\leqslant B_a$ for all $r\geqslant 0$. This means that

$$-\frac{\varepsilon^a}{c_a}|\xi| m_a'(|\xi|\varepsilon) \to |\xi|^{2s} \ \text{ as } \varepsilon\to 0 \quad \text{and} \quad 0<\frac{\varepsilon^a}{c_a}|\xi| m_a'(|\xi|\varepsilon) \leqslant B_a |\xi|^{2s},$$

which, again by dominated convergence, implies that

$$\int_{\mathbb{R}} \left|\frac{\varepsilon^a}{c_a}|\xi| m_a'(|\xi|\varepsilon) + |\xi|^{2s}\right|^2 |\hat{f}(\xi)|^2 \frac{d\xi}{|\xi|^{2s}} \to 0 \quad \text{as } \varepsilon\to 0,$$

and thus establishing the second relation in (3.9).

Next, we prove that $w=\mathcal{E}_a f$ satisfies the partial differential equation (3.8). This can either be shown directly by differentiating (3.1), or using (3.10) and a partial Fourier transform with respect to x. Indeed, the Bessel equation satisfied by $K_{(1-a)/2}$ is equivalent to $(r^a m_a')' = r^a m_a$, which is the same as (3.8) after Fourier transform and scaling.

With (3.9) and (3.8) at hand, it is now easy to show that (3.6) holds. Indeed,

$$\|(-\Delta)^{s/2}f\|_2^2 = \int_{\mathbb{R}} \overline{(-\Delta)^s f(x)} f(x)\,dx = -\frac{1}{c_a}\lim_{\varepsilon\to 0}\varepsilon^a \int_{\mathbb{R}} \overline{\frac{\partial w}{\partial y}(x,\varepsilon)} w(x,\varepsilon)\,dx$$
$$= \frac{1}{c_a}\lim_{\varepsilon\to 0}\iint_{y>\varepsilon} |\nabla w(x,y)|^2 y^a\,dx\,dy.$$

This proves that $\mathcal{E}_a f$ belongs to $\mathcal{H}^{1,a}(\mathbb{R}^2_+)$ and satisfies (3.6). The proof of Proposition 3.5 is now complete. □

For $u\in C_0^\infty(\overline{\mathbb{R}^2_+})$, we denote by $Tu(x):=u(x,0)$ its *trace*. As we shall see, the operator T can be extended by continuity to $\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$, due to the next proposition which also yields a sharp trace inequality. In particular, this auxiliary result identifies the space of functions on $\mathbb{R}$ that arise as traces of functions in $\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$ as the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ when $s=\frac{1}{2}(1-a)$.

PROPOSITION 3.6. *Let $0<s<1$ and $a=1-2s$. Then there is a unique linear bounded operator $T:\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)\to\dot{H}^s(\mathbb{R})$ such that $Tu(x)=u(x,0)$ for $u\in C_0^\infty(\overline{\mathbb{R}^2_+})$. Moreover, for any $u\in\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$, the following inequality holds*:

$$\iint_{\mathbb{R}^2_+} |\nabla u|^2 y^a\,dx\,dy \geqslant c_a\|(-\Delta)^{s/2}Tu\|_2^2, \tag{3.11}$$

with the constant c_a from (3.7). Here equality is attained if and only if $u=\mathcal{E}_a f$ for some $f\in\dot{H}^s(\mathbb{R})$.

Remark 3.7. In [18] inequality (3.11) was derived by different arguments in the range $\frac{1}{2}<s<1$.

Proof. We use a similar argument as in the proof of Proposition 3.5. Let $u\in C_0^\infty(\overline{\mathbb{R}^2_+})$ and $g\in\dot{H}^{-s}(\mathbb{R})$ be given. Note that $f:=(-\Delta)^{-s}g\in\dot{H}^s(\mathbb{R})$. By the same arguments as in the proof of Proposition 3.5, the function $w:=\mathcal{E}_a f$ satisfies (3.8) and (3.9). Hence we conclude

$$\int_{\mathbb{R}} \overline{g(x)}u(x,0)\,dx = \int_{\mathbb{R}} \overline{(-\Delta)^s f(x)}u(x,0)\,dx = -\frac{1}{c_a}\lim_{\varepsilon\to 0}\varepsilon^a\int_{\mathbb{R}} \overline{\frac{\partial w}{\partial y}(x,\varepsilon)}u(x,\varepsilon)\,dx$$
$$= \frac{1}{c_a}\lim_{\varepsilon\to 0}\iint_{y>\varepsilon} \overline{\nabla w(x,y)}\cdot\nabla u(x,y) y^a\,dx\,dy.$$

Next, by the Cauchy–Schwarz inequality,

$$\left|\int_{\mathbb{R}} \overline{g(x)}u(x,0)\,dx\right| \leqslant \frac{1}{c_a}\left(\iint_{\mathbb{R}^2_+} |\nabla w(x,y)|^2 y^a\,dx\,dy\right)^{1/2}\left(\iint_{\mathbb{R}^2_+} |\nabla u(x,y)|^2 y^a\,dx\,dy\right)^{1/2}. \tag{3.12}$$

We also note that, by Proposition 3.5,

$$\limsup_{\varepsilon\to 0}\iint_{y>\varepsilon}|\nabla w(x,y)|^2y^a\,dx\,dy=c_a\|(-\Delta)^{s/2}f\|_2^2=c_a\|(-\Delta)^{-s/2}g\|_2^2.$$

Thus we have shown that

$$\left|r\int_{\mathbb{R}}\overline{g(x)}u(x,0)\,dx\right|\leqslant\frac{1}{\sqrt{c_a}}\|(-\Delta)^{-s/2}g\|_2\left(\iint_{\mathbb{R}^2_+}|\nabla u(x,y)|^2y^a\,dx\,dy\right)^{1/2},$$

which, by duality, is the same as (3.11) for $u\in C_0^\infty(\overline{\mathbb{R}^2_+})$. This allows us to extend the operator T by continuity from $C_0^\infty(\overline{\mathbb{R}^2_+})$ to $\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$, preserving the above inequality, whereas the uniqueness of T follows from the density of $C_0^\infty(\overline{\mathbb{R}^2_+})$.

Moreover, the above argument is valid for any $u\in\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$ and equality in (3.12) is attained if and only if ∇u is a constant multiple of ∇w. Hence u is a weak solution of equation (3.8). By the unique solvability of this equation, u is given as the $\mathcal{E}_a$-extension of its trace. □

For the rest of this section, we will adapt the following convention.

Convention. For $u\in\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)$, we also write $u(x,0)$ to denote its trace $Tu(x)$.

We conclude our preliminary discussion by introducing the 'inhomogeneous' Sobolev space

$$\mathcal{H}^{1,a}(\mathbb{R}^2_+):=\{u\in\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+):u(x,0)\in L^2(\mathbb{R})\},\tag{3.13}$$

endowed with the norm $\|u\|_{\mathcal{H}^{1,a}(\mathbb{R}^2_+)}:=\|u\|_{\dot{\mathcal{H}}^{1,a}(\mathbb{R}^2_+)}+\|Tu\|_{L^2(\mathbb{R})}$. Note that $\mathcal{H}^{1,a}(\mathbb{R}^2_+)$ is in fact a space of functions (even for $-1<a\leqslant 0$). This space will be of use in the next subsection.

3.2. Variational characterization of eigenvalues

Using the results of the previous subsection, we now derive a variational principle for the first $n\geqslant 1$ eigenvalues of a fractional Schrödinger operator $H=(-\Delta)^s+V$ in terms of a *local energy functional*. Apart from requiring that V be in the class K_s, we make the convenient assumption that the bottom of the essential spectrum of $H=(-\Delta)^s+V$ satisfies

$$\inf\sigma_{\mathrm{ess}}(H)\geqslant 0.$$

This can be imposed without loss of generality, by replacing V with $V+c$ where $c\in\mathbb{R}$ is some suitable constant.

We are now ready to formulate our key variational principle for the eigenvalues of H below the essential spectrum.

COROLLARY 3.8. *Let $0<s<1$ and $V\in K_s$. Suppose that $n\geqslant 1$ is an integer and assume that $H=(-\Delta)^s+V$ has at least n eigenvalues*

$$\lambda_1\leqslant\ldots\leqslant\lambda_n<0.$$

Furthermore, let M be an $(n-1)$-dimensional subspace of $L^2(\mathbb{R})$ spanned by eigenfunctions corresponding to the eigenvalues λ_j with $j=1,\ldots,n-1$. Then

$$\lambda_n=\inf\Bigg\{\frac{1}{c_a}\iint_{\mathbb{R}^2_+}|\nabla u|^2y^a\,dx\,dy+\int_{\mathbb{R}}V(x)|u(x,0)|^2\,dx:\\ u\in\mathcal{H}^{1,a}(\mathbb{R}^2_+),\int_{\mathbb{R}}|u(x,0)|^2\,dx=1\text{ and }u(\cdot,0)\perp M\Bigg\},$$

where $a=1-2s$ and $c_a>0$ is the constant from (3.7). Moreover, the infimum is attained if and only if $u=\mathcal{E}_af$, where $\|f\|_2^2=1$ and $f\in M^\perp$ is a linear combination of eigenfunctions corresponding to the eigenvalue λ_n.

Proof. By Proposition 3.6, the infimum on the right-hand side is bounded from below by

$$\inf\Bigg\{\|(-\Delta)^{s/2}f\|_2^2+\int_{\mathbb{R}}V|f|^2\,dx:f\in H^s(\mathbb{R}),\ \|f\|_2^2=1\text{ and }f\perp M\Bigg\},$$

and equality is attained if and only if $u=\mathcal{E}_af$. The assertion now follows from the usual variational characterization for the eigenvalues of $H=(-\Delta)^s+V$. □

3.3. Nodal domain bound on $\mathbb{R}^2_+$

Let $V\in K_s$. Recall that we can always assume that any L^2-eigenfunction ψ of H is real-valued, since $H=(-\Delta)^s+V$ is a real operator. Furthermore, by the remark following Definition 3.1, any such eigenfunction ψ of H is bounded and continuous. Likewise, the extension $\mathcal{E}_a\psi$ belongs to $C^0(\overline{\mathbb{R}^2_+})$ as well. Consider the set $N=\{(x,y)\in\mathbb{R}^2_+:(\mathcal{E}_a\psi)(x,y)=0\}$ which is closed in $\mathbb{R}^2_+$. We define the *nodal domains* of $\mathcal{E}_a\psi$ as the connected components of the open set $\mathbb{R}^2_+\setminus N$ in $\mathbb{R}^2_+$. We have the following bound on the number of nodal domains.

THEOREM 3.9. *Let $0<s<1$, $V\in K_s$ and set $a=1-2s$. Let $n\geqslant 1$ be an integer and assume that $H=(-\Delta)^s+V$ has at least n eigenvalues*

$$\lambda_1\leqslant\ldots\leqslant\lambda_n<0.$$

If $\psi_n\in H^s(\mathbb{R})\cap C^0(\mathbb{R})$ is a real eigenfunction of H with eigenvalue λ_n, then its extension $\mathcal{E}_a\psi_n$ has at most n nodal domains in $\mathbb{R}^2_+$.

Proof. We argue by contradiction. Assume that $\mathcal{E}_a\psi_n$ has nodal domains $\Omega_1, ..., \Omega_m$, where $m \geqslant n+1$. We consider the sets $K_j := \{x \in \mathbb{R} : (x,0) \in \overline{\Omega}_j\}$ for $j=1, ..., m$, where $\overline{\Omega}_j$ is the closure of Ω_j in $\overline{\mathbb{R}^2_+}$. Since $\mathcal{E}_a\psi_n$ is continuous up to the boundary and $\psi_n \not\equiv 0$, we may assume that $K_1 \neq \varnothing$. Furthermore, let M be an $(n-1)$-dimensional subspace of $L^2(\mathbb{R})$ spanned by eigenfunctions corresponding to the eigenvalues λ_j, $j=1, ..., n-1$. Next, we define the function

$$u = (\mathcal{E}_a\psi_n) \sum_{j=1}^{n} \gamma_j \mathbb{1}_{\overline{\Omega}_j}.$$

Note that we can choose the constants $\gamma_j \in \mathbb{R}$, $j=1, ..., n$, in such a way that $u(\cdot, 0) \perp M$ and $\|u(\cdot, 0)\|_2 = 1$. Using standard facts about Sobolev spaces, one can show that

$$u \in \mathcal{H}^{1,a}(\mathbb{R}^2_+) \quad \text{and} \quad \nabla u = (\nabla \mathcal{E}_a\psi_n) \sum_{j=1}^{n} \gamma_j \mathbb{1}_{\overline{\Omega}_j}.$$

By the same argument as in the proof of Proposition 3.6, the function $\mathcal{E}_a\psi_n$ satisfies

$$\frac{1}{c_a} \iint_{\mathbb{R}^2_+} \overline{\nabla v} \cdot (\nabla \mathcal{E}_a\psi_n) y^a \, dx \, dy + \int_{\mathbb{R}} V(x) \overline{v(x,0)} \psi_n(x) \, dx = \lambda_n \int_{\mathbb{R}} \overline{v(x,0)} \psi_n(x) \, dx$$

for any $v \in \mathcal{H}^{1,a}(\mathbb{R}^2_+)$. We can apply this to $v = |\gamma_j|^2 \mathbb{1}_{\overline{\Omega}_j} \mathcal{E}_a\psi_n$, $j=1, ..., n$, (which belong to $\mathcal{H}^{1,a}(\mathbb{R}^2_+)$) and sum over j to obtain

$$\begin{aligned} \frac{1}{c_a} \iint_{\mathbb{R}^2_+} |\nabla u|^2 y^a \, dx \, dy + \int_{\mathbb{R}} V(x) |u(x,0)|^2 \, dx &= \lambda_n \sum_{j=1}^{n} |\gamma_j|^2 \int_{K_j} |\psi_n|^2 \, dx \\ &= \lambda_n \|u(\cdot, 0)\|_2^2 = \lambda_n. \end{aligned}$$

Thus we conclude that equality holds in the variational principle in Corollary 3.8. Hence $u = \mathcal{E}_a f$, where $f \in M^\perp$ is a linear combination of eigenfunctions corresponding to the eigenvalue λ_n. In particular, the non-trivial function u satisfies equation (3.8). Note that $u \equiv 0$ on the open non-empty set $\Omega_{n+1} \subset \mathbb{R}^2_+$. However, we can deduce, by unique continuation of solutions of the elliptic equation (3.8), that $u \equiv 0$ on the upper half-plane $\mathbb{R}^2_+$. Indeed, consider the open connected set $D_\delta = \{(x,y) \in \mathbb{R}^2_+ : \delta < y < 1/\delta\}$, where $0 < \delta < 1$ is a fixed constant. Clearly, the differential operator L on D_δ with $Lu = \operatorname{div}(y^a \nabla u)$ has smooth coefficients, and moreover L is uniformly elliptic on D_δ (with bounds depending on δ). By choosing $\delta_0 > 0$ so small that $\Omega_{n+1} \cap D_{\delta_0} \neq \varnothing$, we deduce by standard unique continuation for $Lu = 0$ that $u \equiv 0$ on the connected set D_{δ_0}. We can repeat this argument for any set $D_\delta \subset \mathbb{R}^2_+$ with $0 < \delta \leqslant \delta_0$ to conclude that $u \equiv 0$ on $\mathbb{R}^2_+$ itself. But this is a contradiction. The proof of Theorem 3.9 is now complete. □

3.4. Proof of Theorem 3.4

We argue by contradiction. Suppose that $\psi_2:\mathbb{R}\to\mathbb{R}$ changes its sign at least three times on $\mathbb{R}$. Thus, after replacing ψ_2 by $-\psi_2$ if necessary, there exist points $x_1<x_2<x_3<x_4$ on the real line such that

$$\psi(x_j)>0 \text{ for } j=1,3 \quad \text{and} \quad \psi(x_j)<0 \text{ for } j=2,4.$$

Next, we consider the extension $\mathcal{E}_a\psi_2$ on $\mathbb{R}^2_+$. Since

$$(\mathcal{E}_a\psi_2)(x_1,0)>0 \quad \text{and} \quad (\mathcal{E}_a\psi_2)(x_2,0)<0,$$

and by continuity of $\mathcal{E}_a\psi_2$ up to the boundary $\partial\mathbb{R}^2_+$, the function $\mathcal{E}_a\psi_2$ has at least two nodal domains in $\mathbb{R}^2_+$. But, in view of Theorem 3.9, we conclude that $\mathcal{E}_a\psi_2$ has exactly two nodal domains in $\mathbb{R}^2_+$, which we denote by Ω_+ and Ω_- from now on.

By continuity of $\mathcal{E}_a\psi_2(x,y)$ again, we deduce that

$$(x_j,\varepsilon)\in\Omega_+ \text{ for } j=1,3 \quad \text{and} \quad (x_j,\varepsilon)\in\Omega_- \text{ for } j=2,4,$$

for all $0<\varepsilon\leqslant\varepsilon_0$, where $\varepsilon_0>0$ is some sufficiently small constant. Note that the connected open sets $\Omega_\pm\subset\mathbb{R}^2_+$ must be arcwise connected. Thus we conclude that there exist two simple continuous curves $\gamma_+,\gamma_-\in C^0([0,1];\overline{\mathbb{R}^2_+})$ with the following properties:

- $\gamma_+(0)=(x_1,0),\gamma_+(1)=(x_3,0)$ and $\gamma_+(t)\in\Omega_+$ for $t\in(0,1)$;
- $\gamma_-(0)=(x_2,0),\gamma_-(1)=(x_4,0)$ and $\gamma_-(t)\in\Omega_-$ for $t\in(0,1)$.

By Lemma D.1 (based on basic topological arguments) we deduce that γ_+ and γ_- must intersect in the upper half-plane $\mathbb{R}^2_+$. But this contradicts $\Omega_+\cap\Omega_-=\varnothing$. Hence the function $\psi_2:\mathbb{R}\to\mathbb{R}$ changes its sign at most twice on $\mathbb{R}$.

Finally, if $\psi_2=\psi_2(|x|)$ is even, then clearly ψ_2 can change its sign on $\{x>0\}$ at most once, since otherwise ψ_2 would change its sign at least four times on $\mathbb{R}$, contradicting the result just derived. By the remark following Theorem 3.4, we deduce that ψ_2 must change its sign at least once on $\mathbb{R}_+$. This completes the proof of Theorem 3.4.

4. Non-degeneracy of ground states

This section is devoted to the proof of Theorem 2.3. That is, we show that (local) non-negative minimizers $Q(x)\geqslant 0$ for the functional $J^{s,\alpha}$ defined in (1.5) have a non-degenerate linearization. In fact, we shall prove a slightly more general result formulated as Lemma 4.1 below.

Let $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$ be fixed throughout this section. Suppose that $Q\in H^s(\mathbb{R})$, with $Q\not\equiv 0$, is a non-negative solution $Q=Q(x)\geqslant 0$ of

$$(-\Delta)^s Q+\lambda Q-Q^{\alpha+1}=0, \tag{4.1}$$

with some positive constant $\lambda>0$. Note that, by Lemma B.4 which is based on the method of moving planes, we have $Q(x)=\widetilde{Q}(|x-x_0|)>0$ for some $x_0\in\mathbb{R}$, where $\widetilde{Q}=\widetilde{Q}(|x|)>0$ is an even and positive function strictly decreasing in $|x|$. Moreover, a simple rescaling argument shows that we could assume $\lambda=1$ without loss of generality. But for the sake of later purpose, we will keep $\lambda>0$ explicit here.

Associated with $Q\in H^s(\mathbb{R})$, we define the linearized operator

$$L_+=(-\Delta)^s+\lambda-(\alpha+1)Q^\alpha \tag{4.2}$$

acting on $L^2(\mathbb{R})$. We record the following (partly immediate) facts about L_+.

- $Q^\alpha\in K_s$, i.e., the potential $V=Q^\alpha$ belongs to the 'Kato-class' with respect to $(-\Delta)^s$. This follows from the remark following Definition 3.1 and Sobolev inequalities. In particular, any L^2-eigenfunction of L_+ is continuous and bounded.
- L_+ is a self-adjoint operator on $L^2(\mathbb{R})$ with quadratic-form domain $H^s(\mathbb{R})$ and operator domain $H^{2s}(\mathbb{R})$.
- The essential spectrum is $\sigma_{\mathrm{ess}}(L_+)=[\lambda,\infty)$.
- The Morse index of L_+ satisfies $\mathcal{N}_-(L_+)\geqslant 1$. Recall that $\mathcal{N}_-(L_+)$ is defined as the number of strictly negative eigenvalues, i.e.,

$$\mathcal{N}_-(L_+)=\#\{e<0: e \text{ is eigenvalue of } L_+ \text{ acting on } L^2(\mathbb{R})\},$$

where multiplicities of eigenvalues are taken into account. To see that indeed $\mathcal{N}_-(L_+)\geqslant 1$, we just use $\langle Q, L_+Q\rangle=-\alpha\|Q\|_{\alpha+2}^{\alpha+2}<0$ by (4.1). Thus, by the min-max principle, the operator L_+ has at least one negative eigenvalue.

- We always have $L_+Q'=0$, and thus $\mathrm{span}\{Q'\}\subseteq\ker L_+$. This follows from differentiating (4.1) with respect to x.
- The lowest eigenvalue $e_0=\inf\sigma(L_+)$ is simple and the corresponding eigenfunction $\psi_0=\psi_0(x)>0$ strictly positive; see Lemma C.2.

To formulate the main result of this section, we now suppose that $Q=Q(|x|)$ is an even function. We introduce the *Morse index of L_+ in the sector of even functions* by defining

$$\mathcal{N}_{-,\mathrm{even}}(L_+):=\#\{e<0: e \text{ is eigenvalue of } L_+ \text{ restricted to } L^2_{\mathrm{even}}(\mathbb{R})\},$$

where multiplicities of eigenvalues are taken into account. Note that $\langle Q, L_+Q\rangle<0$ with $Q=Q(|x|)$ even. Hence we deduce the general lower bound $\mathcal{N}_{-,\mathrm{even}}(L_+)\geqslant 1$ from the min-max principle.

The key non-degeneracy result of this section is now as follows.

LEMMA 4.1. *Let $Q\in H^s(\mathbb{R})$ be an even and positive solution of (4.1) with some $\lambda>0$. Consider its associated linearized operator L_+ acting on $L^2(\mathbb{R})$ and assume that its Morse index in the even sector satisfies $\mathcal{N}_{-,\mathrm{even}}(L_+)=1$. Then we have*

$$\ker L_+ = \operatorname{span}\{Q'\}.$$

Remark 4.2. We will see below that the second-order condition

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} J^{s,\alpha}(Q+\varepsilon\eta)\geqslant 0 \quad \text{for all } \eta\in C_0^\infty(\mathbb{R})$$

will imply that $\mathcal{N}_{-,\mathrm{even}}(L_+)=1$. In particular, any ground state solution $Q=Q(|x|)>0$ will satisfy the assumption of Lemma 4.1.

Proof. By rescaling $Q(x)\mapsto\lambda^{1/\alpha}Q(\lambda^{x/2s})$ (and likewise any element in $\ker L_+$ transforms accordingly), we may assume that $\lambda=1$ holds in (4.1).

Next, we consider the orthogonal decomposition $L^2(\mathbb{R})=L^2_{\mathrm{even}}(\mathbb{R})\oplus L^2_{\mathrm{odd}}(\mathbb{R})$. Since $Q=Q(|x|)$ is an even function, we note that L_+ leaves the subspaces $L^2_{\mathrm{even}}(\mathbb{R})$ and $L^2_{\mathrm{odd}}(\mathbb{R})$ invariant. We treat these subspaces separately as follows.

Recall that $Q'\in L^2_{\mathrm{odd}}(\mathbb{R})$ satisfies $L_+Q'=0$. Moreover, by Lemma B.4, we have that $Q'(x)<0$ for $x>0$. In view of Lemma C.3 applied to L_+, we conclude that Q' is (up to a sign) the unique ground state eigenfunction of L_+ restricted to $L^2_{\mathrm{odd}}(\mathbb{R})$. Hence we see that $\ker L_+|_{L^2_{\mathrm{odd}}}=\operatorname{span}\{Q'\}$.

It remains to show that

$$\ker L_+|_{L^2_{\mathrm{even}}} = \{0\}.$$

To prove this claim, we argue by contradiction. Suppose there exists $v\in L^2_{\mathrm{even}}(\mathbb{R})$, with $v\not\equiv 0$, such that $L_+v=0$. Note that v is continuous and bounded due to the remarks above. Also, since L_+ is a real operator, we may assume that v is real-valued. Next, by assumption, we have $\mathcal{N}_{-,\mathrm{even}}(L_+)=1$, and hence v must be an even eigenfunction of L_+ corresponding to its second eigenvalue. By applying Theorem 3.4 to

$$H=(-\Delta)^s-(\alpha+1)Q^\alpha,$$

we deduce that $v=v(|x|)$ changes its sign exactly once as a function of $|x|$. That is, there exists $r_*>0$ such that the following holds (after multiplying v by -1 if necessary):

$$v(|x|)\geqslant 0 \ \text{ for } |x|\leqslant r_* \quad \text{and} \quad v(|x|)\leqslant 0 \ \text{ for } |x|>r_*, \tag{4.3}$$

where $v\not\equiv 0$ on both sets $\{x:|x|\leqslant r_*\}$ and $\{x:|x|>r_*\}$. Note that we have the same estimate on the number of sign changes of $v=v(|x|)$ on the half-line $\mathbb{R}_+$, as if Sturm–Liouville

oscillation theory for ODEs were applicable. Therefore we can now proceed along the lines of [11], where a simple non-degeneracy proof for non-linear Schrödinger ground states was given based on a result from Sturm–Liouville theory. Adapting this argument to our setting, we notice that a calculation yields

$$L_+Q=-\alpha Q^{\alpha+1} \quad \text{and} \quad L_+R=-2sQ, \tag{4.4}$$

where

$$R:=\frac{d}{d\beta}\bigg|_{\beta=1}\beta^{2s/\alpha}Q(\beta\,\cdot\,)=\frac{2s}{\alpha}Q+xQ'. \tag{4.5}$$

Note that $R\in L^2(\mathbb{R})$, due to the decay estimate stated in Proposition 1.1. By bootstrapping the equation satisfied by R, we further deduce that $R\in H^{2s+1}(\mathbb{R})$ and, in particular, we see that R is in the domain of L_+. Since L_+ is self-adjoint and $v\in\ker L_+$, we obtain from (4.4) that

$$\langle Q^{\alpha+1},v\rangle=\langle Q,v\rangle=0.$$

Next, we consider the even function $f\in\operatorname{ran}L_+$ given by

$$f:=Q^{\alpha+1}-\mu Q=Q(Q^\alpha-\mu),$$

where $\mu\in\mathbb{R}$ is some parameter. Note that $\langle v,f\rangle=0$ for all $\mu\in\mathbb{R}$. Now choose $\mu=Q(r_*)^\alpha$ with $r_*>0$ from (4.3). Since $Q=Q(|x|)>0$ is positive and strictly decreasing in $|x|$, we deduce that

$$f(|x|)>0 \ \text{ for } |x|<r_* \quad \text{and} \quad f(|x|)<0 \ \text{ for } |x|>r_*. \tag{4.6}$$

Combining now (4.6) and (4.3), we see that $vf\geqslant 0$ with $vf\not\equiv 0$. Hence $\langle v,f\rangle>0$. But this violates the orthogonality condition $\langle v,f\rangle=0$. Therefore, the operator L_+ does not have a zero eigenfunction in $L^2_{\mathrm{even}}(\mathbb{R})$. The proof of Lemma 4.1 is now complete. □

Proof of Theorem 2.3

Suppose that $Q=Q(x)>0$ is a positive solution to (4.1) with $\lambda=1$. By Lemma B.4 and translational invariance, we may assume that $Q=Q(|x|)>0$ is even.

Let

$$L_+=(-\Delta)^s+1-(\alpha+1)Q^\alpha$$

be the associated linearized operator. In order to apply Lemma 4.1, it suffices to show that $\mathcal{N}_{-,\mathrm{even}}(L_+)=1$ holds. Indeed, we recall that, by assumption in Theorem 2.3, the function Q is a local minimizer of $J^{s,\alpha}$. Therefore, we have the second-order condition

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0}J^{s,\alpha}(Q+\varepsilon\eta)\geqslant 0 \quad \text{for all } \eta\in C_0^\infty(\mathbb{R}). \tag{4.7}$$

We claim that this implies the upper bound $\mathcal{N}_{-,\mathrm{even}}(L_+)\leqslant\mathcal{N}_-(L_+)\leqslant 1$. To estimate the Morse index, we can adapt an argument for ground states of classical non-linear Schrödinger equations (see [11] and [33]) to our setting as follows.

By Lemma 5.4 below (with $\lambda_s=1$ and $s=s_0$), we obtain the following Pohozaev identities of the form

$$\frac{1}{2a}\int_{\mathbb{R}}|(-\Delta)^{s/2}Q|^2\,dx=\frac{1}{2b}\int_{\mathbb{R}}|Q|^2\,dx=\frac{1}{\alpha+2}\int_{\mathbb{R}}|Q|^{\alpha+2}\,dx=:k,$$

where $a=\alpha/4s>0$, $b=\alpha(2s-1)/4s+1>0$ and $k>0$ are positive constants. Then an elementary (but tedious) calculation shows that inequality (4.7) is equivalent to

$$k\langle\eta,L_+\eta\rangle\geqslant\frac{1}{a}|\langle\eta,(-\Delta)^sQ\rangle|^2+\frac{1}{b}|\langle\eta,Q\rangle|^2-|\langle\eta,Q^{\alpha+1}\rangle|^2.$$

Therefore $\langle\eta,L_+\eta\rangle\geqslant 0$ if $\eta\perp Q^{\alpha+1}$. By the min-max principle, we obtain that $\mathcal{N}_-(L_+)\leqslant 1$, and hence $\mathcal{N}_{-,\mathrm{even}}(L_+)\leqslant\mathcal{N}_-(L_+)\leqslant 1$.

On the other hand, we recall that we always have $\mathcal{N}_{-,\mathrm{even}}(L_+)\geqslant 1$, as remarked in the beginning of this section. Thus we conclude that $\mathcal{N}_{-,\mathrm{even}}(L_+)=1$ holds, whence it follows that $\ker L_+=\mathrm{span}\{Q'\}$, due to Lemma 4.1. The proof of Theorem 2.3 is now complete.

5. Uniqueness of ground states

In this section we prove Theorem 2.4. Our strategy is based on the non-degeneracy result from §4 and an implicit function argument, combined with a global continuation argument. For the reader's orientation, we first give a brief outline of this section as follows.

In §5.1, we fix $0<s_0<1$ and $0<\alpha<\alpha_{\max}(s_0)$. By an implicit function argument, we construct (in some suitable Banach space of even functions) a locally unique branch (Q_s,λ_s) parameterized by s close to s_0 and satisfying

$$(-\Delta)^sQ_s+\lambda_sQ_s-|Q_s|^\alpha Q_s=0.$$

Here the starting point of the branch $(Q_0,\lambda_0)=(Q_{s=s_0},\lambda_{s=s_0})$ is assumed to satisfy some spectral condition; see Proposition 5.2.

Then, in §5.2, we show (as a main result of this section) that the local branch (Q_s,λ_s) can be indeed globally continued to $s=1$, provided that (Q_0,λ_0) satisfies some explicit conditions, such as positivity $Q_0=Q_0(|x|)>0$; see Proposition 5.13. The crucial

and delicate point that allows us to extend to $s=1$ is based on suitable a-priori bounds on regularity and spatial decay for (Q_s, λ_s) of the form

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} Q_s|^2 \, dx \sim 1, \quad \int_{\mathbb{R}} |Q_s|^2 \, dx \sim 1 \quad \text{and} \quad \lambda_s \sim 1,$$

in combination with a uniform pointwise decay estimate $Q_s(|x|) \lesssim |x|^{-1}$ for $|x| \gtrsim 1$. The derivation of all these bounds will cover most of this section and it requires a careful study of the non-linear problem.

Finally, with help of Propositions 5.2 and 5.13, we are able to prove Theorem 2.4 in §5.3 below. That is, we show that the branch (Q_s, λ_s) starting from a ground state $(Q_0, \lambda_0=1)$ exists and is globally unique; in particular, the assumption of having another branch starting from a different ground state $(\widetilde{Q}_0, \tilde{\lambda}_0=1)$ leads to a contradiction. This will follow from the global uniqueness and non-degeneracy for the limiting problem when $s=1$, i.e.,

$$-\Delta Q_* + \lambda_* Q_* - Q_*^{\alpha+1} = 0.$$

5.1. Construction of a local branch

We start with some preliminaries. Let $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$ be given. We consider solutions (Q, λ) with $Q \in L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R})$ and $\lambda \in \mathbb{R}_+$ satisfying

$$(-\Delta)^s Q + \lambda Q - |Q|^\alpha Q = 0. \tag{5.1}$$

In fact, by a bootstrap argument, we see that $Q \in H^{2s+1}(\mathbb{R})$ holds. Nevertheless, it turns out to be convenient to work in the space $L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R})$, which is independent of s. Since we are interested in real-valued and even solutions only, it is convenient to define the (real) Banach space

$$X^\alpha := \{ f \in L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R}) : f \text{ is even and real-valued} \}, \tag{5.2}$$

which we equip with the norm

$$\|f\|_{X^\alpha} := \|f\|_2 + \|f\|_{\alpha+2}.$$

Recall that we make the standard abuse of notation by writing both $f(x)$ and $f(|x|)$ whenever f is an even function on $\mathbb{R}$.

As a next step, we will construct a local branch of solutions $(Q_s, \lambda_s) \in X^\alpha \times \mathbb{R}_+$ of (5.1), which is parameterized by s in some small interval. To this end, we introduce the following assumption.

Assumption 5.1. Let $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$. Suppose that $(Q,\lambda)\in X^\alpha\times\mathbb{R}_+$ satisfies equation (5.1). We assume that the linearized operator

$$L_+=(-\Delta)^s+\lambda-(\alpha+1)|Q|^\alpha$$

acting on $L^2(\mathbb{R})$ has a bounded inverse L_+^{-1} on $L^2_{\mathrm{even}}(\mathbb{R})$.

Remarks. (1) We emphasize that we do not require $Q\in X^\alpha$ to be positive here.

(2) Since $Q\in X^\alpha$ is even, and hence $Q\perp Q'$ in $L^2(\mathbb{R})$, the bounded inverse L_+^{-1} exists on $L^2_{\mathrm{even}}(\mathbb{R})$, provided that L_+ is non-degenerate, i.e., we have $\ker L_+=\mathrm{span}\{Q'\}$ for L_+ acting on $L^2(\mathbb{R})$.

(3) By Sobolev inequalities, the invertibility of L_+^{-1} on $L^2_{\mathrm{even}}(\mathbb{R})$ implies that L_+^{-1} exists on X^α as well.

As a next step, we establish existence and local uniqueness of a branch (Q_s,λ_s) for (5.1) around a solution (Q_0,λ_0) that satisfies Assumption 5.1.

PROPOSITION 5.2. *Let $0<s_0<1$ and $0<\alpha<\alpha_{\max}(s_0)$. Suppose that*

$$(Q_0,\lambda_0)\in X^\alpha\times\mathbb{R}_+$$

satisfies Assumption 5.1 with $s=s_0$ and $\lambda=\lambda_0$. Then, for some $\delta>0$, there exists a map $(Q,\lambda)\in C^1(I;X^\alpha\times\mathbb{R}_+)$, defined on the interval $I=[s_0,s_0+\delta)$, such that the following holds, where we write $(Q_s,\lambda_s)=(Q(s),\lambda(s))$ in the sequel:

(i) *(Q_s,λ_s) solves equation (5.1) with $\lambda=\lambda_s$ for all $s\in I$ and (Q_s,λ_s) satisfies Assumption 5.1.*

(ii) *There exists $\varepsilon>0$ such that (Q_s,λ_s) is the unique solution of (5.1) for $s\in I$ in the neighborhood $\{(Q,\lambda)\in X^\alpha\times\mathbb{R}_+:\|Q-Q_0\|_X+|\lambda-\lambda_0|<\varepsilon\}$. In particular, we have that $(Q_{s_0},\lambda_{s_0})=(Q_0,\lambda_0)$.*

(iii) *For all $s\in I$, we have*

$$\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx=\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx.$$

Remark 5.3. Introducing the function $\lambda_s=\lambda(s)$ ensures that the above "conservation law" for $\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx$ holds. The use of this fact will become evident further below when we derive a-priori bounds.

Proof. We use an implicit function argument as follows. First, we observe that (5.1) can be written as

$$Q=\frac{1}{(-\Delta)^s+\lambda}|Q|^\alpha Q.$$

For some small constant $\delta>0$ chosen below, we consider the mapping

$$F\colon X^\alpha\times\mathbb{R}_+\times[s_0,s_0+\delta)\longrightarrow X^\alpha\times\mathbb{R}, \tag{5.3}$$

which we define as

$$F(Q,\lambda,s):=\begin{pmatrix} Q-\dfrac{1}{(-\Delta)^s+\lambda}|Q|^\alpha Q \\ \|Q\|_{\alpha+2}^{\alpha+2}-\|Q_{s_0}\|_{\alpha+2}^{\alpha+2}\end{pmatrix}. \tag{5.4}$$

As shown in Lemma E.1, the map F is well defined and C^1. Also, by construction, we have that $F(Q_{s_0},\lambda_0,s_0)=0$. To invoke an implicit function argument, we have to show the invertibility of the Fréchet deriviative of F with respect (Q,λ) at (Q_0,λ_0,s_0), which we establish next.

First, we note that the Fréchet derivative of F with respect to (Q,λ) is given by

$$\partial_{(Q,\lambda)}F=\begin{pmatrix} 1-\dfrac{1}{(-\Delta)^s+\lambda}(\alpha+1)|Q|^\alpha & \dfrac{1}{((-\Delta)^s+\lambda)^2}|Q|^\alpha Q \\ (\alpha+2)\langle|Q|^\alpha Q,\cdot\rangle & 0\end{pmatrix}. \tag{5.5}$$

Here $\langle f,\cdot\rangle$ denotes the map $g\mapsto\langle f,g\rangle$. See also Lemma E.1 and its proof.

Now, we claim that the inverse $(\partial_{(Q,\lambda)}F)^{-1}$ exists at (Q_{s_0},λ_0,s_0). Hence we have to show that, for every $f\in X^\alpha$ and $\beta\in\mathbb{R}$ given, there is a unique solution $(\eta,\gamma)\in X^\alpha\times\mathbb{R}$ of the system

$$(1+K)\eta+\gamma g=f, \tag{5.6}$$

$$(\alpha+2)\langle|Q_{s_0}|^\alpha Q_{s_0},\eta\rangle=\beta, \tag{5.7}$$

where we set

$$K:=-\frac{1}{(-\Delta)^{s_0}+\lambda_0}(\alpha+1)|Q_{s_0}|^\alpha \quad\text{and}\quad g:=\frac{1}{((-\Delta)^{s_0}+\lambda_0)^2}|Q_{s_0}|^\alpha Q_{s_0}. \tag{5.8}$$

Next, we note that K is a compact operator on $L^2_{\mathrm{even}}(\mathbb{R})$. Moreover, we see that $-1\notin\sigma(K)$ due to Assumption 5.1. Indeed, assume on the contrary that -1 is in the spectrum $\sigma(K)$ for K acting on $L^2_{\mathrm{even}}(\mathbb{R})$. Then the self-adjoint operator

$$L_+=(-\Delta)^{s_0}+\lambda_0-(\alpha+1)|Q_{s_0}|^\alpha \tag{5.9}$$

would have an even eigenfunction $v\in L^2_{\mathrm{even}}(\mathbb{R})$ such that $L_+v=0$, which contradicts Assumption 5.1.

Thus the operator $1+K$ is invertible on $L^2_{\mathrm{even}}(\mathbb{R})$. Moreover, since $K\colon X^\alpha\to X^\alpha$ holds (see the proof of Lemma E.1 for details), we deduce that $(1+K)^{-1}$ exists on the space X^α as well. Hence we can solve (5.6) uniquely for $\eta\in X^\alpha$ to find that

$$\eta=(1+K)^{-1}f-\gamma(1+K)^{-1}g. \tag{5.10}$$

Plugging this into (5.7) yields

$$(\alpha+2)\langle |Q_{s_0}|^\alpha Q_{s_0}, (1+K)^{-1} g\rangle \gamma = -\beta + (\alpha+2)\langle |Q_{s_0}|^\alpha Q_{s_0}, (1+K)^{-1} f\rangle. \tag{5.11}$$

To deduce unique solvability for $\gamma\in\mathbb{R}$, it remains to show that the coefficient in front of γ does not vanish. To see this, we observe the identity

$$(1+K)^{-1} = L_+^{-1}\circ((-\Delta)^{s_0}+\lambda_0), \tag{5.12}$$

with L_+ given by (5.9). Using this identity together with $L_+ Q_{s_0} = -\alpha |Q_{s_0}|^\alpha Q_{s_0}$ and equation (5.1) satisfied by Q_{s_0}, we now easily deduce that

$$\langle |Q_{s_0}|^\alpha Q_{s_0}, (1+K)^{-1} g\rangle = -\frac{1}{\alpha}\int_{\mathbb{R}} |Q_{s_0}|^2\, dx \neq 0. \tag{5.13}$$

This completes the proof that $\partial_{(Q,\lambda)} F$ is invertible at $(Q_{s_0}, \lambda_0, s_0)$. By applying the implicit function theorem to the map F at $(Q_{s_0}, \lambda_0, s_0)$, we derive the assertions (i)–(iii) provided that $\delta>0$ is sufficiently small.

The proof of Proposition 5.2 is now complete. □

5.2. A-priori bounds and global continuation

Let $0<s_0<1$ and $0<\alpha<\alpha_{\max}(s_0)$ be given. Throughout this subsection, we suppose that $(Q_s,\lambda_s)\in C^1(I; X^\alpha\times\mathbb{R}_+)$ is a local branch defined for $I=[s_0, s_0+\delta)$, as provided by Proposition 5.2.

Now, we consider the corresponding *maximal extension* of the branch (Q_s,λ_s) for $s\in[s_0, s_*)$, where s_* is given by

$$s_* := \sup\{s_0<\tilde{s}<1 : (Q_s,\lambda_s)\in C^1([s_0,\tilde{s}); X^\alpha\times\mathbb{R}_+) \text{ given by Proposition 5.2 and } (Q_s,\lambda_s) \text{ satisfies Assumption 5.1 for } s\in[s_0,\tilde{s})\}.$$

Clearly, we have $s_*\leqslant 1$, and our goal will be to show that $s_*=1$ holds under some suitable assumption on (Q_{s_0},λ_0).

In order to derive suitable a-priori bounds for the maximal branch (Q_s,λ_s), we recall the notation $a\lesssim b$, which means that $a\leqslant Cb$, where $C>0$ is some constant that only depends on the fixed quantities s_0, α and (Q_{s_0},λ_0). As usual, the constant C is allowed to change from inequality to inequality.

As an initial step to derive a-priori bounds, we start with the following *Pohozaev identities* satisfied by Q_s.

LEMMA 5.4. *For all $s\in[s_0,s_*)$ the following identities hold*:

$$\frac{\lambda_s}{2}\int_{\mathbb{R}}|Q_s|^2\,dx=\frac{a_s}{\alpha+2}\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx \quad \textit{and} \quad \frac{1}{2}\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx=\frac{b_s}{\alpha+2}\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx,$$

where $a_s=\alpha(2s-1)/4s+1$ and $b_s=\alpha/4s$.

Proof. By integrating (5.1) against $Q_s\in H^s(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx+\lambda_s\int_{\mathbb{R}}|Q_s|^2\,dx=\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx. \tag{5.14}$$

Next, we integrate (5.1) against xQ_s'. Integration by parts yields that

$$\langle xQ_s',|Q_s|^\alpha Q_s\rangle=-\frac{1}{\alpha+2}\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx \quad \text{and} \quad \langle xQ_s',(-\Delta)^sQ_s\rangle=\frac{2s-1}{2}\langle Q_s,(-\Delta)^sQ_s\rangle,$$

where for the second identity we also use that $[\nabla\cdot x,(-\Delta)^s]=-2s(-\Delta)^s$, which is easily verified in Fourier space. Hence, we deduce that

$$\frac{2s-1}{2}\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx-\frac{\lambda_s}{2}\int_{\mathbb{R}}|Q_s|^2\,dx=-\frac{1}{\alpha+2}\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx. \tag{5.15}$$

(Note that the calculations here involving xQ_s' are well defined, due to the regularity and decay estimates from Proposition 1.1.) By combining equations (5.14) and (5.15), we readily deduce Lemma 5.4. □

Next, we derive the following straightforward a-priori bounds.

LEMMA 5.5. *For all $s\in[s_0,s_*)$ we have the following bounds*:

$$\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx\sim 1, \quad \lambda_s\int_{\mathbb{R}}|Q_s|^2\,dx\sim 1 \quad \textit{and} \quad 1\lesssim\int_{\mathbb{R}}|Q_s|^2\,dx.$$

Proof. By Lemma 5.4, we obtain the desired a-priori bounds for $\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx$ and $\lambda_s\int_{\mathbb{R}}|Q_s|^2\,dx$, since we have $\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx=\text{const.}\neq 0$ along the branch (Q_s,λ_s) and clearly $a_s,b_s\sim 1$ holds for $s\in[s_0,s_*)$.

To derive the lower bound $\int_{\mathbb{R}}|Q_s|^2\,dx\gtrsim 1$, we recall the interpolation estimate (1.4), which yields that

$$\left(\int_{\mathbb{R}}|Q_s|^2\,dx\right)^{\alpha(2s-1)/4s+1}\geqslant\frac{1}{C_{s,\alpha}}\left(\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx\right)\left(\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx\right)^{-\alpha/4s}\geqslant\frac{1}{C},$$

with some constant $C>0$ independent of s. Here we used again that $\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx=\text{const.}$ and $\int_{\mathbb{R}}|(-\Delta)^{s/2}Q_s|^2\,dx\lesssim 1$ from above, as well as the fact that the optimal interpolation constants satisfy $C_{s,\alpha}\leqslant K$ by Lemma A.5 for some $K>0$ uniformly in $s\geqslant s_0>\alpha/2(\alpha+2)$. Here the last strict inequality is due to $\alpha<\alpha_{\max}(s_0)$. □

We now derive an a-priori *upper bound* for $\int_{\mathbb{R}} |Q_s|^2\,dx$ along the branch (λ_s, Q_s). In fact, this result will be one of the key steps in order to extend the branch all the way to $s_*=1$. The proof of the following fact requires substantially more insight into the problem and it will also make use of some auxiliary results, which we derive further below.

LEMMA 5.6. *For all $s\in[s_0, s_*)$, we have the a-priori upper bound*

$$\int_{\mathbb{R}} |Q_s|^2\,dx \lesssim 1.$$

Proof. We will derive the following the differential inequality

$$\frac{d}{ds}\int_{\mathbb{R}} |Q_s|^2\,dx \lesssim \int_{\mathbb{R}} |Q_s|^2\,dx. \tag{5.16}$$

Once this estimate is established, the desired a-priori bound follows from integrating this differential inequality.

To show (5.16), we argue as follows. First, we note that

$$\frac{d}{ds}\int_{\mathbb{R}} |Q_s|^2\,dx = 2\left\langle Q_s, \frac{dQ_s}{ds}\right\rangle.$$

Next, by differentiating the equation (5.1) satisfied by Q_s with respect to s, we see that

$$L_+\frac{dQ_s}{ds} = -(-\Delta)^s \log(-\Delta) Q_s - \frac{d\lambda_s}{ds} Q_s,$$

with $L_+=(-\Delta)^s+\lambda_s-(\alpha+1)|Q_s|^\alpha$. Recall that λ_s is a differentiable function of s. Also, by Proposition 1.1, we note that $Q_s\in H^{2s+1}(\mathbb{R})$ and hence $(-\Delta)^s\log(-\Delta)Q_s\in L^2_{\mathrm{even}}(\mathbb{R})$. Since L_+ is invertible on $L^2_{\mathrm{even}}(\mathbb{R})$ and self-adjoint, we can combine the previous equations to obtain

$$\frac{d}{ds}\int_{\mathbb{R}} |Q_s|^2\,dx = \mathrm{I}+\mathrm{II}, \tag{5.17}$$

where

$$\mathrm{I} = -2\langle L_+^{-1}Q_s, (-\Delta)^s\log(-\Delta)Q_s\rangle \quad \text{and} \quad \mathrm{II} = -2\frac{d\lambda_s}{ds}\langle Q_s, L_+^{-1}Q_s\rangle. \tag{5.18}$$

We start by estimating the term I from above. Here a calculation shows that[3]

$$L_+R = -2s\lambda_s Q_s, \quad \text{with } R := \frac{d}{d\beta}\Big|_{\beta=1} \beta^{2s/\alpha}Q_s(\beta x) = \frac{2s}{\alpha}Q_s + xQ_s'. \tag{5.19}$$

([3]) Note that $R\in L^2(\mathbb{R})$, by the decay estimate in Proposition 1.1. Moreover, we easily deduce that $R\in H^{2s+1}(\mathbb{R})$, by analogous bootstrap arguments as done for Q.

Therefore we conclude that

$$\begin{aligned}
\mathrm{I} &= \frac{1}{s\lambda_s}\langle R, (-\Delta)^s \log(-\Delta) Q_s\rangle \\
&= \frac{1}{s\lambda_s}\left\langle \frac{d}{d\beta}\Big|_{\beta=1} \beta^{2s/\alpha} Q_s(\beta\,\cdot), (-\Delta)^s \log(-\Delta) Q_s \right\rangle \\
&= \frac{1}{2s\lambda_s}\frac{d}{d\beta}\Big|_{\beta=1} \langle \beta^{2s/\alpha} Q_s(\beta\,\cdot), (-\Delta)^s \log(-\Delta) \beta^{2s/\alpha} Q_s(\beta\,\cdot)\rangle \\
&= \frac{1}{2s\lambda_s}\frac{d}{d\beta}\Big|_{\beta=1} \left(\beta^{\alpha/4s+2s-1} \int_{\mathbb{R}} |\widehat{Q}_s(\xi)|^2 |\xi|^{2s} \log(\beta^2|\xi|^2)\, d\xi \right) \\
&= \frac{1}{2s\lambda_s}\left(\left(\frac{4s}{\alpha}+2s-1\right)\langle Q_s, (-\Delta)^s \log(-\Delta) Q_s\rangle + 2\langle Q_s, (-\Delta)^s Q_s\rangle \right).
\end{aligned}$$

In the third equality above, we used the self-adjointness of $(-\Delta)^s \log(-\Delta)$; whereas the fourth equality follows from Plancherel's identity and change of variables. Note that all the manipulations here are well defined, due to the regularity of $Q_s \in H^{2s+1}(\mathbb{R})$.

Next, we apply Lemma 5.7 (derived below) which shows that the a-priori upper bound $\langle Q_s, (-\Delta)^s \log(-\Delta) Q_s\rangle \lesssim 1$ holds. Moreover, we notice that

$$\frac{4s}{\alpha}+2s-1 \geqslant \frac{4s_0}{\alpha}+2s_0-1 > 0,$$

due to the condition that $\alpha < \alpha_{\max}(s_0)$. In summary, we deduce that

$$\mathrm{I} \lesssim \frac{1}{\lambda_s} \lesssim \int_{\mathbb{R}} |Q_s|^2\, dx$$

for $s_0 \leqslant s < s_*$, where the last inequality clearly follows from Lemma 5.5.

It remains to derive an upper bound for II defined in (5.18) above. To this end, we recall the definition of R in (5.19) which shows that

$$\begin{aligned}
\langle Q_s, L_+^{-1} Q_s\rangle &= -\frac{1}{2s\lambda_s}\langle R, Q_s\rangle = -\frac{1}{4s\lambda_s}\frac{d}{d\beta}\Big|_{\beta=1} \beta^{4s/\alpha}\langle Q_s(\beta\,\cdot), Q_s(\beta\,\cdot)\rangle \\
&= \frac{1}{4s\lambda_s}\left(1-\frac{4s}{\alpha}\right)\int_{\mathbb{R}} |Q_s|^2\, dx.
\end{aligned} \tag{5.20}$$

Next, if we differentiate the Pohozaev identities in Lemma 5.4 with respect to s, we obtain

$$\frac{d\lambda_s}{ds}\int_{\mathbb{R}} |Q_s|^2\, dx + \lambda_s \frac{d}{ds}\int_{\mathbb{R}} |Q_s|^2\, dx = \frac{1}{2s^2}\frac{\alpha}{\alpha+2}\int_{\mathbb{R}} |Q_s|^{\alpha+2}\, dx, \tag{5.21}$$

using that

$$\frac{d}{ds}\int_{\mathbb{R}} |Q_s|^{\alpha+2}\, dx = 0.$$

By combining (5.20) and (5.21), we deduce that

$$\begin{aligned}\mathrm{II} &= -2\frac{d\lambda_s}{ds}\langle Q_s, L_+^{-1}Q_s\rangle = \left(\frac{1}{2s}-\frac{2}{\alpha}\right)\left(\frac{d}{ds}\int_{\mathbb{R}}|Q_s|^2\,dx - \frac{1}{2s^2\lambda_s}\frac{\alpha}{\alpha+2}\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx\right)\\ &\leqslant \left(\frac{1}{2s}-\frac{2}{\alpha}\right)\frac{d}{ds}\int_{\mathbb{R}}|Q_s|^2\,dx + \frac{C}{\lambda_s},\end{aligned}$$

for some constant $C>0$ independent of s. In the last step, we used again the fact that $\int_{\mathbb{R}}|Q_s|^{\alpha+2}\,dx$ is a constant. Next, we recall that $\lambda_s^{-1}\lesssim\int_{\mathbb{R}}|Q_s|^2\,dx$ and $\mathrm{I}\lesssim\int_{\mathbb{R}}|Q_s|^2\,dx$. Hence, we get

$$\frac{d}{ds}\int_{\mathbb{R}}|Q_s|^2\,dx = \mathrm{I}+\mathrm{II} \leqslant \left(\frac{1}{2s}-\frac{2}{\alpha}\right)\frac{d}{ds}\int_{\mathbb{R}}|Q_s|^2\,dx + C\int_{\mathbb{R}}|Q_s|^2\,dx,$$

where $C>0$ is some constant independent of s. Noticing again that

$$1-\left(\frac{1}{2s}-\frac{2}{\alpha}\right) \geqslant 1-\left(\frac{1}{2s_0}-\frac{2}{\alpha}\right) > 0,$$

by the condition $\alpha<\alpha_{\max}(s_0)$, we conclude that (5.16) holds. The proof of Lemma 5.6 is now complete. □

Next, we establish the a-priori upper bound on $\langle Q_s, (-\Delta)^s\log(-\Delta)Q_s\rangle$, which was needed in the previous proof.

LEMMA 5.7. *For all $s\in[s_0,s_*)$ we have*

$$\langle Q_s, (-\Delta)^s\log(-\Delta)Q_s\rangle \lesssim 1.$$

Proof. From the identity $Q_s=((-\Delta)^s+\lambda_s)^{-1}|Q_s|^\alpha Q_s$ we deduce that

$$\|(-\Delta)^t Q_s\|_2 = \left\|\frac{(-\Delta)^t}{(-\Delta)^s+\lambda_s}|Q_s|^\alpha Q_s\right\|_2 \leqslant \|(-\Delta)^{t-s}(|Q_s|^\alpha Q_s)\|_2 \tag{5.22}$$

for any $t\geqslant 0$. In particular, we can choose

$$t := s-\frac{\alpha}{4(\alpha+2)},$$

which implies that $s>t>s-\frac{1}{2}s_0\geqslant\frac{1}{2}s_0$, due to the condition $\alpha<\alpha_{\max}(s_0)$.

By our choice of t, the operator $(-\Delta)^{t-s}$ on $\mathbb{R}$ is given by convolution with the singular integral kernel $|x|^{-(\alpha+4)/2(\alpha+2)}$, up to a multiplicative constant C depending only on α. Hence, by the weak Young inequality, we deduce from (5.22) the following bound:

$$\|(-\Delta)^t Q_s\|_2 \lesssim \big\||x|^{-(\alpha+4)/2(\alpha+2)} * (|Q_s|^\alpha Q_s)\big\|_2 \lesssim \big\||Q_s|^{\alpha+1}\big\|_{(\alpha+2)/(\alpha+1)} \lesssim \|Q_s\|_{\alpha+2}^{\alpha+1} \lesssim 1,$$

using the fact that $\int_{\mathbb{R}} |Q_s|^{\alpha+2}\,dx$ is a constant. But the last estimate implies that

$$\begin{aligned}\langle Q_s, (-\Delta)^s \log(-\Delta) Q_s\rangle &= \int_{\mathbb{R}} |\xi|^{2s} \log(|\xi|^2) |\widehat{Q}_s(\xi)|^2\,d\xi \\ &\lesssim \int_{\mathbb{R}} |\xi|^{4t} |\widehat{Q}_s(\xi)|^2\,d\xi \lesssim \|(-\Delta)^t Q_s\|_2^2 \lesssim 1.\end{aligned}$$

Here we used Plancherel's identity together with the inequality

$$\log|\xi|^2 \leqslant C|\xi|^{4t-2s}, \tag{5.23}$$

where the constant $C>0$ only depends on α and s_0. Indeed, this inequality can be simply derived as follows. Clearly it suffices to show that (5.23) holds for $|\xi|\geqslant 1$, since otherwise the left-hand side is negative. Now, we note that

$$4t-2s = 2s - \frac{\alpha}{\alpha+2} \geqslant 2s_0 - \frac{\alpha}{\alpha+2} \geqslant \delta$$

for some constant $\delta>0$ depending only on α and s_0. (To see this, simply use the strict inequality $\alpha/(\alpha+2)<2s_0$ due to the condition on α.) Since $\log z^2 \leqslant 2(\delta/e)^{-1} z^\delta$ for $z\geqslant 1$ and $\delta\leqslant 4t-2s$, we deduce that (5.23) holds for $|\xi|\geqslant 1$. This completes the proof. □

As a first application of our a-priori bounds we shall prove that the linearization of the equation satisfied by Q_s depends continuously (in a sense to be made precise) on s. Along with the operator

$$L_{+,s} = (-\Delta)^s + \lambda_s - (\alpha+1)|Q_s|^\alpha$$

which has already appeared above, we consider

$$L_{-,s} = (-\Delta)^s + \lambda_s - |Q_s|^\alpha.$$

Both are self-adjoint and lower semi-bounded operators in $L^2(\mathbb{R})$. Recall that convergence of a sequence $\{A_n\}_{n=1}^\infty$ of self-adjoint operators to an operator A in norm-resolvent sense means that

$$\left\| \frac{1}{A_n+z} - \frac{1}{A+z} \right\|_{L^2\to L^2} \to 0$$

as $n\to\infty$, where $z\in\mathbb{C}$ with $\operatorname{Im} z\neq 0$.

LEMMA 5.8. *Let $\{s_n\}_{n=1}^\infty \subset [s_0, s_*)$ be a sequence with $s_n\to\tilde{s}<s_*$. Then $L_{\pm,s_n}\to L_{\pm,\tilde{s}}$ in norm-resolvent sense.*

Proof. By straightforward estimates, we find that norm-resolvent convergence will follow from $\lambda_s \to \lambda_{\tilde{s}}$, provided we can show that $V_s = |Q_s|^\alpha$ satisfies

$$\|V_s - V_{\tilde{s}}\|_p \to 0 \quad \text{as } s \to \tilde{s} \tag{5.24}$$

for some $p \geqslant 1$ such that $p > 1/2s_0 \geqslant 1/2s$.

To see that we can always find such p, we argue as follows. By Lemmas 5.5 and 5.6, we have $\|Q_s\|_{H^s} \lesssim 1$, and thus $\|Q_s\|_{H^{s_0}} \lesssim 1$ since $s \geqslant s_0$. As $Q_s \to Q_{\tilde{s}}$ in $L^2(\mathbb{R})$, we deduce from the uniform bound $\|Q_s\|_{H^{s_0}} \lesssim 1$ that

$$\|Q_s - Q_{\tilde{s}}\|_{H^{\sigma_0}} \to 0 \quad \text{as } s \to \tilde{s}$$

for any $0 \leqslant \sigma_0 < s_0$. In particular if $s_0 > \frac{1}{2}$, then $\|Q_s - Q_{\tilde{s}}\|_\infty \to 0$ by Sobolev inequalities. Hence we can choose $p = \infty$ in (5.24) and we conclude that norm-resolvent convergence holds whenever $s_0 > \frac{1}{2}$. Assume now that $s_0 \leqslant \frac{1}{2}$. In this case, by Sobolev inequalities and Hölder's inequality, we deduce that $V_s = |Q_s|^\alpha$ satisfies (5.24) for $p = 2/\alpha(1-2\sigma_0)$ with any $0 \leqslant \sigma_0 < s_0$. But since $2/\alpha(1-2s_0) > 1/2s_0$ due to the condition $\alpha < \alpha_{\max}(s_0)$, we can choose $\sigma_0 < s_0$ sufficiently close to s_0 such that $p = 2/\alpha(1-2\sigma_0) > 1/2s_0$ as well. This shows that norm-resolvent convergence also holds when $s_0 \leqslant \frac{1}{2}$. □

As a result of the norm-resolvent convergence, we shall see that both the Morse index of $L_{+,s}$ and the positivity of Q_s are preserved along the branch. We begin by deriving the first fact.

COROLLARY 5.9. *We have* $\mathcal{N}_{-,\mathrm{even}}(L_{+,s}) = \mathcal{N}_{-,\mathrm{even}}(L_{+,s_0})$ *for all* $s \in [s_0, s_*)$.

Proof. By continuity of the eigenvalues of $L_{+,s}$, any change of the Morse index along the branch would imply that 0 must be an eigenvalue of $L_{+,s}$ acting on $L^2_{\mathrm{even}}(\mathbb{R})$ for some $s \in (s_0, s_*)$. But this contradicts Assumption 5.1. □

As a next step, we wish to analyze sequences $\{Q_{s_n}\}_{n=0}^\infty$ with $s_n \to s_*$. In particular, our goal is to derive strong convergence (along subsequences) for $\{Q_{s_n}\}_{n=1}^\infty$ with respect to the norm $\|\cdot\|_{X^\alpha} = \|\cdot\|_2 + \|\cdot\|_{\alpha+2}$. Recall from Lemmas 5.5 and 5.6 the a-priori bound

$$\|Q_s\|_{H^s}^2 = \int_{\mathbb{R}} |(-\Delta)^{s/2} Q_s|^2 \, dx + \int_{\mathbb{R}} |Q_s|^2 \, dx \lesssim 1 \quad \text{for } s \in [s_0, s_*).$$

Suppose now that $s_n \to s_*$. To turn the uniform bound $\|Q_{s_n}\|_{H^s} \lesssim 1$ into strong convergence of $\{Q_{s_n}\}_{n=1}^\infty$ in some L^p-norm, we need a further ingredient. Indeed, since we consider $d = 1$ space dimension, we recall the well-known fact that the even symmetry of the functions $\{Q_{s_n}\}_{n=1}^\infty$ (unlike for radial symmetry in $d \geqslant 2$ dimensions) is *not sufficient* to gain relative compactness of $\{Q_{s_n}\}_{n=1}^\infty$ in some L^p-norm. To deal with this, we will

now impose that $Q_{s_0}=Q_{s_0}(|x|)>0$ is a positive function. Then the following result shows that $Q_s(|x|)>0$ along the branch. This fact, in turn, will lead to monotonicity of the functions $Q_s(|x|)$ in $|x|$. From this property and the a-priori bound on $\lambda_s\sim 1$, we finally derive a *uniform decay* estimate of the form $Q_s(|x|)\lesssim|x|^{-1}$ for $|x|$ outside a fixed compact set, which will enable us to gain relative compactness in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$.

The fact that positivity of Q_{s_0} implies positivity of Q_s for all s is another consequence of norm-resolvent convergence.

COROLLARY 5.10. *Suppose that $Q_{s_0}=Q_{s_0}(|x|)>0$ is positive. Then $Q_s=Q_s(|x|)>0$ for $x\in\mathbb{R}$ and $s\in[s_0,s_*)$.*

Proof. We divide the proof into two steps as follows.

Step 1. First, we show that positivity of $Q_s(|x|)>0$ is an "open" property along the branch (Q_s,λ_s). That is, if we assume that $Q_{\tilde{s}}=Q_{\tilde{s}}(|x|)>0$ for some $\tilde{s}\in[s_0,s_*)$, then

$$Q_s=Q_s(|x|)>0 \quad \text{for } s\in[s_0,s_*) \text{ with } |s-\tilde{s}|<\varepsilon,$$

where $\varepsilon>0$ is sufficiently small. To prove this claim, we consider the family of self-adjoint operators $L_{-,s}$, introduced before Lemma 5.8, and note that

$$L_{-,s}Q_s=0, \tag{5.25}$$

that is, Q_s is an eigenfunction of $L_{-,s}$ with eigenvalue 0. Furthermore, by Lemma C.2, the lowest eigenvalue of $L_{-,s}$ is non-degenerate and its corresponding eigenfunction can be chosen strictly positive. In particular, the function $Q_{\tilde{s}}(|x|)>0$ is the ground state eigenfunction of $L_{-,\tilde{s}}$ and 0 is the lowest eigenvalue of $L_{-,\tilde{s}}$. Thus, in view of (5.25) and Lemma C.2, it suffices to show that 0 is the lowest eigenvalue of $L_{-,s}$ (for s close to $\tilde{s}$) and finally rule out that $Q_s<0$ holds.

This follows from Lemma 5.8 by standard spectral arguments: Let $\lambda_1(L_{-,s})$ denote the lowest eigenvalue of $L_{-,s}$. By Lemma C.2, the eigenvalue $\lambda_1(L_{-,s})$ is non-degenerate and its corresponding eigenfunction $\psi_{1,s}(x)>0$ is strictly positive. We have already observed that $\lambda_1(L_{-,\tilde{s}})=0$ and $Q_{\tilde{s}}(x)=\psi_{1,\tilde{s}}(x)$. From the norm-resolvent convergence, we conclude that $\lambda_1(L_{-,s})\to\lambda_1(L_{-,\tilde{s}})$ as $s\to\tilde{s}$ and that $\lambda_1(L_{-,s})$ is simple for s close to $\tilde{s}$. (The last statement also follows from Lemma C.2.) Moreover, since $\lambda_1(L_{-,\tilde{s}})$ is isolated, we can find $c>0$ so small that the interval $I_c=(-c,c)$ satisfies $\sigma(L_{-,\tilde{s}})\cap I_c=\{\lambda_1(L_{-,\tilde{s}})\}$. Thus, by the above convergence properties, we deduce that

$$\sigma(L_{-,s})\cap I_c=\{\lambda_1(L_{-,s})\}$$

whenever $|s-\tilde{s}|<\varepsilon$, where $\varepsilon>0$ is sufficiently small. On the other hand, (5.25) shows that 0 is an eigenvalue of $L_{-,s}$, and therefore $\lambda_1(L_{-,s})=0$ for $|s-\tilde{s}|<\varepsilon$. By Lemma C.2

again, we deduce that $Q_s(x)=\sigma(s)\psi_{1,s}(x)$, where $\sigma(s)\in\{1,-1\}$ is some sign depending on s. But since $Q_s\to Q_{\tilde{s}}$ in $L^2(\mathbb{R})$ and $Q_{\tilde{s}}(x)>0$, we conclude that $\sigma(s)=1$ for all s close to s_0. To summarize, we have shown that $Q_s(x)=\psi_{1,s}(x)>0$ for all $|s-\tilde{s}|<\varepsilon$, provided that $\varepsilon>0$ is small and $Q_{\tilde{s}}(x)>0$ holds.

Step 2. Next, we prove that positivity of Q_s along the branch is a "closed" property. That is, if $Q_s(|x|)>0$ for all $s\in[s_0,\tilde{s})$ with some $\tilde{s}<s_*$, then $Q_{\tilde{s}}(|x|)>0$ as well. Indeed, let $\tilde{s}\in(s_0,s_*)$ be given and suppose that $\{s_n\}_{n=1}^{\infty}\subset[s_0,\tilde{s})$ is a sequence with $s_n\to\tilde{s}$ such that $Q_{s_n}(|x|)>0$ for all $n\in\mathbb{N}$. Since $Q_{s_n}\to Q_{\tilde{s}}$ in $L^2(\mathbb{R})$, we see that the limit is a non-negative function $Q_{\tilde{s}}(|x|)\geqslant 0$. We note that $Q_{\tilde{s}}\not\equiv 0$ due to $\|Q_{\tilde{s}}\|_{\alpha+2}=\|Q_{s_0}\|_{\alpha+2}\neq 0$. Hence $Q_{\tilde{s}}=Q_{\tilde{s}}(|x|)\geqslant 0$ is a non-negative and non-trivial solution of

$$Q_{\tilde{s}}=\frac{1}{(-\Delta)^{\tilde{s}}+\lambda_{\tilde{s}}}|Q_{\tilde{s}}|^{\alpha}Q_{\tilde{s}}.$$

From this we deduce that positivity $Q_{\tilde{s}}=Q_{\tilde{s}}(|x|)>0$ holds by using Lemma A.4, which establishes the positivity of the integral kernel for the resolvent $((-\Delta)^s+\lambda)^{-1}$ with $0<s<1$ and $\lambda>0$.

By combining the results of steps 1 and 2 above, the proof is completed. □

Next, we derive a uniform spatial decay estimate along the maximal branch (Q_s,λ_s), provided that $Q_{s_0}(|x|)>0$ holds initially.

LEMMA 5.11. *Suppose that $Q_{s_0}(|x|)>0$ holds. Then we have the uniform decay estimate*

$$0<Q_s(|x|)\lesssim\frac{1}{|x|}$$

for $|x|\geqslant R_0$ and $s\in[s_0,s_*)$. Here $R_0>0$ is some constant independent of s.

Proof. For any $\mu>0$ given, we can rewrite the equation satisfied by Q_s as

$$Q_s=((-\Delta)^s+\mu)^{-1}f_s,\quad \text{with } f_s(x)=Q_s(x)(Q_s^{\alpha}(x)-\lambda_s+\mu).$$

Note that $|Q|_s^{\alpha}=Q_s^{\alpha}$, since $Q_s(|x|)>0$ for $s\in[s_0,s_*)$, by Corollary 5.10.

By Lemmas 5.5 and 5.6, we have the uniform lower bound $\lambda_s\gtrsim 1$. In particular, we can choose $\mu>0$ fixed and independent of s such that $\lambda_s\geqslant 2\mu$ for $s\in[s_0,s_*)$. Next, we claim that the positive part $f_s^+:=\max\{f_s,0\}$ has compact support such that

$$f_s^+(x)\equiv 0\quad \text{for } |x|\geqslant\tfrac{1}{2}R_0, \tag{5.26}$$

where $R_0>0$ is some large constant independent of s. Indeed, the functions $Q_s=Q_s(|x|)$ are even and positive. Hence, by Lemma B.4, we deduce that each function $Q_s(|x|)$ is

decreasing in $|x|$. Also, we recall the uniform bound $\|Q_s\|_2 \lesssim 1$ from Lemma 5.6. Hence, for any $|x|>0$,

$$|x|\,|Q_s(x)|^2 \leqslant \frac{1}{2}\int_{|y|\leqslant|x|} |Q_s(y)|^2\,dy \leqslant \frac{1}{2}\int_{\mathbb{R}} |Q_s(y)|^2\,dy \lesssim 1.$$

Therefore $Q_s(y)\lesssim |x|^{-1/2}$ for $|x|>0$. Moreover, we have $-\lambda_s+\mu\leqslant -\mu<0$ and $Q_s(|x|)>0$. These facts imply that (5.26) holds with some large constant $R_0>0$ independent of s.

Next, by Lemma A.4, we conclude that the kernel $G_{s,\mu}$ of the resolvent $((-\Delta)^s+\mu)^{-1}$ is given by a positive function $G_{s,\mu}(x)>0$ that satisfies the uniform bound

$$0<G_{s,\mu}(x)\lesssim \frac{1}{|x|} \quad \text{for } |x|>0. \tag{5.27}$$

Since $|x-y|\geqslant \frac{1}{2}|x|$ when $|x|\geqslant R_0$ and $|y|\leqslant \frac{1}{2}R_0$, we can combine (5.26) and (5.27) to find the bound

$$0<Q_s(|x|)\leqslant \int_{|y|\leqslant R_0/2} G_{s,\mu}(x-y) f_s^+(y)\,dy \lesssim \frac{1}{|x|}\int_{|y|\leqslant R_0/2} f_s^+(y)\,dy \lesssim \frac{1}{|x|}$$

for $|x|\geqslant R_0$. In the last step, we used the fact that $f_s\leqslant Q_s^{\alpha+1}$ and the uniform bound $\|Q_s\|_{\alpha+2}\lesssim 1$ together with Hölder's inequality to obtain that

$$\int_{|y|\leqslant R_0/2} f_s^+(y)\,dy \leqslant R_0^{1/(\alpha+2)}\|Q_s\|_{\alpha+2}^{\alpha+1}\lesssim 1.$$

This completes the proof of Lemma 5.11. □

We are now in the position to derive the following key fact.

LEMMA 5.12. *Let $\{s_n\}_{n=1}^{\infty}\subset[s_0,s_*)$ be a sequence such that $s_n\to s_*$. Furthermore, we suppose that $Q_{s_n}=Q_{s_n}(|x|)>0$ are positive functions. Then (after possibly passing to a subsequence) we have that*

$$Q_{s_n}\to Q_* \ \text{ in } L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R}) \quad \text{and} \quad \lambda_{s_n}\to\lambda_*,$$

where $\lambda_*>0$ and $Q_*=Q_*(|x|)>0$ satisfy

$$(-\Delta)^{s_*}Q_*+\lambda_* Q_* - Q_*^{\alpha+1}=0.$$

Moreover, we have the following estimates on the Morse index:

$$\liminf_{n\to\infty}\mathcal{N}_{-,\mathrm{even}}(L_{+,n})\geqslant \mathcal{N}_{-,\mathrm{even}}(L_{+,*})\geqslant 1,$$

where $L_{+,n}=(-\Delta)^{s_n}+\lambda_n-(\alpha+1)Q_{s_n}^{\alpha}$ and $L_{+,*}=(-\Delta)^{s_*}+\lambda_*-(\alpha+1)Q_*^{\alpha}$ denote the linearized operators associated with Q_{s_n} and Q_*, respectively.

Remarks. (1) One key step in the proof of Lemma 5.12 will be to establish *strong convergence* of $\{Q_{s_n}\}_{n=1}^\infty$ in $L^2(\mathbb{R})$. Here, the pointwise decay bound from Lemma 5.11 will guarantee this fact. Note that the (immediate) uniform decay estimate

$$Q_{s_n}(x) \lesssim |x|^{-1/2}$$

(see the proof of Lemma 5.11) is not sufficient to conclude strong convergence of $\{Q_{s_n}\}_{n=1}^\infty$ in $L^2(\mathbb{R})$. Generally speaking, the gain of relative compactness of $\{Q_{s_n}\}_{n=1}^\infty$ in $L^2(\mathbb{R})$ is due to the fact that we can derive a better uniform decay estimate of the form

$$Q_{s_n}(|x|) \lesssim |x|^{-1},$$

which is square-integrable at infinity.

(2) By bootstrapping arguments, we can derive strong convergence of $\{Q_{s_n}\}_{n=1}^\infty$ in $H^{2s_*}(\mathbb{R})$, once strong convergence in $L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R})$ is known. However, we do not need this refinement and hence we omit its proof.

(3) Note that we do not claim nor need uniqueness of $Q_* = Q_*(|x|) > 0$ at this point.

Proof. Define the sequences $\{Q_n\}_{n=1}^\infty$ with $Q_n = Q_{s_n}$ and $\{\lambda_n\}_{n=1}^\infty$ with $\lambda_n = \lambda_{s_n}$.

First, by combining Lemmas 5.5 and 5.6, we obtain the uniform bounds $\lambda_n \sim 1$. Thus, after passing to a subsequence, we may assume that $\lambda_n \to \lambda_*$ with some positive limit $\lambda_* > 0$.

From Lemmas 5.5 and 5.6 we have the a-priori bound $\|Q_n\|_{H^{s_n}} \lesssim 1$. Since $s_n \geqslant s_0$, this implies in particular that $\|Q_n\|_{H^{s_0}} \lesssim 1$ holds. Hence (after passing to a subsequence) we may assume that $Q_n \rightharpoonup Q_*$ weakly in $H^{s_0}(\mathbb{R})$ and, by local Rellich–Kondratchov compactness, we deduce that $Q_n \to Q_*$ in $L^2_{\mathrm{loc}}(\mathbb{R})$. To upgrade this fact to strong convergence in $L^2(\mathbb{R})$ itself, we recall that Lemma 5.11 implies the uniform decay estimate

$$|Q_n(x)| \lesssim \frac{1}{|x|} \quad \text{for } |x| \geqslant R_0 \text{ and } n \geqslant 1, \tag{5.28}$$

where $R_0 > 0$ is independent of n. Using this uniform decay, we easily derive strong convergence of $\{Q_n\}_{n=1}^\infty$ in $L^2(\mathbb{R})$. Indeed, let $\varepsilon > 0$ be given. Choose $R_\varepsilon \geqslant R_0$ so large that $\int_{|x|>R_\varepsilon} |x|^{-2}\, dx \leqslant \varepsilon^2$ and $\int_{|x|>R_\varepsilon} |Q_*|^2\, dx \leqslant \varepsilon^2$. Since moreover $Q_n \to Q_*$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, there exists $n_0 \geqslant 1$ such that $\int_{|x|\leqslant R_\varepsilon} |Q_n - Q_*|^2\, dx \leqslant \varepsilon^2$ for $n \geqslant n_0$. Using the pointwise bound (5.28) and the triangle inequality, we thus conclude that

$$\|Q_n - Q_*\|_{L^2(\mathbb{R})} \leqslant \|Q_n - Q_*\|_{L^2(|x|\leqslant R_\varepsilon)} + \|Q_n - Q_*\|_{L^2(|x|>R_\varepsilon)} \lesssim \varepsilon$$

for all $n \geqslant n_0$. This shows that $Q_n \to Q_*$ strongly in $L^2(\mathbb{R})$.

In order to see that $Q_n \to Q_*$ strongly in $L^{\alpha+2}(\mathbb{R})$, we first recall the uniform bound $\|Q_n\|_{H^{s_0}} \lesssim 1$. Using the condition $\alpha < \alpha_{\max}(s_0)$ and Sobolev inequalities, we deduce that $\|Q_n\|_p \lesssim 1$ for some $p > \alpha+2$ (with $p \leqslant 2/(1-2s_0)$ if $s_0 < \frac{1}{2}$). Thus, by Hölder's inequality, we deduce that $Q_n \to Q_*$ in $L^{\alpha+2}(\mathbb{R})$ as well.

Using now the convergence properties of $\{Q_n\}_{n=1}^{\infty}$ derived above, we conclude that the limit $Q_* \in L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R})$ satisfies the equation

$$(-\Delta)^{s_*} Q_* + \lambda_* Q_* - Q_*^{\alpha+1} = 0. \tag{5.29}$$

Note that $Q_n = Q_n(|x|) > 0$ and $Q_n \to Q_*$ in $L^2(\mathbb{R})$, which implies that $Q_* = Q_*(|x|) \geqslant 0$. Furthermore, $Q_n \to Q_*$ in $L^{\alpha+2}(\mathbb{R})$ and $\|Q_n\|_{\alpha+2} = \|Q_0\|_{\alpha+2} \neq 0$ for all $n \in \mathbb{N}$. Hence $Q_* \not\equiv 0$ as well. Finally, we deduce $Q_*(x) > 0$ by noting that $Q_* = ((-\Delta)^{s_*} + \lambda_*)^{-1} Q_*^{\alpha+1}$ and using the positivity of the integral kernel of the resolvent $((-\Delta)^{s_*} + \lambda_*)^{-1}$; see Lemma A.4.

It remains to show the claimed estimate for the Morse index of $L_{+,*}$ in the sector of even functions. Consider the sequence $\{L_{+,n}\}_{n=1}^{\infty}$ of self-adjoint operators defined in the lemma. Note that $\|Q_n\|_{H^{s_0}} \lesssim 1$ and $Q_n \to Q_*$ in $L^2(\mathbb{R})$. Then, by adapting the proof of Lemma 5.8, we deduce that $Q_n \to Q_*$ in $L^p(\mathbb{R})$ for some $p > 1/2s_0 \geqslant 1/2s_*$. In particular, this implies that $L_{+,n} \to L_{+,*}$ in the norm-resolvent sense. Since the Morse index is lower semi-continuous with respect to the norm-resolvent topology, we conclude that

$$\liminf_{n\to\infty} \mathcal{N}_{-,\mathrm{even}}(L_{+,n}) \geqslant \mathcal{N}_{-,\mathrm{even}}(L_{+,*}).$$

Furthermore, an elementary calculation shows that

$$\langle Q_*, L_{+,*} Q_* \rangle = -\alpha \int_{\mathbb{R}} |Q_*|^{\alpha+2}\, dx < 0.$$

Hence, by the min-max principle, we conclude that $L_{+,*}$ acting on $L^2_{\mathrm{even}}(\mathbb{R})$ has at least one strictly negative eigenvalue. Therefore, we obtain that

$$\mathcal{N}_{-,\mathrm{even}}(L_{+,*}) \geqslant 1.$$

The proof of Lemma 5.12 is now complete. □

As one of the main results of this section, we now prove that any maximal branch (Q_s, λ_s) extends to $s_* = 1$, provided that Q_{s_0} satisfies some explicit conditions (which in particular hold true if Q_{s_0} is a ground state).

PROPOSITION 5.13. *Let $0 < s_0 < 1$ and $0 < \alpha < \alpha_{\max}(s_0)$ be given. Suppose that*

$$(Q_0, \lambda_0) \in X^{\alpha} \times \mathbb{R}_+$$

satisfies Assumption 5.1 with $s=s_0$ *and* $\lambda=\lambda_0$. *Furthermore, assume that* $Q_0=Q_0(|x|)$ *is positive and that the corresponding linearized operator* $L_{+,0}$ *satisfies* $\mathcal{N}_{-,\mathrm{even}}(L_{+,0})=1$.

Then the corresponding maximal branch $(Q_s,\lambda_s)\in C^1([s_0,s_*);X^\alpha\times\mathbb{R}_+)$ *extends to* $s_*=1$. *Moreover, we have that*

$$Q_s\to Q_* \quad \text{in } L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R}) \quad \text{and} \quad \lambda_s\to\lambda_*$$

as $s\to 1$, *where* $Q_*=Q_*(|x|)>0$ *is the unique solution of*

$$\begin{cases} -\Delta Q_*+\lambda_* Q_*-Q_*^{\alpha+1}=0, \\ Q_*=Q_*(|x|)>0, \\ Q_*\in L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R}), \end{cases}$$

and $\lambda_*>0$ *is given by*

$$\lambda_*=\left(\frac{\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx}{\int_{\mathbb{R}}|P|^{\alpha+2}\,dx}\right)^{2\alpha/(\alpha+4)}.$$

Here $P=P(|x|)>0$ *denotes the unique positive even solution in* $C^2(\mathbb{R})$ *that satisfies*

$$-\Delta P+P-P^{\alpha+1}=0$$

with $P\to 0$ *as* $|x|\to\infty$.

Remarks. (1) The function $P(x)$ is in fact known in closed form, i.e., we have that

$$P(x)=\frac{(\sigma+1)^{1/2\sigma}}{\cosh^{1/\sigma}(\sigma x)},$$

where $\sigma=\frac{1}{2}\alpha$. However, this fact has no relevance in the proof below.

(2) Note that $\lambda_*>0$ only depends on α and the quantity $\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx$.

Proof. Let $(Q_s,\lambda_s)\in C^1([s_0,s_*);X^\alpha\times\mathbb{R}_+)$ be the maximal branch with $s_*\in(s_0,1]$. Let $\{s_n\}_{n=1}^\infty\subset[s_0,s_*)$ be such that $s_n\to s_*$. Define the sequences $\{Q_n\}_{n=1}^\infty\subset X^\alpha$ and $\{\lambda_n\}_{n=1}^\infty\subset\mathbb{R}_+$ by $Q_n=Q_{s_n}$ and $\lambda_n=\lambda_{s_n}$ for $n\in\mathbb{N}$. By Lemma 5.12 and after passing to a subsequence if necessary, we may assume that $Q_n\to Q_*$ in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ and $\lambda_n\to\lambda_*$ for some $Q_*(|x|)>0$ and $\lambda_*>0$. For the Morse index of $L_{+,*}=(-\Delta)^{s_*}+\lambda_*-(\alpha+1)Q_*^\alpha$ on even functions, we claim that

$$\mathcal{N}_{-,\mathrm{even}}(L_{+,*})=1. \tag{5.30}$$

Indeed, by assumption, by Corollary 5.9 and Lemma 5.12 we have

$$1=\mathcal{N}_{-,\mathrm{even}}(L_{+,s_0})=\liminf_{n\to\infty}\mathcal{N}_{-,\mathrm{even}}(L_{+,n})\geqslant\mathcal{N}_{-,\mathrm{even}}(L_{+,*})\geqslant 1.$$

Next, we shall use (5.30) to deduce that $s_*=1$. Suppose on the contrary that $s_*<1$ was true. Then by (5.30) we can apply Lemma 4.1 to deduce that $L_{+,*}$ is invertible on $L^2_{\mathrm{even}}(\mathbb{R})$. Hence $(Q_*,\lambda_*)\in X^\alpha\times\mathbb{R}_+$ satisfies Assumption 5.1. Thus, by Proposition 5.2, we can extend the branch (Q_s,λ_s) beyond s_*, which contradicts the maximality property of s_*. This contradiction proves that $s_*=1$.

By Lemma 5.12, $Q_*=Q_*(|x|)>0$ solves the non-linear equation

$$-\Delta Q_*+\lambda_* Q_*-Q_*^{\alpha+1}=0. \tag{5.31}$$

Note that, by bootstrapping this equation for $Q_*\in L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$, we conclude that $Q_*\in H^2(\mathbb{R})$. Using this fact, we deduce that in fact $Q_*\in C^2(\mathbb{R})$ holds. Now we recall the well-known fact that $-\Delta P+P-P^{\alpha+1}=0$ has a unique positive solution $P=P(|x|)>0$ in $C^2(\mathbb{R})$ with $P\to 0$ as $|x|\to\infty$; see the remark above. By a simple scaling argument, we infer that

$$Q_*(x)=\lambda_*^{1/\alpha}P(\lambda_*^{1/2}x). \tag{5.32}$$

Furthermore, since $\int_{\mathbb{R}}|Q_n|^{\alpha+2}\,dx=\int_{\mathbb{R}}|Q_{s_0}|^{\alpha+2}\,dx$ for all $n\geqslant 1$ and $Q_n\to Q_*$ in $L^{\alpha+2}(\mathbb{R})$, we find that

$$\int_{\mathbb{R}}|Q_*|^{\alpha+2}\,dx=\int_{\mathbb{R}}|Q_{s_0}|^{\alpha+2}\,dx. \tag{5.33}$$

Using now (5.32) and (5.33), we see that $\lambda_*>0$ is given by the formula in Proposition 5.13. In particular, this shows that the limit $\lambda_*>0$ is independent of the sequence $\{s_n\}_{n=1}^\infty$. Furthermore, by uniqueness of Q_* with $\lambda_*>0$ given, we conclude that the limit Q_* is also independent of $\{s_n\}_{n=1}^\infty$. Hence $Q_s\to Q_*$ in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ and $\lambda_s\to\lambda_*$ as $s\to 1$.

The proof of Proposition 5.13 is now complete. □

5.3. Proof of Theorem 2.4

First, we prove uniqueness of ground states and we argue by contradiction as follows.

Let $0<s_0<1$ and $0<\alpha<\alpha_{\max}(s_0)$ be given. Recall our definition of the (real) Banach space X^α of real-valued and even functions in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$; see (5.2).

Suppose that $Q=Q_0(|x|)>0$ and $\widetilde{Q}_0=\widetilde{Q}_0(|x|)>0$ are two ground states for problem (1.2) such that $Q_0\not\equiv\widetilde{Q}_0$. First, we observe that we always have

$$\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx=\int_{\mathbb{R}}|\widetilde{Q}_0|^{\alpha+2}\,dx. \tag{5.34}$$

Indeed, from Lemma 5.4 we find that $Q_0\in H^{s_0}(\mathbb{R})$ satisfies the Pohozaev idenitites

$$\frac{1}{2}\int_{\mathbb{R}}|Q_{s_0}|^2\,dx=\frac{a_{s_0}}{\alpha+2}\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx\ \text{ and }\ \frac{1}{2}\int_{\mathbb{R}}|(-\Delta)^{s_0/2}Q_0|^2\,dx=\frac{b_{s_0}}{\alpha+2}\int_{\mathbb{R}}|Q_0|^{\alpha+2}\,dx,$$

where $a_{s_0}=\alpha(2s_0-1)/4s_0+1$ and $b_{s_0}=\alpha/4s_0$. Moreover, by assumption, the ground state $Q_0\in H^{s_0}(\mathbb{R})$ optimizes the interpolation estimate (1.4). Thus, we also find that

$$\int_{\mathbb{R}} |Q_0|^{\alpha+2}\,dx = C_{\alpha,s_0}\left(\int_{\mathbb{R}} |(-\Delta)^{s_0/2}Q_0|_2\,dx\right)^{\alpha/4s_0}\left(\int_{\mathbb{R}} |Q_0|^2\,dx\right)^{\alpha(2s_0-1)/4s_0+1},$$

with $C_{\alpha,s_0}>0$ being the optimal constant for (1.4) when $s=s_0$. Combining now the last three equations, we conclude that $\int_{\mathbb{R}} |Q_0|^{\alpha+2}\,dx=f(\alpha,s_0)$, for some function $f(\alpha,s_0)$ that only depends on α and s_0. By repeating the same arguments for the ground state $\widetilde{Q}_{s_0}$, we thus deduce that $\int_{\mathbb{R}} |Q_0|^{\alpha+2}\,dx=\int_{\mathbb{R}} |\widetilde{Q}_0|^{\alpha+2}\,dx$ holds.

Next, by Theorem 2.3, we see that both $Q_0\in X^\alpha$ and $\widetilde{Q}_0\in X^\alpha$ satisfy Assumption 5.1 with $s=s_0$ and $\lambda=1$. Hence, by Proposition 5.13, there exist two global branches

$$(Q_s,\lambda_s)\in C^1([s_0,1);X^\alpha\times\mathbb{R}_+) \quad\text{and}\quad (\widetilde{Q}_s,\tilde{\lambda}_s)\in C^1([s_0,1);X^\alpha\times\mathbb{R}_+),$$

which solve equation (5.1) and we have $(Q_{s_0},\lambda_{s_0})=(Q_0,1)$ and $(\widetilde{Q}_{s_0},\tilde{\lambda}_{s_0})=(\widetilde{Q}_0,1)$. Note that, by the local uniqueness stated in Proposition 5.2, the branches (Q_s,λ_s) and $(\widetilde{Q}_s,\tilde{\lambda}_s)$ cannot intersect. Moreover, by Proposition 5.13, we have the following facts.

- $Q_s\to Q_*$ in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ and $\lambda_s\to\lambda_*$ as $s\to 1$.
- $\widetilde{Q}_s\to\widetilde{Q}_*$ in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ and $\tilde{\lambda}_s\to\tilde{\lambda}_*$ as $s\to 1$.

Here $\lambda_*>0$ and $\tilde{\lambda}_*>0$ are given by the formula in Proposition 5.13. Furthermore, the functions $Q_*=Q_*(|x|)>0$ and $\widetilde{Q}_*=\widetilde{Q}_*(|x|)>0$ are the unique even and positive solutions in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ of the non-linear equations

$$-\Delta Q_*+\lambda_* Q_*-Q_*^{\alpha+1}=0 \quad\text{and}\quad -\Delta\widetilde{Q}_*+\tilde{\lambda}_*\widetilde{Q}_*-\widetilde{Q}_*^{\alpha+1}=0,$$

respectively. However, by Proposition 5.13 and equation (5.34), we deduce that $\lambda_*=\tilde{\lambda}_*$, which implies that

$$Q_*\equiv\widetilde{Q}_*,$$

due to the uniqueness result for the limiting equation as stated in Proposition 5.13. Next, we remark that Q_* has a non-degenerate linearized operator

$$L_+=-\frac{d^2}{dx^2}+\lambda_*-(\alpha+1)Q_*^\alpha;$$

see, e.g., [11]. Hence, we can invoke an implicit function argument around (Q_*,λ_*) to construct a locally unique branch $(Q_s,\lambda_s)\in C^1((1-\delta,1];X^\alpha\times\mathbb{R}_+)$, with some $\delta>0$ small, such that

$$(-\Delta)^s Q_s+\lambda_s Q_s-Q_s^{\alpha+1}=0,$$

and (Q_s,λ_s) is the unique solution for $s\in(1-\delta,1]$ in the neighborhood

$$N=\{(Q,\lambda)\in X^\alpha\times\mathbb{R}_+ : \|Q-Q_*\|_{X^\alpha}+|\lambda-\lambda_*|<\varepsilon\},$$

where $\varepsilon>0$ is a small constant. Since $Q_s\to Q_*$ and $\widetilde{Q}_s\to Q_*$ in $L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ and both $\lambda_s\to\lambda_*$ and $\tilde{\lambda}_s\to\lambda_*$ as $s\to 1$, we conclude that the branches (Q_s,λ_s) and $(\widetilde{Q}_s,\tilde{\lambda}_s)$ must intersect at some $s\in[s_0,1)$. But this is a contradiction to the local uniqueness of the branches (Q_s,λ_s) and $(\widetilde{Q}_s,\tilde{\lambda}_s)$, as given by Proposition 5.2. This proves uniqueness of ground states as stated in Theorem 2.4.

Finally, we establish uniqueness of optimizers for the Gagliardo–Nirenberg inequality (1.4). Here we simply note that, by rearrangement inequalities, we have

$$J^{s,\alpha}(v^*)\leqslant J^{s,\alpha}(v), \tag{5.35}$$

where $v^*=v^*(|x|)\geqslant 0$ denotes the symmetric-decreasing rearrangement of $v\in H^s(\mathbb{R})$. By [16] we see that strict inequality holds in (5.35), unless $v(x)$ equals $v^*(|x|)$ up to a complex phase and spatial translation. Since v minimizes $J^{s,\alpha}$ and so does v^*, we deduce that $v^*=v^*(|x|)\geqslant 0$ solves the corresponding Euler–Lagrange equation

$$(-\Delta)^s v^*+\lambda v^*-\mu(v^*)^{\alpha+1}=0,$$

with some positive constants $\lambda>0$ and $\mu>0$. By a simple rescaling argument and uniqueness of the ground state $Q=Q(|x|)>0$, we see that $v^*(|x|)=aQ(b|x|)$ for some constants $a>0$ and $b>0$.

The proof of Theorem 2.4 is now complete.

Appendix A. Some uniform bounds

In this appendix, we derive some uniform bounds (with respect to s) for the heat kernel $e^{-t(-\Delta)^s}$ with $0<s<1$. Moreover, as a a direct consequence, we obtain corresponding uniform bounds for the resolvent $((-\Delta)^s+\lambda)^{-1}$.

Although many of the following bounds can be directly inferred from the literature for each $0<s<1$ individually, we were not able to find a reference which yields the desired bounds in a uniform fashion for $s_0\leqslant s<1$ with $s_0>0$ fixed. Also, we mention that it is straightforward to generalize the following arguments to any space dimension. However, due to notational convenience, we have decided to focus on the 1-dimensional case in what follows.

Consider the heat kernel $e^{-t(-\Delta)^s}$ on $\mathbb{R}$ with $0<s<1$. That is, we consider the convolution operator with convolution kernel given by

$$P^{(s)}(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-t|\xi|^{2s}}e^{-i\xi x}\,d\xi, \tag{A.1}$$

where $t>0$ is a parameter. Note the scaling property

$$P^{(s)}(x,t)=t^{-1/2s}P^{(s)}(t^{-1/2s}x,1) \quad \text{for } x\in\mathbb{R} \text{ and } t>0.$$

Moreover, it is obvious that $P^{(s)}(x,t)$ is an even function of x. We first record the following known (but not completely obvious) positivity and monotonicity result.

LEMMA A.1. *Let $0<s<1$ and $t>0$ be fixed. Then*

$$P^{(s)}(x,t)>0 \text{ for } x\in\mathbb{R} \quad \text{and} \quad \frac{d}{dx}P^{(s)}(x,t)<0 \text{ for } x>0.$$

Proof. We give the following (fairly simple) proof, which mainly rests on Bernstein's theorem about the Laplace transform.

First, by the scaling property of $P^{(s)}(x,t)$, we may assume that $t=1$ holds. Now we consider the non-negative function $f(E)=E^s$ on the positive real line $(0,\infty)$. Using that $0<s<1$, it is easy to see that $f'(E)$ is completely monotone (i.e., we have $(-1)^n f^{(n)}(E)\leqslant 0$ for all $n\in\mathbb{N}$). This fact, in turn, implies that the map $E\mapsto e^{-f(E)}$ is completely monotone as well. Hence, by Bernstein's theorem, we infer that

$$e^{-f(E)}=\int_0^\infty e^{-\tau E}\,d\mu_f(\tau)$$

for some non-negative measure μ_f depending on f. Setting $E=|\xi|^2$ and recalling the inverse Fourier transform of the Gaussian $e^{-\tau|\xi|^2}$, we obtain the following "*subordination formula*" given by

$$P^{(s)}(x,1)=\int_0^\infty \frac{1}{\sqrt{2\tau}}e^{-x^2/4\tau}\,d\mu_s(\tau) \tag{A.2}$$

with some non-negative measure $\mu_s\geqslant 0$ and $\mu_s\not\equiv 0$. From this formula we readily deduce that $P^{(s)}(x,1)>0$ for $x\in\mathbb{R}$ and

$$\frac{d}{dx}P^{(s)}(x,1)<0 \quad \text{for } x>0.$$

As remarked above, this yields the desired result for all $t>0$. □

Next, we derive the following pointwise estimate for $P^{(s)}(t,x)$.

LEMMA A.2. *For $0<s_0<1$ fixed, we have the pointwise bound*

$$P^{(s)}(x,t)\leqslant C\min\{t^{-1/2s},|x|^{-1}\}$$

for $x\in\mathbb{R}$, $t>0$ and $s_0\leqslant s<1$. Here the constant $C>0$ depends only on s_0.

Remark A.3. By a classical result in [8], we can obtain the following bounds

$$\frac{A}{|x|^{1+2s}} \leqslant P^{(s)}(x,t=1) \leqslant \frac{B}{|x|^{1+2s}} \quad \text{for } |x| \geqslant 1,$$

where the constants $A>0$ and $B>0$ depend on s. However, the arguments given there do not provide any insight on how to obtain uniform decay bounds with respect to $s \geqslant s_0 > 1$.

Proof. First, we easily obtain the bound

$$P^{(s)}(x,t) \leqslant \int_{\mathbb{R}} e^{-t|\xi|^{2s}}\, d\xi = \frac{1}{s}\Gamma\left(\frac{1}{2s}\right) t^{-1/2s} \leqslant C t^{-1/2s},$$

with some constant $C>0$ depending only on s_0. Furthermore, integration by parts yields that

$$xP^{(s)}(x,t) = -i \int_{\mathbb{R}} \left(\frac{d}{d\xi} e^{-t|\xi|^{2s}}\right) e^{-ix\xi}\, d\xi.$$

Hence, we find that

$$|xP^{(s)}(x,t)| \leqslant \int_{\mathbb{R}} 2st|\xi|^{2s-1} e^{-t|\xi|^{2s}}\, d\xi = \int_{\mathbb{R}} e^{-|u|}\, du = 2,$$

which completes the proof. □

Now, we consider the kernel for the resolvent $((-\Delta)^s+\lambda)^{-1}$ on $\mathbb{R}$ with $\lambda>0$. By functional calculus, we have the general formula

$$\frac{1}{(-\Delta)^s+\lambda} = \int_0^\infty e^{-\lambda t} e^{-t(-\Delta)^s}\, dt. \tag{A.3}$$

We have the following properties of the integral kernel associated with

$$\frac{1}{(-\Delta)^s+\lambda}.$$

LEMMA A.4. *For $0<s<1$ and $\lambda>0$ set*

$$G_{s,\lambda}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{|\xi|^{2s}+\lambda} e^{-i\xi x}\, dx.$$

Then the following properties hold:

(i) *$G_{s,\lambda} \in L^p(\mathbb{R})$ for $1<p<\infty$ with $1-1/p<2s$.*

(ii) *$G_{s,\lambda}(x)>0$ for $x \in \mathbb{R}$ and $G_{s,\lambda}(x)$ is an even function which is strictly decreasing in $|x|$.*

(iii) *For $0<s_0<1$ fixed, we have*

$$\|G_{s,\lambda}\|_p \leqslant C\left(\frac{p}{p-1}\right)^{1/p} \lambda^{(1-1/p)/2s-1} \Gamma\left(1-\frac{1}{2s}\left(1-\frac{1}{p}\right)\right)$$

for $s_0 \leqslant s<1$ and $1<p<\infty$ with $1-1/p<2s$. Here the constant $C>0$ only depends on s_0.

(iv) *For $0<s_0<1$ fixed, we have the pointwise bound*

$$0<G_{s,\lambda}(x)\leqslant\frac{C}{\lambda|x|},$$

for $|x|>0$ and $s_0\leqslant s<1$, where the constant $C>0$ only depends on s_0.

Proof. As for property (i), this will clearly follow once we have deduced that (iii) holds. To see that (ii) holds, we simply recall formula (A.3) and use the corresponding properties of $P^{(s)}(t,x)$ in Lemma A.1. To prove (iii), we note that (A.3) yields

$$\|G_{s,\lambda}\|_p\leqslant\int_0^\infty e^{-\lambda t}\|P^{(s)}(\cdot,t)\|_p\,dt.$$

From Lemma A.2 we conclude that

$$\|P^{(s)}(\cdot,t)\|_p\leqslant C\biggl(\int_{|x|<t^{1/2s}}t^{-p/2s}\,dx+\int_{|x|\geqslant t^{1/2s}}|x|^{-p}\,dx\biggr)^{1/p}\leqslant C\biggl(\frac{p}{p-1}\biggr)^{1/p}t^{-(1-1/p)/2s},$$

with $1<p<\infty$ and where $C>0$ only depends on s_0. A straightforward combination of these bounds yields the desired estimate, provided that $1-1/p<2s$ holds.

To establish the pointwise bound stated in (iv), we simply use (A.3) in combination with Lemma A.2. □

We conclude this section by deriving a uniform bound for the optimal constants $C_{\alpha,s}>0$ in the Gagliardo–Nirenberg inequality (in $d=1$ dimension)

$$\int_{\mathbb{R}}|f|^{\alpha+2}\,dx\leqslant C_{\alpha,s}\biggl(\int_{\mathbb{R}}|(-\Delta)^{s/2}f|^2\,dx\biggr)^{\alpha/4s}\biggl(\int_{\mathbb{R}}|f|^2\,dx\biggr)^{\alpha(2s-1)/4s+1},$$

where $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$. We have the following uniform bound.

LEMMA A.5. *Let $0<\alpha<\infty$ be given. Then there is a constant $K_\alpha>0$ such that $C_{\alpha,s}\leqslant K_\alpha$ for $\alpha/2(\alpha+2)\leqslant s<1$.*

Proof. Let $s_0=\alpha/2(\alpha+2)$ and note that $0<s_0<\frac12$. By Sobolev inequalities, we have $\|f\|_{\alpha+2}\leqslant\widetilde{K}\|(-\Delta)^{s_0/2}f\|_2$ for some constant $\widetilde{K}>0$ depending only on α. Next, we use that $\varrho H\leqslant\theta H^{1/\theta}+(1-\theta)\varrho^{1/(1-\theta)}$ for any non-negative operator $H\geqslant0$ and any real numbers $\varrho>0$ and $0<\theta<1$. Evaluating this operator inequality on a function f and optimizing with respect to ϱ, we find that

$$\langle f,Hf\rangle\leqslant\langle f,H^{1/\theta}f\rangle^\theta\|f\|_2^{2(1-\theta)}.$$

Given $s>s_0$, we apply this to $H=(-\Delta)^{s_0}$ with $\theta=s_0/s$. This yields

$$\|f\|_{\alpha+2}\leqslant\widetilde{K}\|(-\Delta)^{s_0/2}f\|_2\leqslant\widetilde{K}\|(-\Delta)^{s/2}f\|_2^{s_0/s}\|f\|_2^{1-s_0/s},$$

whence the result follows with $K_\alpha=\widetilde{K}^{\alpha+2}$. □

Appendix B. Regularity, symmetry and monotonicity

Let $0<s<1$ and $0<\alpha<\alpha_{\max}(s)$ be fixed throughout this section. We consider (not necessarily real-valued) solutions $Q=Q(x)$ in the distributional sense of the equation

$$(-\Delta)^s Q+\lambda Q-|Q|^\alpha Q=0, \tag{B.1}$$

where $\lambda>0$ is given. Again, we could assume that $\lambda=1$ by a rescaling argument, but we keep $\lambda>0$ explicit below.

We start with a simple regularity result used in §5.

LEMMA B.1. *If $Q\in L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$ solves (B.1), then $Q\in H^s(\mathbb{R})$.*

Remark. Formally, this regularity result follows from integrating (B.1) against Q. However, this argument is not legitimate, since we only assume that $(-\Delta)^s Q\in H^{-2s}(\mathbb{R})$ a priori.

Proof. Using that $Q=((-\Delta)^s+\lambda)^{-1}|Q|^\alpha Q$ for $Q\in L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$, we deduce that

$$\|(-\Delta)^{s/2}Q\|_2=\left\|\frac{(-\Delta)^{s/2}}{(-\Delta)^s+\lambda}|Q|^\alpha Q\right\|_2\lesssim_\lambda\left\|\frac{1}{(-\Delta)^{s/2}+1}|Q|^\alpha Q\right\|_2.$$

Now, we invoke Lemma A.4, part (iii), with $s_0=\frac{1}{2}s$ and use Young's inequality. Indeed, since $1+\frac{1}{2}=1/p+(\alpha+1)/(\alpha+2)$ implies that $1-1/p=\alpha/(\alpha+2)<2s$ as $\alpha<\alpha_{\max}(s)$, we deduce that

$$\left\|\frac{1}{(-\Delta)^{s/2}+1}|Q|^\alpha Q\right\|_2\lesssim_s\big\||Q|^\alpha Q\big\|_{(\alpha+2)/(\alpha+1)}=\|Q\|_{\alpha+2}^{\alpha+1}.$$

Therefore $Q\in L^2(\mathbb{R})$ satisfies $\|(-\Delta)^{s/2}Q\|_2<\infty$ and hence $Q\in H^s(\mathbb{R})$. □

Next, we proceed with the following improved regularity result.

LEMMA B.2. *If $Q\in H^s(\mathbb{R})$ solves (B.1), then $Q\in H^{2s+1}(\mathbb{R})$.*

Remark B.3. If $\alpha\geqslant 1$ is an integer in equation (B.1), it is easy to see that $Q\in H^k(\mathbb{R})$ for all $k\geqslant 1$. See also [22] for an analyticity result of $Q(x)$ in this case.

Proof. First, we remark that $Q\in L^\infty(\mathbb{R})$. Of course, this fact immediately follows if $s>\frac{1}{2}$, due to Sobolev inequalities. To see that $Q\in L^\infty(\mathbb{R})$ also when $0<s\leqslant\frac{1}{2}$, we can use the L^p-bounds for the resolvent $((-\Delta)^s+\lambda)^{-1}$ derived in Lemma A.4. Then, by iterating the identity $Q=((-\Delta)^s+\lambda)^{-1}|Q|^\alpha Q$ sufficiently many times, we conclude that $\|Q\|_\infty<\infty$. (Alternatively, we could use that $Q^\alpha\in K_s$ and use the remarks in §3 to infer that $Q\in L^\infty(\mathbb{R})$ holds.)

Given $Q\in L^\infty(\mathbb{R})$, we can now show that $Q\in H^{2s+1}(\mathbb{R})$ as follows. Since $Q\in L^2(\mathbb{R})$, it remains to derive the bound $\|(-\Delta)^{s+1/2}Q\|_2<\infty$. We treat the cases $s\geqslant\frac{1}{2}$ and $0<s<\frac{1}{2}$ separately as follows.

Case $s\geqslant\frac{1}{2}$. As usual, this case is straightforward to handle. Indeed, we notice that

$$\|(-\Delta)^s Q\|_2=\left\|\frac{(-\Delta)^s}{(-\Delta)^s+\lambda}|Q|^\alpha Q\right\|_2\lesssim_\lambda \big\||Q|^\alpha Q\big\|_2\lesssim_\lambda \|Q\|_\infty^\alpha\|Q\|_2<\infty.$$

Hence we have $Q\in H^{2s}(\mathbb{R})$ and in particular $Q\in H^1(\mathbb{R})$, since $s\geqslant\frac{1}{2}$ by assumption. Next, we proceed to find that

$$\|(-\Delta)^{s+1/2}Q\|_2=\left\|\frac{(-\Delta)^{s+1/2}}{(-\Delta)^s+\lambda}|Q|^\alpha Q\right\|_2\lesssim_{\lambda,\alpha}\big\|\nabla(|Q|^\alpha Q)\big\|_2\lesssim_\lambda\|Q\|_\infty^\alpha\|\nabla Q\|_2<\infty,$$

where we used that $\big|\nabla(|Q|^\alpha Q)\big|\leqslant(\alpha+1)|Q|^\alpha|\nabla Q|$ a.e. in $\mathbb{R}$. Thus, we have shown that $Q\in H^{2s+1}(\mathbb{R})$, provided that $s\geqslant\frac{1}{2}$ holds.

Case $0<s<\frac{1}{2}$. First, we recall that the well-known identity

$$\|(-\Delta)^{\sigma/2}u\|_2^2=\frac{2^{2\sigma-1}}{\sqrt{\pi}}\frac{\Gamma\big(\frac{1}{2}(1+2\sigma)\big)}{|\Gamma(-\sigma)|}\iint_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2\sigma}}\,dx\,dy$$

for any $0<\sigma<1$. From this we conclude that

$$\|(-\Delta)^{\sigma/2}(|Q|^\alpha Q)\|_2\lesssim_\alpha\|Q\|_\infty^\alpha\|(-\Delta)^{\sigma/2}Q\|_2,\tag{B.2}$$

where we use the pointwise inequality

$$\big||Q|^\alpha(x)Q(x)-|Q|^\alpha(y)Q(y)\big|\leqslant\alpha\max\{|Q|^\alpha(x),|Q|^\alpha(y)\}|Q(x)-Q(y)|.$$

Recall that $0<s<\frac{1}{2}$ by assumption, and let $N\geqslant2$ be the unique integer such that $1/(N+1)\leqslant s<1/N$. By using estimate (B.2) and $Q\in L^\infty(\mathbb{R})$, we conclude that

$$\begin{aligned}\|(-\Delta)^{(k+1)s/2}Q\|_2&=\left\|\frac{(-\Delta)^{s/2}(-\Delta)^{ks/2}}{(-\Delta)^s+\lambda}|Q|^\alpha Q\right\|_2\lesssim_{k,s,\lambda,\alpha}\|(-\Delta)^{ks/2}(|Q|^\alpha Q)\|_2\\&\lesssim_{k,s,\lambda,\alpha}\|Q\|_\infty^\alpha\|(-\Delta)^{ks/2}Q\|_2\lesssim_{k,s,\lambda,\alpha}\|(-\Delta)^{ks/2}Q\|_2\end{aligned}$$

for $k=1,...,N$. By iteration and since $Q\in L^2(\mathbb{R})$, we thus obtain that

$$\|Q_s\|_{H^{(N+1)s}}\lesssim_{s,k,\alpha}\|Q_s\|_{H^s}<\infty.$$

Since $(N+1)s\geqslant1$, we deduce that $Q\in H^1(\mathbb{R})$ holds. Given this fact, we can now conclude that $Q\in H^{2s+1}(\mathbb{R})$ in the same fashion as done above for $s\geqslant\frac{1}{2}$.

The proof of Lemma B.2 is now complete. □

Next, we turn to symmetry and monotonicity results about solutions of (B.1). Indeed, by adapting the recent moving plane arguments developed by L. Ma and L. Zhao in [26] for the non-local Pekar–Choquard equation, we can derive the following symmetry and monotonicity result.

LEMMA B.4. *If $Q\in H^s(\mathbb{R})$, with $Q\geqslant 0$ and $Q\not\equiv 0$, solves (B.1), then we have*

$$Q(x)=\widetilde{Q}(|x-x_0|)$$

with some $x_0\in\mathbb{R}$ and the function $\widetilde{Q}(r)$ satisfies $\widetilde{Q}(r)>0$ and $\widetilde{Q}'(r)<0$ for $r>0$.

Proof. To deduce Lemma B.4 with the slightly weaker statement that $\widetilde{Q}(r)$ is (not necessarily strictly) decreasing, we can directly apply the moving plane arguments developed in [26]. More precisely, by following [26, §5], we only have to verify that the kernel $K=K(x-y)$ for the resolvent $((-\Delta)^s+1)^{-1}$ on $\mathbb{R}$ satisfies the following conditions:

(1) $K(|z|)$ is real-valued and even;

(2) $K(|z|)>0$ for $z\in\mathbb{R}$;

(3) $K(|z|)$ is monotone decreasing in $|z|$.

Indeed, we have all these facts about $K(x-y)$ due to Lemma A.4, which is based on the properties of the heat kernel $e^{-t(-\Delta)^s}$ on $\mathbb{R}$.

Finally, we show that $\widetilde{Q}'(r)<0$ for $r>0$. Without loss of generality, we may assume that $x_0=0$, and hence $Q(x)=\widetilde{Q}(|x|)>0$. By differentiating (B.1) with respect to x, we obtain

$$L_+Q'=0,$$

where

$$L_+=(-\Delta)^s+1-(\alpha+1)Q^\alpha.$$

Note that $Q'\in L^2_{\mathrm{odd}}(\mathbb{R})$ and $Q'(x)=\widetilde{Q}'(r)\leqslant 0$ for $x=r>0$, since $\widetilde{Q}(r)$ is monotone decreasing. In view of Lemma C.3 applied to L_+, we deduce that $Q'\in L^2_{\mathrm{odd}}(\mathbb{R})$ is the ground state eigenfunction of L_+ restricted to $L^2_{\mathrm{odd}}(\mathbb{R})$. Thus we either have $Q'(x)<0$ or $Q'(x)>0$ for $x>0$, where the latter alternative is clearly ruled out. Hence $Q'(x)=\widetilde{Q}'(r)<0$ for $x=r>0$. □

Appendix C. The Kato class K_s and Perron–Frobenius theory for $H=(-\Delta)^s+V$

In this section, we collect some basic results about fractional Schrödinger operators

$$H=(-\Delta)^s+V \quad \text{acting on } L^2(\mathbb{R}).$$

Although most of our discussion generalizes to higher space dimensions $d\geqslant 1$, we shall content ourselves with the 1-dimensional case. In §3, we defined K_s as the *Kato-class* with respect to $(-\Delta)^s$; see Definition 3.1. In particular, the condition $V\in K_s$ guarantees that the heat semi-group e^{-tH} maps $L^2(\mathbb{R})$ to $L^\infty(\mathbb{R})\cap C^0(\mathbb{R})$ for $t>0$. In particular, any L^2-eigenfunction of H is bounded and continuous. See [10] for more details.

First, we derive the following sufficient condition in terms of L^p-spaces for a potential V to be in K_s. (Although the following result may be known in the literature, we were not able to find a suitable reference.)

LEMMA C.1. *Let $0<s<1$ and $V:\mathbb{R}\to\mathbb{R}$ be given. Then the following holds.*

If $0<s\leqslant\frac{1}{2}$ and $V\in L^p(\mathbb{R})$ for some $p>1/2s$, then $V\in K_s$. If $\frac{1}{2}<s<1$ and $V\in L^p(\mathbb{R})$ for some $p\geqslant 1$, then $V\in K_s$.

Proof. In view of Definition 3.1, we have to show that

$$\lim_{E\to\infty}\big\|((-\Delta)^s+E)^{-1}|V|\big\|_{L^\infty\to L^\infty}=0. \tag{C.1}$$

Using that

$$((-\Delta)^s+E)^{-1}=\int_0^\infty e^{-Et}e^{-t(-\Delta)^s}\,dt$$

for $E>0$ and Hölder's inequality, we obtain

$$\big\|((-\Delta)^s+E)^{-1}|V|\big\|_{L^\infty\to L^\infty}\leqslant\|V\|_p\int_0^\infty e^{-Et}\|e^{-t(-\Delta)^s}\|_{L^p\to L^\infty}\,dt.$$

Next, let $P^{(s)}(x,t)$ denote the kernel of $e^{-t(-\Delta)^s}$ on $\mathbb{R}$. By Young's inequality, we have $\|e^{-t(-\Delta)^s}\|_{L^p\to L^\infty}\leqslant\|P^{(s)}(\cdot,t)\|_q$ with $1/p+1/q=1$. Next, we find that

$$\|P^{(s)}(\cdot,t)\|_q\leqslant C\bigg(\int_{|x|<t^{1/2s}}t^{-q/2s}\,dx+\int_{|x|\geqslant t^{1/2s}}\frac{t^q}{|x|^{q(1+2s)}}\,dx\bigg)^{1/q}\leqslant Ct^{-(q-1)/2sq},$$

where the constant $C>0$ only depends on s. Indeed, this follows from the simple bound $P^{(s)}(x,t)\leqslant Ct^{-1/2s}$ for all $x\in\mathbb{R}$ from Lemma A.2, combined with the s-dependent bound stated in Remark A.3 and the scaling property $P^{(s)}(x,t)=t^{-1/2s}P^{(s)}(t^{-1/2s}x,1)$ for $t>0$. Since $1/p=(q-1)/q$, the previous bound for $\|P^{(s)}(\cdot,t)\|_q$ implies that

$$\big\|((-\Delta)^s+E)^{-1}|V|\big\|_{L^\infty\to L^\infty}\leqslant C\|V\|_p\int_0^\infty e^{-Et}t^{-1/2sp}\,dt.$$

From this we deduce that (C.1) holds if $p>1/2s$ for $s\leqslant\frac{1}{2}$, or if $p\geqslant 1$ for $s>\frac{1}{2}$. □

As a next result, we show that fractional Schrödinger operators $H=(-\Delta)^s+V$ enjoy the following Perron–Frobenius property.

LEMMA C.2. *Let $0<s<1$ and consider $H=(-\Delta)^s+V$ acting on $L^2(\mathbb{R})$, where we assume that $V\in K_s$. Suppose that $e=\inf\sigma(H)$ is an eigenvalue. Then e is simple and its corresponding eigenfunction $\psi=\psi(x)>0$ is positive (after replacing ψ by $-\psi$ if necessary).*

Proof. By Lemma A.1, the operator $e^{-t(-\Delta)^s}$ acting on $L^2(\mathbb{R})$ is *positivity improving* for $t>0$. By this, we mean that if $f\geqslant 0$ and $f\not\equiv 0$, then $e^{-t(-\Delta)^s}f>0$.

Next, we consider $H=(-\Delta)^s+V$ acting on $L^2(\mathbb{R})$. Since $V\in K_s$, it follows that V is an infinitesimally bounded perturbation of $(-\Delta)^s$. Hence we can apply standard Perron–Frobenius-type arguments (see, e.g., [32]) to deduce that the largest eigenvalue of e^{-tH} is simple and its corresponding eigenfunction strictly positive. By functional calculus, this fact is equivalent to saying that the lowest eigenvalue of H is simple and has a positive eigenfunction. □

LEMMA C.3. *Let $H=(-\Delta)^s+V$ be as in Lemma C.2. Moreover, we assume that $V=V(|x|)$ is even and let H_{odd} denote the restriction of H to $L^2_{\mathrm{odd}}(\mathbb{R})$. If $e=\inf\sigma(H_{\mathrm{odd}})$ is an eigenvalue, then e is simple and the corresponding odd eigenfunction $\psi=\psi(x)$ satisfies $\psi(x)>0$ for $x>0$ (after replacing ψ by $-\psi$ if necessary).*

Proof. This result follows by a slight twist of standard abstract Perron–Frobenius arguments.

Let $(-\Delta)^s_{\mathrm{odd}}$ denote the restriction of $(-\Delta)^s$ to $L^2_{\mathrm{odd}}(\mathbb{R})$. By odd symmetry, we find that $e^{-t(-\Delta)^s_{\mathrm{odd}}}$ acts on $f\in L^2_{\mathrm{odd}}(\mathbb{R})$ according to

$$(e^{-t(-\Delta)^s_{\mathrm{odd}}}f)(x)=\int_0^\infty K_{t,s}(x,y)f(y)\,dy. \tag{C.2}$$

Here the integral kernel $K_{t,s}(x,y)$ is given by

$$K_{t,s}(x,y)=P^{(s)}(x-y,t)-P^{(s)}(x+y,t), \tag{C.3}$$

with $P^{(s)}(x,t)$ denoting the Fourier transform of $e^{-t|\xi|^{2s}}$ in $\mathbb{R}$. Now, we claim that $K_t(x,y)>0$ holds for $0<x,y<\infty$. Indeed, recall that $P^{(s)}(x,t)$ is even in x, positive and strictly decreasing with respect to $|x|$; see Lemma A.1. Hence if we write $z=x-y$ and $z'=x+y$ for $x,y>0$, we easily check that $|z|<|z'|$ holds. Therefore we deduce that $K_{t,s}(x,y)>0$ is a strictly positive kernel on $L^2(\mathbb{R}_+)$. Hence $e^{-t(-\Delta)^s_{\mathrm{odd}}}$ can be identified with a positivity improving operator on $L^2(\mathbb{R}_+)$.

Now, we consider $H_{\mathrm{odd}}=(-\Delta)^s_{\mathrm{odd}}+V$ with $V=V(|x|)$ even. Using standard Perron–Frobenius arguments (see the proof of Lemma C.2 and the reference there), we deduce that the largest eigenvalue of $e^{-tH_{\mathrm{odd}}}$ on $L^2(\mathbb{R}_+)$ is simple and its corresponding eigenfunction satisfies $\psi_0=\psi_0(x)>0$ for $x>0$. By functional calculus, this fact now implies Lemma C.3 about H_{odd}. □

Appendix D. A topological lemma

The following auxiliary result was needed in §3.

LEMMA D.1. *Let $x_1<x_2<x_3<x_4$ be real numbers. Suppose that $\gamma,\widetilde{\gamma}\colon[0,1]\to\overline{\mathbb{R}^2_+}$ are simple (i.e. injective) continuous curves such that*

(i) *$\gamma(0)=(x_1,0)$, $\gamma(1)=(x_3,0)$ and $\gamma(t)\in\mathbb{R}^2_+$ for $t\in(0,1)$;*

(ii) *$\widetilde{\gamma}(0)=(x_2,0)$, $\gamma(1)=(x_4,0)$ and $\gamma(t)\in\mathbb{R}^2_+$ for $t\in(0,1)$.*

Then γ and $\widetilde{\gamma}$ intersect in $\mathbb{R}^2_+$, i.e., we have $\gamma(t)=\widetilde{\gamma}(t_*)$ for some $t,t_*\in(0,1)$.

Proof. We define the continuous curve $\widehat{\gamma}\colon[0,1]\to\overline{\mathbb{R}^2_+}$ by setting

$$\widehat{\gamma}(t):=\begin{cases}\gamma(2t) & \text{for } 0\leqslant t\leqslant\frac{1}{2},\\ ((2t-1)(x_1-x_3)+x_3,0) & \text{for } \frac{1}{2}<t\leqslant 1.\end{cases}$$

Note that $\widehat{\gamma}(0)=\widehat{\gamma}(1)=(x_1,0)$. Clearly $\widehat{\gamma}$ is a Jordan curve (i.e., a simple and closed continuous curve) in $\mathbb{R}^2$. By Jordan's curve theorem (see [27] for a simple proof based on Brouwer's fixed point theorem), the set $A=\mathbb{R}^2\setminus\widehat{\gamma}([0,1])$ has exactly two open connected components in $\mathbb{R}^2$. Let us denote these two components by B and C in what follows. Moreover, we have that B, say, is bounded, whereas the component C is unbounded. Finally, the Jordan curve theorem states that $\widehat{\gamma}([0,1])=\partial B=\partial C$ holds. Next, we consider the sets

$$N_{\varepsilon,+}(x_2)=\{(x,y)\in\mathbb{R}^2:\sqrt{(x-x_2)^2+y^2}<\varepsilon \text{ and } y>0\},$$
$$N_{\varepsilon,-}(x_2)=\{(x,y)\in\mathbb{R}^2:\sqrt{(x-x_2)^2+y^2}<\varepsilon \text{ and } y<0\},$$

where $\varepsilon>0$ is given. Since $x_1<x_2<x_3$ by assumption and by construction of $\widehat{\gamma}$, we have that $(x_2,0)\in\widehat{\gamma}([0,1])$. Suppose now that $(\tilde{x},\tilde{y})\in N_{\varepsilon,-}(x_2)$, where $\varepsilon>0$ is arbitrary. Clearly, we can connect the point $(\tilde{x},\tilde{y})$ with $(x_4,0)$ by a continuous curve in the lower half-plane without intersecting the Jordan curve $\widehat{\gamma}$. Furthermore, it is obvious that $(x_4,0)$ belongs to the unbounded component C (by connecting it to (x_4,y) with $y\to-\infty$ without intersecting $\widehat{\gamma}$). Hence, we conclude that $N_{\varepsilon,-}(x_2)\subset C$ for any $\varepsilon>0$. On the other hand, we recall that $\partial A=\widehat{\gamma}([0,1])$. Since $N_{\varepsilon,-}(x_2)\cap B=\varnothing$ for all $\varepsilon>0$, we find that $N_{+,\varepsilon}(x_2)\subset B$ for some $\varepsilon>0$ sufficiently small.

Now we conclude as follows. First, we note that $N_{\varepsilon,+}(x_4)\subset C$ for $\varepsilon>0$ sufficiently small, since C is open and $(x_4,0)\in C$. Second, from $\widetilde{\gamma}(0)=(x_2,0)$ and $\widetilde{\gamma}(1)=(x_4,0)$ and by continuity, we deduce from $\widetilde{\gamma}(t)\in\mathbb{R}^2_+$ for $t\in(0,1)$ that

$$\widetilde{\gamma}(t)\in N_{+,\varepsilon}(x_2)\subset B \text{ for } t \text{ close to } 0 \quad\text{and}\quad \widetilde{\gamma}(t)\in N_{\varepsilon,+}(x_4)\subset C \text{ for } t \text{ close to } 1$$

for some $\varepsilon>0$ sufficiently small. Hence there exists $t_*\in(0,1)$ such that $\widetilde{\gamma}(t_*)\in\widehat{\gamma}([0,1])$. But since $\widetilde{\gamma}(t_*)$ lies in the upper half-plane $\mathbb{R}^2_+$, we actually deduce that $\widetilde{\gamma}$ must intersect γ in $\mathbb{R}^2_+$. □

Appendix E. Regularity of F

We define the map

$$F(Q,\lambda,s) := \begin{pmatrix} Q - \dfrac{1}{(-\Delta)^s+\lambda}|Q|^\alpha Q \\ \|Q\|_{\alpha+2}^{\alpha+2} - c_0 \end{pmatrix}, \tag{E.1}$$

for $Q \in L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R})$, $s_0 \leqslant s < 1$ and $\lambda > 0$. Here $c_0 \in \mathbb{R}$ is some fixed constant.

LEMMA E.1. *Let $0 < s_0 < 1$ and $0 < \alpha < \alpha_{\max}(s_0)$. Consider the real Banach space $X^\alpha = L^2(\mathbb{R}) \cap L^{\alpha+2}(\mathbb{R})$ equipped with the norm $\|\cdot\|_{X^\alpha} = \|\cdot\|_2 + \|\cdot\|_{\alpha+2}$. Define $F(Q,\lambda,s)$ as above. Then the map $F\colon X^\alpha \times \mathbb{R}_+ \times [s_0,1) \to X^\alpha \times \mathbb{R}$ is C^1.*

Proof. We start by showing that $F\colon X^\alpha \times \mathbb{R}_+ \times [s_0,1) \to X^\alpha \times \mathbb{R}$ is well defined. From Lemma A.4 together with Young's and Hölder's inequalities we find that

$$\left\| \frac{1}{(-\Delta)^s+\lambda}|Q|^\alpha f \right\|_q \lesssim_{\lambda,s,p} \|Q\|_r^\alpha \|f\|_r, \tag{E.2}$$

where $1 < p < \infty$ and $1 \leqslant q, r \leqslant \infty$ satisfy

$$\frac{1}{q} + 1 - \frac{1}{p} = \frac{\alpha+1}{r} \quad \text{and} \quad 1 - \frac{1}{p} < 2s. \tag{E.3}$$

In particular, if we choose $r = \alpha+2$ and $q = 2$, we find that

$$1 - \frac{1}{p} = \frac{\alpha}{2(\alpha+2)} < s_0 < 2s,$$

since $\alpha < \alpha_{\max}(s_0)$. Furthermore, by setting $r = \alpha+2$ and $q = \alpha+2$, we see that

$$1 - \frac{1}{p} = \frac{\alpha}{\alpha+2} < 2s_0 \leqslant 2s,$$

due to $\alpha < \alpha_{\max}(s_0)$. Hence we can apply (E.2) to conclude that $F(Q,\lambda,s)$ is well defined.

Next, we turn to the Fréchet differentiability of F. (Recall that we restrict to real-valued functions.) First, we consider the second component of the map $F = (F_1, F_2)$, which is given by

$$F_2(Q,\lambda,s) := \|Q\|_{\alpha+2}^{\alpha+2} - c_0,$$

with some fixed constant $c_0 \in \mathbb{R}$. It is easy to see that $F_2(Q,\lambda,s)$ is Fréchet differentiable with

$$\frac{\partial F_2}{\partial Q} = (\alpha+2)\langle |Q|^\alpha Q, \cdot \rangle, \quad \frac{\partial F_2}{\partial \lambda} = 0 \quad \text{and} \quad \frac{\partial F_2}{\partial s} = 0,$$

where $\langle f,\cdot\rangle$ denotes the map $g\mapsto\langle f,g\rangle$. Moreover, it is straightforward to check that $\partial F_2/\partial Q$ depends continuously on Q with respect to the topology in $X=L^2(\mathbb{R})\cap L^{\alpha+2}(\mathbb{R})$. Let us now turn to the Fréchet differentiability of the first component

$$F_1(Q,\lambda,s):=Q-\frac{1}{(-\Delta)^s+\lambda}|Q|^\alpha Q.$$

We claim that

$$\frac{\partial F_1}{\partial Q}=1-\frac{1}{(-\Delta)^s+\lambda}(\alpha+1)|Q|^\alpha,\quad \frac{\partial F_1}{\partial\lambda}=\frac{1}{((-\Delta)^s+\lambda)^2}|Q|^\alpha Q$$

and

$$\frac{\partial F_1}{\partial s}=\frac{(-\Delta)^s\log(-\Delta)}{((-\Delta)^s+\lambda)^2}|Q|^\alpha Q.$$

Indeed, it follows from standard arguments (e.g., Sobolev embeddings and Hölder's inequality) combined with (E.2) that the derivatives $\partial F_1/\partial Q$, $\partial F_1/\partial\lambda$ and $\partial F_1/\partial s$ exist and are given as above. For instance, to prove this claim for $\partial F_1/\partial s$ we argue as follows. Let $(Q,\lambda,s)\in X\times\mathbb{R}_+\times[s_0,1)$ be fixed and suppose that $s+h\in[s_0,1)$ with $h\in\mathbb{R}$ and $h\neq0$. We have to show that

$$F_1(Q,\lambda,s+h)-F_1(Q,\lambda,s)=\frac{\partial F_1}{\partial s}(Q,\lambda,s)h+r(h),$$

where $|h|^{-1}r(h)\to0$ in X^α as $h\to0$. To show this fact, we consider the function

$$f(\xi,s):=\frac{1}{|\xi|^{2s}+\lambda}\quad\text{for }\xi\in\mathbb{R}\text{ and }s\in[s_0,1).$$

An elementary calculation yields

$$\frac{\partial f}{\partial s}=-\frac{|\xi|^{2s}\log|\xi|^2}{(|\xi|^{2s}+\lambda)^2}\quad\text{and}\quad\frac{\partial^2 f}{\partial s^2}=2\frac{|\xi|^{4s}(\log|\xi|^2)^2}{(|\xi|^{2s}+\lambda)^3}-\frac{|\xi|^{2s}(\log|\xi|^2)^2}{(|\xi|^{2s}+\lambda)^2}.$$

In particular, for any $\frac12 s_0>\sigma>0$ and $s\geqslant s_0$, we have the following bounds:

$$\begin{aligned}\left|\frac{\partial f}{\partial s}\right|&\lesssim_{s_0,\sigma}\frac{|\xi|^{2s+\sigma}+1}{(|\xi|^{2s}+\lambda)^2}\lesssim_{\sigma,s_0,\lambda}\frac{1}{|\xi|^{2s_0-\sigma}+1},\\ \left|\frac{\partial^2 f}{\partial s^2}\right|&\lesssim_{s_0,\sigma}\frac{|\xi|^{4s+\sigma}+1}{(|\xi|^{2s}+\lambda)^3}+\frac{|\xi|^{2s+\sigma}+1}{(|\xi|^{2s}+\lambda)^2}\lesssim_{\delta,s_0,\sigma}\frac{1}{|\xi|^{2s_0-\sigma}+1}.\end{aligned}$$

Next, by Sobolev inequalities, we obtain

$$\|u\|_{X^\alpha}\lesssim_\alpha\|((-\Delta)^{s_\alpha/2}+1)u\|_2\quad\text{with }s_\alpha=\frac{\alpha}{2(\alpha+2)}.\tag{E.4}$$

Note that $s_\alpha<s_0$ since $\alpha<\alpha_{\max}(s_0)$. Now, by Plancherel's identity and Taylor's theorem applied to $f(\xi,s)$ and estimate (E.2), we deduce (with $\partial F_1/\partial s$ given above) the following estimate:

$$\begin{aligned}\Big\|F_1(Q,\lambda,s+h)-F_1(Q,\lambda,s)-\frac{\partial F_1}{\partial s}(Q,\lambda,s)h\Big\|_{X^\alpha}&\\ \lesssim_\alpha h^2\sup_{\xi\in\mathbb{R}}\Big|(|\xi|^{s_\alpha}+1)\frac{\partial^2 f}{\partial s^2}(\xi,s)(|\xi|^{s_\alpha+\varepsilon}+1)\Big|\,\Big\|\frac{1}{(-\Delta)^{(s_\alpha+\varepsilon)/2}+1}|Q|^\alpha Q\Big\|_2&\\ \lesssim_{\alpha,\lambda,s_0,\sigma} h^2\sup_{\xi\in\mathbb{R}}\Big(\frac{|\xi|^{2s_\alpha+\sigma}+1}{|\xi|^{2s_0-\sigma}+1}\Big)\|Q\|_{\alpha+2}^{\alpha+1}\lesssim_{\alpha,\lambda,s_0,\sigma} h^2\|Q\|_{X^\alpha}^{\alpha+1},&\end{aligned}$$

with some small constant $\sigma>0$ such that $s_0>s_\alpha+2\sigma$ holds, which is possible since $s_\alpha<s_0$. Also, we used above that $\big\|((-\Delta)^{(s_\alpha+\varepsilon)/2}+1)^{-1}|Q|^\alpha|Q|\big\|_2\lesssim_{s_\alpha,\varepsilon}\|Q\|_{\alpha+2}^{\alpha+1}$ by (E.2). Thus we conclude that $\partial F_1/\partial s$ exists and is given as claimed.

Let us now turn to the continuity of $\partial F_1/\partial Q$, $\partial F_1/\partial\lambda$ and $\partial F_1/\partial s$. Again, this follows from standard arguments in combination with (E.2). For example, to show that $\partial F_1/\partial Q$ depends continuously on (Q,λ,s), we can argue as follows. Let $(Q,\lambda,s)\in X^\alpha\times\mathbb{R}_+\times[s_0,1)$ be fixed and suppose that $\varepsilon>0$ is given. We have to find $\delta>0$ such that

$$\Big\|\Big(\frac{\partial F_1}{\partial Q}(Q,\lambda,s)-\frac{\partial F_1}{\partial Q}(\widetilde{Q},\tilde\lambda,\tilde s)\Big)f\Big\|_{X^\alpha}\leqslant\varepsilon\|f\|_{X^\alpha},\tag{E.5}$$

whenever $\|Q-\widetilde{Q}\|_{X^\alpha}+|\lambda-\tilde\lambda|+|s-\tilde s|\leqslant\delta$ and $(\widetilde{Q},\tilde\lambda,\tilde s)\in X^\alpha\times\mathbb{R}_+\times[s_0,1)$. Indeed, by using (E.4), we see that (E.5) follows if we can show that

$$\|(A_{s,\lambda}|Q|^\alpha-A_{\tilde s,\tilde\lambda}|\widetilde{Q}|^\alpha)f\|_2\leqslant\varepsilon\|f\|_{X^\alpha},\tag{E.6}$$

where we set

$$A_{s,\lambda}:=\frac{(-\Delta)^{s_\alpha/2}+1}{(-\Delta)^s+\lambda}.$$

Next, we note that $A_{\tilde s,\tilde\lambda}=A_{\tilde s,\lambda}-(\lambda-\tilde\lambda)B_{\tilde s,\lambda,\tilde\lambda}$ with

$$B_{\tilde s,\lambda,\tilde\lambda}=\frac{(-\Delta)^{s_\alpha/2}+1}{((-\Delta)^{\tilde s}+\tilde\lambda)((-\Delta)^{\tilde s}+\lambda)}.$$

Furthermore, we observe that

$$A_{s,\lambda}|Q|^\alpha-A_{\tilde s,\lambda}|\widetilde{Q}|^\alpha=(A_{s,\lambda}-A_{\tilde s,\lambda})|Q|^\alpha+A_{s,\lambda}(|\widetilde{Q}|^\alpha-|Q|^\alpha).$$

Hence the left-hand side of (E.6) can be estimated as follows

$$\text{LHS of (E.6)}\leqslant\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where

$$\mathrm{I}=\big\|(A_{s,\lambda}-A_{\tilde{s},\lambda})|Q|^\alpha f\big\|_2,\quad \mathrm{II}=\big\|A_{s,\lambda}(|\widetilde{Q}|^\alpha-|Q|^\alpha)f\big\|_2$$

and

$$\mathrm{III}=\big\|(\lambda-\tilde{\lambda})B_{\tilde{s},\lambda,\tilde{\lambda}}|\tilde{Q}|^\alpha f\big\|_2.$$

To estimate I, we recall the bounds for $f(\xi,s)$ derived above and we find (with $\sigma>0$ so small that $s_0>s_\alpha+2\sigma$) the following bound

$$\begin{aligned}\mathrm{I}&\leqslant\sup_{\xi\in\mathbb{R}}\big|(f(\xi,s)-f(\xi,\tilde{s}))(|\xi|^{s_\alpha}+1)(|\xi|^{s_\alpha+\sigma}+1)\big|\left\|\frac{1}{(-\Delta)^{(s_\alpha+\sigma)/2}+1}|Q|^\alpha f\right\|_2\\&\lesssim_{s_0,\alpha,\sigma,\lambda}|s-\tilde{s}|\sup_{\xi\in\mathbb{R}}\left(\frac{|\xi|^{2s_\alpha+\sigma}+1}{|\xi|^{2s_0-\sigma}+1}\right)\|Q\|_{\alpha+2}^\alpha\|f\|_{\alpha+2}\\&\lesssim_{s_0,\alpha,\sigma,\lambda}|s-\tilde{s}|\,\|Q\|_{X^\alpha}^\alpha\|f\|_{X^\alpha}\\&\leqslant\tfrac{1}{3}\varepsilon\|f\|_{X^\alpha},\end{aligned}$$

provided that $|s-\tilde{s}|\leqslant\delta$ for some $\delta>0$. Here we also used (E.2).

To control II, we choose again $\sigma>0$ so small that $s_0>s_\alpha+\sigma$, which yields

$$\begin{aligned}\mathrm{II}&\lesssim_{s_0,\alpha,\sigma,\lambda}\sup_{\xi\in\mathbb{R}}\left|\frac{(|\xi|^{s_\alpha}+1)(|\xi|^{s_\alpha+\sigma}+1)}{|\xi|^{2s_0-\sigma}+\lambda}\right|\left\|\frac{1}{(-\Delta)^{(s_\alpha+\sigma)/2}+1}(|\widetilde{Q}|^\alpha-|Q|^\alpha)f\right\|_2\\&\lesssim_{s_0,\alpha,\sigma,\lambda}\big\||\widetilde{Q}|^\alpha-|Q|^\alpha\big\|_{(\alpha+2)/\alpha}\|f\|_{\alpha+2}.\end{aligned}$$

Suppose now that $0<\alpha\leqslant1$. Then $\big||\widetilde{Q}|^\alpha-|Q|^\alpha\big|\leqslant|\widetilde{Q}-Q|^\alpha$ a.e. in $\mathbb{R}$. On the other hand, if we have $\alpha>1$, we deduce that $\big||\widetilde{Q}|^\alpha-|Q|^\alpha\big|\lesssim_\alpha(|\widetilde{Q}|^{\alpha-1}+|Q|^{\alpha-1})|\widetilde{Q}-Q|$ a.e. in $\mathbb{R}$. Hence, in either case, we can apply Hölder's inequality to conclude that

$$\mathrm{II}\lesssim_{s_0,\alpha,\sigma,\lambda,\|Q\|_{X^\alpha}}\|\widetilde{Q}-Q\|_{X^\alpha}^{\min\{\alpha,1\}}\|f\|_{X^\alpha}\leqslant\tfrac{1}{3}\varepsilon\|f\|_{X^\alpha},$$

provided that $\|\widetilde{Q}-Q\|_{X^\alpha}\leqslant\delta$ for some $\delta>0$.

Finally, we remark that we readily deduce that

$$\mathrm{III}\lesssim_{s,\alpha,\lambda}|\tilde{\lambda}-\lambda|\,\|Q\|_{\alpha+2}^\alpha\|f\|_{\alpha+2}\leqslant\tfrac{1}{3}\varepsilon\|f\|_{X^\alpha},$$

provided that $|\tilde{\lambda}-\lambda|\leqslant\delta$ for some $\delta>0$. This completes the proof that $\partial F_1/\partial Q$ depends continuously on (Q,λ,s).

The arguments that show continuity for the derivatives $\partial F_1/\partial\lambda$ and $\partial F_2/\partial s$ are very similar to the estimates given above. Therefore we omit the details, and the proof of Lemma E.1 is now complete. □

Note added in proof

Recently, the results of this paper have been generalized to $N \geqslant 2$ dimensions by the authors together with L. Silvestre in [15].

References

[1] Abdelouhab, L., Bona, J. L., Felland, M. & Saut, J.-C., Nonlocal models for nonlinear, dispersive waves. *Phys. D*, 40 (1989), 360–392.

[2] Abramowitz, M. & Stegun, I. A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards Applied Mathematics Series, 55. U.S. Government Printing Office, Washington, DC, 1972.

[3] Albert, J. P. & Bona, J. L., Total positivity and the stability of internal waves in stratified fluids of finite depth. *IMA J. Appl. Math.*, 46 (1991), 1–19.

[4] Albert, J. P., Bona, J. L. & Saut, J.-C., Model equations for waves in stratified fluids. *Proc. Roy. Soc. London Ser. A*, 453 (1997), 1233–1260.

[5] Amick, C. J. & Toland, J. F., Uniqueness and related analytic properties for the Benjamin–Ono equation—a nonlinear Neumann problem in the plane. *Acta Math.*, 167 (1991), 107–126.

[6] Bañuelos, R. & Kulczycki, T., The Cauchy process and the Steklov problem. *J. Funct. Anal.*, 211 (2004), 355–423.

[7] Bennett, D. P., Brown, R. W., Stansfield, S. E., Stroughair, J. D. & Bona, J. L., The stability of internal solitary waves. *Math. Proc. Cambridge Philos. Soc.*, 94 (1983), 351–379.

[8] Blumenthal, R. M. & Getoor, R. K., Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95 (1960), 263–273.

[9] Caffarelli, L. & Silvestre, L., An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32 (2007), 1245–1260.

[10] Carmona, R., Masters, W. C. & Simon, B., Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions. *J. Funct. Anal.*, 91 (1990), 117–142.

[11] Chang, S.-M., Gustafson, S., Nakanishi, K. & Tsai, T.-P., Spectra of linearized operators for NLS solitary waves. *SIAM J. Math. Anal.*, 39 (2007/08), 1070–1111.

[12] Chang, S.-Y. A. & González, M. d. M., Fractional Laplacian in conformal geometry. *Adv. Math.*, 226 (2011), 1410–1432.

[13] Chen, W., Li, C. & Ou, B., Classification of solutions for an integral equation. *Comm. Pure Appl. Math.*, 59 (2006), 330–343.

[14] Elgart, A. & Schlein, B., Mean field dynamics of boson stars. *Comm. Pure Appl. Math.*, 60 (2007), 500–545.

[15] Frank, R. L., Lenzmann, E. & Seiringer, R., Uniqueness of radial solutions for the fractional Laplacian. Preprint, 2013. arXiv:1302.2652 [math.AP].

[16] Frank, R. L. & Seiringer, R., Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.*, 255 (2008), 3407–3430.

[17] Fröhlich, J. & Lenzmann, E., Blowup for nonlinear wave equations describing boson stars. *Comm. Pure Appl. Math.*, 60 (2007), 1691–1705.

[18] González, M. d. M., Gamma convergence of an energy functional related to the fractional Laplacian. *Calc. Var. Partial Differential Equations*, 36 (2009), 173–210.

[19] Graham, C. R. & Zworski, M., Scattering matrix in conformal geometry. *Invent. Math.*, 152 (2003), 89–118.

[20] Kenig, C. E., Martel, Y. & Robbiano, L., Local well-posedness and blow-up in the energy space for a class of L^2 critical dispersion generalized Benjamin–Ono equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 853–887.
[21] Kwong, M. K., Uniqueness of positive solutions of $\Delta u-u+u^p=0$ in $\mathbf{R}^n$. *Arch. Ration. Mech. Anal.*, 105 (1989), 243–266.
[22] Li, Y. A. & Bona, J. L., Analyticity of solitary-wave solutions of model equations for long waves. *SIAM J. Math. Anal.*, 27 (1996), 725–737.
[23] Li, Y. Y., Remark on some conformally invariant integral equations: the method of moving spheres. *J. Eur. Math. Soc.* (*JEMS*), 6 (2004), 153–180.
[24] Lieb, E. H. & Yau, H.-T., The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. *Comm. Math. Phys.*, 112 (1987), 147–174.
[25] Lin, Z., Instability of nonlinear dispersive solitary waves. *J. Funct. Anal.*, 255 (2008), 1191–1224.
[26] Ma, L. & Zhao, L., Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Ration. Mech. Anal.*, 195 (2010), 455–467.
[27] Maehara, R., The Jordan curve theorem via the Brouwer fixed point theorem. *Amer. Math. Monthly*, 91 (1984), 641–643.
[28] Majda, A. J., McLaughlin, D. W. & Tabak, E. G., A one-dimensional model for dispersive wave turbulence. *J. Nonlinear Sci.*, 7 (1997), 9–44.
[29] Maz′ya, V. G., *Sobolev Spaces*. Springer Series in Soviet Mathematics. Springer, Berlin–Heidelberg, 1985.
[30] McLeod, K., Uniqueness of positive radial solutions of $\Delta u+f(u)=0$ in $\mathbf{R}^n$. II. *Trans. Amer. Math. Soc.*, 339 (1993), 495–505.
[31] McLeod, K. & Serrin, J., Uniqueness of positive radial solutions of $\Delta u+f(u)=0$ in $\mathbf{R}^n$. *Arch. Ration. Mech. Anal.*, 99 (1987), 115–145.
[32] Reed, M. & Simon, B., *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, New York, 1978.
[33] Weinstein, M. I., Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16 (1985), 472–491.
[34] — Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation. *Comm. Partial Differential Equations*, 12 (1987), 1133–1173.

Rupert L. Frank
Department of Mathematics
Princeton University
Washington Road
Princeton, NJ 08544
U.S.A.
rlfrank@math.princeton.edu

Current address:
Mathematics 253-37
Caltech
Pasadena, CA 91125
U.S.A.
rlfrank@caltech.edu

Enno Lenzmann
Institute for Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
lenzmann@math.ku.dk

Current address:
Department of Mathematics
and Computer Sciences
University of Basel
Rheinsprung 21
CH-4051 Basel
Switzerland
enno.lenzmann@unibas.ch

Received May 24, 2011