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Classification of Levi degenerate homogeneous CR-manifolds in dimension 5

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Contents

1. Introduction

The main topic of this paper is the study of real-analytic CR-manifolds M with everywhere degenerate Levi form. In particular, for homogeneous manifolds of this type, we develop methods for the computation of the Lie algebras $\mathfrak{hol}(M,a)$ of infinitesimal CR-transformations at every $a\in M$. We also classify, up to local CR-equivalence, all locally homogeneous degenerate CR-manifolds in dimension 5.

In this context, a well-studied example of a homogeneous Levi degenerate CR-manifold is the quadratic hypersurface

$$\mathcal{M} := \{z \in \mathbb{C}^3 : (\operatorname{Re} z_1)^2 + (\operatorname{Re} z_2)^2 = (\operatorname{Re} z_3)^2 \text{ and } \operatorname{Re} z_3 > 0\};$$

compare, e.g., [13], [16], [23] and [31]. This 5-dimensional CR-manifold has several remarkable properties and serves as motivation for various considerations in this paper. Notice that $\mathcal{M}$ can also be written as a tube manifold

$$\mathcal{M} = \mathcal{F} + i\mathbb{R}^3 \subset \mathbb{C}^3, \quad \text{where } \mathcal{F} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2 \text{ and } x_3 > 0\}$$

is the future light cone in 3-dimensional space-time. A glance at this description shows that $\mathcal{M}$ is homogeneous under a group of complex-affine transformations. It is less obvious that the Lie algebras of *global* and *local* infinitesimal CR-transformations at $a \in \mathcal{M}$, $\mathfrak{hol}(\mathcal{M})$ and $\mathfrak{hol}(\mathcal{M}, a)$, are both isomorphic to $\mathfrak{so}(2,3)$, and hence have dimension 10; compare [23]. Also the following 'globalization' is known: the group $\mathrm{SO}(2,3)$ acts on the complex quadric $\mathbf{Q}_3 \subset \mathbb{P}_4(\mathbb{C})$ by biholomorphic transformations and has a hypersurface orbit that contains $\mathcal{M}$ as a dense domain.

The cone $\mathcal{F}$ clearly is a disjoint union of affine half-lines. Therefore, $\mathcal{M}$ is a disjoint union of complex half-planes, actually $\mathcal{M}$ is a fiber bundle with typical fiber $\mathbb{H}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. However, the total space $\mathcal{M}$ is not even locally CR-equivalent to a product $\mathbb{H}^+ \times M'$, with M' a CR-manifold. Notice that the Levi form in both cases, that is, for $\mathcal{M}$ and for a product of $\mathbb{H}^+$ with a Levi nondegenerate 3-dimensional CR-manifold M', has exactly one nonzero eigenvalue at every point. Hence, one needs more invariants to distinguish those CR-manifolds. While every product $\mathbb{C} \times M'$ is holomorphically degenerate, the crucial fact here is that the light cone tube $\mathcal{M}$ is non-degenerate in a higher order sense: To be precise, $\mathcal{M}$ is 2-nondegenerate at every point, and we refer to [4] and also to [3, §11.1], for the notion of k-nondegeneracy.

In the nonhomogeneous setting, for every $k \in \mathbb{N}$ and fixed manifold dimension it is not difficult to construct large classes of CR-manifolds, even hypersurfaces, which are k-nondegenerate at some points, but are Levi nondegenerate in a dense open subset. It seems to be much harder to construct CR-manifolds which are k-nondegenerate everywhere for $k \geqslant 2$. Note that the CR-dimension of a *homogeneous* M is an upper bound for the degree k of nondegeneracy. Hence, the lowest manifold dimension for which everywhere 2-nondegenerate CR-manifolds can exist is 5. This raises the intriguing question whether, besides the light cone tube, there exist more 2-nondegenerate homogeneous CR-manifolds in dimension 5. So far, compare e.g. [14], [23], [16] and [5], all known examples in dimension 5 finally turned out to be locally CR-isomorphic to $\mathcal{M}$. Even the announced 'new example' in [17] revealed itself as locally equivalent to $\mathcal{M}$ as shown in [16]. Therefore the desire arose to find examples that are not locally CR-isomorphic to $\mathcal{M}$.

The main objective of this paper is to show that there are actually infinitely many locally mutually inequivalent examples and to provide a full classification. The starting point is the following simple observation: Suppose that $F\subset\mathbb{R}^n$ is an affinely homogeneous (connected) submanifold of dimension say $d<n$. Then the tube $M:=F+i\mathbb{R}^n$ is a generic CR-submanifold of $\mathbb{C}^n$ of CR-dimension d and is homogeneous under a group of complex-affine transformations. Indeed, every real-affine transformation leaving F invariant extends to a complex-affine transformation leaving M invariant and, in addition, M is invariant under all translations $z\mapsto z+iv$ with $v\in\mathbb{R}^n$. Clearly, the crucial question is the following: When is M k-nondegenerate and when are two tubes M and M' of this type locally CR-equivalent?

The classification of all affinely homogeneous surfaces $F\subset\mathbb{R}^3$ can be found in [11] and [12]. In particular, a complete list (up to local affine equivalence and in a slightly different formulation) of all degenerate types that are not a cylinder, is given by the following examples:

(1) $F=\mathcal{F}$, the future light cone as above;

(2a) $F=\{r(\cos t,\sin t,e^{\omega t})\in\mathbb{R}^3 : r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}$ with $\omega>0$ arbitrary;

(2b) $F=\{r(1,t,e^{t})\in\mathbb{R}^3 : r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}$;

(2c) $F=\{r(1,e^{t},e^{\theta t})\in\mathbb{R}^3 : r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}$ with $\theta>2$ arbitrary;

(3) $F=\{c(t)+rc'(t)\in\mathbb{R}^3 : r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}$, where $c(t):=(t,t^2,t^3)$ parameterizes the *twisted cubic* $\{(t,t^2,t^3):t\in\mathbb{R}\}$ in $\mathbb{R}^3$ and $c'(t)=(1,2t,3t^2)$.

Notice that the limit case $\omega=0$ in (2a) gives the future light cone $\mathcal{F}$, while the limit case $\theta=2$ in (2c) gives the linearly homogeneous surface $\{x\in\mathbb{R}^3 : x_1x_3=x_2^2 \text{ and } x_1,x_2>0\}$, which is locally, but not globally, linearly equivalent to $\mathcal{F}$. In fact, $\mathcal{F}$ is linearly equivalent to the cone $\{x\in\mathbb{R}^3 : x_1x_3=x_2^2 \text{ and } x_1+x_3>0\}$.

As our first main result we show the following theorem (compare Propositions 8.8 and 8.9 for details).

THEOREM I. *For every surface F in* (1)–(3) *the corresponding tube manifold $M:=F+i\mathbb{R}^3$ is a homogeneous* 2*-nondegenerate CR-submanifold of $\mathbb{C}^3$ and any two of them are pairwise locally CR-inequivalent. Furthermore, for every F in* (2a)–(3) *and every $a\in M=F+i\mathbb{R}^3$ the following hold*:

(i) *the Lie algebra $\mathfrak{hol}(M,a)$ is solvable and has dimension* 5;

(ii) *the stability group* $\mathrm{Aut}(M,a)$ *is trivial*;

(iii) *every homogeneous real-analytic CR-manifold M', that is locally CR-equivalent to M, is already globally CR-equivalent to M.*

Notice that, a priori, there is no reason why the F in (1)–(3), although known to be locally affinely inequivalent, should have locally CR-inequivalent tubes (for nondegenerate affinely homogeneous surfaces in $\mathbb{R}^3$, for instance, there are counterexamples).

We actually prove an analog of Theorem I in every dimension $n\geqslant 3$, where the same trichotomy occurs as above. Consider the following surfaces $F\subset\mathbb{R}^n$:

(1′) $F=\mathcal{F}^n:=\{x\in\mathbb{R}^n: x_1>0, x_2>0 \text{ and } x_j=x_1^{2-j}x_2^{j-1} \text{ for } 3\leqslant j\leqslant n\}$;

(2′) $F=\{re^{t\varphi}(a)\in\mathbb{R}^n: r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}$, where φ is an endomorphism of $\mathbb{R}^n$ having $a\in\mathbb{R}^n$ as cyclic vector and the n eigenvalues of φ do not form an arithmetic progression in $\mathbb{C}$;

(3′) $F=\{c(t)+rc'(t): r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}$, where $c(t):=(t,t^2,...,t^n)\in\mathbb{R}^n$ parameterizes the twisted n-ic in $\mathbb{R}^n$.

In §6 and §7 we show, among other statements, the following: For every F and F' from (1′)–(3′) the tube manifolds $M:=F+i\mathbb{R}^n$ and $M':=F'+i\mathbb{R}^n$ are affinely homogeneous generic 2-nondegenerate submanifolds of $\mathbb{C}^n$ with CR-dimension 2. Furthermore, M and M' are locally CR-equivalent if and only if F and F' are globally affinely equivalent, and this holds if and only if for given $a\in M$ and $a'\in M'$ the Lie algebras $\mathfrak{hol}(M,a)$ and $\mathfrak{hol}(M',a')$ are isomorphic. In case $F=\mathcal{F}^n$, the Lie algebra $\mathfrak{hol}(M,a)$ contains a copy of $\mathfrak{gl}(2,\mathbb{R})$ and hence is not solvable. In all other cases, that is, F comes from (2′) or (3′), the Lie algebra $\mathfrak{hol}(M,a)$ is solvable of dimension $n+2$ and the stability group $\mathrm{Aut}(M,a)$ has order at most 2.

Let us briefly comment on the proof of Theorem I. Once the defining equations for an F under consideration are explicitly known (this is quite obvious for the types (1)–(3), compare §8, but seems to be hard for the types (2′) and (3′)), one can compute, by standard methods, the order k of nondegeneracy. However, the amount of calculation necessary to determine k in such a way grows very fast with k and with the dimension of $M\subset\mathbb{C}^n$. This is one of the reasons, especially with an eye on possible generalizations, to choose a different approach, which does not use explicit equations. For instance, given an arbitrary submanifold $F\subset\mathbb{R}^n$ which is (locally) *affinely homogeneous*, we present a simple criterion (Proposition 3.7) which allows us to determine quickly the order k of nondegeneracy for the corresponding tube manifold.

The hard part of the proof is to show that the various tubes $F+i\mathbb{R}^3$ are mutually locally inequivalent as CR-manifolds. Recall that for real-analytic, not necessarily homogeneous, *hypersurfaces with nondegenerate Levi form* there exist local invariants which determine each M up to local CR-equivalence due to the fundamental work of Cartan, Tanaka and Chern–Moser, compare [8], [9] and [32]. However, an analogous approach is not available for M of higher codimensions or when the Levi form is degenerate. To distinguish the various tubes $F+i\mathbb{R}^3$, we develop a method (valid also in greater generality) which enables us to determine explicitly the Lie algebras $\mathfrak{hol}(M,a)$ of infinitesimal CR-transformations of the various CR-germs (M,a) (see §2 for basic definitions and §6 for further details).

The CR-manifolds occurring in the previous theorem are quite special as they all are tube manifolds. Moreover, all but one (namely the twisted cubic case in (3)) are actually conical. In the Levi nondegenerate case, many (homogeneous) examples are known which are not locally CR-equivalent to any tube manifold. For instance, the unit sphere subbundle of TS^3 with its canonical CR-structure is such an example. Therefore, our second main result came quite unexpected to us.

THEOREM II. *Every* 5*-dimensional locally homogeneous* 2*-nondegenerate CR-manifold M is locally CR-equivalent to $F+i\mathbb{R}^3$ with F being one of the surfaces in* (1)–(3).

For the precise definition of local homogeneity we refer to §2. A priori, locally homogeneous CR-manifolds might exist which are not locally CR-equivalent to any globally homogeneous one. As a by-product of the above two results we get that such a pathology does not happen in the case under consideration. A word concerning the regularity in Theorem II: According to our general assumption in this paper, Theorem II is formulated and proved in the category of real-analytic CR-manifolds. But then it automatically holds in the smooth situation as well, due to the following well-known fact: If M is a smooth CR-manifold that is locally homogeneous under a *finite-dimensional* Lie algebra $\mathfrak{g}$ of smooth infinitesimal CR-transformations, then there is a real-analytic atlas on M such that all vector fields in $\mathfrak{g}$ become real-analytic.

Theorem II also gives a classification of all (abstract) 2-nondegenerate locally homogeneous CR-manifolds in dimension 5 up to local CR-equivalence. In fact, using Cartan's classification [8] of the 3-dimensional Levi nondegenerate homogeneous CR-manifolds, the following result follows.

CLASSIFICATION. *Every* 5*-dimensional locally homogeneous CR-manifold M with degenerate Levi form is locally CR-equivalent to one of the following*:

(i) $M=F+i\mathbb{R}^3\subset\mathbb{C}^3$, *where F is one of the surfaces* (1)–(3) *from above*;

(ii) $M=\mathbb{C}\times M'$, *where M' is one of the* 3*-dimensional Levi nondegenerate homogeneous CR-manifolds from Cartan's list in* [8];

(iii) $M=\mathbb{C}^2\times\mathbb{R}$ *or* $M=\mathbb{C}\times\mathbb{R}^3$ *with* $\mathbb{R}^3$ *totally real.*

The manifolds in (i) are all 2-nondegenerate and those in (ii) and (iii) are holomorphically degenerate. Also, the manifolds in (iii) are just the Levi flat ones.

With Theorem II the following question arises naturally for higher codimension: *Are there*, *up to local CR-equivalence*, *other locally homogeneous* 2*-nondegenerate CR-manifolds of CR-dimension* 2 *besides those that are tubes over the surfaces F in* $(1')$–$(3')$

above? Notice in this context that every locally homogeneous 2-nondegenerate CR-manifold of dimension 6 necessarily has CR-dimension 2.

For 5-dimensional CR-hypersurfaces with *nondegenerate* Levi form, that is, when Chern–Moser invariants are available, there already exists a partial classification: Locally homogeneous CR-hypersurfaces in $\mathbb{C}^3$ with stability groups of positive dimension have been classified by Loboda in terms of local equations in normal form; see [25], [26], [27].

The paper is organized as follows. After recalling some necessary preliminaries in §2, we discuss in §3 tube manifolds $M=F\oplus i\mathbb{R}^n\subset\mathbb{C}^n$ over real-analytic submanifolds $F\subset\mathbb{R}^n$. It turns out that the CR-structure of M is closely related to the real-affine structure of the base F. For instance, the Levi form of M is essentially the sesquilinear extension of the second fundamental form of the submanifold $F\subset\mathbb{R}^n$. Generalizing the notion of the second fundamental form, we define higher-order invariants for F (see Definition 3.4). In the uniform case these invariants precisely detect the k-nondegeneracy of the corresponding CR-manifold $M=F\oplus i\mathbb{R}^n$. It is known that the (uniform) k-nondegeneracy of a real-analytic CR-manifold M together with minimality ensures that the Lie algebras $\mathfrak{hol}(M,a)$ are finite-dimensional, and is equivalent to this in the special class of locally homogeneous CR-manifolds. For submanifolds $F\subset\mathbb{R}^n$, which in addition are homogeneous under a group of affine transformations, a simple criterion for k-nondegeneracy of the associated tube manifold M is given in Proposition 3.7.

In §4 these results are applied to the case where F is conical in $\mathbb{R}^n$, that is, locally invariant under dilations $z\mapsto tz$ for t near $1\in\mathbb{R}$. In this case M is always *Levi degenerate*. Assuming that $\mathfrak{hol}(M,a)$ is finite-dimensional (which automatically is the case for minimal and finitely nondegenerate CR-manifolds), we develop some basic techniques for the explicit computation of $\mathfrak{hol}(M,a)$. The main results in §4 are the following: We prove that under the finiteness assumption, $\mathfrak{hol}(M,a)$ consists only of polynomial vector fields and carries a natural graded structure; see Proposition 4.2. We prove (under the same assumptions, see Proposition 4.4 (ii)) that local CR-equivalences between two such tube manifolds are always rational maps (even if these manifolds are not real-algebraic). Furthermore, jet determination estimates are provided (Proposition 4.4 (iv)). In the special situation where $\mathfrak{hol}(M,a)$ consists of affine germs only, these results are further strengthened.

In §5 we illustrate by examples how our methods can be applied. In Example 5.1 we present for every $c\geqslant 1$ and $k\in\{2,3,4\}$ a homogeneous submanifold of $\mathbb{C}^{k+c}$ which is k-nondegenerate and has codimension c. We close this section investigating for all $1<p<n$, $q:=n-p$ and $\alpha\in\mathbb{R}^*$ the tubes $M^\alpha_{p,q}$ over the cones

$$F^\alpha_{p,q}:=\left\{x\in(\mathbb{R}^+)^n:\sum_{j=1}^{p}x_j^\alpha=\sum_{j=p+1}^{n}x_j^\alpha\right\}.$$

Using our methods from the preceding section, we explicitly determine the Lie algebras $\mathfrak{hol}(M^{\alpha}_{p,q}, a)$ for arbitrary integers $\alpha \geqslant 2$.

In §6, we construct homogeneous CR-submanifolds $M = M^{\varphi,d} \subset \mathbb{C}^n$ of tube type, depending on the choice of an endomorphism $\varphi \in \mathrm{End}(\mathbb{R}^n)$, an integer $1 < d < n$ and a cyclic vector $a \in \mathbb{R}^n$ for φ in the following way: The powers $\varphi^0, \varphi^1, ..., \varphi^{d-1}$ span an abelian Lie algebra $\mathfrak{h}$ and, in turn, give a cone $F := \exp(\mathfrak{h})(a) \subset \mathbb{R}^n$. The corresponding tube manifold $M^{\varphi,d} = F + i\mathbb{R}^n$ is 2-nondegenerate, minimal and of CR-dimension d. The key result here is the explicit determination of the CR-invariant $\mathfrak{hol}(M^{\varphi,d}, a)$ for φ in 'general position' (Propositions 6.5 and 6.17). The precise meaning for φ of being in 'general position' is stated in Lemma 6.8. Further results are, again for φ in general position, that the tube manifold M is simply connected and has trivial stability group at every point. As a consequence, the manifolds M of this type have the following remarkable property: *Every homogeneous* (*real-analytic*) *CR-manifold locally CR-equivalent to M is globally CR-equivalent to M.*

In §7 the results from §6 are further refined in the case of homogeneous CR-manifolds $M^{\varphi} := M^{\varphi,2} := F^{\varphi} + i\mathbb{R}^n \subset \mathbb{C}^n$ of CR-dimension 2 but without restrictions on the codimension. In fact, every minimal and locally homogeneous tube CR-manifold $M := S + i\mathbb{R}^n$ with a conical 2-dimensional $S \subset \mathbb{R}^n$ is locally CR-equivalent to M^{φ} for some cyclic φ. In this section also the case is treated, where φ is not in 'general position', that is, the characteristic roots do form an arithmetic progression (see Lemma 6.8). The main results here are: Whether a cyclic φ is in general position or not, the Lie algebras $\mathfrak{hol}(M^{\varphi}, a)$ (Proposition 7.3) and the global automorphism groups $\mathrm{Aut}(M^{\varphi})$ (Proposition 7.5) are determined. Furthermore, the problem of local and global CR-equivalence among the M^{φ}'s is solved (Propositions 7.6 and 7.7) and a moduli space is constructed (§7.8).

Part I of the paper is concluded with §8, where the examples (1)–(3) from Theorem I are presented in more detail. The results of the preceding section are applied to this case of 5-dimensional tube manifolds M. In particular, Propositions 8.8 and 8.11 contain some additional information to that stated in Theorem I and also complete the proof of Theorem I.

Part II of the paper is mainly devoted to prove Theorem II. In the preliminary §9 we explain how the geometric properties such as k-nondegeneracy, minimality or the CR-dimension of an arbitrary locally homogeneous CR-germ (M, o) can be encoded in a pure Lie algebraic object, the associated CR-algebra $(\mathfrak{g}, \mathfrak{q})$. This is the key for our classification and is based on results taken from [15]. Specified to 5-dimensional CR-germs, we formulate the precise algebraic conditions on a CR-algebra $(\mathfrak{g}, \mathfrak{q})$ ensuring that the associated CR-germ (M, o) is 2-nondegenerate.

Once the classification of 2-nondegenerate 5-dimensional locally homogeneous CR-germs (M,o) is reduced to a classification problem of certain CR-algebras, we begin in §10 to carry out the details of the proof. It is subdivided into several sections, lemmata and claims and will only be completed in §16. Our proof relies on a quite subtle analysis of Lie algebraic properties of CR-algebras and uses basic structure theory of Lie algebras and Lie groups. The methods are quite general and can be adapted to handle also higher-dimensional cases.

A more detailed outline of the proof can be found in the first part of §10.

Part I. Levi degenerate CR-manifolds

2. Preliminaries

In the following let E always be a complex vector space of finite dimension and $M\subset E$ an immersed connected real-analytic submanifold. In most cases M will be locally closed in E. Due to the canonical identifications $T_aE=E$, for every $a\in M$ we consider the tangent space T_aM as an $\mathbb{R}$-linear subspace of E. Then $H_aM:=T_aM\cap iT_aM$ is the largest complex linear subspace of E contained in T_aM. The manifold M is called a *CR-submanifold* if $\dim_{\mathbb{C}} H_aM$ does not depend on $a\in M$. This dimension is called the *CR-dimension* of M and H_aM is called the *holomorphic tangent space* at a; compare [3] as general reference for CR-manifolds. Given another real-analytic CR-submanifold M' of a complex vector space E', a smooth mapping $g\colon M\to M'$ is called *CR* if for every $a\in M$ the differential $dg_a\colon T_aM\to T_{ga}M'$ maps the corresponding holomorphic tangent spaces in a complex-linear way to each other. Keeping in mind the identification $T_aE=E$, a vector field on M is a mapping $f\colon M\to E$ with $f(a)\in T_aM$ for all $a\in M$. For better distinction we also write $\xi=f(z)\partial/\partial z$ instead of f and ξ_a instead of $f(a)$; compare the convention (2.1) in [16].

An *infinitesimal CR-transformation* on M is by definition a real-analytic vector field $f(z)\partial/\partial z$ on M such that the corresponding local flow consists of CR-transformations. Let us denote by $\mathfrak{hol}(M)$ the space of all such vector fields, which is a real Lie algebra with respect to the usual bracket. For every $f(z)\partial/\partial z\in\mathfrak{hol}(M)$ and every $a\in M$, there exist an open neighbourhood $U\subset M$ of a with respect to the manifold topology on M, an open neighbourhood W of a with respect to E, and a holomorphic mapping $h\colon W\to E$ with $f(z)=h(z)$ for all $z\in U\cap W$; compare [2] or [3, Proposition 12.4.22].

Further, for every $a\in M$, we denote by $\mathfrak{hol}(M,a)$ the Lie algebra of all germs of infinitesimal CR-transformations defined on arbitrary open neighbourhoods of a with respect to the manifold topology of M. For simplicity and without essential loss of generality, we always assume that the CR-submanifold M is *generic* in E, that is,

$E{=}T_aM\oplus iT_aM$ for all $a\in M$. This assumption allows us to consider $\mathfrak{hol}(M,a)$ in a canonical way as a real Lie subalgebra of the complex Lie algebra $\mathfrak{hol}(E,a)$, which we always do in the following. The CR-manifold M is called *holomorphically nondegenerate* at a if $\mathfrak{hol}(M,a)$ is totally real in $\mathfrak{hol}(E,a)$, i.e. if $\mathfrak{hol}(M,a)\cap i\,\mathfrak{hol}(M,a){=}0$ in $\mathfrak{hol}(E,a)$. This condition together with the minimality of M at a implies that $\dim\mathfrak{hol}(M,a)<\infty$; see [3, Theorem 12.5.3]. Here, the CR-submanifold $M\subset E$ is called *minimal* at $a\in M$ if $T_aR{=}T_aM$ for every submanifold $R\subset M$ with $a\in R$, and $H_zM\subset T_zR$ for all $z\in R$. By [3, Proposition 15.5.1], $\dim\mathfrak{hol}(M,a)<\infty$ implies that M is holomorphically nondegenerate at a.

The CR-manifold M is called *locally homogeneous* at the point $a\in M$ if there exists a Lie subalgebra $\mathfrak{g}\subset\mathfrak{hol}(M,a)$ of finite dimension such that the canonical evaluation map $\mathfrak{g}\to T_aM$ is surjective. By [33] this is already satisfied if the canonical evaluation map $\mathfrak{hol}(M,a)\to T_aM$ is surjective. In the global setting the situation is quite different: In [21, p. 69], an example of a domain $D\subset\mathbb{C}^2$ was given, on which the group $\mathrm{Aut}(D)$ of all biholomorphic automorphisms acts transitively but on which no connected Lie group of finite dimension can act transitively. In case the CR-manifold M is locally homogeneous at a, the condition $\dim\mathfrak{hol}(M,a)<\infty$ is equivalent to M being holomorphically nondegenerate and minimal at a.

By $\mathfrak{aut}(M,a):=\{\xi\in\mathfrak{hol}(M,a):\xi_a{=}0\}$ we denote the *isotropy subalgebra* at $a\in M$. Clearly, $\mathfrak{aut}(M,a)$ has finite codimension in $\mathfrak{hol}(M,a)$. Furthermore, we denote by $\mathrm{Aut}(M,a)$ the group of all *germs* of real-analytic CR-isomorphisms $h\colon W\to\widetilde{W}$ with $h(a){=}a$, where W and $\widetilde{W}$ are arbitrary open neighbourhoods of a in M. It is known that every germ in $\mathrm{Aut}(M,a)$ can be represented by a holomorphic map $U\to E$, where U is an open neighbourhood of a in E; compare, e.g., [3, Corollary 1.7.13]. Furthermore, $\mathrm{Aut}(M)$ denotes the group of all global real-analytic CR-automorphisms $h\colon M\to M$ and $\mathrm{Aut}(M)_a$ its isotropy subgroup at a. There is a canonical group monomorphism $\mathrm{Aut}(M)_a\hookrightarrow\mathrm{Aut}(M,a)$ as well as an exponential map $\exp\colon\mathfrak{aut}(M,a)\to\mathrm{Aut}(M,a)$ for every $a\in M$.

By $\mathfrak{aff}(M)\subset\mathfrak{hol}(M)$ we denote the Lie subalgebra of all (complex) affine infinitesimal CR-transformations on M. For every $a\in M$ furthermore $\mathfrak{aff}(M,a)\subset\mathfrak{hol}(M,a)$ is the Lie subalgebra of all affine germs. The canonical embedding $\mathfrak{aff}(M)\hookrightarrow\mathfrak{aff}(M,a)$ is an isomorphism for every $a\in M$.

Suppose that $g\colon U\to U'$ is a biholomorphic mapping between open subsets $U,U'\subset E$. Then

$$g_*(f(z)\partial/\partial z)=g'(g^{-1}z)(f(g^{-1}z))\partial/\partial z \tag{2.1}$$

defines a complex Lie algebra isomorphism $g_*\colon\mathfrak{hol}(U)\to\mathfrak{hol}(U')$, where $g'(u)\in\mathrm{End}(E)$, for every $u\in U$, is the derivative of g at u. For real-analytic CR-submanifolds $M,M'\subset E$ every CR-isomorphism $g\colon (M,a)\to(M',a')$ of manifold germs induces a Lie algebra isomorphism $g_*\colon \mathfrak{g}\to\mathfrak{g}'$, where $\mathfrak{g}:=\mathfrak{hol}(M,a)$ and $\mathfrak{g}':=\mathfrak{hol}(M',a')$. From (2.1) it is clear that g_* extends to a complex Lie algebra isomorphism $\mathfrak{l}\to\mathfrak{l}'$, where the sums $\mathfrak{l}:=\mathfrak{g}\oplus i\mathfrak{g}\subset\mathfrak{hol}(E,a)$ and $\mathfrak{l}':=\mathfrak{g}'\oplus i\mathfrak{g}'\subset\mathfrak{hol}(E,a')$ are not necessarily direct. In particular, $g\mapsto g_*$ defines a group homomorphism $\mathrm{Aut}(M,a)\to\mathrm{Aut}(\mathfrak{g})$.

A basic invariant of a CR-manifold is the (vector-valued) *Levi form*. Its definitions found in the literature may differ by a constant factor. Here we choose the following one: It is well-known that for every point a in the CR-manifold M there is a well-defined alternating $\mathbb{R}$-bilinear map

$$\omega_a\colon H_aM\times H_aM\longrightarrow E/H_aM$$

satisfying $\omega_a(\xi_a,\zeta_a)\equiv[\xi,\zeta]_a \bmod H_aM$, where ξ and ζ are arbitrary smooth vector fields on M with $\xi_z,\zeta_z\in H_zM$ for all $z\in M$. We define the Levi form

$$\mathcal{L}_a\colon H_aM\times H_aM\longrightarrow E/H_aM \tag{2.2}$$

to be twice the sesquilinear part of ω_a. By *sesquilinear* we always mean 'conjugate linear in the first and complex linear in the second variable', that is,

$$\mathcal{L}_a(v,w)=\omega_a(v,w)+i\omega_a(iv,w)\quad\text{for all }\ v,w\in H_aM.$$

In particular, the vectors $\mathcal{L}_a(v,v)$, $v\in H_aM$, are contained in iT_aM/H_aM, which can be identified in a canonical way with the normal space E/T_aM to $M\subset E$ at a. The following remark follows immediately from the way the Levi form is defined.

Remark 2.3. Suppose that Z is a complex manifold, $\varphi\colon Z\to M$ is a smooth CR-mapping and $a=\varphi(c)$ for some $c\in Z$. Then every vector $v\in d\varphi_c(T_cZ)\subset H_aM$ satisfies $\mathcal{L}_a(v,v)=0$. In general, v is not contained in the *Levi kernel*

$$K_aM:=\{v\in H_aM:\mathcal{L}_a(v,w)=0\text{ for all }w\in H_aM\}.$$

The CR-manifold M is called *Levi nondegenerate* at a if $K_aM=0$. Generalizing that, the notion of *k-nondegeneracy* for M at a has been introduced for every integer $k\geqslant 1$ (see [3] and [4]). As shown in [3, Theorem 11.5.1], a real-analytic and connected CR-manifold M is holomorphically nondegenerate at a (equivalently, at every $z\in M$) if and only if there exists a $k\geqslant 1$ such that M is k-nondegenerate at some point $b\in M$. For $k=1$ this notion is equivalent to M being Levi nondegenerate at $b\in M$.

In the second part of our paper we also need a more general notion of a (real-analytic) CR-manifold. This is a connected real-analytic manifold M together with a subbundle $HM\subset TM$ (called the 'holomorphic subbundle') and a bundle endomorphism J of HM, with $J^2=-\mathrm{id}$, such that (HM,J) is involutive; compare [6, §7.4]. By a theorem in [1], there exists an embedding $M\hookrightarrow Z$ into a complex manifold Z, such that H_zM corresponds to $T_zM\cap iT_zM$, where $T_zZ\to T_zZ$ is simply the multiplication by the imaginary unit i (here the real-analyticity is necessary). The bundle homomorphism $J\colon HM\to HM$ is then the restriction to H_zM, for every $z\in M$, of that multiplication by i. For local considerations one can always assume that Z is (an open subset of) $\mathbb{C}^n$.

3. Tube manifolds

Let V be a real vector space of finite dimension and $E:=V\oplus iV$ its complexification. Let furthermore $F\subset V$ be a connected real-analytic submanifold and $M:=F+iV\subset E$ be the corresponding *tube manifold*. M is a generic CR-submanifold of E, invariant under all translations $z\mapsto z+iv$, $v\in V$. In case V' is another real vector space of finite dimension, E' its complexification, $F'\subset V'$ a real-analytic submanifold and $\varphi\colon V\to V'$ an affine mapping with $\varphi(F)\subset F'$, then clearly φ extends in a unique way to a complex-affine mapping $E\to E'$ with $\varphi(M)\subset M'$. However, it should be noted that higher-order real-analytic maps $\psi\colon F\to F'$ also extend locally to holomorphic maps $\psi\colon U\to E'$, U open in E. But in contrast to the affine case, we have in general $\psi(M\cap U)\not\subset M'$. We may therefore ask how the CR-structure of M is related to the real affine structure of the submanifold $F\subset V$.

For every $a\in F$ let $T_aF\subset V$ be the *tangent space* and $N_aF:=V/T_aF$ be the *normal space* to F at a. Then $T_aM=T_aF\oplus iV$ for the corresponding tube manifold M, and N_aF can be canonically identified with the normal space $N_aM=E/T_aM$ of M in E. Define the map $l_a\colon T_aF\times T_aF\to N_aF$ in the following way: For $v,w\in T_aF$ choose a smooth map $f\colon V\to V$ with $f(a)=v$ and $f(x)\in T_xF$ for all $x\in F$ (actually it suffices to choose such an f only in a small neighborhood of a). Then put

$$l_a(v,w):=f'(a)(w) \mod T_aF, \tag{3.1}$$

where the linear operator $f'(a)\in\mathrm{End}(V)$ is the derivative of f at a. One shows that l_a does not depend on the choice of f and is a symmetric bilinear map. We mention that if V is provided with a flat Riemannian metric and N_aF is identified with $T_aF^\perp$, then l is nothing but the second fundamental form of F (see [30, §II.3.3]). The form l_a can also be read off from local equations for F; more precisely, suppose that $U\subset V$ is an open subset, W is a real vector space and $h\colon U\to W$ is a real-analytic submersion with $F=h^{-1}(0)$. Then the derivative $h'(a)\colon V\to W$ induces a linear isomorphism $N_aF\cong W$, and modulo this identification l_a is nothing but the second derivative $h''(a)\colon V\times V\to W$

at a, restricted to $T_aF\times T_aF$. By

$$K_aF:=\{w\in T_aF: l_a(v,w)=0 \text{ for all } v\in T_aF\}$$

we denote the *kernel* of l_a. The manifold F is called (affinely) *nondegenerate* at a if $K_aF=0$ holds. The following statement follows directly from the definition of l_a.

LEMMA 3.2. *Suppose that $\varphi\in\mathrm{End}(V)$ satisfies $\varphi(x)\in T_xF$ for all $x\in F$. Then $\varphi(a)\in K_aF$ if and only if $\varphi(T_aF)\subset T_aF$.*

Lemma 3.2 applies in particular for $\varphi=\mathrm{id}$ in case F is a *cone*, that is, $rF=F$ for all real $r>0$. More generally, we call the submanifold $F\subset V$ *conical* if $x\in T_xF$ for all $x\in F$. Then $\mathbb{R}a\subset K_aF$ holds for all $a\in F$.

In the remaining part of this section we explain how the CR-structure of the tube manifold M is related to the real objects l_a, TF, KF and K^rF, to be defined below, which depend only on the affine geometry of F. In general it needs some effort to check whether a given CR-manifold M is k-nondegenerate at a point $a\in M$ (in the sense of [4]). For affinely homogeneous tube manifolds, however, there are simple criteria, see Propositions 3.5 and 3.7. We start with some preparations. For every $a\in F\subset M$,

$$H_aM=T_aF\oplus iT_aF\subset E \tag{3.3}$$

is the holomorphic tangent space at a, and E/H_aM can be canonically identified with $N_aF\oplus iN_aF$. It is easily seen that the Levi form $\mathcal{L}_a$ of M at a, compare (2.2), is nothing but the sesquilinear extension of the form l_a from $T_aF\times T_aF$ to $H_aM\times H_aM$. In particular,

$$K_aM=K_aF\oplus iK_aF$$

is the *Levi kernel* of M at a. In case the dimension of K_aF does not depend on $a\in F$, these spaces form a subbundle $KF\subset TF$. In this case, for every $v\in K_aF$ there exists a smooth function $f\colon V\to V$ with $f(a)=v$ and $f(x)\in K_xF$ for all $x\in F$, that is, the tangent vector v extends to a smooth section in KF. In any case, let us define inductively linear subspaces K_a^rF of T_aF as follows.

Definition 3.4. For every real-analytic submanifold $F\subset V$, every $a\in F$ and every $r\in\mathbb{N}$, put

(i) $K_a^0F:=T_aF$, and define

(ii) $K_a^{r+1}F$ to be the space of all vectors $v\in K_a^rF$ such that there is a smooth mapping $f\colon V\to V$ with $f'(a)(T_aF)\subset K_a^rF$, $f(a)=v$ and $f(x)\in K_x^rF$ for all $x\in F$.

It is clear that $K_a^1F=K_aF$ holds. Let us call F of *uniform degeneracy* or *uniformly degenerate* if for every $r\in\mathbb{N}$ the dimension of K_a^rF does not depend on $a\in F$. In this case it can be shown that for every $v\in K_a^rF$ the outcome of the condition $f'(a)(T_aF)\subset K_a^rF$ in (ii) does not depend on the choice of the smooth mapping $f\colon V\to V$ satisfying $f(a)=v$ and $f(x)\in K_x^rF$ for all $x\in F$. For instance, F is of uniform degeneracy if F is locally affinely homogeneous, that is, if there exists a Lie algebra $\mathfrak{a}$ of affine vector fields on V such that every $\xi\in\mathfrak{a}$ is tangent to F and such that the canonical evaluation map $\mathfrak{a}\to T_aF$ is surjective for every $a\in F$. Clearly, if F is locally affinely homogeneous in the above sense, then the corresponding tube manifold $M=F+iV$ is locally homogeneous as CR-manifold.

Recall our convention that every smooth map $f\colon V\to V$ is considered as the smooth vector field $\xi=f(x)\partial/\partial x$ on V. Our computations below are considerably simplified by the obvious fact that every smooth vector field ξ on V has a unique smooth extension to E that is invariant under all translations $z\mapsto z+iv$, $v\in V$. In case ξ is tangent to $F\subset V$, the extension satisfies $\xi_z\in H_zM$ for all $z\in M$.

In case the submanifold $F\subset V$ is uniformly degenerate in a neighbourhood of $a\in F$, we call F *affinely k-nondegenerate* at a if $K_a^kF=0$ and $k\geqslant 1$ is minimal with respect to this property. It can be seen that 'affine k-nondegeneracy' is invariant under affine coordinate changes. As a consequence of [23] (compare the last five lines in the appendix therein) we state the following results.

Proposition 3.5. *Suppose that F is uniformly degenerate in a neighbourhood of $a\in F$. Then the corresponding tube manifold $M=F+iV$ is k-nondegenerate as a CR-manifold at $a\in M$ if and only if F is affinely k-nondegenerate at a.*

Corollary 3.6. *Suppose that* $\dim F\geqslant 2$ *and $K_xF=\mathbb{R}x$ for all $x\in F$. Then F is affinely* 2*-nondegenerate at every point.*

Proof. The map $f=\mathrm{id}$ has the property that $f(x)\in K_xF$ for every $x\in F$. Hence, the relation $f'(x)(T_xF)=T_xF\not\subset K_xF$ implies that $x\notin K_x^2F$ and thus $K_x^2F=0$ as well as $x\neq 0$. In particular, F is uniformly degenerate. $\square$

For locally affinely homogeneous submanifolds $F\subset V$ the spaces K_a^rF can easily be characterized. For each affine vector field $\xi=h(x)\partial/\partial x$ on V denote by $\xi^{\mathrm{lin}}:=h-h(0)\in\mathrm{End}(V)$ the *linear part* of ξ.

Proposition 3.7. *Suppose that $\mathfrak{A}$ is a linear space of affine vector fields on V such that every $\xi\in\mathfrak{A}$ is tangent to F and the canonical evaluation mapping $\mathfrak{A}\to T_aF$ is a linear isomorphism. Then, given any $r\in\mathbb{N}$, the vector $v\in K_a^rF$ is in $K_a^{r+1}F$ if and only if $\xi^{\mathrm{lin}}(v)\in K_a^rF$ for every $\xi\in\mathfrak{A}$.*

Proof. By the implicit function theorem, there exist open neighbourhoods Y of $0\in\mathfrak{A}$ and X of $a\in M$ such that $g(y):=\exp(y)a$ defines a diffeomorphism $g: Y\to X$. Define the smooth map $f: X\to V$ by $f(x)=\mu_y(v)$, where μ_y for $y:=g^{-1}(x)$ is the linear part of the affine transformation $\exp(y)$. Then $f(a)=v$ and $f(x)\in K_x^rF$ for every $x\in X$. A simple computation shows that

$$f'(a)(g'(0)\xi)=\xi^{\mathrm{lin}}(v) \quad \text{for every } \xi\in\mathfrak{A}.$$

In view of Definition 3.4 (ii), this identity implies the claim. □

It is easily seen that a necessary condition for M being minimal as a CR-manifold is that F is not contained in an affine hyperplane of V. For later use in Proposition 4.10 we state the following sufficient condition.

PROPOSITION 3.8. *Suppose that $\mathfrak{A}$ has the same properties as in Proposition 3.7 and denote by $\Lambda\subset\mathrm{End}(V)$ the* associative *real subalgebra generated by $\{\xi^{\mathrm{lin}}:\xi\in\mathfrak{A}\}$. Then the tube manifold $M=F+iV$ is minimal at a if V is the linear span of all vectors $\lambda(v)$ with $v\in T_aF$ and $\lambda\in\Lambda$.*

Proof. Without loss of generality we assume that the canonical evaluation mapping $\mathfrak{A}\to T_xF$ is a linear isomorphism for every $x\in F$. We also assume that V is the linear span of all $\lambda(v)$ as above. Define inductively for every integer $r\geqslant 1$ the subbundle $H^rM\subset TM$ in the following way: $H^1M:=HM$ and every $H_z^{r+1}M$, $z\in M$, is the linear span of H_z^rM together with all vectors $[\xi,\eta]_z$, where ξ and η are arbitrary smooth sections in H^rM over M. For the proof, it is enough to show that $T_aM=H_a^\infty M:=\bigcup_{r\geqslant 1}T_aM$.

From $T_aF\oplus iT_aF\subset H_a^rM\subset T_aF\oplus iV$ we see that for every $1\leqslant r\leqslant\infty$ there is a unique linear subspace $H_a^rF\subset V$ with $H_a^rM=T_aF\oplus iH_a^rF$ and $H_a^1F=T_aF$. Therefore, it is enough to show that $H_a^\infty F=V$. We claim that $\xi^{\mathrm{lin}}(H_a^rF)\subset H_a^{r+1}F$ holds for all $\xi\in\mathfrak{A}$. To see this, fix an arbitrary $w\in H_a^rF$ and an arbitrary vector field $\xi\in\mathfrak{A}$. Choose a smooth section η over F in the bundle iH_a^rF with $\eta_a=iw$ and extend η as well as ξ in the unique way to smooth vector fields on M that are invariant under all translations $z\mapsto z+iv$ with $v\in V$. Then ξ and η are sections in H^rM, and $[\xi,\eta]_a=i\xi^{\mathrm{lin}}(w)\in H_a^{r+1}M$ implies that $\xi^{\mathrm{lin}}(w)\in H_a^{r+1}F$ as required. Now define inductively the linear subspaces $W^r\subset V$ by $W^1:=H_a^1F=T_aF$ and letting W^{r+1} be the linear span of W^r together with all $\xi^{\mathrm{lin}}(W^r)$, $\xi\in\mathfrak{A}$. Then $V=\bigcup_{r\geqslant 1}W^r$ by assumption and $W^r\subset H_a^rF$ by induction give $V\subset H_a^\infty M\subset V$ as desired. □

LEMMA 3.9. *Suppose that $F\subset V$ is a submanifold such that for every $c\in V$ with $c\neq 0$ there exists a linear transformation $g\in\mathrm{GL}(V)$ with $g(F)=F$ and $g(c)\neq c$ (this condition is automatically satisfied if F is a cone). Then for $M=F+iV$ the CR-automorphism group $\mathrm{Aut}(M)$ has trivial center.*

Proof. Let an element in the center of $\mathrm{Aut}(M)$ be given and let $h\colon U\to E$ be its holomorphic extension to an appropriate connected open neighbourhood U of M. Since h commutes with every translation $z\mapsto z+iv$, $v\in V$, it is a translation itself. Indeed, for $a\in F$ fixed and $c:=h(a)-a$, the translation $\tau(z):=z+c$ coincides with h on $a+iV$ and hence on U by the identity principle. For every $g\in\mathrm{GL}(V)\cap\mathrm{Aut}(M)$ the identity $gh=hg$ implies $g(c)=c$. This forces $c=0$ by our assumption, that is, $h(z)\equiv z$. □

PROPOSITION 3.10. *Suppose that the homogeneous CR-manifold M is simply connected and that* $\mathrm{Aut}(M)$ *has trivial center. In case the stability group* $\mathrm{Aut}(M,a)$ *is trivial for some* (*and hence every*) $a\in M$, *the following properties hold*:

(i) *Let M' be an arbitrary homogeneous CR-manifold and $D\subset M$ and $D'\subset M'$ be nonempty domains. Then every real-analytic CR-isomorphism $h\colon D\to D'$ extends to a real-analytic CR-isomorphism $M\to M'$.*

(ii) *Let M' be an arbitrary locally homogeneous CR-manifold and $D'\subset M'$ be a domain that is CR-equivalent to M. Then $D'=M'$.*

Proof. (i) Fix a point $a\in D$. For every $g\in G:=\mathrm{Aut}(M)$ with $g(a)\in D$ there exists a transformation $g'\in G':=\mathrm{Aut}(M')$ with $hg(a)=g'h(a)$. Since $\mathrm{Aut}(M',a')=\{\mathrm{id}\}$, the transformation g' is uniquely determined by g and satisfies $hg=g'h$ in a neighbourhood of a. Since the Lie group G is simply connected, $g\mapsto g'$ extends to a group homomorphism $G\to G'$ and h extends to a CR-covering map $h\colon M\to M'$. The deck transformation group $\Gamma:=\{g\in G: gh=h\}$ is in the center of G and hence is trivial by assumption. Therefore, $h\colon M\to M'$ is a CR-isomorphism.

(ii) The proof is essentially the same as of Proposition 6.3 in [16]. □

The condition 'locally homogeneous' in Proposition 3.10 (ii) cannot be omitted. A counterexample is given for every integer $k\geqslant 3$ by the tube $M'\subset\mathbb{C}^3$ over

$$F':=\{x\in\mathbb{R}^3: x_2^k=x_1^{k-1}x_3 \text{ and } x_1^2+x_2^2>0\}.$$

Then with $\mathbb{R}^+:=e^{\mathbb{R}}$ the tube M over $F:=F'\cap(\mathbb{R}^+)^3$ is the Example 8.4 below for $\theta=k$, and M and M' satisfy for $D'=M$ the assumption of Proposition 3.10 (ii).

4. Tube manifolds over cones

In this section we always assume that the (connected) submanifold $F\subset V$ is conical (that is, $x\in T_xF$ for every $x\in F$) and that $a\in F$ is a given point. Then, for $M:=F+iV$, the Lie algebra $\mathfrak{g}:=\mathfrak{hol}(M,a)$ contains the Euler vector field $\delta:=z\partial/\partial z$. Denote by $\mathfrak{P}$ the complex Lie algebra of all polynomial holomorphic vector fields $f(z)\partial/\partial z$ on E, that is, $f\colon E\to E$ is a polynomial map. Then $\mathfrak{P}$ has the $\mathbb{Z}$-grading

$$\mathfrak{P}=\bigoplus_{k\in\mathbb{Z}}\mathfrak{P}_k,\quad [\mathfrak{P}_k,\mathfrak{P}_l]\subset\mathfrak{P}_{k+l}, \tag{4.1}$$

where $\mathfrak{P}_k$ is the k-eigenspace of $\operatorname{ad}(\delta)$ in $\mathfrak{P}$. Then $\mathfrak{P}_k$ is the subspace of all $(k+1)$-homogeneous vector fields in $\mathfrak{P}$ if $k\geqslant-1$ and is 0 otherwise. Define $\mathfrak{g}_k:=\mathfrak{g}\cap\mathfrak{P}_k$. Clearly, $\bigoplus_{k\in\mathbb{Z}}\mathfrak{g}_k$ is a graded, in general proper subalgebra of $\mathfrak{g}$, and $\mathfrak{aff}(M,a)=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$.

Proposition 4.2. *Retaining the above notation, suppose that* $\mathfrak{g}=\mathfrak{hol}(M,a)$ *has finite dimension. Then, the following properties hold:*

(i) $\mathfrak{g}\subset\mathfrak{P}$, *that is, every* $f(z)\partial/\partial z\in\mathfrak{g}$ *is a polynomial vector field on* $E=V\oplus iV$. *Furthermore,* $f(iV)\subset iV$.

(ii) $\mathfrak{g}=\bigoplus_{k\geqslant-1}\mathfrak{g}_k$, $[\mathfrak{g}_k,\mathfrak{g}_l]\subset\mathfrak{g}_{k+l}$ *and* $\mathfrak{g}_{-1}=\{iv\partial/\partial z:v\in V\}$.

(iii) *For every* $z\in M$ *the canonical map* $\mathfrak{hol}(M)\to\mathfrak{hol}(M,z)$ *is a Lie algebra isomorphism.*

(iv) $\mathfrak{g}_k=0$ *for some* $k\in\mathbb{N}$ *implies that* $\mathfrak{g}_j=0$ *for every* $j\geqslant k$.

Proof. Consider $\mathfrak{l}:=\mathfrak{g}\oplus i\mathfrak{g}\subset\mathfrak{hol}(E,a)$, which contains the vector field $\eta:=(z-a)\partial/\partial z$. We first show that $\mathfrak{l}\subset\mathfrak{P}$. Fix an arbitrary $\xi:=f(z)\partial/\partial z\in\mathfrak{l}$. Then in a certain neighbourhood of $a\in E$ there exists a unique expansion $\xi=\sum_{k\in\mathbb{N}}\xi_k$, where $\xi_k=p_k(z-a)\partial/\partial z$ for a k-homogeneous polynomial map $p_k\colon E\to E$. It is easily verified that the vector field $\operatorname{ad}(\eta)\xi\in\mathfrak{l}$ has the expansion $\operatorname{ad}(\eta)\xi=\sum_{k\in\mathbb{N}}(k-1)\xi_k$. Now assume that for $d:=\dim\mathfrak{l}$ there exist indices $k_0<k_1<...<k_d$ such that $\xi_{k_l}\neq0$ for $0\leqslant l\leqslant d$. Since the Vandermonde matrix $((k_l-1)^j)$ is nonsingular, we get that the vector fields $(\operatorname{ad}(\eta))^j\xi=\sum_{k\in\mathbb{N}}(k-1)^j\xi_k$, $0\leqslant j\leqslant d$, are linearly independent in $\mathfrak{l}$, a contradiction. This implies that $\xi\in\mathfrak{P}$ as claimed.

Since $\mathfrak{g}\subset\mathfrak{P}$ has finite dimension, every $\eta\in\mathfrak{g}$ is a finite sum $\eta=\sum_{k=-1}^m\eta_k$ with $\eta_k\in\mathfrak{P}_k$ and $m\in\mathbb{N}$ not depending on η. For every polynomial $p\in\mathbb{R}[X]$ then

$$p(\operatorname{ad}(\delta))\eta=\sum_{k=-1}^m p(k)\eta_k$$

shows that $\eta_k\in\mathfrak{g}_k$ for all k, that is, $\mathfrak{g}=\bigoplus\mathfrak{g}_k$. The identity $\mathfrak{g}_{-1}=\{iv\partial/\partial z:v\in V\}$ follows from the fact that $\mathfrak{g}_{-1}$ is totally real in $\mathfrak{P}_{-1}$ and this implies that $f(iV)\subset iV$ for all $f(z)\partial/\partial z\in\mathfrak{g}_k$, by $[\mathfrak{g}_{-1},\mathfrak{g}_k]\subset\mathfrak{g}_{k-1}$ and induction on k. For the proof of the last claim, assume that $\mathfrak{g}_k=0$ for some $k\geqslant0$. Then $[\mathfrak{P}_{-1},\mathfrak{g}_{k+1}]\subset(\mathfrak{g}_k\oplus i\mathfrak{g}_k)=0$ implies $\mathfrak{g}_{k+1}\subset\mathfrak{g}_{-1}$ and hence $\mathfrak{g}_{k+1}=0$. $\square$

Corollary 4.3. *In case* $\mathfrak{g}=\mathfrak{hol}(M,a)$ *has finite dimension, the CR-manifold* $M=F+iV$ *is locally homogeneous at* a *if and only if* F *is locally linearly homogeneous at* $a\in F$.

Proof. From Proposition 4.2 follows that $W:=\{\xi_a:\xi\in\bigoplus_{k\in\mathbb{N}}\mathfrak{g}_{2k}\}$ is a subspace of V, while $\{\xi_a:\xi\in\bigoplus_{k\in\mathbb{N}}\mathfrak{g}_{2k-1}\}=iV$. Hence, M is locally homogeneous at a if and only if $W=T_aF$. But for every $k\in\mathbb{N}$ and every $\xi\in\mathfrak{g}_{2k}$ the vector field $\eta:=(\mathrm{ad}(ia\partial/\partial z))^{2k}\xi$ is in $\mathfrak{g}_0$ and satisfies $\eta_a=(-1)^k(2k+1)!\xi_a$, that is, $W=\{\xi_a:\xi\in\mathfrak{g}_0\}$. □

Notice that the conclusion $\mathfrak{g}\subset\mathfrak{P}$ together with an eigenspace decomposition as in Proposition 4.2 can for $V=\mathbb{R}^n$ be obtained in the same way if instead of 'F conical' it is only assumed for $M=F+i\mathbb{R}^n$ that $\mathfrak{g}=\mathfrak{hol}(M,a)$ contains a vector field $\alpha_1z_1\partial/\partial z_1+\alpha_2z_2\partial/\partial z_2+...+\alpha_nz_n\partial/\partial z_n$ with $\alpha_k>0$ for all k; compare e.g. (7.13).

PROPOSITION 4.4. *Assume that $\mathfrak{g}:=\mathfrak{hol}(M,a)$ has finite dimension, that $F'\subset V$ is another conical submanifold with tube manifold $M'=F'+iV$, and that $\mathfrak{g}':=\mathfrak{hol}(M',a')$ for some $a'\in F'$. Assume that the CR-germs (M,a) and (M',a') are isomorphic and let $g,\tilde{g}\colon(M,a)\to(M',a')$ be arbitrary CR-isomorphisms. Then, the following properties hold*:

(i) $\dim\mathfrak{g}_k=\dim\mathfrak{g}'_k$ *for all $k\in\mathbb{Z}$, where $\mathfrak{g}_k$ and $\mathfrak{g}'_k$ are given by the decomposition in Proposition* 4.2 (ii).

(ii) *g is represented by a rational transformation on E.*

(iii) *In case $\mathfrak{g}_1=0$, g is represented by a linear transformation in* $\mathrm{GL}(V)\subset\mathrm{GL}(E)$ *mapping every K_a^rF onto $K_{a'}^rF'$.*

(iv) *$g=\tilde{g}$ if and only if g and $\tilde{g}$ have the same d-jet at a, where*

$$d:=\min\{k\in\mathbb{N}:\mathfrak{g}_k=0\}.$$

Proof. Let $\mathfrak{l}:=\mathfrak{g}\oplus i\mathfrak{g}$ and $\mathfrak{l}_k:=\mathfrak{l}\cap\mathfrak{P}_k$ for all k. The Lie algebra automorphism $\Psi:=\exp(\mathrm{ad}(a\partial/\partial z))$ of $\mathfrak{l}$ maps every $f(z)\partial/\partial z$ to $f(z+a)\partial/\partial z$. For every k denote by $\mathfrak{l}^k\subset\mathfrak{l}$ the subspace of all vector fields that vanish of order at least $k+1$ at a. Then $\Psi(\mathfrak{l}^k)=\bigoplus_{j\geqslant k}\mathfrak{l}_j$ implies that $\dim\mathfrak{l}_k=\dim\mathfrak{l}^k/\mathfrak{l}^{k+1}$. As a consequence, $\dim_{\mathbb{R}}\mathfrak{g}_k=\dim_{\mathbb{C}}\mathfrak{l}_k$ is a CR-invariant of the germ (M,a) for every k.

For the proof of (ii) and (iii), put $\mathfrak{l}':=\mathfrak{g}'+i\mathfrak{g}'$ and extend g to a biholomorphic mapping $g\colon U\to U'$ with $g(a)=a'$ and $g(U\cap M)=U'\cap M'$ for suitable connected open neighbourhoods U and U' of a and a' in E. Consider the induced Lie algebra isomorphism $g_*\colon\mathfrak{l}\to\mathfrak{l}'$; compare (2.1). Its inverse $\Theta:=g_*^{-1}$ is given by

$$\Theta(f(z)\partial/\partial z)=g'(z)^{-1}f(g(z))\partial/\partial z. \tag{4.5}$$

Since $\mathfrak{l}$ consists of polynomial vector fields (Proposition 4.2), there exist *polynomial* maps $p\colon E\to E$ and $q\colon E\to\mathrm{End}(E)$ such that

$$\Theta(z\partial/\partial z)=p(z)\partial/\partial z\quad\text{and}\quad\Theta(e\partial/\partial z)=(q(z)e)\partial/\partial z$$

for all $e\in E$. Then (4.5) implies that $g'(z)^{-1}=q(z)$ and $g'(z)^{-1}g(z)=p(z)$, that is,

$$g(z)=q(z)^{-1}p(z) \tag{4.6}$$

in a neighbourhood of $a\in E$ and, in particular, g is rational.

Now suppose $\mathfrak{g}_1=0$. Then also $\mathfrak{g}_k=\mathfrak{g}'_k=0$ for all $k\geqslant 1$ by (i) and Proposition 4.2 (iv). Clearly $\Theta(\mathfrak{l}'_{a'})=\mathfrak{l}_a$, where $\mathfrak{l}_a:=\{\xi\in\mathfrak{l}:\xi_a=0\}$, and similarly $\mathfrak{l}'_{a'}\subset\mathfrak{l}'$, are the isotropy subalgebras at a and a'. Also, $\delta_a:=(z-a)\partial/\partial z$ is the unique element in $\mathfrak{l}_a$ such that $\mathrm{ad}(\delta_a)$ induces the negative identity on the factor space $\mathfrak{l}/\mathfrak{l}_a$. As $\delta_{a'}$ has the same uniqueness property for $\mathfrak{l}'_{a'}$, we must have $\Theta(\delta_{a'})=\delta_a$. Since $\delta=z\partial/\partial z$ is the $\mathfrak{g}$-component of δ_a as well as of $\delta_{a'}$ in $\mathfrak{g}\oplus i\mathfrak{g}$, we actually get $\Theta(\delta)=\delta$, that is, $p(z)\equiv z$. Also $\mathfrak{l}_{-1}$ is Θ-invariant, implying that q is constant. Therefore $g=q(a)^{-1}$.

For the proof of (iv) we assume without loss of generality that $M=M'$, $a=a'$ and that $g,\mathrm{id}\in\mathrm{Aut}(M,a)$ have the same d-jet at a. This implies that $g(z)-z$ vanishes of order $>d$ and $g'(z)-\mathrm{id}$ vanishes of order $\geqslant d$ at a. Therefore also $g'(z)^{-1}-\mathrm{id}=q(z)-\mathrm{id}$ vanishes of order $\geqslant d$ at a. Since q is a polynomial of degree $\leqslant d$, there is a d-homogeneous polynomial $s: E\rightarrow\mathrm{End}(V)$ with $q(z)=\mathrm{id}+s(z-a)$. Consider the vector field $\eta:=(z-a)\partial/\partial z\in\mathfrak{l}$ and define the holomorphic mappings $h,r: U\rightarrow E$ by

$$h(z):=q(z)(g(z)-a)=(z-a)+r(z).$$

Then $\Theta(\eta)=h(z)\partial/\partial z\in\mathfrak{g}$ shows that h and r are polynomials of degree $\leqslant d$. But

$$r(z)=(g(z)-z)+s(z-a)(g(z)-a)$$

vanishes of order $>d$ at a, that is, $r=0$ and $\Theta(\eta)=\eta$. This implies that $\Theta(\mathfrak{l}_{-1})=\mathfrak{l}_{-1}$, since $\mathfrak{l}_{-1}$ is the (-1)-eigenspace of $\mathrm{ad}(\eta)$. Therefore g is an affine transformation on E. From $g(a)=a$ and $g'(a)=\mathrm{id}$ we finally get $g=\mathrm{id}$. □

COROLLARY 4.7. *Let $M:=F+iV$ and $M':=F'+iV$, with $F,F'\subset V$ being conical submanifolds, and let $a\in F$ and $a'\in F'$ be arbitrary points. Assume furthermore that $\mathfrak{hol}(M,a)=\mathfrak{aff}(M,a)$ holds. Then, the following conditions are equivalent*:

(i) *the manifold germs (M,a) and (M',a') are CR-equivalent*;

(ii) *the manifold germs (M,a) and (M',a') are affinely equivalent*;

(iii) *the manifold germs (F,a) and (F',a') are linearly equivalent.*

Recall that $\mathfrak{aut}(M,a)\subset\mathfrak{g}=\mathfrak{hol}(M,a)$ is defined as the isotropy subalgebra at a and $\mathrm{Aut}(M,a)$ is the CR-automorphism group of the manifold germ (M,a), also called the *stability group* at $a\in M$.

PROPOSITION 4.8. *Let* $F \subset V$ *be conical and let* $M = F + iV$. *In case* $\mathfrak{g} = \mathfrak{hol}(M, a)$ *has finite dimension, the following conditions are equivalent*:

(i) $\mathfrak{g}_1 = 0$;

(ii) $\mathfrak{g} = \mathfrak{aff}(M, a)$;

(iii) *the tangential representation* $g \mapsto g'(a)$ *induces a group monomorphism*

$$\mathrm{Aut}(M, a) \hookrightarrow \mathrm{GL}(V).$$

Each of these conditions is satisfied if $\mathfrak{aut}(M, a) = 0$.

Proof. Let $\mathfrak{l} := \mathfrak{g} \oplus i\mathfrak{g} \subset \mathfrak{hol}(E, a)$ and $\mathfrak{l}_k := \mathfrak{g}_k \oplus i\mathfrak{g}_k$ for all k.

(i) $\Rightarrow$ (ii) Follows from the last claim in Proposition 4.2.

(ii) $\Rightarrow$ (iii) By Proposition 4.4 (iii), every $g \in \mathrm{Aut}(M, a)$ is represented by a linear transformation on V.

(iii) $\Rightarrow$ (i) Let $\xi \in \mathfrak{g}_1$ be an arbitrary vector field. Then there exists a unique symmetric bilinear map $b: E \times E \to E$ with $\xi = b(z, z)\partial/\partial z$. Now,

$$(\mathrm{ad}(ia\partial/\partial z))^2 \xi = -2b(a, a)\partial/\partial z \in \mathfrak{g},$$

i.e., $\eta := h(z)\partial/\partial z$ is in $\mathfrak{aut}(M, a)$, where $h(z) := b(z, z) - b(a, a)$. For every $t \in \mathbb{R}$ therefore the transformation $\psi_t := \exp(t\eta) \in \mathrm{Aut}(M, a)$ has derivative $\psi'_t(a) = \exp(th'(a)) \in \mathrm{GL}(E)$ in a. But $\psi'_t(a) \in \mathrm{GL}(V)$ by Proposition 4.4 (iii) and thus $2b(a, v) = h'(a)v \in V$ for all $v \in V$. On the other hand, $b(a, v) \in iV$ by Lemma 3.9, implying that $\psi'_t(a) = \mathrm{id}$ for all $t \in \mathbb{R}$. By the injectivity of the tangential representation, therefore, η_t does not depend on t, and we get $\xi = 0$. This proves (i) and thus the equivalence of (i)–(iii).

Suppose $\mathfrak{aut}(M, a) = 0$ and that there exists a nonzero vector field $\xi \in \mathfrak{g}_1$. Then $\xi_a \in iV$ and there exists an $\eta \in \mathfrak{g}_{-1}$ with $\xi - \eta \in \mathfrak{aut}(M, a)$, a contradiction. □

Remark 4.9. Notice that condition (iii) in Proposition 4.8 states that the tangential representation takes its values in the subgroup $\mathrm{GL}(V) \subset \mathrm{GL}(E)$. In general, the tangential representation is not injective and also takes values outside $\mathrm{GL}(V)$. The tube $\mathcal{M}$ over the future light cone can serve as a counterexample for both of these phenomena.

PROPOSITION 4.10. *Suppose that* $M = F + iV$, *with* $F \subset V$ *being a conical submanifold, is locally homogeneous and that* $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ *for* $\mathfrak{g} = \mathfrak{hol}(M, a)$. *Then the tangential representation at* a *induces a group isomorphism*

$$\mathrm{Aut}(M, a) \cong \{g \in \mathrm{GL}(V) : g\mathfrak{g}_0 g^{-1} = \mathfrak{g}_0 \textit{ and } g(a) = a\},$$

where $\mathfrak{g}_0$ *is considered in the canonical way as a linear subspace of* $\mathrm{End}(V)$.

Proof. The assumptions imply that $H(a)\cap F$ is a neighbourhood of a in F for $H:=\exp(\mathfrak{g}_0)\subset \mathrm{GL}(V)$. Let $g\in\mathrm{GL}(V)$ be an arbitrary linear transformation with $g(a)=a$ and $g\mathfrak{g}_0g^{-1}=\mathfrak{g}_0$. Then $gHg^{-1}=H$ and hence $gH(a)=Hg(a)=H(a)$, that is $g\in\mathrm{Aut}(M,a)$. Conversely, by Proposition 4.4 (iii), every element of $\mathrm{Aut}(M,a)$ can be represented by some $g\in\mathrm{GL}(V)$ with $g(a)=a$. Since δ is invariant under the Lie algebra automorphism $\Theta=g_*$ of $\mathfrak{g}$, also $\mathfrak{g}_0$ is invariant under Θ. As a consequence, $\Theta(\varphi)=g\varphi g^{-1}$ for every $\varphi\in\mathfrak{g}_0$, that is, $g\mathfrak{g}_0g^{-1}=\mathfrak{g}_0$. □

The real Lie algebra structure of $\mathfrak{hol}(M,a)$ is a CR-invariant for the manifold germ (M,a). For certain classes of conical tube manifolds this gives a complete invariant.

PROPOSITION 4.11. *Let $F,F'\subset V$ be conical submanifolds for which the corresponding tubes $M:=F+iV$ and $M':=F'+iV$ are locally homogeneous CR-manifolds. Let $a\in F$ and $a'\in F'$ be arbitrary points and assume that for $\mathfrak{g}:=\mathfrak{hol}(M,a)$ the spaces $\mathfrak{g}_k$ occurring in the grading of Proposition* 4.2 (ii) *satisfy $\mathfrak{g}_k=[\mathfrak{g}_0,\mathfrak{g}_0]=0$. Then, the following conditions are equivalent*:

(i) *the germs (M,a) and (M',a') are CR-equivalent*;

(ii) *$\mathfrak{hol}(M,a)$ and $\mathfrak{hol}(M',a')$ are isomorphic as real Lie algebras.*

Proof. Only the implication (ii) ⇒ (i) is not obvious. Suppose that

$$\Theta\colon \mathfrak{g}\to\mathfrak{g}':=\mathfrak{hol}(M',a')$$

is a Lie algebra isomorphism. We use the same symbol for the complex linear extension:

$$\Theta\colon \mathfrak{l}\to\mathfrak{l}'.$$

Our first step is to show that Θ can be modified in such a way that it respects the gradings. To begin with, $[\mathfrak{g}',\mathfrak{g}']$ is abelian since our assumption implies that $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}_{-1}$ has this property. With $\mathfrak{g}'=\bigoplus_{k\in\mathbb{Z}}\mathfrak{g}'_k$ being the grading for $\mathfrak{g}'$ as in Proposition 4.2 (ii), assume that there exists a minimal integer $k\geqslant 1$ with $\mathfrak{g}'_k\neq 0$. Then $\mathfrak{g}'_{-1}=[\delta,\mathfrak{g}'_{-1}]$ and $\mathfrak{c}:=[\mathfrak{g}'_{-1},\mathfrak{g}'_k]\neq 0$ imply that $\mathfrak{g}'_{-1},\mathfrak{c}\subset[\mathfrak{g}',\mathfrak{g}']$ together with $[\mathfrak{g}'_{-1},\mathfrak{c}]\neq 0$, a contradiction. Therefore $\mathfrak{g}'=\mathfrak{g}'_{-1}\oplus\mathfrak{g}'_0$, with $[\mathfrak{g}',\mathfrak{g}']=\mathfrak{g}'_{-1}$, and, as a consequence, $\Theta(\mathfrak{g}_{-1})=\Theta([\mathfrak{g},\mathfrak{g}])=[\mathfrak{g}',\mathfrak{g}']=\mathfrak{g}'_{-1}$. Since $\mathrm{ad}\colon\mathfrak{g}'_0\to\mathfrak{gl}(\mathfrak{g}'_{-1})$ is injective, $\delta+\mathfrak{g}'_{-1}$ is precisely the set of all $\xi\in\mathfrak{g}'$ such that $\mathrm{ad}(\xi)$ induces the negative identity on $\mathfrak{g}'_{-1}$. Therefore, there exists $\eta\in\mathfrak{g}'_{-1}$ with $\Theta(\delta)=\delta-\eta$. Replacing Θ by $\exp(\mathrm{ad}(\eta))\Theta=(\mathrm{id}+\mathrm{ad}(\eta))\Theta$, we get $\Theta(\delta)=\delta$ and finally $\Theta(\mathfrak{g}_k)=\mathfrak{g}'_k$ for all k.

There exists a linear operator $\theta\in\mathrm{GL}(V)\subset\mathrm{GL}(E)$ with $\Theta(e\partial/\partial z)=\theta(e)\partial/\partial z$ for all $e\in E=V\oplus iV$. We claim that $H'=\theta H\theta^{-1}$ holds for the abelian subgroups $H:=\exp\mathfrak{g}_0$ and $H':=\exp\mathfrak{g}'_0$ of $\mathrm{GL}(V)$. Indeed, applying Θ to $[e\partial/\partial z,\lambda(z)\partial/\partial z]=\lambda(e)\partial/\partial z$ yields

$$\tilde\lambda\theta=\theta\lambda\quad\text{for all } \lambda(z)\partial/\partial z\in\mathfrak{g}_0 \text{ and } \tilde\lambda(z)\partial/\partial z:=\Theta(\lambda(z)\partial/\partial z)\in\mathfrak{g}'_0.$$

We may therefore assume (possibly after replacing F by θF and a by θa) that $H=H'$, $F=H(a)$ and $F'=H(a')$. Denote by $\Lambda\subset\mathrm{End}(V)$ the associative subalgebra generated by all $\lambda\in\mathrm{End}(V)$ with $\lambda(z)\partial/\partial z\in\mathfrak{g}_0$. Then Λ is abelian, contains the identity of $\mathrm{End}(V)$ and $H\subset\Lambda$. Since $\dim\mathfrak{g}<\infty$, the CR-manifold $F+iV$ is minimal and consequently F cannot be contained in a hyperplane of V. This implies that $\Lambda(a)=V$, and thus the existence of a $g\in\Lambda$ with $a'=g(a)$. From $gH=Hg$ we get $F'=g(F)$. Since $g(F)$ also cannot be contained in a hyperplane of V, finally $g\in\mathrm{GL}(V)$ follows. □

In Propositions 6.5 and 6.17 a large class of linearly homogeneous conical submanifolds $F\subset V$ is given for which the corresponding tubes M satisfy the condition $\mathfrak{g}_1=[\mathfrak{g}_0,\mathfrak{g}_0]=0$ in Proposition 4.11.

We note that we do not know a single example with $\dim\mathfrak{g}<\infty$ and $\dim\mathfrak{g}_k>\dim\mathfrak{g}_{-k}$ for some $k\in\mathbb{N}$. We also do not know any pair M,M' of holomorphically nondegenerate conical tube manifolds, that are locally CR-equivalent but are not locally affinely equivalent. For Levi nondegenerate tube manifolds (which necessarily cannot be conical) such examples can be found in [10]. In [20] even two affinely homogeneous examples are contained which are locally affinely inequivalent but whose associated tube manifolds are locally CR-equivalent.

5. Some examples

In this section we present two classes of examples. To our knowledge, the only known example of a homogeneous k-nondegenerate CR-manifold with $k\geqslant3$ occurs in [15] for the case $k=3$: this is a hypersurface M in a 7-dimensional compact complex manifold, on which the simple Lie group $\mathrm{SO}(3,4)^0$ acts by biholomorphic transformations with orbit M. In our first example we give, for arbitrary CR-codimension $c\geqslant1$, a minimal homogeneous 3-nondegenerate as well as a minimal homogeneous 4-nondegenerate CR-manifold. The second class of examples deals with tubes M over cones of the form

$$\Big\{x\in(\mathbb{R}^+)^n:\sum_{j\leqslant p}x_j^\alpha=\sum_{j>p}x_j^\alpha\Big\},$$

with $1\leqslant p<n$ and $\alpha\neq0,1$. Using results from the preceding section, we explicitly determine all Lie algebras $\mathfrak{g}=\mathfrak{hol}(M,a)$ for certain α. Among these are all hyperquadrics (that is, $\alpha=2$) of signature (p,q) with $q:=n-p$, where $\mathfrak{g}$ turns out to be isomorphic to $\mathfrak{so}(p+1,q+1)$.

The first example of CR-manifolds introduced below consists of tubes $M=\Gamma(a)+iV$ over certain group orbits $\Gamma(a)$, where the connected group $\Gamma:=\{g\in\mathrm{GL}(2,\mathbb{R}):\det(g)>0\}$

acts linearly on the real vector space V. In that way we obtain homogeneous k-nondegenerate CR-manifolds for $k\in\{2,3,4\}$. For dimensional reasons, it is impossible to construct CR-manifolds of higher nondegeneracy, employing the group $\Gamma=\mathrm{GL}(2,\mathbb{R})^0$. We do not know how to construct k-nondegenerate homogeneous tube manifolds with $k\geqslant 5$ (should these exist) with suitable other groups, either.

Example 5.1. For fixed integers $k\in\{2,3,4\}$ and $c\geqslant 1$ let $V\subset\mathbb{R}[u,v]$ be the subspace of all homogeneous polynomials of degree $m:=k+c-1$. Then the group Γ (see above) acts irreducibly on V by $p\mapsto p\circ g^{-1}$ for all $g\in\Gamma$, and has the subgroup $\{g\in\mathbb{R}\,\mathrm{id}: g^m=\mathrm{id}\}$ as kernel of ineffectivity. For $a:=\sum_{j=0}^{k-2} u^j v^{m-j}\in V$ the orbit $F=\mathcal{F}^{k,c}:=\Gamma(a)$ is a connected conical submanifold of dimension k in V. With Proposition 3.7 it is easily seen that

$$K_a^r F=\sum_{j=0}^{k-1-r}\mathbb{R}u^j v^{m-j}\quad\text{for all }\ r\geqslant 0.$$

In particular, the tube manifold $\mathcal{M}^{k,c}:=F+iV$ is a k-nondegenerate homogeneous CR-manifold of CR-dimension k and CR-codimension c. The manifolds $\mathcal{M}^{2,c}$ will be discussed in more detail in §7. In particular, $\mathcal{M}^{2,1}$ is linearly equivalent to the future light cone tube $\mathcal{M}$. For easier handling let us identify $\mathbb{R}^{m+1}$ with V via

$$(x_0,x_1,\dots,x_m)\leftrightsquigarrow\sum_{j=0}^{m}x_j\binom{m}{j}u^j v^{m-j}.$$

Since Γ acts on $\mathcal{M}^{k,c}$ by (linear) CR-automorphisms, the linear part $\mathfrak{g}_0$ of $\mathfrak{g}=\mathfrak{hol}(\mathcal{M}^{k,c},a)$ (compare Proposition 4.2 (ii)) contains a copy of $\mathfrak{gl}(2,\mathbb{R})$. More explicitly, this subalgebra is spanned by the vector fields

$$\begin{aligned}\zeta_1&:=\sum_{j=1}^{m}jz_j\partial/\partial z_j, &\quad \xi^{-1,1}&:=\sum_{j=0}^{m-1}(m-j)z_{j+1}\partial/\partial z_j,\\ \xi^{1,-1}&:=\sum_{j=1}^{m}jz_{j-1}\partial/\partial z_j, &\quad \zeta_2&:=\sum_{j=0}^{m-1}(m-j)z_j\partial/\partial z_j.\end{aligned}\tag{5.2}$$

In particular, the vector fields $\xi^{1,-1}$, $\xi^{-1,1}$ and $\xi^{0,0}:=\zeta_1-\zeta_2=\sum_{j=0}^{m}(2j-m)z_j\partial/\partial z_j$ span a copy of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{g}_0$. In case $k=2$, a straightforward computation shows that

$$\Gamma_a=\left\{\begin{pmatrix}\alpha&\beta\\0&\varepsilon\end{pmatrix}\in\Gamma:\varepsilon^m=1\right\}$$

is the isotropy subgroup at $a\in\mathcal{M}^{2,c}$. Hence,

$$\left\{\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}\in\Gamma\right\}\cong\mathbb{C}^*$$

acts transitively on F, that is, $\mathcal{M}^{2,c}$ is diffeomorphic to $\mathbb{R}^n\times\mathbb{C}^*$, where $n:=m+1=\dim V$.

It seems quite hard to find explicit global equations for $\mathcal{M}^{k,c}$. For $k{=}2$ see (7.2) and for $k{=}3$ we have the following: Denote by $S{\subset}V{=}\mathbb{R}^n$, $n{=}3{+}c{=}m{+}1$, the union of all Γ-orbits in V that have dimension $\leqslant 3$. Then $\mathcal{M}^{3,c}{\subset}S{+}i\mathbb{R}^n$ and $\mathcal{M}^{2,c+1}$ is contained in the closure of $\mathcal{M}^{3,c}$ in $S{+}i\mathbb{R}^n$. Consider the $4{\times}n$-matrix

$$D(x):=\begin{pmatrix} 0 & x_1 & 2x_2 & \dots & (m-1)x_{m-1} & mx_m \\ mx_1 & (m-1)x_2 & (m-2)x_3 & \dots & x_m & 0 \\ 0 & x_0 & 2x_1 & \dots & (m-1)x_{m-2} & mx_{m-1} \\ mx_0 & (m-1)x_1 & (m-2)x_2 & \dots & x_{m-1} & 0 \end{pmatrix}$$

formed by the (real parts of the) coefficients of the four vector fields in (5.2). Then, for every $x{\in}\mathbb{R}^n$, the rank of $D(x)$ is just the dimension of the orbit $\Gamma(x)$. In particular, the algebraic subvariety $S{\subset}\mathbb{R}^n$ is given by the simultaneous vanishing of all minors of order 4 in $D(x)$. For $k{=}4$ an explicit equation of the 4-nondegenerate hypersurface $\mathcal{M}^{4,1}$ can be found in [22].

For the rest of the example we concentrate on the hypersurface $\mathcal{M}^{3,1}$. Then

$$S=\{x\in\mathbb{R}^4:\varrho(x)=0\}$$

for

$$\varrho(x):=\det D(x)=9(x_0^2x_3^2+4x_0x_2^3-6x_0x_1x_2x_3-3x_1^2x_2^2+4x_1^3x_3).$$

Every connected component U of $\mathbb{R}^4\backslash S$ is an open Γ-orbit and $U{+}i\mathbb{R}^4$ is an affinely homogeneous tube domain in $\mathbb{C}^4$.

Now consider the subgroup $\Sigma{:=}\mathrm{SL}(2,\mathbb{R}){\subset}\Gamma$. As Σ is normal in Γ and $\Sigma(a)$ is open in $\Gamma(a)$, the orbits $\Sigma(a)$ and $\Gamma(a)$ coincide. As every vector field in $\mathfrak{sl}(2,\mathbb{R}){\subset}\mathfrak{g}_0$ annihilates the function ϱ, every hypersurface $S_\alpha{:=}\{x{\in}\mathbb{R}^4{:}\varrho(x){=}\alpha\}$, $\alpha{\in}\mathbb{R}$, is invariant under the group Σ. For every $b{\in}\mathbb{R}^4\backslash S$ the orbit $\Gamma(b)$ has dimension 4, and hence the orbit $\Sigma(b)$ has dimension 3. As a consequence, for every $\alpha{\neq}0$ the variety S_α is nonsingular and every orbit $\Sigma(b)$, $b{\notin}S$, is a closed hypersurface in $\mathbb{R}^4$. Actually it can be shown that for every $b{\in}V\backslash S$ near a, the homogeneous tube manifold $\Sigma(b){+}i\mathbb{R}^4$ is Levi nondegenerate and has indefinite Levi form.

Example 5.3. Fix integers $p{\geqslant}q{\geqslant}1$ with $n{:=}p{+}q{\geqslant}3$ and a real number α with $\alpha^2{\neq}\alpha$. Then

$$F=F^\alpha_{p,q}:=\left\{x\in(\mathbb{R}^+)^n:\sum_{j=1}^p x_j^\alpha=\sum_{j=p+1}^n x_j^\alpha\right\}$$

is a hypersurface in $V{:=}\mathbb{R}^n$. Furthermore, F is a cone and therefore $\dim K_aF{\geqslant}1$ for every $a{\in}F$. On the other hand, the second derivative at a of the defining equation for F gives a nondegenerate symmetric bilinear form on $V{\times}V$, whose restriction to

$T_aF\times T_aF$ then has a kernel of dimension $\leqslant 1$. Therefore $\dim K_aF=1$ for every $a\in F$ and, by Corollary 3.6, the CR-manifold $M=M^\alpha_{p,q}:=F^\alpha_{p,q}+iV$ is everywhere 2-nondegenerate (compare [13, Example 4.2.5] for the special case $n=\alpha=3$). Since M as a hypersurface is also minimal, $\mathfrak{g}=\mathfrak{hol}(M,a)$ has finite dimension.

For the special case $\alpha=2$ and $q=1$, the above cone $F=F^2_{n-1,1}$ is an open piece of the future light cone

$$\{x\in\mathbb{R}^n: x_1^2+...+x_{n-1}^2=x_n^2 \text{ and } x_n>0\}$$

in n-dimensional space-time, which is affinely homogeneous. In [23] it has been shown that for the corresponding tube manifold M the Lie algebra $\mathfrak{g}=\mathfrak{hol}(M,a)$ is isomorphic to $\mathfrak{so}(n,2)$ for every $a\in M$. In case $q>1$, the following result seems to be new.

Case $\alpha=2$. Consider on $\mathbb{C}^n$ the symmetric bilinear form

$$\langle z|w\rangle:=\sum_{j\leqslant p} z_jw_j-\sum_{j>p} z_jw_j.$$

Then F is an open piece of the hypersurface $\{x\in\mathbb{R}^n:x\neq 0 \text{ and } \langle x|x\rangle=0\}$, on which the reductive group $\mathbb{R}^*\cdot\mathrm{O}(p,q)$ acts transitively. Therefore, $\mathfrak{s}_0:=\mathbb{R}\delta\oplus\mathfrak{so}(p,q)$ is contained in $\mathfrak{g}_0$. One checks that

$$\mathfrak{s}:=\mathfrak{g}_{-1}\oplus\mathfrak{s}_0\oplus\mathfrak{s}_1,\quad \mathfrak{s}_1:=\{(2i\langle c|z\rangle z-i\langle z|z\rangle c)\partial/\partial z: c\in\mathbb{R}^n\}$$

is a Lie subalgebra of $\mathfrak{g}$. The radical $\mathfrak{r}$ of $\mathfrak{s}$ is $\mathrm{ad}(\delta)$-invariant and hence of the form $\mathfrak{r}=\mathfrak{r}_{-1}\oplus\mathfrak{r}_0\oplus\mathfrak{r}_1$ for $\mathfrak{r}_k:=\mathfrak{r}\cap\mathfrak{g}_k$. From $\mathfrak{so}(p,q)$ being semisimple we conclude that $\mathfrak{r}_0\subset\mathbb{R}\delta$. But δ cannot be in $\mathfrak{r}$ since otherwise $\mathfrak{g}_{-1}\subset\mathfrak{r}$ would give the false statement $[\mathfrak{g}_{-1},\mathfrak{s}_1]\subset\mathbb{R}\delta$. Therefore $\mathfrak{r}_0=0$, and $[\mathfrak{g}_{-1},\mathfrak{r}_1]=[\mathfrak{r}_{-1},\mathfrak{s}_1]=0$ implies $\mathfrak{r}=0$. Now, [23, Proposition 3.8] implies that $\mathfrak{g}=\mathfrak{s}$, and, in particular, that $\mathfrak{g}$ has dimension $\binom{n+2}{2}$. In fact, it can be seen that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p+1,q+1)$.

Case α *an integer* $\geqslant 3$. Then F is an open piece of the real-analytic submanifold

$$S:=\left\{x\in\mathbb{R}^n: x\neq 0 \text{ and } \sum_{j=1}^{p} x_j^\alpha=\sum_{j=p+1}^{n} x_j^\alpha\right\} \tag{5.4}$$

which is connected in case $q>1$ and has two connected components otherwise. For every $x\in\mathbb{R}^n$ let $d(x)\in\mathbb{N}$ be the cardinality of the set $\{j:x_j=0\}$. It is easily seen that $\dim K_xS=1+d(x)$ holds for every $x\in S$. Now consider the group

$$\mathrm{GL}(F):=\{g\in\mathrm{GL}(V):g(F)=F\}.$$

Every $g\in\mathrm{GL}(F)$ leaves S and hence also $H:=\{x\in\mathbb{R}^n:d(x)>0\}$ invariant, that is, g is the product of a diagonal matrix and a permutation matrix. Inspecting the action of $\mathrm{GL}(F)$ on $\{c\in\overline{F}:d(c)=n-2\}$, we see that $\mathrm{GL}(F)$, as a group, is generated by $\mathbb{R}^+\cdot\mathrm{id}$ and certain coordinate permutations. As a consequence, $\mathfrak{g}_0=\mathbb{R}\delta$. Now suppose that there exists a nonzero vector field $\xi\in\mathfrak{g}_1$. Then $\xi=q(z,z)\partial/\partial z$ for some symmetric bilinear map $q:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ with $q(c,z)\partial/\partial z\in\mathbb{C}\delta$ for every $c\in\mathbb{C}^n$. As $q\neq0$ is symmetric, any two vectors in $\mathbb{C}^n$ must be linearly dependent, which contradicts $n\geqslant3$. Therefore $\mathfrak{g}_1=0$ and hence $\mathfrak{g}_k=0$ for all $k\geqslant1$, by Proposition 4.4 (iv). In particular, $\dim\mathfrak{g}=n+1<\dim M$ for the tube manifold $M=F+iV$, that is, M is not locally homogeneous. For $n=3$ this gives an alternative proof for [14, Proposition 6.36].

PROPOSITION 5.5. *Let $a,a'\in F=F^{\alpha}_{p,q}$ be arbitrary points. Then in case $3\leqslant\alpha\in\mathbb{N}$ the CR-manifold germs (M,a) and (M,a') are CR-equivalent if and only if $a'\in\mathrm{GL}(F)(a)$.*

Proof. Suppose that $g:(M,a)\to(M,a')$ is an isomorphism of CR-manifold germs. From $\mathfrak{g}'=\mathfrak{g}'_{-1}\oplus\mathbb{R}\delta$ for $\mathfrak{g}':=\mathfrak{hol}(M',a')$, Proposition 4.4 (iii) implies that g is represented by a linear transformation in $\mathrm{GL}(V)$ that we also denote by g. But then $g(F)\subset S$ with S defined in (5.4). Because $g(F)$ has empty intersection with H, we actually have $g(F)\subset F$. Replacing g by its inverse, we get the opposite inclusion, that is $g\in\mathrm{GL}(F)$. □

6. Levi degenerate CR-manifolds associated with an endomorphism

The lowest CR-dimension for which there exist homogeneous CR-manifolds that are Levi degenerate but not holomorphically degenerate is 2. The construction recipe below will give, up to local affine equivalence, all affinely homogeneous conical tube submanifolds of $\mathbb{C}^n$ with CR-dimension 2. Indeed, it is based on the following simple observation. Suppose that $F\subset V:=\mathbb{R}^n$ is a conical locally linearly homogeneous submanifold of dimension 2. Denote by $\mathfrak{a}$ the Lie algebra of all linear vector fields on V that are tangent to F. Then, fixing a point $a\in F$, there exists a $\varphi\in\mathrm{End}(V)$ with $\varphi(x)\partial/\partial x\in\mathfrak{a}$ and $T_aF=\mathbb{R}a\oplus\mathbb{R}\varphi(a)$. Therefore, the orbit $H(a)$ under the subgroup $H:=\{\exp(r\,\mathrm{id}+t\varphi):r,t\in\mathbb{R}\}$ is an (immersed) surface in V having the same germ at a as F.

6.1. Construction recipe

Throughout this section, let $1<d<n$ be arbitrary integers and V be a real vector space of dimension n. Let furthermore $\varphi\in\mathrm{End}(V)$ be a fixed endomorphism and $\mathfrak{h}\subset\mathrm{End}(V)$ the linear span of all powers φ^k for $k=0,1,...,d-1$. Then $H:=\exp(\mathfrak{h})\subset\mathrm{GL}(V)$ is an abelian subgroup, and for given $a\in V$ the orbit $F:=H(a)$ is a cone and an immersed submanifold

of V (not necessarily locally closed in case $n\geqslant 4$). Furthermore, the tube $M=F+iV\subset E$ is an immersed CR-submanifold of E.

6.2. Cyclic endomorphisms and vectors

A vector $a\in V$ is called *cyclic* with respect to $\varphi\in\mathrm{End}(V)$ if the $\varphi^k(a)$, $k\geqslant 0$, span V. This is equivalent to $a, \varphi(a), ..., \varphi^{n-1}(a)$ being a basis of V. We call $\varphi\in\mathrm{End}(V)$ *cyclic* if it has a cyclic vector and denote by $\mathrm{Cyc}(V)\subset\mathrm{End}(V)$ the subset of all cyclic endomorphisms. If $a,b\in V$ both are cyclic vectors of φ then there exists a transformation $g\in\mathbb{R}[\varphi]\subset\mathrm{End}(V)$ with $b=g(a)$. But g commutes with every element of the group $H=\exp(\mathfrak{h})$ and hence maps the orbit $H(a)$ onto $H(b)$. In particular, the CR-isomorphism type of $M=H(a)+iV$ only depends on φ and d, but not on the choice of the cyclic vector a. To emphasize this dependence, we also write $M^{\varphi,d}$ for M and $F^{\varphi,d}$ for $H(a)$, but only if φ is cyclic. We tacitly assume that a choice for a cyclic vector a has been made. In case $d=2$ we even write M^{φ} and F^{φ} instead of $M^{\varphi,2}$ and $F^{\varphi,2}$, respectively. The following proposition shows the relevance of these manifolds in our discussion.

PROPOSITION 6.3. *For $F=H(a)$ and $M=F+iV$ as in* §6.1, *the following conditions are equivalent*:

(i) $\mathfrak{hol}(M,a)$ *has finite dimension*;

(ii) *a is a cyclic vector of φ.*

If these conditions are satisfied, M is a minimal 2*-nondegenerate homogeneous CR-manifold with CR-dimension d and Levi kernel $K_aM=\mathbb{C}a$.*

Proof. (i) $\Rightarrow$ (ii) Condition (i) together with the homogeneity of M implies that M is minimal. Let $W\subset V$ be the linear span of all vectors $\varphi^k(a)$, $k\geqslant 0$. Then $H\subset\mathbb{R}[\varphi]$ implies $H(a)\subset W$ and hence $W=V$ by the minimality of M. Therefore, a is a cyclic vector, and the $\varphi^k(a)$, $0\leqslant k<d$, form a basis of the tangent space T_aF. In particular, F has dimension d, which is also the CR-dimension of M.

(ii) $\Rightarrow$ (i) Suppose that a is a cyclic vector of φ. Lemma 3.2 gives $\mathbb{R}a\subset K_aF$, since F is a cone in V. For the proof of the opposite inclusion fix an arbitrary $w\in T_aF$ with $w\notin\mathbb{R}a$. Then $w=\sum_{j=0}^m c_j\varphi^j(a)$ with $c_m\neq 0$ for some $1\leqslant m<d$ and $\varphi^{d-m}(w)\notin T_aF$ shows that $w\notin K_aF$ by Proposition 3.7. Therefore, M is 2-nondegenerate by Corollary 3.6 and $K_aM=\mathbb{C}a$. It remains to show that M is minimal at a. But this immediately follows from Proposition 3.8. $\square$

For the manifolds $M=M^{\varphi,d}$ it is possible to compute the Lie algebras $\mathfrak{g}=\mathfrak{hol}(M,a)$ in 'most cases'. Clearly, the Lie algebra $\mathfrak{g}_0$ contains the d-dimensional Lie algebra $\mathfrak{h}$ (as before, $\mathfrak{g}_0$ is canonically identified with a linear subspace of $\mathrm{End}(V)$). In Propositions 6.5 and 6.17 we will show that the equality $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{h}$ holds for all φ in 'general position'.

Suppose that $\varphi\in\mathrm{End}(V)$ has the cyclic vector $a\in V$ and that the integer d satisfies $1<d<n$. Let

$$\chi_\varphi := \prod_{j=1}^{s} (X-\lambda_j)^{n_j} \in \mathbb{R}[X], \quad n_j \geqslant 1, \tag{6.4}$$

with mutually distinct eigenvalues $\lambda_1, ..., \lambda_s \in \mathbb{C}$, be the characteristic polynomial of φ. Let furthermore $\alpha_1, ..., \alpha_n$ be the family of all roots of χ_φ, that is, each λ_j occurs n_j times in this family. As usual, $\alpha_1, ..., \alpha_n$ is called an *arithmetic progression* in $\mathbb{C}$ if there exists a $\beta\in\mathbb{C}$ such that, after a suitable permutation, $\alpha_j=\alpha_1+(j-1)\beta$ for all $1\leqslant j\leqslant n$.

PROPOSITION 6.5. *Let $M=M^{\varphi,d}$ and $\mathfrak{g}:=\mathfrak{hol}(M,a)$ for a cyclic vector $a\in V\cap M$ of φ. Then $\mathfrak{g}_0=\mathfrak{h}$ holds if one of the following conditions is satisfied*:

(i) $d=2$ *and the characteristic roots of φ do not form an arithmetic progression*;

(ii) $d\geqslant 3$ *and* $s>d$.

For the proof we need several preparations. To simplify the notation at various places let us introduce

$$S := \{1, ..., s\}, \quad m := d-1 \quad \text{and} \quad m_k := n_k-1 \text{ for all } k\in S. \tag{6.6}$$

For every $K\subset S$ define furthermore

$$\Delta_K := \{\beta_{jk} : j\in S \text{ and } k\in K\} \quad \text{with} \quad \beta_{jk} := (\lambda_j-\lambda_k, \lambda_j^2-\lambda_k^2, ..., \lambda_j^m-\lambda_k^m) \in \mathbb{C}^m, \tag{6.7}$$

and denote by $\mathscr{K}$ the set of all nonempty subsets $K\subset S$ such that there exist subsets $P\subset Q\subset S\setminus K$ with the following two properties, where the maximum over the empty set here is defined to be 0:

(i) $\sum_{k\in K} n_k+\sum_{q\in Q} n_q\geqslant d+\max\{n_p-1 : p\in P\}$;

(ii) for every $\beta\in\Delta_Q\setminus\Delta_K$ there exist uniquely determined $j\in S$ and $q\in Q$ with $\beta=\beta_{jq}$ such that $n_j<n_q$ and $q\in P$.

Notice that $\mathscr{K}$ contains every subset $K\subset S$ with $\sum_{k\in K} n_k\geqslant d$ (just take $Q=\varnothing$). In the following lemma we show that (i) or (ii) in Proposition 6.5 will follow from a more general technical condition that will allow us to give a uniform proof of Proposition 6.5 for all d.

LEMMA 6.8. *Suppose that $s\geqslant d$ and that $\bigcap_{K\in\mathscr{K}} \Delta_K=\{0\}$ holds. Then one of the conditions* (i) *and* (ii) *in Proposition* 6.5 *is satisfied.*

Proof. Case $d=2$. Assume that the characteristic roots $\alpha_1, ..., \alpha_n$ do not form an arithmetic progression and that there exists a nonzero $\beta\in\bigcap_{K\in\mathscr{K}} \Delta_K$. We claim that

$\lambda_1,...,\lambda_s$ is an arithmetic progression. Otherwise $s\geqslant 3$ and there exists a k with $1<k<s$ such that, without loss of generality, $\lambda_j=\lambda_1+(j-1)\beta$ for all $1\leqslant j\leqslant k$ and $\lambda_1-\beta\neq\lambda_r\neq\lambda_k+\beta$ for all $r>k$. For every $j>k$ the set $\Delta_{\{k,j\}}$ contains the number β, that is, there is an $r\in S$ with $\lambda_r=\lambda_k+\beta$ or $\lambda_r=\lambda_j+\beta$. The first possibility violates our assumptions. In the second case necessarily $r>k$ must hold, since $r\leqslant k$ would imply $\lambda_j=\lambda_1-\beta$, and thus the second possibility cannot be true for *all* j with $k<j\leqslant s$. This proves that $\lambda_1,...,\lambda_s$ is an arithmetic progression, and also $s<n$ by assumption. In particular, $n_j>1$ for some $j\in S$. Next, we claim that $\{1\}\in\mathscr{K}$ and $\{s\}\in\mathscr{K}$. To see that $\{1\}\in\mathscr{K}$, let $k\in\{1,...,s\}$ be minimal with $n_k>1$. With $P:=Q:=\{k\}\setminus\{1\}$, condition (ii) after (6.7) is fulfilled and the first part of the claim follows. A similar argument proves also that $\{s\}\in\mathscr{K}$. But then $\Delta_{\{1\}}\cap\Delta_{\{s\}}=\{0\}$ gives a contradiction.

Case $d\geqslant 3$. Suppose that $0\neq\beta_{jk}\in\bigcap_{K\in\mathscr{K}}\Delta_K$ and that $s>d$. Then $L:=S\setminus\{k\}\in\mathscr{K}$ (with $Q=\varnothing$) and there are $l\in L$ and $r\in S$ with $\beta_{jk}=\beta_{rl}$. Since $d\geqslant 3$, the equation $\beta_{jk}=\beta_{rl}$ implies $k=l$, a contradiction. □

Proof of Proposition 6.5. It is enough to assume that the assumption of Lemma 6.8 is satisfied. Consider the decomposition $E=E_1\oplus...\oplus E_s$ with E_k being the kernel of $(\varphi-\lambda_k)^n$ for every $k\in S$. Let $\pi_k\colon E\to E_k$ be the canonical projection and let $\varepsilon_k\colon E_k\to E$ be the canonical injection. Then $a_k:=\pi_k(a)$ is a cyclic vector for $\varphi_k:=\pi_k\varphi\varepsilon_k\in\operatorname{End}(E_k)$. Furthermore, if we put $a_k^j:=(\varphi_k-\lambda_k)^j(a_k)$ for all $j\geqslant 0$, then $a_k^0,...,a_k^{m_k}$ is a basis of E_k. With $m=d-1$ as defined above let $\Phi:=(\varphi^1,...,\varphi^m)\in\operatorname{End}(V)^m$.

For every real (or complex) vector space W and all tuples $t=(t_1,...,t_m)\in\mathbb{R}^m$ and $w=(w_1,...,w_m)\in W^m$ let us write as shorthand $t\cdot w:=\sum_{j=1}^m t_jw_j$. For every $t\in\mathbb{R}^m$ the point $e^{t\cdot\Phi}(a)$ is contained in $F=M\cap V$.

Now fix an arbitrary $\mu\in\mathfrak{g}_0\subset\operatorname{End}(V)$. Since the vector field μ is tangent to F, for every $t\in\mathbb{R}^m$ there exist real coefficients $r_0,r_1,...,r_m$ with

$$\mu e^{t\cdot\Phi}(a)=Re^{t\cdot\Phi}(a)\quad\text{for }\ R:=\sum_{l=0}^m r_l\varphi^l. \tag{6.9}$$

Actually, every r_l has to be considered as a real-valued function on $\mathbb{R}^m$. Put

$$\mu_k:=\pi_k\mu\quad\text{and}\quad N_k:=(\varphi_k-\lambda_k,...,\varphi_k^m-\lambda_k^m)\in\operatorname{End}(E)^m$$

for all $k\in S$. Applying π_k to (6.9) gives

$$\begin{aligned}Re^{t\cdot N_k}(a_k)&=e^{t\cdot N_k}R(a_k)=\sum_{j=1}^s e^{t\cdot\beta_{jk}}\mu_k e^{t\cdot N_j}(a_j)\quad\text{with}\\ R(a_k)&=\sum_{j=0}^{m_k}\varrho_{k,j}a_k^j\quad\text{for}\quad\varrho_{k,j}(t):=\sum_{l=j}^m\binom{l}{j}\lambda_k^{l-j}r_l(t).\end{aligned}\tag{6.10}$$

For every subset $B \subset \mathbb{C}^m$ denote by $\mathscr{F}(B) \subset \mathcal{C}(\mathbb{R}^m, \mathbb{C})$ the smallest linear subspace containing all functions $h(t)e^{t\cdot\beta}$ with $\beta \in B$ and $h \in \mathbb{C}[t] = \mathbb{C}[t_1, ..., t_m]$ being a polynomial function on $\mathbb{R}^m$. Then, it is well known that every $f \in \mathscr{F}(B)$ has a unique representation $f = \sum_{\beta\in B} f^{[\beta]}e^{t\cdot\beta}$ with $f^{[\beta]} \in \mathbb{C}[t]$ and $\{\beta \in B : f^{[\beta]} \neq 0\}$ finite. Since $t\cdot N_j$ is nilpotent, i.e., $e^{\pm t\cdot N_j}$ is polynomial, as a consequence of the identities (6.10) we get for every $K \subset S$,

$$\varrho_{k,j} \in \mathscr{F}(\Delta_K) \quad \text{for every } k \in K \text{ and } 0 \leqslant j \leqslant m_k. \tag{6.11}$$

Denote by $\mathscr{B}$ the set of all subsets $B \subset \mathbb{C}^m$ with $r_l \in \mathscr{F}(B)$ for all l.

Claim 1. *$\bigcap_{K\in\mathscr{K}} \Delta_K \in \mathscr{B}$, that is, $r_l \in \mathbb{C}[t]$ for all l.*

Proof. Fix an arbitrary $K \in \mathscr{K}$ and let $P \subset Q$ be as in the definition of $\mathscr{K}$. Since $\mathscr{B}$ is closed under intersections, it is enough to show that $\Delta_K \in \mathscr{B}$. Assume on the contrary that this is not true. Consider the following linear system of equations for the r_l (compare also Remark 6.16):

$$\varrho_{k,j} = \sum_{l=j}^{m} \binom{l}{j} \lambda_k^{l-j} r_l \in \mathscr{F}(\Delta_{K\cup Q}) \quad \text{for all } k \in K \cup Q \text{ and } 0 \leqslant j \leqslant m_k. \tag{6.12}$$

The coefficient matrix is of generalized Vandermonde type and hence has rank $d = m+1$, since by the definition of $\mathscr{K}$ the number of equations is at least d. This implies that $r_l \in \mathscr{F}(\Delta_K \cup \Delta_Q)$ for all l. Since, by assumption, not all r_l are in $\mathscr{F}(\Delta_K)$, there is a $\beta \in \Delta_Q \setminus \Delta_K$ such that the β-components $r_l^{[\beta]} \in \mathbb{C}[t]$ do not vanish for all l simultaneously. By the definition of $\mathscr{K}$, there are uniquely determined $p \in S$ and $q \in P$ with $\beta = \beta_{pq}$. Define $L := K \cup Q \setminus \{q\}$. Since $\beta \notin \Delta_L$, we get from (6.10) the linear system

$$\varrho_{k,j}^{[\beta]} = \sum_{l=j}^{m} \binom{l}{j} \lambda_k^{l-j} r_l^{[\beta]} = 0, \quad \text{where } k \text{ runs through } L \text{ and } 0 \leqslant j \leqslant m_k. \tag{6.13}$$

By the very definition of K, P and Q, it follows that the above linear system consists of at least $d-1$ equations. Consequently, we can write it in the form

$$\sum_{l=j}^{m-1} \binom{l}{j} \lambda_k^{l-j} r_l^{[\beta]} \in \mathbb{C} r_m^{[\beta]} \quad \text{for all } k \in L \text{ and } 0 \leqslant j \leqslant m_k.$$

Since its coefficient matrix is of generalized Vandermonde type, every $r_l^{[\beta]}$ is a complex multiple of $r_m^{[\beta]}$ and, in particular, $r_m^{[\beta]} \neq 0$. We claim that $\varrho_{q,0}^{[\beta]} \neq 0$. Indeed, otherwise we could add the equation $\varrho_{q,0}^{[\beta]} = 0$ to the linear system (6.13), which then has a coefficient matrix of generalized Vandermonde type with rank d, contradicting $r_m^{[\beta]} \neq 0$. Denote by D

the degree of $\varrho_{q,0}^{[\beta]}=\sum_{l=0}^{m}\lambda_q^l r_l^{[\beta]}$. Then $D=\deg\varrho_{q,0}^{[\beta]}=\deg r_m^{[\beta]}\geqslant 0$ and all r_l and $\varrho_{q,j}^{[\beta]}$ have degree $\leqslant D$. The equations (6.10) imply (replace k by q, form the β-components for $\beta{:=}\beta_{pq}$ and carry out the multiplication by $e^{t\cdot N_q}$) that

$$\mu_q e^{t\cdot N_p}(a_p)=e^{t\cdot N_q}\sum_{j=0}^{m_q}\varrho_{q,j}^{[\beta]}a_q^j=\sum_{j=0}^{m_q}(f_j\varrho_{q,0}^{[\beta]}+g_j)a_q^j \tag{6.14}$$

for certain polynomials $f_j,g_j\in\mathbb{C}[t]$ with $\deg(g_j)<\deg(f_j)=j$. Comparing degrees on both sides in (6.14) and keeping in mind that $n_p<n_q$ (by the definition of $\mathscr{K}$), we get

$$D+m_q=\deg(f_{m_q}\varrho_{q,m_q}^{[\beta]}+g_{m_q})\leqslant m_p<m_q.$$

This contradicts $r_m^{[\beta]}\neq 0$ and Claim 1 is proved. □

Claim 2. *Every r_l is a constant polynomial.*

Proof. Fix a $k\in S$ with $D:=\deg(\varrho_{k,0})=\max_{j\in S}\deg(\varrho_{j,0})$. With $s\geqslant d$, a Vandermonde argument applied to the linear system $\varrho_{j,0}=\sum_{l=0}^{m}\lambda_j^l r_l$, $j\in S$, gives that every r_l has degree $\leqslant D$. As in (6.14), we have

$$\mu_k e^{t\cdot N_k}(a_k)=e^{t\cdot N_k}\sum_{j=0}^{m_k}\varrho_{k,j}a_k^j=\sum_{j=0}^{m_k}(f_j\varrho_{k,0}+g_j)a_k^j \tag{6.15}$$

for polynomials $f_j,g_j\in\mathbb{C}[t]$ with $\deg(g_j)<\deg(f_j)=j$. All coefficient polynomials in (6.15) in front of the a_k^j have degree $\leqslant m_k$, that is $D+m_k\leqslant m_k$, and hence $D\leqslant 0$. This proves Claim 2. □

The proof of Proposition 6.5 is now complete. Indeed, since F contains a basis of V, the endomorphism μ is uniquely determined by the function tuple (r_l), that is, $\dim\mathfrak{g}_0\leqslant d=\dim\mathfrak{h}$. □

Remark 6.16. For given tuples $\lambda_1,...,\lambda_s\in\mathbb{C}$ and $n_1,...,n_s\in\mathbb{N}$, with $n_k\geqslant 1$ for all k and $n:=\sum_{k=1}^{s}n_k$, let $L:=\{(k,j):1\leqslant k\leqslant s \text{ and } 0\leqslant j<n_k\}$ be endowed with the lexicographic order. Then it can be seen that the following $n\times n$-matrix of *generalized Vandermonde type* (every entry with $l<j$ is zero)

$$\left(\binom{l}{j}\lambda_k^{l-j}\right)_{0\leqslant l<n,\,(k,j)\in L}$$

has determinant

$$\prod_{p<q}(\lambda_q-\lambda_p)^{n_pn_q}.$$

Notice from the proof of Proposition 6.5 that the condition $\bigcap_{K\in\mathscr{K}}\Delta_K=\{0\}$ guarantees that every $\mu\in\mathfrak{g}_0$ leaves every generalized eigenspace E_k of φ invariant, while $s\geqslant d$ guarantees that every such μ actually is in $\mathfrak{h}$. In the next section we will see that condition (i) in Proposition 6.5 for $d=2$ is optimal (compare Proposition 7.3).

PROPOSITION 6.17. *Let $M=M^{\varphi,d}$ and assume that $\mathfrak{g}_0=\mathfrak{h}$ for $\mathfrak{g}=\mathfrak{hol}(M,a)$. Then $\mathfrak{g}=\mathfrak{aff}(M,a)$ and $\mathfrak{aut}(M,a)=0$.*

Proof. For the proof of $\mathfrak{g}=\mathfrak{aff}(M,a)$, it is enough to show that $\mathfrak{g}_1=0$ by Proposition 4.2 (iv). This is more easily done in the more general complex setting (compare the following Lemma 6.18). Finally, counting dimensions yields $\mathfrak{aut}(M,a)=0$. □

It remains to show the next lemma. As before,we identify the spaces $\mathrm{End}(E)$ and $\mathfrak{P}_0$; see (4.1).

LEMMA 6.18. *Let $\varphi\in\mathrm{Cyc}(E)$ be an arbitrary cyclic endomorphism. For a given integer $d<n=\dim E$, let furthermore $\mathfrak{h}$ be the complex linear span of all powers φ^j, $0\leqslant j<d$. Then*

$$\{\xi\in\mathfrak{P}_1:[\mathfrak{P}_{-1},\xi]\subset\mathfrak{h}\}=0.$$

Proof. We identify E with $\mathbb{C}^n$ in such a way that the matrix Φ of φ is in Jordan normal form. More precisely, Φ is a block diagonal matrix with Jordan blocks $J_1,...,J_s$, where each block J_l is lower triangular, has the eigenvalue λ_l on its main diagonal and is of size $n_l\times n_l$ for some $n_l\geqslant 1$. We also introduce an equivalence relation on $\{1,...,n\}$ in the following way: put $j\sim k$ if the jth row and the kth column in Φ intersect in one of the Jordan blocks.

Now suppose that there exists a nonzero vector field $\xi\in\mathfrak{P}_1$ with $[\mathfrak{P}_{-1},\xi]\subset\mathfrak{h}$. This ξ has a unique representation

$$\xi=\sum_{j,k,p=1}^{n}c_p^{jk}z_jz_k\partial/\partial z_p\quad\text{with } c_p^{jk}=c_p^{kj}\in\mathbb{C}.$$

For every $j\leqslant n$ the vector field $\partial/\partial z_j$ is contained in $\mathfrak{P}_{-1}$. Therefore

$$\xi_j:=\frac{1}{2}[\partial/\partial z_j,\xi]=\sum_{k,p=1}^{n}c_p^{jk}z_k\partial/\partial z_p$$

is contained in $\mathfrak{h}$, implying that $c_p^{jk}=0$ if $k\not\sim p$ and, by symmetry, that $c_p^{jk}=0$ if $j\not\sim p$. This implies that

$$\xi_j=\sum_{p\sim j}\sum_{k\sim j}c_p^{jk}z_k\partial/\partial z_p. \tag{6.19}$$

By assumption, $\xi\neq 0$, and therefore $\xi_r\neq 0$ for some $r\leqslant n$. Without loss of generality, we may assume that $r\sim 1$. Replacing φ by $\varphi-\lambda_1$, we may also assume that $\lambda_1=0$. Put $b:=n_1$, that is, the Jordan block J_1 has size $b\times b$ and is nilpotent. Define, for all $1\leqslant p,k\leqslant b$,

$$\begin{aligned}\eta_p &:= z_1\partial/\partial z_p+z_2\partial/\partial z_{p+1}+\ldots+z_{b+1-p}\partial/\partial z_b,\\ \mathfrak{D}_k &:= \sum_{j=k}^{b}\mathbb{C}\eta_j,\\ \psi_k &:= \varphi^{k-1}\prod_{l=2}^{s}(\varphi-\lambda_l)^{n_l}\in \operatorname{End}(E).\end{aligned}$$

From (6.19), we know that $\xi_j\in\mathfrak{D}_1\cap\mathfrak{h}$ for all $j\leqslant b$. In particular,

$$\xi_j=\sum_{p=1}^{b}c_p^{j1}\eta_p,\quad \text{where } c_p^{jk}=\begin{cases} c_{p+1-k}^{j1}, & \text{if } k\leqslant p,\\ 0, & \text{otherwise,}\end{cases}$$

and hence

$$\xi_j=c_1^{11}\eta_j+c_2^{11}\eta_{j+1}+\ldots+c_{b+1-j}^{11}\eta_b$$

for all j, due to the symmetry of c_p^{jk} in the upper indices. As a consequence, there exists a minimal $q\leqslant b$ with $q\geqslant 1$ and $c_q^{11}\neq 0$. But then $\xi_{b+1-q}=c_q^{11}\eta_b$ implies that $\eta_b\in\mathfrak{D}_b\cap\mathfrak{h}$. We show that this cannot be true: Since ψ_1 as polynomial in φ has constant term $\lambda_2\lambda_3\ldots\lambda_s\neq 0$, we get that ψ_k spans $\mathfrak{D}_k$ over $\mathfrak{D}_{k+1}$ for every $k\geqslant 1$. This implies that $\eta_b=\sum_{j=1}^{n}e_j\varphi^{j-1}$ for suitable real coefficients e_j, with $e_n\neq 0$, and thus $\eta_b\notin\mathfrak{h}$. □

For the application of Proposition 3.10 to manifolds of the type $M=M^{\varphi,d}$, it is necessary to know when M is simply connected and when $\operatorname{Aut}(M,a)$ is the trivial group.

A sufficient condition for $M^{\varphi,d}$ (and $F^{\varphi,d}=H(a)$) to be simply connected is the following: *There exist eigenvalues* $\lambda_1,\ldots,\lambda_d$ *of* φ *such that* $\det(A+\bar{A})\neq 0$, *where*

$$A=(\lambda_j^{k-1})_{1\leqslant j,k\leqslant d}$$

is the corresponding Vandermonde matrix.

To get a partial answer to the second question, suppose that $g\in\mathrm{GL}(V)$ satisfies $g(a)=a$ and $g\varphi g^{-1}=\varepsilon\varphi$ for some $\varepsilon\in\mathbb{R}$. From $g\varphi^k g^{-1}=(\varepsilon\varphi)^k$ we get $g(\varphi^k(a))=\varepsilon^k\varphi^k(a)$ and thus $g\exp(t\varphi^k)g^{-1}=\exp(t\varepsilon^k\varphi^k)$ for all $t\in\mathbb{R}$ and $k\geqslant 0$. This means that $gHg^{-1}=H$ for the group $H=\exp(\mathfrak{h})$. But then $g(F)=F$ for $F=H(a)$ and consequently $g\in\operatorname{Aut}(M,a)$.

As an example, suppose that φ, with cyclic vector a, is nilpotent. For every $t\in\mathbb{R}$ there is a unique $g_t\in\mathrm{GL}(V)$ with $g_t(\varphi^k(a))=e^{kt}\varphi^k(a)$ for all $k\geqslant 0$. Then $g_t\in\operatorname{Aut}(M,a)$ shows that $\mathfrak{g}_0\neq\mathfrak{h}$ as well as $\mathfrak{aut}(M,a)\neq 0$.

Notice that we always may assume without loss of generality that $\varphi\in\operatorname{Cyc}(V)$ has trace 0 (otherwise replace φ by $\varphi-c\,\mathrm{id}\in\operatorname{End}(V)$ for $c:=n^{-1}\operatorname{tr}(\varphi)$, since this procedure does not change the algebra $\mathfrak{h}$).

Lemma 6.20. *Suppose that $2d\leqslant n+1$ for $n=\dim V$ and that φ is trace-free. Suppose in addition that $\mathfrak{aut}(M,a)=0$ holds for the cyclic vector $a\in V$ and $M:=M^{\varphi,d}$. Then*

$$\mathrm{Aut}(M,a)=\{g\in \mathrm{GL}(V): g(a)=a \text{ and } g\varphi g^{-1}=\pm\varphi\}.$$

In particular, $\mathrm{Aut}(M,a)$ *has always order* $\leqslant 2$ *and is trivial*, *for instance*, *if the spectrum of φ in $\mathbb{C}$ is not symmetric with respect to the origin of $\mathbb{C}$.*

Proof. By Proposition 4.8, the assumption $\mathfrak{aut}(M,a)=0$ implies that $\mathfrak{g}_1=0$ and $\mathfrak{g}_0=\mathfrak{h}$. As a consequence of Proposition 4.10, therefore,

$$\mathrm{Aut}(M,a)=\{g\in \mathrm{GL}(V): g(a)=a \text{ and } g\mathfrak{h}g^{-1}=\mathfrak{h}\}$$

holds. Let $g\in\mathrm{Aut}(M,a)$ be an arbitrary automorphism. Then $g\varphi g^{-1}=\sum_{j=0}^{m}c_j\varphi^j$ for some real coefficients c_j and $m:=d-1$. We show, by induction on k, that $c_j=0$ holds for all $j>m/k$ and all $1\leqslant k\leqslant m$. For $k=1$ this is obvious. So fix a $k>1$ with $k\leqslant m$. By the induction hypothesis, $g\varphi^k g^{-1}=(\sum_{j=0}^{m}c_j\varphi^j)^k=\sum_{l=0}^{2m}e_l\varphi^l\in\mathfrak{h}$, with real coefficients e_l. Since $2m<n$ by assumption, we must have $e_l=0$ for all $l>m$, that is $c_j=0$ for all $j>m/k$. For $k=m$ this implies that $c_j=0$ for all $j>1$. Taking traces finally gives $g\varphi g^{-1}=\varepsilon\varphi$ for $\varepsilon:=c_1$. By the above example, φ cannot be nilpotent, that is, φ has nonzero spectrum in $\mathbb{C}$. Since this spectrum is invariant under multiplication by ε, necessarily $\varepsilon=\pm1$ holds. □

7. Homogeneous 2-nondegenerate manifolds of CR-dimension 2

In this section we specialize to homogeneous tube manifolds $M=F+iV$ in $E=V\oplus iV$ of CR-dimension 2, that is, where $F\subset V$ is a surface of dimension 2. We begin with manifolds of type $M^{\varphi}=M^{\varphi,2}$ that are obtained by the construction recipe in §6.1. Since Propositions 6.5 and 6.17 of the preceding section do not cover the case where the characteristic roots $\alpha_1,...,\alpha_n$ of $\varphi\in\mathrm{Cyc}(V)$, $n=\dim V$, form an arithmetic progression, let us discuss this case first.

Possibly after replacing φ by $\varphi-r\,\mathrm{id}$ with an appropriately chosen constant r, we may assume without loss of generality that φ is trace-free. Since multiplication of φ by any nonzero real number does not change the algebra $\mathfrak{h}=\mathbb{R}\,\mathrm{id}\oplus\mathbb{R}\varphi$, there are essentially three different cases for φ with characteristic roots forming an arithmetic progression: either φ is nilpotent or φ has pairwise different characteristic roots in $\mathbb{R}$ or in $i\mathbb{R}$. Consider the manifold $M:=\mathcal{M}^{2,n-2}=\mathcal{F}^{2,n-2}+iV$ from Example 5.1. As already remarked in Example 5.1, the conformal subgroup $H^{\mathrm{i}}:=\mathbb{R}^+\cdot\mathrm{SO}(2)\subset\mathrm{GL}(2,\mathbb{R})$ acts transitively on $\mathcal{F}^{2,n-2}$. The corresponding Lie algebra is $\mathfrak{h}_{\mathrm{i}}:=\mathbb{R}\,\mathrm{id}\oplus\mathbb{R}\varphi_{\mathrm{i}}$, where $\varphi_{\mathrm{i}}:=\xi^{1,-1}-\xi^{-1,1}\in\mathfrak{g}_0$ is a trace-free semisimple endomorphism with eigenvalues in $i\mathbb{R}$; see Example 5.2 for the notation. Next consider the subgroup $H^{\mathrm{r}}:=\mathbb{R}^{+}\cdot\mathrm{SO}(1,1)\subset\mathrm{GL}(2,\mathbb{R})$ with Lie algebra $\mathfrak{h}^{\mathrm{r}}:=\mathbb{R}\,\mathrm{id}\oplus\mathbb{R}\varphi_{\mathrm{r}}$, where $\varphi_{\mathrm{r}}:=\xi^{1,-1}+\xi^{-1,1}\in\mathfrak{g}_0$ is a trace-free semisimple endomorphism with real eigenvalues. In particular, $H^{\mathrm{r}}(a)$ is open in $H^{\mathrm{i}}(a)$. Finally, consider the solvable subgroup

$$H^{\mathrm{z}}:=\left\{\begin{pmatrix}\alpha & 0\\ \beta & \alpha\end{pmatrix}\in\mathrm{GL}(2,\mathbb{R}):\alpha>0\right\}\quad\text{with Lie algebra }\ \mathfrak{h}^{\mathrm{z}}:=\mathbb{R}\,\mathrm{id}\oplus\mathbb{R}\varphi_{\mathrm{z}},$$

where $\varphi_{\mathrm{z}}:=\xi^{1,-1}$ has only zero eigenvalues. Again, the orbit $H^{\mathrm{z}}(a)$ is open in $H^{\mathrm{i}}(a)$. This implies that the manifolds $M^{\varphi_{\mathrm{r}}}$ and $M^{\varphi_{\mathrm{z}}}$ are open subsets of $M^{\varphi_i}=\mathcal{M}^{2,n-2}$, and hence that all three of them are locally CR-equivalent. Since $\mathcal{M}^{2,n-2}$ is minimal as a CR-manifold, the endomorphisms φ_{i}, φ_{r} and φ_{z} have a as cyclic vector by Proposition 6.3 (which also can easily be verified directly). By construction, the characteristic roots of the endomorphisms φ_{i}, φ_{r} and φ_{z} form an arithmetic progression and represent the three types with imaginary, real and zero characteristic roots. The estimate $\dim\mathfrak{g}_0\geqslant 4>\dim\mathfrak{h}^{\psi}$ for $\psi=\varphi_{\mathrm{i}},\varphi_{\mathrm{r}},\varphi_{\mathrm{z}}$, together with Propositions 6.5 and 6.17, implies that the characteristic roots of all three endomorphisms form an arithmetic progression. Summing up, we have proved the following result.

PROPOSITION 7.1. *Let $\varphi,\varphi'\in\mathrm{End}(V)$ be endomorphisms with cyclic vectors $a,a'\in V$. Assume that for both endomorphisms the families of characteristic roots $\alpha_1,...,\alpha_n$ and $\alpha_1',...,\alpha_n'$ form arithmetic progressions. Then the germs (M^{φ},a) and $(M^{\varphi'},a')$ are CR-equivalent.*

We can use Proposition 7.1 to get explicit global equations for every M^{φ} where the characteristic roots of $\varphi\in\mathrm{Cyc}(V)$ form an arithmetic progression. Indeed, we may take $\varphi:=\xi^{1,-1}$ from (5.2) on $\mathbb{C}^{m+1}$ with coordinates $(z_0,z_1,...,z_m)$ and $a:=(1,0,...,0)$. Then φ is nilpotent and $S:=\exp(\mathbb{R}\varphi)(a)=\{(1,t,t^2,...,t^m):t\in\mathbb{R}\}$. Consequently, $M^{\varphi}=F+iV$ is the tube over the cone F generated by S (that is $F=\mathbb{R}^{+}\cdot S$). As a consequence, F is an open piece of the algebraic surface given by the following explicit system of quadratic equations on $\mathbb{R}^{m+1}$ with coordinates $(x_0,x_1,...,x_m)$:

$$x_0x_{j+1}=x_1x_j\quad\text{for }\ 0<j<m. \tag{7.2}$$

This can be reformulated also in the following slightly different form. Let

$$C:=\{(t,t^2,...,t^n):t\in\mathbb{R}\}$$

be the *twisted n-ic* in $\mathbb{R}^n$ (also called twisted cubic, quartic, etc.; see [18] for interesting properties of these curves). Then the cone $\mathbb{R}^{+}\cdot C$ generated by C is a nonsingular surface outside the origin and the corresponding tube manifold is locally CR-equivalent to M^φ, with φ as in Proposition 7.1. The twisted n-ic will also show up in another type of examples (compare Proposition 7.10 below).

Next, we extend Propositions 6.5 and 6.17 to the case $d=2$ where the characteristic roots of φ do form an arithmetic progression. Recall that for the light cone tube $\mathcal{M}\cong\mathcal{M}^{2,1}$ the Lie algebra $\mathfrak{g}=\mathfrak{hol}(\mathcal{M},a)$ is isomorphic to $\mathfrak{so}(2,3)$; compare [23], [16]. In particular, $\mathfrak{g}_0\cong\mathfrak{gl}(2,\mathbb{R})$ and $\dim\mathfrak{g}_1=3$ in this case.

PROPOSITION 7.3. *Assume that the characteristic roots of* $\varphi\in\mathrm{Cyc}(V)$ *form an arithmetic progression. Then for* $M=M^\varphi$, $n=\dim V$ *and* $\mathfrak{g}=\mathfrak{hol}(M,a)$, *the following properties hold*:

(i) $\mathfrak{g}_0$ *is isomorphic to* $\mathfrak{gl}(2,\mathbb{R})$ *and hence has dimension* 4.

(ii) $\mathfrak{g}=\mathfrak{aff}(M,a)$ *in case* $n\geqslant 4$. *In particular*, $\dim\mathfrak{g}=n+4$ *in this case.*

Proof. (i) We may assume that $\lambda_j=\frac{1}{2}(1-n)+(j-1)$ for all $j\in S$ using the notation in (6.6) and (6.7). Then $\bigcap_{K\in\mathscr{K}}\Delta_K=\{-1,0,1\}$. Solving (6.9) for r_0 and r_1 gives that all pairs

$$\begin{aligned} r_0&=(n-1)(ue^{-t}+v_0-we^t),\\ r_1&=2(ue^{-t}+v_1+we^t), \end{aligned}\tag{7.4}$$

with $u,v_0,v_1,w\in\mathbb{R}$ arbitrary, form the solution space. This implies that $\dim\mathfrak{g}_0=4$. Since M is locally CR-equivalent to $\mathcal{M}^{2,n-2}$, the Lie algebra $\mathfrak{g}_0$ contains a copy of $\mathfrak{gl}(2,\mathbb{R})$, that is $\mathfrak{g}_0\cong\mathfrak{gl}(2,\mathbb{R})$.

(ii) Let $n\geqslant 4$. We may assume that for $m=n-1$ the Lie algebra $\mathfrak{g}_0$ is the linear span of the vector fields (5.2). For every $\nu\in\mathbb{Z}^2$ let $\mathfrak{g}^\nu:=\{\xi\in\mathfrak{g}:[\zeta_j,\xi]=\nu_j\xi \text{ for } j=1,2\}$. Then $\mathfrak{g}^\nu\subset\mathfrak{g}_k$ for $k=(\nu_1+\nu_2)/m$ and

$$\mathfrak{g}=\bigoplus_{\nu\in\mathbb{Z}^2}\mathfrak{g}^\nu \quad\text{with}\quad [\mathfrak{g}^\nu,\mathfrak{g}^\mu]\subset\mathfrak{g}^{\nu+\mu},$$

compare also [16, (3.5)]. Clearly $i\partial/\partial z_k\in\mathfrak{g}^{-k,k-m}$ for all $0\leqslant k\leqslant m$. Because of Proposition 4.2 (iv), it is enough to show that $\mathfrak{g}_1=0$. Assume on the contrary that there exists a nonzero $\xi\in\mathfrak{g}_1$. Then we may assume, without loss of generality, that $\xi\in\mathfrak{g}^{k,m-k}$ for some $k\in\mathbb{Z}$. Let c be the cardinality of $\{0\leqslant j\leqslant m:[\partial/\partial z_j,\xi]\neq 0\}$. From $\mathfrak{g}_0=\mathfrak{g}^{-1,1}\oplus\mathfrak{g}^{0,0}\oplus\mathfrak{g}^{1,-1}$ and $[i\partial/\partial z_j,\xi]\in\mathfrak{g}^{k-j,j-k}$, we see that $c\leqslant 3$. Assume that $c=3$, which implies $0<k<m$. From $[\partial/\partial z_j,\xi]\neq 0$, for $j=k\pm 1$ and the special form of $\xi^{-1,1}$ and $\xi^{1,-1}$ in (5.2), we see that ξ must depend on all n variables, a contradiction to $n>3$. But $c\leqslant 2$ also gives a contradiction since in all spaces $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$ and $\mathfrak{g}^{1,-1}$ every nonzero vector field must depend on at least $n-2$ variables. □

Recall that $\mathrm{Aut}(M)$ is the group of all global CR-automorphisms and $\mathrm{Aff}(M)$ is the subgroup of all affine transformations of M.

PROPOSITION 7.5. *Let* $\varphi\in\mathrm{End}(V)$ *be a trace-free cyclic endomorphism. Then the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Aff}(M)$ *coincide. Furthermore, with* $n=\dim V$, *the following dimension estimates hold*:

(i) $\dim\mathrm{Aut}(M)=n+4$ *if the characteristic roots of* φ *are pairwise distinct and form an arithmetic progression in* $i\mathbb{R}$;

(ii) $\dim\mathrm{Aut}(M)=n+3$ *if* φ *is nilpotent*;

(iii) $\dim\mathrm{Aut}(M)=n+2$ *in all other cases.*

Proof. Let $F:=M\cap V$ and denote by $\mathfrak{a}\subset\mathfrak{g}=\mathfrak{hol}(M,a)$ the Lie algebra of $\mathrm{Aut}(M)$. Since $\mathfrak{a}$ contains the Euler vector field, we have $\mathfrak{a}=\mathfrak{a}_{-1}\oplus\mathfrak{a}_0\oplus\mathfrak{a}_1$ for $\mathfrak{a}_j:=\mathfrak{a}\cap\mathfrak{g}_j$. We first determine $\dim\mathfrak{a}_0$. Because of $\mathfrak{h}:=\mathbb{R}\,\mathrm{id}\oplus\mathbb{R}\varphi\subset\mathfrak{a}_0$, we have $\dim\mathfrak{a}_0\geqslant 2$.

In case (i), M is CR-equivalent to $\mathcal{M}^{2,n-2}$; compare Example 5.1. Therefore $\mathrm{GL}(2,\mathbb{R})$ acts transitively on F and $\dim\mathfrak{a}_0=4$ by Proposition 7.3.

Next consider case (ii), that is, φ is nilpotent. Then $\mathfrak{a}_0$ consists of all $\xi\in\mathfrak{g}_0\subset\mathrm{End}(V)$ with $\xi(c)\in\mathbb{R}c$ for all $c\in\partial F:=\overline{F}\backslash F$, where $\overline{F}$ is the closure of F in V. We may assume that $a=(1,0,...,0)\in\mathbb{R}^n$ and $\varphi=\xi^{1,-1}$ in the notation of (5.2). This implies that

$$F=\{e^s(1,t,t^2,...,t^{n-1})\in\mathbb{R}^n:s,t\in\mathbb{R}\},$$

and hence $\partial F=\mathbb{R}c$ for $c:=(0,...,0,1)$. Therefore, $\mathfrak{a}_0$ is the linear span of ζ_1, ζ_2 and $\xi^{1,-1}$ and $\dim\mathfrak{a}_0=3$ in this situation.

Next consider the case $V=V_1\oplus V_2\oplus...\oplus V_n$, where every V_j is the $\bigl(\tfrac{1}{2}(1-n)+j-1\bigr)$ eigenspace of φ, that is, the characteristic roots of φ form an arithmetic progression in $\mathbb{R}$. Here $\partial F=V_1\cup V_n$ is easily verified. The vector fields in $\mathfrak{g}_0$ are characterized by the function tuples (r_0,r_1) in (7.4). The condition $\xi(V_j)\subset V_j$ for $j=1,n$ implies that r_0 and r_1 are constant for every $\xi\in\mathfrak{a}_0$, that is $\mathfrak{a}_0=\mathfrak{h}$. On the other hand, if the characteristic roots of φ do not form an arithmetic progression, then also $\mathfrak{g}_0=\mathfrak{h}$ by Proposition 6.5 (i), that is, $\dim\mathfrak{a}_0=2$ always holds in case (iii).

Next we show that $\mathfrak{a}_1=0$ in all cases. For $n\geqslant 4$ this follows from $\mathfrak{g}_1=0$, see Proposition 7.3 (ii). In case (iii) we have $\mathfrak{a}_0=\mathfrak{h}$ and the claim follows by Lemma 6.18. Therefore we only have to consider cases (i) and (ii) for $n=3$. In case (i), M is the future light cone tube $\mathcal{M}$ and $\mathrm{Aut}(\mathcal{M})=\mathrm{Aff}(\mathcal{M})$ follows as a special case of [23, Proposition 6.9]. In case (ii), M is a proper domain in $\mathcal{M}$: we realize $\mathcal{M}=\mathcal{F}+i\mathbb{R}^3$ in $\mathbb{C}^3$ with coordinates (z_0,z_1,z_2) as $\mathcal{F}=\{x\in\mathbb{R}^3:x_0x_2=x_1^2$ and $x_0+x_2>0\}$. Then $\mathfrak{hol}(\mathcal{M})$ is the linear span of the vector fields (3.5) and (3.7) in [16]. We may assume without loss of generality that $\varphi=\xi^{-1,1}=2z_1\partial/\partial z_0+z_2\partial/\partial z_1$. This implies that $\mathcal{M}\backslash M=\mathbb{R}^{+}\cdot c+i\mathbb{R}^3$ for

$c{:=}(1,0,0)\in\mathcal{F}$. We know already that $\mathfrak{a}$ is the linear span of the vector fields ζ_1, ζ_2 and $\xi^{-1,1}$. From [16, Figure 1 and (3.7)] we therefore derive that either $\mathfrak{a}_1=0$ or $\mathfrak{a}_1=\mathbb{R}\xi^{0,2}$ for $\xi^{0,2}{:=}iz_1^2\partial/\partial z_0+iz_1z_2\partial/\partial z_1+iz_2^2\partial/\partial z_2$. The latter possibility cannot occur, since $\xi^{0,2}$ is not tangent to $\mathcal{M}\backslash M$: check, for instance, the point $(1,i,0)$. This proves that $\mathfrak{a}=\mathfrak{g}_{-1}\oplus\mathfrak{a}_0$ in all cases. As in the proof of Proposition 4.4 (iii), it is shown that this implies that $\mathrm{Aut}(M)=\mathrm{Aff}(M)$. The above dimension estimates for $\mathfrak{a}_0$ imply the dimension estimates in (i)–(iii). □

Next we solve the local as well as the global CR-equivalence problem for all manifolds M^φ. Because of Proposition 7.1, in the local situation only the case has to be considered where the characteristic roots of φ do not form an arithmetic progression. Recall that without loss of generality we always may assume that φ is trace-free.

PROPOSITION 7.6. *Let $\varphi,\varphi'\in\mathrm{End}(V)$ be trace-free cyclic endomorphisms with characteristic roots $\alpha_1,...,\alpha_n$ and $\alpha'_1,...,\alpha'_n$ respectively. Suppose that $\alpha_1,...,\alpha_n$ do not form an arithmetic progression. Then for given cyclic vectors $a,a'\in V$ and corresponding $M=M^\varphi$ and $M'=M^{\varphi'}$, the Lie algebras $\mathfrak{hol}(M,a)$ and $\mathfrak{aff}(M,a)$ coincide. Furthermore, the following conditions are equivalent*:

(i) *the Lie algebras $\mathfrak{hol}(M,a)$ and $\mathfrak{hol}(M',a)$ are isomorphic*;

(ii) *the germs (M,a) and (M',a') are CR-equivalent*;

(iii) *$g\varphi'g^{-1}=r\varphi$ for some suitable $g\in\mathrm{GL}(V)$ and $r\in\mathbb{R}^*$*;

(iv) *there exists a permutation $\pi\in\mathfrak{S}_n$ and an $r\in\mathbb{R}^*$ with $\alpha'_j=r\alpha_{\pi(j)}$ for all j.*

Proof. The fact that $\mathfrak{g}{:=}\mathfrak{hol}(M,a)=\mathfrak{aff}(M,a)$ and $\dim\mathfrak{g}=n+2$ follows from Propositions 6.5 and 6.17.

(i) ⇒ (ii) With $\mathfrak{g}$, also $\mathfrak{g}'{:=}\mathfrak{hol}(M',a')$ has dimension $n+2$. Therefore $\alpha'_1,...,\alpha'_n$ do not form an arithmetic progression either; see Proposition 7.3. This implies that $\mathfrak{g}_0=\mathfrak{h}$ and $\mathfrak{g}'_0=\mathfrak{h}'$, and (ii) follows by Proposition 4.11.

(ii) ⇒ (iii) Let g be a CR-isomorphism $(M,a)\to(M',a')$. Then $g\in\mathrm{GL}(V)$ as a consequence of Proposition 4.4 (iii), and clearly $g\mathfrak{h}g^{-1}=\mathfrak{h}'$. Since $\mathbb{R}\varphi'\subset\mathfrak{h}'$ is precisely the subset of all trace-free endomorphisms, (iv) follows.

The remaining implications are easy to check and left to the reader. □

PROPOSITION 7.7. *Let $\varphi,\varphi'\in\mathrm{End}(V)$ be trace-free cyclic endomorphisms and let $M{:=}M^\varphi$ and $M'{:=}M^{\varphi'}$. Then, the following conditions are equivalent*:

(i) *the groups* $\mathrm{Aff}(M)$ *and* $\mathrm{Aff}(M')$ *are isomorphic*;

(ii) *M and M' are globally CR-equivalent*;

(iii) *$g\varphi'g^{-1}=r\varphi$ for suitable $g\in\mathrm{GL}(V)$ and $r\in\mathbb{R}^*$.*

Proof. (i) ⇒ (iii) Suppose that (i) holds. By Proposition 7.5, we may assume that $\dim\mathrm{Aut}(M)=n+2$ without loss of generality. Then (iii) follows from Proposition 7.6 if, at least for one of φ and φ', the characteristic roots do not form an arithmetic progression. In the remaining cases the claim follows from Proposition 7.5, since for both endomorphisms the characteristic roots form an arithmetic progression in $\mathbb{R}$ and are pairwise distinct.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) This is obvious, since $\mathrm{Aut}(M)=\mathrm{Aff}(M)$ and $\mathrm{Aut}(M')=\mathrm{Aff}(M')$ by Proposition 7.5. $\square$

7.8. Some moduli spaces

For fixed $n=\dim V$, let $\mathscr{M}$ be the space of all global CR-equivalence classes $[M^\varphi]$ of manifolds M^φ with $\varphi\in\mathrm{Cyc}(V)$, that is, every $[M^\varphi]$ is the set of all M^ψ that are globally CR-equivalent to M^φ. Furthermore, put

$$\Phi:=\{\varphi\in\mathrm{Cyc}(V):\mathrm{tr}(\varphi)=0\}\quad\text{and}\quad\mathscr{M}^*:=\{[M^\varphi]\in\mathscr{M}:\varphi\in\Phi\text{ and }\varphi^n\neq0\}.$$

The reductive group $\mathbb{R}^*\times\mathrm{GL}(V)$ acts on $\mathrm{End}(V)$ by

$$\varphi\longmapsto rg\varphi g^{-1}\quad\text{for every }(r,g)\in\mathbb{R}^*\times\mathrm{GL}(V)$$

and leaves the cone Φ invariant. By Proposition 7.7, $\mathscr{M}$ can be identified, as a set, with the quotient $\Phi/(\mathbb{R}^*\times\mathrm{GL}(V))$. This quotient can be built in several steps: For every j let $\sigma_j(\varphi)\in\mathbb{R}$ be the jth elementary symmetric function in n variables evaluated on the characteristic roots of φ, that is,

$$X^n+\sum_{j=2}^{n}(-1)^j\sigma_j(\varphi)X^{n-j}\in\mathbb{R}[X]$$

is the characteristic polynomial of $\varphi\in\Phi$. Let $W:=\mathbb{R}^{n-1}$ with coordinates $(x_2,...,x_n)$, and denote by $\sigma:\Phi\to W$ the mapping given by $\varphi\mapsto(\sigma_2(\varphi),...,\sigma_n(\varphi))$. Since every real polynomial factors into a product of linear and quadratic real polynomials, the map σ is surjective and $\mathscr{M}$ can be canonically identified, as a set, with the quotient $W/\mathbb{R}^*$, where $\mathbb{R}^*$ acts on W by $(x_2,x_3,...,x_n)\mapsto(t^2x_2,t^3x_3,...,t^nx_n)$ for every $t\in\mathbb{R}^*$. The subgroup $\{\pm1\}\subset\mathbb{R}^*$ leaves the sphere $S^{n-2}=\left\{x\in W:\sum_{j=2}^n x_j^2=1\right\}$ invariant, and $\mathscr{M}^*$ can be identified with the quotient $Q^{n-2}:=S^{n-2}/\{\pm1\}$. In general, Q^{n-2} can be stratified into a finite number of manifolds. For instance, Q^1 is a compact line segment and Q^2 is homeomorphic to the sphere S^2. At this point a word of caution is necessary: we do not give a topology on $\mathscr{M}$, the topology on Q^{n-2} only serves for the readers imagination.

Instead of $\mathscr{M}$, we can also consider the space of *local* CR-equivalence classes for manifolds of type M^φ. By our results, this space is of the form $\mathscr{M}/\!\sim$, where the equivalence relation $\sim$ on $\mathscr{M}$ just identifies the three equivalence classes $[M^\varphi]\in\mathscr{M}$ such that the characteristic roots of φ form an arithmetic progression. Clearly, $\mathscr{M}/{\sim}=\mathscr{M}^*/{\sim}$ can be obtained by identifying two points in Q^{n-2}. In the special case $n{=}3$, the endpoints of the line segment Q^1 have to be identified, that is, $\mathscr{M}$ can be thought of in this case as the circle $\bar{\mathbb{R}}:=\mathbb{R}\cup\infty$ (without topology), where the point ∞ corresponds to the class represented by the future light cone tube $\mathcal{M}$. To be more specific, in case $n{=}3$, for every $\varphi\in\Phi$ call

$$\mu(M^\varphi) := -\frac{\sigma_2(\varphi)^3}{\sigma_3(\varphi)^2} \in \bar{\mathbb{R}} \tag{7.9}$$

the *modulus* of the CR-manifold M^φ (with $t/0:=\infty$ for all $t\in\mathbb{R}$). It is clear that M^φ and M^ψ, in case $n{=}3$, are locally CR-equivalent if and only if they have the same modulus. A special meaning has the modulus $\mu_0:=\frac{27}{4}$: For real moduli $>\mu_0$ the endomorphism φ has three distinct real eigenvalues, while in case of real moduli $<\mu_0$ the endomorphism φ has one real and two purely imaginary eigenvalues.

7.10. Another type of examples

For $k{=}3$ and $c\geqslant 1$ let $m:=c+2$ and V be as in Example 5.1. Then also the subgroup

$$\Sigma := \left\{ \begin{pmatrix} r & 0 \\ t & 1 \end{pmatrix} : r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R} \right\} \subset \mathrm{GL}(2,\mathbb{R}) \tag{7.11}$$

acts on V by $p\mapsto p\circ g^{-1}$, and the Lie algebra of Σ corresponds to the linear span of the two vector fields ζ_1 and $\xi^{1,-1}$ in Example 5.2. For $a:=v^m+muv^{m-1}\in V$ the orbit $F:=\Sigma(a)$ is a surface in V, and for $o:=v^m$ the orbit $C:=\Sigma(o)$ is a curve in the closure of F. Identifying $\mathbb{R}^{m+1}$ with coordinates $(x_0,x_1,...,x_m)$ and V as in Example 5.1, we get $o=(1,0,...,0)$, $a=(1,1,0,...,0)$ and

$$C = \{(1,t,t^2,...,t^m) : t\in\mathbb{R}\}.$$

Furthermore $T_oC=\mathbb{R}\cdot b$, with $b:=(0,1,0,...,0)$, for the tangent space at $o\in C$. On the other hand, the affine half-line $o+\mathbb{R}^+\cdot b$ is contained in F. The geometric meaning of this is the following: The development $S:=\bigcup_{c\in C}(c+T_cC)$ of the curve C is divided by C into two Σ-orbits, one of which is F (compare [12, p. 45] for the special case $m{=}3$). Now identify the Σ-invariant hyperplane $W:=\{x\in V : x_0=1\}$ with $\mathbb{R}^m$ by 'dropping the coordinate' x_0. Then C becomes the twisted m-ic $\{(t,t^2,...,t^m):t\in\mathbb{R}\}$, o becomes the origin and a the first basis vector $(1,0,...,0)$ in $\mathbb{R}^m$. In the coordinates of $\mathbb{R}^m$ the vector fields ζ_1 and $\xi^{1,-1}$ are affine and have the forms

$$\zeta_1 = \sum_{j=1}^m jz_j\partial/\partial z_j \quad \text{and} \quad \xi^{1,-1} = \partial/\partial z_1 + \sum_{j=2}^m jz_{j-1}\partial/\partial z_j. \tag{7.12}$$

By Proposition 3.7, it is easily verified that $K_a^r F=\{x\in\mathbb{R}^m : x_j=0 \text{ if } j+r>2\}$ for all $r\geqslant 0$. Since the twisted m-ic is not contained in any hyperplane of $\mathbb{R}^m$, we therefore get that the tube $M:=C+i\mathbb{R}^m$ is a homogeneous minimal 2-nondegenerate submanifold of $\mathbb{C}^m$ with CR-dimension 2 and CR-codimension $m-2$. Notice that for the cone $\mathbb{R}^+\cdot F$ generated by F in V, the tube $\mathbb{R}^+\cdot F+iV$ is an open piece of $\mathcal{M}^{3,n-3}$, $n:=m+1$, and hence is 3-nondegenerate.

For every integer j, denote by $\mathfrak{g}^{(j)}$ the k-eigenspace of $\operatorname{ad}(\zeta_1)$ in $\mathfrak{g}:=\mathfrak{hol}(M,a)$. Then every $\xi\in\mathfrak{g}^{(j)}$ is a complex linear combination of monomial vector fields $z_1^{\nu_1} z_2^{\nu_2} \dots z_m^{\nu_m}\partial/\partial z_p$ with $\nu_1+2\nu_2+\dots+m\nu_m=j+p$. As in the proof of Proposition 4.2, it is shown that $\mathfrak{g}$ has the $\mathbb{Z}$-grading

$$\mathfrak{g}=\bigoplus_{j\geqslant -m}\mathfrak{g}^{(j)}. \tag{7.13}$$

It can be seen that $\mathfrak{g}$ is the linear span of all $i\partial/\partial z_j$, $1\leqslant j\leqslant m$, as well as ζ_1 and $\xi^{1,-1}$. In particular, $\mathfrak{g}$ is a solvable Lie algebra of dimension $m+2$, coincides with $\mathfrak{aff}(M,a)$ and has commutator subgroup of dimension $m+1$. A proof will be sketched for the special case $m=3$ in Example 8.5 below.

8. Homogeneous 2-nondegenerate CR-manifolds in dimension 5

In this section we specialize the examples of the previous section to the case $V=\mathbb{R}^3$. We start with manifolds of type $M=M^\varphi=F^\varphi+i\mathbb{R}^3$ in $\mathbb{C}^3$. Then the local CR-equivalence classes of these manifolds are parameterized by the modules $\mu(M)\in\overline{\mathbb{R}}$; compare (7.9). The φ occurring in the following examples are not necessarily trace-free but easily could be transformed to be so. As defined in the previous section, let $\mu_0:=\frac{27}{4}$.

Example 8.1. ($\mu=\infty$) Let $F:=\{x\in\mathbb{R}^3 : x_1^2+x_2^2=x_3^2 \text{ and } x_3>0\}$ be the future light cone. This surface occurs as F^φ for $\varphi:=x_2\partial/\partial x_1-x_1\partial/\partial x_2$ having spectrum $\{\pm i,0\}$.

Example 8.2. ($\mu<\mu_0$) For $\omega>0$, let $F\subset\mathbb{R}^3$ be the orbit of $(1,0,1)$ under the group of all linear transformations $x\mapsto r(\cos tx_1-\sin tx_2, \sin tx_1+\cos tx_2, e^{\omega t}x_3)$, $r\in\mathbb{R}^+, t\in\mathbb{R}$.

With $r:=(x_1^2+x_2^2)^{1/2}$, the manifold F is given in $\{x\in\mathbb{R}^3 : r>0\}$ by the explicit equations

$$x_3=r\exp(\omega\cos^{-1}(x_1/r))=r\exp(\omega\sin^{-1}(x_2/r)),$$

where locally always one of these suffices. A suitable choice is

$$\varphi=x_1\partial/\partial x_2-x_2\partial/\partial x_1+\omega x_3\partial/\partial x_3$$

with spectrum $\{\pm i,\omega\}$.

Example 8.3. ($\mu=\mu_0$) Let $F\subset\mathbb{R}^3$ be the orbit of $(1,0,1)$ under the group of all linear transformations $x\mapsto r(x_1, x_2+tx_1, e^t x_3)$ with $r\in\mathbb{R}^+$ and $t\in\mathbb{R}$, that is,

$$F=\{x\in\mathbb{R}^3 : x_1>0 \text{ and } x_3=x_1 e^{x_2/x_1}\}.$$

Here $\varphi=x_1\partial/\partial x_2+x_3\partial/\partial x_3$ has characteristic roots 0, 0 and 1.

Example 8.4. ($\mu_0<\mu<\infty$) For $\theta>2$ let

$$F:=\{x\in(\mathbb{R}^+)^3 : x_3=x_1(x_2/x_1)^\theta\}.$$

Here $\varphi=x_2\partial/\partial x_2+\omega x_3\partial/\partial x_3$ has eigenvalues $\{0,1,\omega\}$.

The following is Example 7.10 specialized to $m=3$.

Example 8.5. Let Σ be the group generated by the following two one-parameter groups of affine transformations on $\mathbb{R}^3$:

$$x\mapsto(e^t x_1, e^{2t}x_2, e^{3t}x_3) \quad\text{and}\quad x\mapsto(x_1+t, x_2+2tx_1+t^2, x_3+3tx_2+3t^2x_1+t^3). \tag{8.6}$$

Then Σ is isomorphic to the group defined in (7.11). For $a:=(1,0,0)$, the orbit $F:=\Sigma(a)$ is

$$F=\{(t,t^2,t^3)+r(1,2t,3t^2)\in\mathbb{R}^3 : r\in\mathbb{R}^+ \text{ and } t\in\mathbb{R}\}.$$

The tube $M:=F+i\mathbb{R}^3$ is an affinely homogeneous 2-nondegenerate CR-manifold. The Lie algebra of Σ is spanned by the affine vector fields

$$\zeta_1:=z_1\partial/\partial z_1+2z_2\partial/\partial z_2+3z_3\partial/\partial z_3 \quad\text{and}\quad \xi^{1,-1}:=\partial/\partial z_1+z_1\partial/\partial z_2+z_2\partial/\partial z_3,$$

compare also (7.12). The Lie algebra $\mathfrak{g}:=\mathfrak{hol}(M,a)$ is of finite dimension and has the grading (7.13) for $m=3$, where $\mathfrak{g}^{(k)}$ is the k-eigenspace of $\operatorname{ad}(\zeta_1)$. We claim that $\mathfrak{g}$ has dimension 5 and coincides with $\mathfrak{aff}(M,a)$. The proof consists of several elementary steps which we only sketch here. To begin with, define $\mathfrak{a}^{(k)}\subset\mathfrak{g}^{(k)}$ by

$$\mathfrak{a}^{(-3)}:=\mathbb{R}i\partial/\partial z_3, \quad \mathfrak{a}^{(-2)}:=\mathbb{R}i\partial/\partial z_2, \quad \mathfrak{a}^{(-1)}:=\mathbb{R}i\partial/\partial z_1\oplus\mathbb{R}\varphi, \quad \mathfrak{a}^{(0)}:=\mathbb{R}\zeta$$

and $\mathfrak{a}^{(k)}:=0$ for all other k. By induction on k, it is seen that $\mathfrak{g}^{(k)}=\mathfrak{a}^{(k)}$ holds for all k. For $k\leqslant-3$ this is obvious. For $k=-2$, suppose that there exists a $\xi\in\mathfrak{g}^{(-2)}\setminus\mathfrak{a}^{(-2)}$. Then $\xi=\alpha\partial/\partial z_2+\beta z_1\partial/\partial z_3$ for some $\alpha,\beta\in\mathbb{C}$. From $[\xi,\mathfrak{a}^{(-1)}]\subset\mathfrak{a}^{(-3)}$ we get $\beta\in\mathbb{R}$, and then $\xi_a\in T_aM$ implies $\beta=0$. But then $\xi\in\mathfrak{a}^{(-2)}$ since $\mathfrak{a}^{(-2)}\oplus i\mathfrak{a}^{(-2)}\not\subset\mathfrak{g}^{(-2)}$, a contradiction. For $k\geqslant-1$ the procedure is as follows. Suppose that there exists $\xi\in\mathfrak{g}^{(k)}\setminus\mathfrak{a}^{(k)}$. Then write ξ as a complex linear combination of monomial vector fields as mentioned above and subtract from ξ a suitable element of $\mathfrak{a}^{(k)}$, thus killing as many coefficients in front of monomial terms of the form $f(z)\partial/\partial z_1$ as possible. By induction hypothesis, $[\mathfrak{g}^{(k)},\mathfrak{a}^{(j)}]\subset\mathfrak{a}^{(k+j)}$ holds for all $j<0$ and gives $\xi=0$, a contradiction. This proves the claim and also the first part of the following statement.

LEMMA 8.7. *Let* $M{:=}F+i\mathbb{R}^3$, *with* $F{=}H(a)\subset\mathbb{R}^3$, *as in Example* 8.5. *Then* $\mathfrak{g}{=}\mathfrak{hol}(M,a)$ *is a solvable Lie algebra of dimension* 5 *with commutator algebra* $[\mathfrak{g},\mathfrak{g}]$ *of dimension* 4. *Furthermore*, $\mathrm{Aut}(M,a)=\{\mathrm{id}\}$.

Proof. Fix $a{:=}(1,0,0)\in F$ and write $\xi_j{:=}i\partial/\partial z_j$ for $j{=}1,2,3$. Let $h\in\mathrm{Aut}(M,a)$ and Θ be the induced Lie algebra automorphism of $\mathfrak{l}{:=}\mathfrak{g}\oplus i\mathfrak{g}$. With $\mathfrak{g}$, the subspaces

$$\mathfrak{n} := [\mathfrak{g},\mathfrak{g}] = \bigoplus_{k<0}\mathfrak{g}^{(k)}, \quad [\mathfrak{n},\mathfrak{n}] = \mathbb{R}\xi_2\oplus\mathbb{R}\xi_3 \quad \text{and} \quad [\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = \mathbb{R}\xi_3$$

are stable under Θ and hence also

$$\mathfrak{n}\cap K_aM = \mathbb{R}\xi_1 \quad \text{and} \quad [\mathfrak{n},\mathfrak{n}]\cap H_aM = \mathbb{R}\xi_2,$$

where $\mathfrak{g}$ and the tangent space T_aM are identified via the evaluation map. In particular, $\Theta(\mathfrak{P}_{-1})=\mathfrak{P}_{-1}$ and $h(z)=g(z)+c$ for a diagonal matrix $g\in\mathrm{GL}(3,\mathbb{R})$ with $c=g(a)-a$; compare (2.1). Taking commutators of h with all elements in the second 1-parameter group of (8.6), and then taking the derivative by t at $t{=}0$, gives $c\partial/\partial z\in\mathfrak{g}$. From $\mathfrak{g}\cap i\mathfrak{g}{=}0$, we conclude that $c{=}0$ and thus $g_{11}{=}1$ for the diagonal matrix g. Now, ψ is the unique vector field in $\mathfrak{g}$ that has the value $\partial/\partial z$ at both points 0 and a, implying that $\Theta(\psi)=\psi$. Therefore g is the unit matrix and h is the identity in $\mathrm{Aut}(M,a)$. □

For every $M{=}F+i\mathbb{R}^3$, with $F\subset\mathbb{R}^3$ being one of the cones from Examples 8.1–8.4, the commutator of $\mathfrak{hol}(M,a)$ has either dimension 10 (Example 8.1) or dimension 3 (all the others). As a consequence of Proposition 7.6 and Lemma 8.7 we therefore get the following result.

PROPOSITION 8.8. *The CR-manifolds* $M{=}F+i\mathbb{R}^3$, *with* $F\subset\mathbb{R}^3$ *occurring in Examples* 8.1–8.5, *are all homogeneous and* 2-*nondegenerate. Furthermore*, *they are mutually locally CR-inequivalent.*

By an argument from [16] together with [19, Theorem 2.5.10], a holomorphic extension property for *global* continuous CR-functions f on $M{=}F\oplus i\mathbb{R}^3\subset\mathbb{C}^3$, F being one of the cones from Examples 8.1–8.5, can be obtained. Every such f has a unique continuous extension to the convex hull $\widehat{M}$ of M in $\mathbb{C}^3$ that is holomorphic on the interior of $\widehat{M}$ with respect to $\mathbb{C}^3$. Since M is completely contained in the interior of $\widehat{M}$ in case F belongs to Example 8.2, every global continuous CR-function on such an M is real-analytic.

For the tube $\mathcal{M}$ over the future light cone (that is, Example 8.1) there exist many (even simply-connected) homogeneous CR-manifolds that are all locally CR-equivalent to $\mathcal{M}$ but are mutually nondiffeomorphic; compare [23]. In contrast to this, using already Theorem II from §9 below, we can state the following global result.

PROPOSITION 8.9. *Let M be a homogeneous 2-nondegenerate CR-manifold that is not locally CR-equivalent to the tube $\mathcal{M}$ over the future light cone. Then M is simply connected and* $\mathrm{Aut}(M)$ *is a solvable Lie group of dimension* 5 *acting transitively and freely on M. For every $a\in M$ the stability group* $\mathrm{Aut}(M,a)$ *is trivial and every homogeneous real-analytic CR-manifold M' which is locally CR-equivalent to M is already globally CR-equivalent to M.*

Proof. By Theorem II, $M=F+iV$ for F as in one of the Examples 8.2–8.5. It is easily checked that F and hence M is simply connected. In case F is a cone, the claim follows by Lemma 6.20. Therefore, we may assume that F is the submanifold of Example 8.5. But then $\mathrm{Aut}(M,a)$ is the trivial group, by Lemma 8.7, and $\mathrm{Aut}(M)$ has trivial center, by Proposition 3.9. But then the claim follows from Proposition 3.10. $\square$

The affinely homogeneous surfaces $F\subset\mathbb{R}^3$ of Examples 8.1–8.5 already occur in [12, p. 43]. There the surfaces are presented in their affine normal forms. Our Example 8.1 ($\mu=\infty$) corresponds to P4, Examples 8.2–8.4 ($\mu\in\mathbb{R}$) to P3 and Example 8.5 to P1. The remaining degenerate types in [12], types P2$^{\pm}$ and P5, do not show up among our examples, since the associated tube manifolds are holomorphically degenerate.

Let us consider the type P3 in [12, p. 43] a little bit closer. This is the family of local surfaces in $\mathbb{R}^3$ given by the local equations in affine normal form

$$x_3=x_1^2+x_1^2x_2+x_1^2x_2^2+x_1^5+x_1^2x_2^3+4x_1^5x_2+x_1^2x_2^4+ax_1^7+10x_1^5x_2^2+x_1^2x_2^5+\mathrm{O}(8), \qquad (8.10)$$

where $a\in\mathbb{R}$ is an arbitrary parameter and $\mathrm{O}(8)$ for every fixed a is a convergent power series in $x=(x_1,x_2,x_3)$ vanishing of order $\geqslant 8$ at the origin and uniquely determined by the requirement, that (8.10) defines, near the origin of $\mathbb{R}^3$, a locally affinely homogeneous surface. Different values of $a\in\mathbb{R}$ give locally affinely inequivalent surfaces, and the associated tubes in $\mathbb{C}^3$ correspond in a one-to-one way to our Examples 8.2–8.4. It is not difficult to see that the modulus μ of every such surface is related to the parameter a in (8.10) by the formula $100\mu=(28a)^3$.

In [11] and [12] all locally affinely homogeneous surfaces in $\mathbb{R}^3$ have been classified up to local affine equivalence. Inspecting the degenerate surfaces in these classifications gives together with our results the following proposition.

PROPOSITION 8.11. *Let F be a locally affinely homogeneous surface in $\mathbb{R}^3$ and assume that the corresponding tube $M:=F+i\mathbb{R}^3$ in $\mathbb{C}^3$ is* 2*-nondegenerate. Then*

(i) *M is locally CR-equivalent to a manifold occurring in Examples* 8.1–8.5;

(ii) *for any other locally affinely homogeneous surface F' in $\mathbb{R}^3$, the corresponding tubes M and $M':=F'+i\mathbb{R}^3$ are locally CR-equivalent if and only if F and F' are locally affinely equivalent.*

Part II. The classification

9. Lie-theoretic characterization of locally homogeneous CR-manifolds

In this part of the paper we classify all homogeneous 5-dimensional 2-nondegenerate CR-manifolds up to local CR-equivalence, that is, we carry out the proof of the following theorem.

THEOREM II. *Let M be a locally homogeneous* 2*-nondegenerate real-analytic CR-manifold of dimension* 5*. Then M is locally CR-equivalent to a tube $F+i\mathbb{R}^3\subset\mathbb{C}^3$, where $F\subset\mathbb{R}^3$ is one of the affinely homogeneous surfaces occurring in Examples* 8.1–8.5.

We call (in accordance with §2) a real-analytic CR-manifold M *locally homogeneous* at a point $o\in M$ if there exists a Lie subalgebra $\mathfrak{g}\subset\mathfrak{hol}(M,o)$ of finite dimension such that the canonical evaluation map $\mathfrak{g}\to T_oM$ is surjective, that is, such that the tangent vectors $\xi_o, \xi\in\mathfrak{g}$ span the tangent space T_oM. If this is the case, we also call the corresponding CR-germ (M,o) *locally homogeneous*. If a particular locally transitive $\mathfrak{g}\subset\mathfrak{hol}(M,o)$ has been fixed, we also say that the germ (M,o) is $\mathfrak{g}$-homogeneous.

The proof of Theorem II relies on a natural equivalence (see [15, Proposition 4.1]) between the category of CR-manifold germs with a locally transitive Lie algebra action and a certain purely algebraically defined category. Before we briefly outline the main steps of our proof, we recall the notion of a CR-algebra, taken from [28], and introduce some notation.

Definition 9.1. A *CR-algebra* is a pair $(\mathfrak{g},\mathfrak{q})$, where $\mathfrak{g}$ is a real Lie algebra of *finite* dimension and $\mathfrak{q}$ is a complex Lie subalgebra of the complexification $\mathfrak{l}:=\mathfrak{g}\oplus i\mathfrak{g}$. The CR-algebra $(\mathfrak{g},\mathfrak{q})$ is called *effective* if 0 is the only ideal of $\mathfrak{g}$ contained in $\mathfrak{g}\cap\mathfrak{q}$.

Remarks 9.2. (i) In [28] also the case is allowed where $\mathfrak{g}$ has infinite dimension, but $\mathfrak{q}$ has to have finite codimension in $\mathfrak{l}$. In this part of the paper, however, only finite-dimensional Lie algebras occur.

(ii) The CR-algebras form a category in an obvious way: A morphism $(\mathfrak{g},\mathfrak{q})\to(\mathfrak{g}',\mathfrak{q}')$ of CR-algebras is a Lie algebra homomorphism $\mathfrak{g}\to\mathfrak{g}'$ in the usual sense whose complex linear extension $\mathfrak{l}\to\mathfrak{l}'$ maps $\mathfrak{q}$ to $\mathfrak{q}'$. Unfortunately, the resulting notion of isomorphism between CR-algebras is too strict for our purposes. We therefore mainly work with the coarser notion of geometric equivalence between CR-algebras to be introduced later.

(iii) The geometric situation behind the notion of a CR-algebra is the following. Let Z be a complex manifold, homogeneous under a complex Lie group L, let $o\in Z$ be a point with isotropy subgroup $Q\subset L$ at o and let $G\subset L$ be a connected real form of L, that is, the connected identity component of the fixed point set L^σ for an involutive antiholomorphic automorphism σ of L. Then each G-orbit M in Z is a generic (immersed) CR-submanifold. Let $\mathfrak{g}$ and $\mathfrak{q}$ be the Lie algebras of G and Q, respectively. Then $(\mathfrak{g},\mathfrak{q})$ is a CR-algebra that completely describes the CR-germ (M,o) together with the local action of G near o.

(iv) In general, however, not every CR-algebra $(\mathfrak{g},\mathfrak{q})$ can be obtained from a global situation as described in (iii), only from a more general local setting. Nevertheless, the set $\mathcal{G}$ of all CR-equivalence classes of locally homogeneous CR-germs and the set $\mathcal{A}$ of all geometric equivalence classes of CR-algebras stand in a canonical one-to-one correspondence; compare also [15, §4]. For convenience of the reader, we briefly describe below this correspondence in both directions. As a reference for a general discussion of 'local' and 'infinitesimal' actions we refer to the original paper of Palais, [29].

In the following let M always be a locally homogeneous real-analytic CR-manifold with base point $o\in M$ that *locally* can be embedded in some $\mathbb{C}^n$ (equivalently, M is an (abstract) real-analytic *involutive* CR-manifold as defined at the end of §2). Each such CR-manifold can globally and generically be embedded into a complex manifold Z [1]. Since we are only interested in the local structure of M at o and therefore mostly deal with CR-germs (M,o), we may assume without loss of generality that M is embedded in a complex vector space $E\cong\mathbb{C}^n$ as a locally closed generic CR-submanifold.

Next we describe the interplay between locally homogeneous CR-germs and CR-algebras more closely. In particular, we give two canonical constructions that induce the one-to-one correspondence between the sets $\mathcal{G}$ and $\mathcal{A}$ mentioned in Remark 9.2 (iv) and also allow the precise definition of 'geometric equivalence' for CR-algebras.

Let (M,o) be a CR-germ and let $\mathfrak{g}\subset\mathfrak{hol}(M,o)$ be a locally transitive Lie subalgebra of finite dimension. Then an effective CR-algebra $(\mathfrak{g},\mathfrak{q})$ can be associated in the following way. To begin with, realize $\mathfrak{hol}(M,o)$ in the canonical way as a real Lie subalgebra of the complex Lie algebra $\mathfrak{hol}(E,o)$. This is possible, since we assumed M to be generic in E and we can use [3, Proposition 12.4.22]. As in Definition 9.1, we always denote by $\mathfrak{l}=\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\oplus i\mathfrak{g}$ the formal complexification of $\mathfrak{g}$. Now $\Xi(\xi+i\eta):=\xi+J\eta$ defines a Lie algebra homomorphism $\Xi\colon\mathfrak{l}\to\mathfrak{hol}(E,o)$, where J denotes the complex structure tensor $J\colon TE\to TE$. We will not make a notational distinction between the complex structures in $\mathfrak{l}$ or TZ, and write 'i' for it. The homomorphism Ξ is in general not injective (it is, if M is holomorphically nondegenerate). Let $\mathfrak{q}\subset\mathfrak{l}$ be the Ξ-preimage of the isotropy subalgebra $\{\xi\in\mathfrak{hol}(E,o):\xi_o=0\}$. Then the CR-algebra $(\mathfrak{g},\mathfrak{q})$ is called a *CR-algebra associated* with the locally homogeneous CR-germ (M,o). It is obvious that $\mathfrak{g}\cap\mathfrak{q}$ is nothing but the isotropy subalgebra $\mathfrak{g}_o=\{\xi\in\mathfrak{g}:\xi_o=0\}$ and the tangent space T_oM can be canonically identified with $\mathfrak{g}/\mathfrak{g}_o$. Also, the holomorphic tangent space H_oM and the partial complex structure $J\colon H_oM\to H_oM$ (equivalently: the decomposition $H^{1,0}_oM\oplus H^{0,1}_oM$ of the complexification $H_o^{\mathbb{C}}M = H_oM\otimes\mathbb{C}$) can be read off the CR-algebra $(\mathfrak{g},\mathfrak{q})$: Let σ be the conjugate linear involution of $\mathfrak{l}$ with $\mathfrak{l}^\sigma := \mathrm{Fix}(\sigma) = \mathfrak{g}$. Then it is easily verified that $\mathfrak{H} := (\mathfrak{q}+\sigma\mathfrak{q})^\sigma$ coincides with $\{\xi\in\mathfrak{g} : \xi_o\in iT_oM\}$, that is, $\mathfrak{H}/\mathfrak{g}_o$ is canonically isomorphic to H_oM (the capital letter for $\mathfrak{H}$ is chosen to indicate that $\mathfrak{H}$ in general is not a Lie algebra, only a linear subspace). Further, $H_o^{0,1}M = \mathfrak{q}/(\mathfrak{q}\cap\sigma\mathfrak{q})$ and $H_o^{1,0}M = \sigma\mathfrak{q}/(\mathfrak{q}\cap\sigma\mathfrak{q})$. In [15] it has been shown that the geometric properties of the CR-structure of the CR-germ (M,o) like minimality, k-nondegeneracy and holomorphic degeneracy can be read off every CR-algebra $(\mathfrak{g},\mathfrak{q})$ associated with (M,o). The facts relevant for our classification will be discussed below.

There is also a canonical way to associate a locally homogeneous CR-manifold germ (M,o) with a given CR-algebra $(\mathfrak{g},\mathfrak{q})$ (not necessarily effective): Choose a complex Lie group L with Lie algebra $\mathfrak{l} = \mathfrak{g}\oplus i\mathfrak{g}$ and a complex linear subspace $E\subset\mathfrak{l}$ with $\mathfrak{l} = \mathfrak{q}\oplus E$. Then there exist open neighbourhoods U of $0\in E$, V of $0\in\mathfrak{q}$ and R of $\mathrm{id}\in L$ such that $(u,v)\mapsto\exp(u)\exp(v)$ defines a biholomorphic mapping $\varphi: U\times V\to R$. Choose an open neighbourhood P of $0\in\mathfrak{g}$ with $\exp(P)\subset R$. Then, in our particular situation, with $\pi: U\times V\to U$ being the canonical projection, the mapping $\psi := \pi\circ\varphi^{-1}\circ\exp: P\to U$ has constant rank. Without loss of generality, we therefore may assume that $M := \psi(P)$ is a connected real-analytic submanifold of E containing the origin $0\in E$. The Lie algebra $\mathfrak{l}$ can be identified with the Lie algebra of all right-invariant vector fields on L, and every $\xi\in\mathfrak{l}$ can be projected along $\pi\circ\varphi^{-1}: R\to U$ to a holomorphic vector field $\tilde{\xi}\in\mathfrak{hol}(U)$. Thus the real subalgebra $\tilde{\mathfrak{g}} := \{\tilde{\xi} : \xi\in\mathfrak{g}\}\subset\mathfrak{hol}(U)$ is a homomorphic image of $\mathfrak{g}$ and spans at every $x\in M$ the tangent space T_xM. In particular, M is a generic CR-submanifold of E and $\tilde{\mathfrak{g}}$ is a locally transitive subalgebra. It is not difficult so see that $\tilde{\mathfrak{g}}$ is obtained from $\mathfrak{g}$ by factoring out the kernel of ineffectivity. More precisely, let $\mathfrak{j}$ be the largest ideal in $\mathfrak{g}$ with $\mathfrak{j}\subset\mathfrak{g}\cap\mathfrak{q}$. Then $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}/\mathfrak{j}$. If there exists a Lie group L with Lie algebra $\mathfrak{l}$ such that the subalgebra $\mathfrak{q}$ corresponds to a *closed* complex subgroup $Q\subset L$, then we may take L/Q for U and the G-orbit through $[Q]\in L/Q$ for M. We call (M,o) the CR-*germ associated* with the CR-algebra $(\mathfrak{g},\mathfrak{q})$.

Definition 9.3. The CR-algebras $(\mathfrak{g},\mathfrak{q})$ and $(\mathfrak{g}',\mathfrak{q}')$ are called *geometrically equivalent* if the associated CR-germs are CR-equivalent.

Notice that CR-algebras are always geometrically equivalent if they are isomorphic in the categorical sense of Remark 9.2 (ii), but not conversely in general. Notice also that every CR-algebra is geometrically equivalent to an effective one. In the following §§9.4–9.8, we fix *for the rest of the paper* the basic setup and notation, which are mainly taken from [15].

9.4. Notation

Given a CR-algebra $(\mathfrak{g},\mathfrak{q})$, let (M,o) be the associated CR-germ. Write $\mathfrak{l}:=\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\oplus i\mathfrak{g}$ for the complexification and $\sigma:\mathfrak{l}\to\mathfrak{l}$ for the complex conjugation with $\mathfrak{l}^{\sigma}=\mathfrak{g}$. Then

(i) $\mathfrak{g}_o:=\mathfrak{g}\cap\mathfrak{q}$ is called the *real isotropy subalgebra*. Define $\mathfrak{l}_o:=\mathfrak{q}^{(\infty)}:=\mathfrak{q}\cap\sigma\mathfrak{q}$ and note that $\mathfrak{l}_o$ is the complexification $(\mathfrak{g}_o)^{\mathbb{C}}$ of $\mathfrak{g}_o$.

(ii) $\mathfrak{g}/\mathfrak{g}_o$ and $\mathfrak{H}/\mathfrak{g}_o\subset\mathfrak{g}/\mathfrak{g}_o$, for $\mathfrak{H}:=(\mathfrak{q}+\sigma\mathfrak{q})^{\sigma}\subset\mathfrak{g}$, are called the *real* and the *holomorphic tangent space*, respectively.

(iii) The descending chain $\mathfrak{q}^{(0)}\supset\mathfrak{q}^{(1)}\supset\mathfrak{q}^{(2)}\supset\ldots\supset\mathfrak{q}^{(\infty)}$ of complex subalgebras is inductively defined by $\mathfrak{q}^{(0)}:=\mathfrak{q}$, $\mathfrak{q}^{(\infty)}:=\mathfrak{q}\cap\sigma\mathfrak{q}$ and $\mathfrak{q}^{(k+1)}:=\{w\in\mathfrak{q}^{(k)}:[w,\sigma\mathfrak{q}]\subset\mathfrak{q}^{(k)}+\sigma\mathfrak{q}\}$ for $k\in\mathbb{N}$.

If (M,o) is the manifold germ associated with the CR-algebra in §9.4, then the holomorphic tangent space $\mathfrak{H}/\mathfrak{g}_o\subset\mathfrak{g}/\mathfrak{g}_o$ in the sense of (ii) can be canonically identified with the holomorphic tangent space H_oM in the geometric sense. As shown in [15], the mapping $\mathfrak{q}\to\mathfrak{H}$, $w\mapsto w+\sigma w$, induces a complex linear isomorphism $\mathfrak{q}/\mathfrak{q}^{(\infty)}\cong\mathfrak{H}/\mathfrak{g}_o$. The former quotient is canonically isomorphic to $H_o^{0,1}M$ (similarly, $H_o^{1,0}M\cong\sigma\mathfrak{q}/\mathfrak{q}^{(\infty)}$). The Levi kernel K_oM and its higher-order analogues K_o^rM can be considered as complex linear subspaces of $\mathfrak{q}/\mathfrak{q}^{(\infty)}$. We will make extensive use of some of the main results of [15], such as Theorem 5.10.

9.5. Algebraic characterization of *k*-nondegeneracy

For every $r\geqslant 0$ the space $\mathfrak{q}^{(r)}$ is a Lie subalgebra of $\mathfrak{q}$ and the rth Levi kernel K_o^rM is isomorphic to $\mathfrak{q}^{(r)}/\mathfrak{q}^{(\infty)}$. In particular, for every $k\geqslant 1$ the locally homogeneous CR-manifold M is k-nondegenerate if and only if

$$\mathfrak{q}^{(k-1)}\neq\mathfrak{q}^{(k)}=\mathfrak{q}^{(\infty)}.$$

To handle the 5-dimensional case we also introduce the following abbreviations:

(iv) $\mathfrak{f}:=\mathfrak{q}^{(1)}=\{v\in\mathfrak{q}:[v,\sigma\mathfrak{q}]\subset\mathfrak{q}+\sigma\mathfrak{q}\}$ and $\mathfrak{F}:=(\mathfrak{f}+\sigma\mathfrak{f})^{\sigma}$.

Then $\mathfrak{g}_o\subset\mathfrak{F}\subset\mathfrak{H}\subset\mathfrak{g}$ are real subspaces stable under $\operatorname{ad}(\mathfrak{g}_o)$ and $\mathfrak{l}_o\subset\mathfrak{f}\subset\mathfrak{q}$ are complex subalgebras. In [15, Lemma 5.9] it has been shown that actually $\mathfrak{F}$ is a Lie algebra and it coincides with $N_{\mathfrak{g}}(\mathfrak{H})\cap\mathfrak{H}$ (here, given $W\subset\mathfrak{g}$, $N_{\mathfrak{g}}(W):=\{v\in\mathfrak{g}:[v,W]\subset W\}$).

Summarizing the above discussion, the following result is the key for our classification.

Proposition 9.6. *Let (M,o) be a $\mathfrak{g}$-homogeneous CR-germ and let $(\mathfrak{g},\mathfrak{q})$ be the corresponding CR-algebra. Then M is 5-dimensional, minimal and 2-nondegenerate if and only if*

$$\operatorname{codim}_{\mathfrak{g}}(\mathfrak{g}_0)=5 \quad \textit{and} \quad \mathfrak{q}\neq\mathfrak{f}\neq\mathfrak{q}^{(2)}=\mathfrak{q}^{(\infty)}:=\mathfrak{q}\cap\sigma\mathfrak{q}.$$

Lemma 9.7. *The Lie algebraic terms in Proposition* 9.6 *are equivalent to the following set of conditions*:

(I) $\dim \mathfrak{F}/\mathfrak{g}_o = \dim \mathfrak{H}/\mathfrak{F} = 2$ *and* $\dim_{\mathbb{R}} \mathfrak{g}/\mathfrak{H} = 1 = \dim_{\mathbb{C}} \mathfrak{f}/\mathfrak{l}_o = \dim_{\mathbb{C}} \mathfrak{q}/\mathfrak{f}$;

(II) $[\mathfrak{g}_o, \mathfrak{F}] \subset \mathfrak{F}$, $[\mathfrak{F}, \mathfrak{F}] \subset \mathfrak{F}$ *and* $[\mathfrak{F}, \mathfrak{H}] \subset \mathfrak{H}$;

(III) $[\mathfrak{q}, \sigma\mathfrak{q}] \not\subset \mathfrak{q} + \sigma\mathfrak{q}$, *if M is not Levi flat*;

(IV) $[\mathfrak{f}, \sigma\mathfrak{q}] \subset \mathfrak{q} + \sigma\mathfrak{q}$ *and* $\mathfrak{f} \neq \mathfrak{l}_o$, *if M is Levi degenerate*;

(V) $[\mathfrak{f}, \sigma\mathfrak{q}] \not\subset \mathfrak{f} + \sigma\mathfrak{q}$ *if M is* 2*-nondegenerate.*

We will frequently use the fact that the condition '$[\mathfrak{F}, \mathfrak{H}] \subset \mathfrak{F}$' (instead of $\subset \mathfrak{H}$) violates condition (V).

By the canonical bijection between the classes $\mathcal{G}$ and $\mathcal{A}$ mentioned in Remark 9.2 (iv) and made precise above, our classification problem is transferred to the classification of certain effective CR-algebras up to geometric equivalence. Unfortunately, with a given locally homogeneous CR-germ (M, o) there may be associated many CR-algebras $(\mathfrak{g}, \mathfrak{q})$ for which the $\mathfrak{g}$'s are nonisomorphic. For instance, if $M = \mathcal{M}$ is the tube over the future light cone, then with $(\mathcal{M}, o)$ there are associated CR-algebras $(\mathfrak{g}, \mathfrak{q})$ with $\mathfrak{g} \cong \mathfrak{so}(2,3)$, $\mathfrak{g} \cong \mathfrak{so}(1,3)$ and $\mathfrak{g} \cong \mathfrak{so}(2,2)$ together with a bunch of other Lie algebras that are not semisimple; see [16] for explicit realizations. Therefore, the best we can do in the following is to consider only CR-algebras $(\mathfrak{g}, \mathfrak{q})$ such that $\dim \mathfrak{g}$ is minimal in the geometric equivalence class of $(\mathfrak{g}, \mathfrak{q})$. But, also then, we are still left in the example $(\mathcal{M}, o)$ with several nonisomorphic solvable Lie algebras of dimension 5 as well as one nonsolvable Lie algebra of dimension 5 with 2-dimensional nonabelian radical and Levi part $\cong \mathfrak{sl}(2, \mathbb{R})$.

9.8. Fundamental assumption

In the following, every CR-algebra $(\mathfrak{g}, \mathfrak{q})$ under consideration is assumed to satisfy the condition in Proposition 9.6 (equivalently, (I)–(V)) as well as the following additional condition:

(VI) for every CR-algebra $(\mathfrak{g}', \mathfrak{q}')$, which is geometrically CR-equivalent to $(\mathfrak{g}, \mathfrak{q})$, the dimension estimate $\dim \mathfrak{g}' \geqslant \dim \mathfrak{g}$ holds.

Condition (VI) implies, in particular, the following two conditions:

$(\mathsf{VI})_1$ $(\mathfrak{g}, \mathfrak{q})$ is effective;

$(\mathsf{VI})_2$ there is no proper subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ with $\mathfrak{g}' + \mathfrak{g}_o = \mathfrak{g}$ for $\mathfrak{g}_o = \mathfrak{g} \cap \mathfrak{q}$.

Indeed, $(\mathfrak{g}', \mathfrak{q}')$ with $\mathfrak{q}' := (\mathfrak{g}' + i\mathfrak{g}') \cap \mathfrak{q}$ is a CR-algebra that is geometrically equivalent to $(\mathfrak{g}, \mathfrak{q})$ in case $\mathfrak{g}' + \mathfrak{g}_o = \mathfrak{g}$.

9.9. Basic structure theory

In the following we frequently use standard facts concerning reductive Lie algebras and parabolic subalgebras; see [24, Chapters VI and VII] as a general reference. To fix our notation, let $\mathfrak{s}$ be a complex reductive Lie algebra and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{s}$. Denote by $\Phi=\Phi(\mathfrak{s},\mathfrak{t})\subset\mathfrak{t}^*$ the corresponding root system and by $\Pi\subset\Phi$ the subset of simple roots. A subalgebra $\mathfrak{r}\subset\mathfrak{s}$ is called *parabolic* if it contains a maximal solvable subalgebra (also called a *Borel subalgebra*) $\mathfrak{b}$ of $\mathfrak{s}$. Conjugacy classes of parabolic subalgebras in $\mathfrak{s}$ are parameterized by the subsets of Π. For every $\mathscr{P}\subset\Pi$, set $\langle\!\langle\mathscr{P}\rangle\!\rangle:=\Phi\cap\bigoplus_{\alpha\in\mathscr{P}}\mathbb{Z}\alpha$. Then, the corresponding parabolic subalgebra is defined by

$$\mathfrak{r}=\mathfrak{r}_{\mathscr{P}}:=\mathfrak{r}^{\mathrm{red}}\oplus\mathfrak{r}^{\mathrm{nil}}\quad\text{with}\quad\mathfrak{r}^{\mathrm{red}}:=\mathfrak{t}\oplus\bigoplus_{\alpha\in\langle\!\langle\mathscr{P}\rangle\!\rangle}\mathfrak{s}_\alpha\quad\text{and}\quad\mathfrak{r}^{\mathrm{nil}}:=\bigoplus_{\alpha\notin\langle\!\langle\mathscr{P}\rangle\!\rangle}\mathfrak{s}_\alpha. \tag{9.10}$$

The case of a reductive *real* Lie algebra $\mathfrak{s}$ is a little bit more sophisticated. In contrast to the complex situation there may exist several conjugacy classes of Cartan subalgebras. Among these, the class most suitable for our purposes consists of the so-called *maximally split Cartan subalgebras* $\mathfrak{t}\subset\mathfrak{s}$, defined as follows. Select a Cartan decomposition $\mathfrak{s}=\mathfrak{k}\oplus\mathfrak{p}$ and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Consider the centralizer $\mathfrak{m}:=C_{\mathfrak{k}}(\mathfrak{a})$ and put $\mathfrak{t}:=\mathfrak{a}\oplus\mathfrak{t}_{\mathfrak{m}}$, where $\mathfrak{t}_{\mathfrak{m}}\subset\mathfrak{m}$ is a maximal abelian subalgebra. The conjugacy classes of the real parabolic subalgebras in $\mathfrak{s}$ are parameterized by the subsets of the simple roots $\Pi\subset\Phi(\mathfrak{s},\mathfrak{a})$ in the restricted root system $\Phi(\mathfrak{s},\mathfrak{a})\subset\mathfrak{a}^*$. Each parabolic subalgebra $\mathfrak{r}$ in $\mathfrak{s}$ has the decomposition $\mathfrak{r}=\mathfrak{r}^{\mathrm{red}}\ltimes\mathfrak{r}^{\mathrm{nil}}$ into the reductive and nilpotent parts (once a maximally split Cartan subalgebra $\mathfrak{t}\subset\mathfrak{r}$ is selected, the reductive factor with $\mathfrak{r}^{\mathrm{red}}\supset\mathfrak{t}$ is unique). Following a common convention, the roots λ occurring in the root space decomposition of the nilpotent ideal $\mathfrak{r}^{\mathrm{nil}}=\bigoplus_{\lambda\in\Lambda}\mathfrak{s}_\lambda$ are negative, and we write $\mathfrak{r}^{-\mathrm{n}}:=\mathfrak{r}^{\mathrm{nil}}$, $\mathfrak{r}^{\mathrm{r}}:=\mathfrak{r}^{\mathrm{red}}$ and $\Phi^{-\mathrm{n}}:=\Phi(\mathfrak{r}^{\mathrm{nil}},\mathfrak{a})=\Lambda$. Each parabolic subalgebra $\mathfrak{r}$ containing $\mathfrak{t}$ determines the decomposition

$$\mathfrak{s}=\mathfrak{r}^{\mathrm{n}}\oplus\mathfrak{r}^{\mathrm{r}}\oplus\mathfrak{r}^{-\mathrm{n}}\quad\text{with}\ \mathfrak{r}^{\mathrm{n}}:=\bigoplus_{\lambda\in\Phi(\mathfrak{r}^{\mathrm{nil}},\mathfrak{a})}\mathfrak{s}_{-\lambda},\quad\text{and we put}\ \mathfrak{r}^{\mathrm{opp}}:=\mathfrak{r}^{\mathrm{r}}\ltimes\mathfrak{r}^{\mathrm{n}}. \tag{9.11}$$

We call $\mathrm{rk}(\mathfrak{s}):=\dim\mathfrak{t}$ the *rank* of $\mathfrak{s}$ and $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}):=\dim\mathfrak{a}$, with $\mathfrak{a}\subset\mathfrak{p}$ as above, the *real rank* of $\mathfrak{s}$.

10. The Lie algebra $\mathfrak{g}$ has small semisimple part

In the preceding section we have explained how 2-nondegeneracy can be expressed in pure Lie algebraic terms. In this section we start with the actual proof of Theorem II. Since this proof will be quite involved, we subdivide it into several sections, lemmata and claims until the final step is completed in §16. For the convenience of the reader, we briefly outline the main steps.

As explained above, the classification can be reduced to the determination of all CR-algebras satisfying the fundamental assumption in §9.8. Once all possible CR-algebras are known, we have to identify the underlying CR-germs. In general, it may happen that algebraically inequivalent CR-algebras give rise to equivalent CR-germs. For this last part of the proof we use results from §8.

Our proceeding will be to show that the assumption in §9.8 severely restricts the possibilities for $(\mathfrak{g}, \mathfrak{q})$. This will be achieved by a detailed structural study of the Lie algebras $\mathfrak{g}$ occurring in $(\mathfrak{g}, \mathfrak{q})$. Recall that every Lie algebra $\mathfrak{h}$ has a Levi–Mal'tsev decomposition $\mathfrak{h}=\mathfrak{h}^{\mathrm{ss}} \ltimes \operatorname{rad}(\mathfrak{h})$, where $\mathfrak{h}^{\mathrm{ss}}$ is semisimple and is uniquely determined up to an inner automorphism of $\mathfrak{h}$. Furthermore, $\operatorname{rad}(\mathfrak{h})$ is the radical of $\mathfrak{h}$, i.e., the unique maximal solvable ideal in $\mathfrak{h}$. In the first part of the proof, we investigate the various possibilities for $\mathfrak{g}^{\mathrm{ss}}$, where $(\mathfrak{g}, \mathfrak{q})$ satisfies certain conditions stated in the previous section. To be precise: *For the rest of the paper the fundamental assumption in* §9.8 *remains in force for* all *CR-algebras* $(\mathfrak{g}, \mathfrak{q})$ *under consideration.* In particular, every $(\mathfrak{g}, \mathfrak{q})$ is effective (condition $(\mathsf{VI})_1$, and therefore $\mathfrak{g}$ can be considered as a transitive Lie subalgebra of $\mathfrak{hol}(M, o)$). Furthermore, condition $(\mathsf{VI})_2$ states that there is no proper Lie subalgebra of $\mathfrak{g}$ that also is transitive on (M, o).

Let an arbitrary CR-algebra $(\mathfrak{g}, \mathfrak{q})$ subject to the assumption in §9.8 be given, and let $\mathfrak{g}^{\mathrm{ss}} \ltimes \operatorname{rad}(\mathfrak{g})$ be a Levi–Mal'tsev decomposition of $\mathfrak{g}$. In this and in the following few sections we assume that $\mathfrak{g}^{\mathrm{ss}} \neq 0$ and investigate which simple factors can occur in $\mathfrak{g}^{\mathrm{ss}}$. Thereby we use the following notation: We fix a simple ideal $\mathfrak{s}$ in $\mathfrak{g}^{\mathrm{ss}}$ and denote the corresponding complementary ideal by $\mathfrak{s}'$, that is,

$$\mathfrak{g} = \mathfrak{g}^{\mathrm{ss}} \ltimes \operatorname{rad} \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^{\mathrm{ss}} = \mathfrak{s} \times \mathfrak{s}'. \tag{10.1}$$

In this section we show that $\mathfrak{g}^{\mathrm{ss}}$ only can contain simple factors isomorphic to one of the Lie algebras $\mathfrak{so}(2,3)$, $\mathfrak{so}(1,3)$, $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$. This result is obtained by analyzing which simple real Lie algebras can contain proper subalgebras of very low codimensions. In the next sections we exclude further possibilities for $\mathfrak{s}$. In §11 we show that the factors $\mathfrak{so}(2,3)$ and $\mathfrak{so}(1,3)$ cannot occur in $\mathfrak{g}^{\mathrm{ss}}$ as it will turn out that their existence in the Levi factor would violate the minimality assumption (VI). Nevertheless notice that there exist CR-algebras $(\mathfrak{so}(2,3), \mathfrak{q})$ and $(\mathfrak{so}(1,3), \mathfrak{q})$ satisfying (I)–(V) and $(\mathsf{VI})_1$. All the underlying CR-germs of such CR-algebras are locally CR-equivalent to the light cone tube $\mathcal{M}$. In §12 we find the first examples of CR-algebras which satisfy the assumption in §9.8. In these cases $\mathfrak{g}^{\mathrm{ss}}$ contains a simple factor $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{R})$, and then necessarily $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}$, where $\mathfrak{r}$ is a 2-dimensional nonabelian Lie algebra. Also, all such CR-algebras give rise only to CR-germs locally CR-equivalent to the tube over the light cone. In §13 we finally eliminate the possibility $\mathfrak{s}\cong\mathfrak{su}(2)$ for the simple factor $\mathfrak{s}$ in $\mathfrak{g}^{\mathrm{ss}}$. At that stage of the proof, we will have proved the following dichotomy: Let $(\mathfrak{g},\mathfrak{q})$ be an arbitrary CR-algebra which obeys the assumption in §9.8. If $\mathfrak{g}^{\mathrm{ss}}\neq 0$ then $\mathfrak{g}^{\mathrm{ss}}\cong\mathfrak{sl}(2,\mathbb{R})$ and furthermore the underlying CR-germ is locally CR-equivalent to the light cone tube $\mathcal{M}$. Or $\mathfrak{g}^{\mathrm{ss}}=0$, i.e., $\mathfrak{g}$ is solvable.

For this reason, from §14 on, we only consider CR-algebras $(\mathfrak{g},\mathfrak{q})$ with $\mathfrak{g}$ solvable, and show that then necessarily $\dim\mathfrak{g}=5=\dim M$. In §15 we look closer at the nilcenter $\mathfrak{z}$ of $\mathfrak{g}$ and find out that $\dim\mathfrak{z}\in\{1,3\}$. The Main lemma 15.14, which might be of interest in itself, gives a sufficient condition for a CR-algebra to be associated with a tube $F+i\mathbb{R}^3\subset\mathbb{C}^3$ over an affinely homogeneous surface $F\subset\mathbb{R}^3$. In the last section, we show by ad-hoc methods that indeed every 5-dimensional solvable Lie algebra $\mathfrak{g}$ occurring in the CR-algebra $(\mathfrak{g},\mathfrak{q})$ under consideration fulfills the assumption of the main lemma. Hence, due to the aforementioned assertions and since (up to local affine equivalence) all affinely homogeneous surfaces in $\mathbb{R}^3$ occur among Examples 8.1–8.5, this completes the proof of the classification theorem.

We now begin with the proof of Theorem II.

LEMMA 10.2. *Let $(\mathfrak{g},\mathfrak{q})$ be a CR-algebra subject to the assumption in §9.8. Then the simple Lie subalgebra $\mathfrak{s}\subset\mathfrak{g}^{\mathrm{ss}}$ can only be isomorphic to $\mathfrak{so}(2,3)$, $\mathfrak{so}(1,3)$, $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$.*

Proof. The proof is carried out in several reduction steps. To begin with, we write as shorthand

$$\mathfrak{h}_o:=\mathfrak{g}_o\cap\mathfrak{h},\quad \mathfrak{h}_{\mathfrak{F}}:=\mathfrak{F}\cap\mathfrak{h}\quad\text{and}\quad \mathfrak{h}_{\mathfrak{H}}:=\mathfrak{H}\cap\mathfrak{h}$$

for every subalgebra $\mathfrak{h}\subset\mathfrak{g}$. Notice that $\mathfrak{h}_o\subset\mathfrak{h}_{\mathfrak{F}}$ are subalgebras and $\mathfrak{h}_{\mathfrak{H}}$ is a linear $\mathrm{ad}(\mathfrak{h}_{\mathfrak{F}})$-stable subspace of $\mathfrak{h}$.

Consider the subalgebras $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}$ of $\mathfrak{s}$. The case $\mathfrak{s}_o=\mathfrak{s}_{\mathfrak{F}}=\mathfrak{s}$, that is $\mathfrak{s}\subset\mathfrak{g}_o$, can be ruled out since then $\mathfrak{s}'\oplus\mathrm{rad}(\mathfrak{g})$ would be a proper locally transitive subalgebra of $\mathfrak{g}$, contradicting assumption $(\mathsf{VI})_2$. Therefore, at least one of the inclusions $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}\subset\mathfrak{s}$ is proper. Consequently, there is always a proper subalgebra of codimension $\leqslant 3$ in $\mathfrak{s}$. Indeed, in case $\mathfrak{s}_{\mathfrak{F}}\neq\mathfrak{s}$ the subalgebra $\mathfrak{s}_{\mathfrak{F}}$ has this property, and in case $\mathfrak{s}_{\mathfrak{F}}=\mathfrak{s}$ the proper subalgebra $\mathfrak{s}_o$ has codimension $\leqslant 2$. Hence, there exists a maximal proper subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ with either $\mathfrak{s}_{\mathfrak{F}}\subset\mathfrak{h}$ or $\mathfrak{s}_o\subset\mathfrak{h}$. Such a maximal subalgebra $\mathfrak{h}$ has codimension $\leqslant 3$ in $\mathfrak{s}$. Due to [7, §VIII.10, Corollary 1], every maximal proper subalgebra of $\mathfrak{s}$ is either reductive or parabolic. In the following claims we list all simple Lie algebras $\mathfrak{s}$ which admit proper maximal subalgebras of such low codimensions. We discuss the reductive and parabolic cases separately.

CLAIM 1. *Let $\mathfrak{k}$ be a simple real algebra and $\mathfrak{h}\subset\mathfrak{k}$ reductive with $0<\mathrm{codim}_{\mathfrak{s}}\,\mathfrak{h}\leqslant 3$. Then $\mathfrak{k}$ can only be isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$ or $\mathfrak{so}(1,3)$.*

Proof. Let $\mathfrak{h}=\mathfrak{h}_1\times...\times\mathfrak{h}_p\times\mathfrak{z}$ be the decomposition of the reductive subalgebra $\mathfrak{h}$ into the simple factors $\mathfrak{h}_j$ and the center $\mathfrak{z}$. Weyl's theorem implies the existence of an $\mathrm{ad}(\mathfrak{h})$-stable complement $\mathfrak{v}\subset\mathfrak{s}$. Let $\varrho:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{v})$ be the induced adjoint representation. Every restriction $\varrho_j:\mathfrak{h}_j\to\mathfrak{gl}(\mathfrak{v})$ must be faithful since otherwise $\mathfrak{h}_j$ would be an ideal in $\mathfrak{s}$. The crucial condition here is $\dim\mathfrak{v}\leqslant 3$ which, in turn, implies that each $\mathfrak{h}_j$ is isomorphic either to $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(3)$ or $\mathfrak{sl}(3,\mathbb{R})$. As a consequence,

$$\dim\mathfrak{h}\leqslant\begin{cases}8k, & \text{if } r=2k,\\ 8k+3, & \text{if } r=2k+1,\end{cases}\qquad \text{for } r:=\mathrm{rk}(\mathfrak{h})\leqslant\mathrm{rk}(\mathfrak{s}).$$

On the other hand, a glance at the classification of simple Lie algebras shows that

$$\dim\mathfrak{s}\geqslant\begin{cases}2k^2+4k=\dim_{\mathbb{R}}\mathfrak{sl}(k+1,\mathbb{C}), & \text{if } r=2k,\\ 4k^2+8k+3=\dim\mathfrak{sl}(2k+2,\mathbb{R}), & \text{if } r=2k+1.\end{cases}$$

Putting both inequalities together and bearing in mind that $\dim\mathfrak{s}-\dim\mathfrak{h}\leqslant 3$ shows that the rank of $\mathfrak{s}$ can only be one of the numbers 1, 2 and 4. Since $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ are the only simple real Lie algebras of rank 1, we may assume that $\mathfrak{s}$ has rank 2 or 4. The case $\mathrm{rk}(\mathfrak{s})=4$ can be ruled out in the following way. Consider first the situation where $\mathfrak{s}$ is *of complex type,* that is, $\mathfrak{s}$ is the underlying real Lie algebra of a complex simple Lie algebra $\mathfrak{c}$ of (complex) rank 2. Then $\mathfrak{c}$ is either $\mathfrak{sl}(3,\mathbb{C})$ or $\mathfrak{so}(5,\mathbb{C})$. But in both cases a proper reductive real subalgebra has at least (real) codimension 4 (in fact, 8). If $\mathfrak{s}$ is *of real type* then the above estimates give $\dim\mathfrak{h}\leqslant 16$ and $\dim\mathfrak{s}\geqslant 24=\dim\mathfrak{sl}(5,\mathbb{R})$. For the remaining case $\mathrm{rk}(\mathfrak{s})=2$, either $\mathfrak{s}\cong\mathfrak{so}(1,3)$, which is in the list of the claim, or $\mathfrak{s}$ is isomorphic to a real form of $\mathfrak{sl}(3,\mathbb{C})$ or $\mathfrak{so}(5,\mathbb{C})$. In both cases every proper reductive complex subalgebra has at least codimension 4. This proves Claim 1. □

CLAIM 2. *Let $\mathfrak{k}$ be a simple real Lie algebra and $\mathfrak{h}\subset\mathfrak{k}$ be parabolic with $0<\mathrm{codim}_{\mathfrak{s}}\mathfrak{h}\leqslant 3$. Then $\mathfrak{s}$ is isomorphic to $\mathfrak{so}(1,3)$, $\mathfrak{sl}(4,\mathbb{R})$, $\mathfrak{su}(2)$ or to a noncompact real form of $\mathfrak{sl}(3,\mathbb{C})$ or $\mathfrak{so}(5,\mathbb{C})$.*

Proof. Since every parabolic subalgebra of a compact Lie algebra is trivial, apart from $\mathfrak{s}\cong\mathfrak{su}(2)$ we only have to consider the case where $\mathfrak{s}$ is noncompact. The estimate $\mathrm{rk}(\mathfrak{s})\leqslant\dim\mathfrak{s}-\dim\mathfrak{h}$ (compare §9.9) implies that $\mathrm{rk}(\mathfrak{s})\leqslant 3$. We work out the various cases separately.

$\mathrm{rk}(\mathfrak{s})=3$. The complexification $\mathfrak{s}^{\mathbb{C}}$ of $\mathfrak{s}$ is one of the Lie algebras $\mathfrak{sl}(4,\mathbb{C})$, $\mathfrak{so}(7,\mathbb{C})$ or $\mathfrak{sp}(3,\mathbb{C})$. The latter two can be immediately ruled out since a glance at the corresponding Satake diagrams shows that every proper parabolic subalgebra of them is at least of codimension 4. In the remaining case $\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(4,\mathbb{C})$, only the normal real form $\mathfrak{sl}(4,\mathbb{R})$ has a parabolic subalgebra of codimension 3. (A glance at the Satake diagrams for the remaining noncompact real forms of $\mathfrak{sl}(4,\mathbb{C})$ excludes further possibilities).

$\mathrm{rk}(\mathfrak{s})=1,2$. The rank conditions imply that $\mathfrak{s}$ is the underlying real Lie algebra of $\mathfrak{sl}(2,\mathbb{C})$ or a real form of $\mathfrak{sl}(3,\mathbb{C})$ or $\mathfrak{so}(5,\mathbb{C})$. The Lie algebra $\mathfrak{so}(1,3)$, as well as every noncompact real form of $\mathfrak{sl}(3,\mathbb{C})$ or $\mathfrak{so}(5,\mathbb{C})$, contains a parabolic subalgebra of codimension less than or equal to 3. Since $\dim\mathfrak{sl}(2,\mathbb{R})=\dim\mathfrak{su}(2)=3$, all simple Lie algebras of rank 1 also contain parabolic subalgebras of the required codimension.

In order to further reduce the list of possibilities for $\mathfrak{s}$, we have to look more closely at the particular real forms obtained in Claims 1 and 2, but not in the list of Lemma 10.2.

Elimination of $\mathfrak{s}\cong\mathfrak{sl}(4,\mathbb{R})$. Up to an automorphism of $\mathfrak{sl}(4,\mathbb{R})$, there is only one such parabolic subalgebra $\mathfrak{h}$ of codimension 3. Let $\mathfrak{t}=\mathfrak{a}\subset\mathfrak{h}$ be the split Cartan subalgebra and $\mathfrak{h}=\mathfrak{h}^{\mathrm{r}}\ltimes\mathfrak{h}^{-\mathrm{n}}$ be the decomposition as in (9.11). Note that here the reductive factor $\mathfrak{h}^{\mathrm{r}}\cong\mathfrak{gl}(3,\mathbb{R})$ acts irreducibly on $\mathfrak{h}^{\pm\mathrm{n}}\cong\mathbb{R}^3$. The only possibility for the flag $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}\subset\mathfrak{s}_{\mathfrak{H}}\subset\mathfrak{s}$, which cannot be trivially excluded, is when $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}=\mathfrak{h}\subsetneq\mathfrak{s}_{\mathfrak{H}}\subsetneq\mathfrak{s}$ and $\mathrm{codim}_{\mathfrak{h}}(\mathfrak{s}_o)\leqslant 2$. This cannot be true since $\dim\mathfrak{s}_{\mathfrak{H}}/\mathfrak{s}_{\mathfrak{F}}=2$ and $[\mathfrak{s}_{\mathfrak{F}}:\mathfrak{s}_{\mathfrak{H}}]\subset\mathfrak{s}_{\mathfrak{H}}$ (condition (II)), but $\mathfrak{h}=\mathfrak{s}_{\mathfrak{F}}$ acts irreducibly on the 3-dimensional space $\mathfrak{s}/\mathfrak{s}_{\mathfrak{F}}\cong\mathfrak{h}^{\mathrm{n}}$.

Elimination of $\mathfrak{s}\cong\mathfrak{su}(1,2)$. In this case of real rank 1, there exists, up to conjugacy, only one parabolic subalgebra $\mathfrak{h}$. This is the minimal one $\mathfrak{h}=\mathfrak{h}_{\varnothing}=\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$, which is solvable with $\mathrm{codim}_{\mathfrak{s}}\mathfrak{h}=3$. The reductive part $\mathfrak{h}^{\mathrm{r}}=\mathfrak{m}\oplus\mathfrak{a}=:\mathfrak{t}$ of $\mathfrak{h}$ is a maximally split Cartan subalgebra. As before, the only situation which cannot be trivially disposed of is when $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}=\mathfrak{h}$ and $c:=\mathrm{codim}_{\mathfrak{h}}(\mathfrak{s}_o)\in\{0,1,2\}$. Recall that $\mathfrak{h}$ yields the decomposition $\mathfrak{s}=\mathfrak{h}^{\mathrm{n}}\oplus\mathfrak{t}\oplus\mathfrak{h}^{-\mathrm{n}}$. If $c=0$, that is $\mathfrak{s}_o=\mathfrak{h}$, then $\mathfrak{h}^{\mathrm{n}}\oplus\mathrm{rad}(\mathfrak{g})$ would be a proper locally transitive subalgebra of $\mathfrak{g}$ in contradiction with assumption $(\mathsf{VI})_2$. If $c=1$ then either $\mathfrak{t}\cap\mathfrak{s}_o\neq\mathfrak{t}$, and then $\mathfrak{h}^{\mathrm{opp}}\oplus\mathrm{rad}(\mathfrak{g})$ would be a proper locally transitive subalgebra of $\mathfrak{g}$ (again contradicting $(\mathsf{VI})_2$), or $\mathfrak{t}\subset\mathfrak{s}_o$. But taking into account the particular structure of $\mathfrak{h}^{-\mathrm{n}}=\mathfrak{s}_{-\lambda}\oplus\mathfrak{s}_{-2\lambda}$, this also does not occur, since otherwise $\mathfrak{s}_o\cap\mathfrak{h}^{-\mathrm{n}}$ would be a $\mathfrak{t}$-stable 2-dimensional subalgebra, which is impossible since $\mathrm{ad}(\mathfrak{t})$ acts irreducibly on the 2-dimensional root spaces $\mathfrak{s}_{\pm\lambda}$ and $[\mathfrak{s}_{\lambda},\mathfrak{s}_{\lambda}]=\mathfrak{s}_{2\lambda}$.

The case $c=2$ remains, that is, $\mathfrak{s}=\mathfrak{g}\cong\mathfrak{su}(1,2)$ is locally transitive. Then

$$\mathfrak{l}=\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(3,\mathbb{C})$$

and $\mathfrak{q}$ is a subalgebra of complex dimension 5. Consequently $\mathfrak{q}$ is contained in a maximal (6-dimensional) parabolic subalgebra $\mathfrak{h}\cong\mathfrak{gl}(2,\mathbb{C})\ltimes\mathbb{C}^2$ of $\mathfrak{l}$, that is, either $\mathfrak{q}$ coincides with a Borel subalgebra $\mathfrak{b}$, or is conjugate to a subalgebra $\mathfrak{j}\cong\mathfrak{sl}(2,\mathbb{C})\ltimes\mathbb{C}^2$. In both cases the subgroups Q of $L=\mathrm{SL}(3,\mathbb{C})$ corresponding to $\mathfrak{b}$ or $\mathfrak{j}$ are closed and the underlying CR-germ (M,o) is locally CR-equivalent to an $\mathrm{SU}(1,2)$-orbit either in $L/B\cong\mathbb{F}(\mathbb{C}^3)$, the complex manifold of full flags in $\mathbb{C}^3$, or in the $\mathbb{C}^*$-principal bundle L/J over $\mathbb{P}_2(\mathbb{C})$. A direct check shows that in neither case there exist 2-nondegenerate $\mathrm{SU}(1,2)$-orbits.

Elimination of $\mathfrak{s}\cong\mathfrak{sl}(3,\mathbb{R})$. Then $3\leqslant\dim\mathfrak{s}_o\leqslant 6$ holds except for the trivial case $\mathfrak{s}_o=\mathfrak{s}$. If $\dim\mathfrak{s}_o=3$ and hence $\mathfrak{s}$ is locally transitive, then as in the previous situation each CR-germ (M,o) associated with a CR-algebra $(\mathfrak{sl}(3,\mathbb{R}),\mathfrak{q})$ is locally CR-equivalent to an $\mathrm{SL}(3,\mathbb{R})$-orbit either in $\mathrm{SL}(3,\mathbb{C})/B$ or in $\mathrm{SL}(3,\mathbb{C})/J$ (as discussed above, with $\mathrm{SL}(3,\mathbb{R})$ in place of $\mathrm{SU}(1,2)$). Again, none of these orbits is 2-nondegenerate. The case $\dim\mathfrak{s}_o\geqslant 4$ remains. But then there always exists a parabolic (proper) subalgebra $\mathfrak{h}\subset\mathfrak{s}$ with $\mathfrak{s}_o+\mathfrak{h}=\mathfrak{s}$, that is, $\mathfrak{h}\oplus\mathrm{rad}(\mathfrak{g})$ is a proper locally transitive subalgebra of $\mathfrak{g}$ excluding the case $\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(3,\mathbb{C})$.

Elimination of $\mathfrak{s}\cong\mathfrak{so}(1,4)$. There exists up to conjugacy a unique parabolic subalgebra $\mathfrak{h}=\mathfrak{h}^{\mathrm{r}}\ltimes\mathfrak{h}^{-\mathrm{n}}\subset\mathfrak{s}$ of codimension 3, and we have to investigate only the cases $\mathfrak{s}\supsetneq\mathfrak{s}_{\mathfrak{H}}\supsetneq\mathfrak{s}_{\mathfrak{F}}=\mathfrak{h}\supset\mathfrak{s}_o$. A close look at the minimal (and maximal proper) parabolic subalgebra $\mathfrak{h}=\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{s}_\lambda$ shows that $\mathfrak{m}\cong\mathfrak{so}(3)$, and $\mathfrak{m}$ acts irreducibly on the 3-dimensional nilpotent ideal $\mathfrak{h}^{-\mathrm{n}}=\mathfrak{s}_{-\lambda}$. Consequently, $\mathfrak{h}$ acts irreducibly on $\mathfrak{s}/\mathfrak{h}=\mathfrak{s}/\mathfrak{s}_{\mathfrak{F}}\cong\mathbb{R}^3$. This leads to a contradiction as $[\mathfrak{s}_{\mathfrak{F}},\mathfrak{s}_{\mathfrak{H}}]\subset\mathfrak{s}_{\mathfrak{H}}$, i.e., $\mathfrak{s}_{\mathfrak{H}}/\mathfrak{s}_{\mathfrak{F}}$ would be a 2-dimensional stable subspace.

This completes the proof of Claim 2. $\square$

The proof of Lemma 10.2 is now complete. $\square$

In [5, Theorem 6] it is claimed that for every 2-nondegenerate real-analytic hypersurface $M\subset\mathbb{C}^3$ the Lie algebra $\mathfrak{hol}(M,a)$ has dimension $\leqslant 11$ at every point. Using this result would save a few arguments in the proof of Lemma 10.2. Instead, we preferred to present a self-contained proof of the proposition.

11. The cases $\mathfrak{s}\cong\mathfrak{so}(2,3)$ and $\mathfrak{s}\cong\mathfrak{so}(1,3)$

We continue the proof of Theorem II. So far we have proved that for the CR-algebra $(\mathfrak{g},\mathfrak{q})$ under consideration the simple factor $\mathfrak{s}$ of $\mathfrak{g}^{\mathrm{ss}}$, see (10.1), can only be isomorphic to one of the simple Lie algebras listed in Lemma 10.2. In this section we show that among these, the possibilities $\mathfrak{s}\cong\mathfrak{so}(2,3)$ and $\mathfrak{s}\cong\mathfrak{so}(1,3)$ cannot occur. Here and in the following, upper case roman numerals refer to the conditions in §9.5 and §9.8.

We would like to mention that for the tube $\mathcal{M}$ over the future light cone one has $\mathfrak{hol}(\mathcal{M},o)\cong\mathfrak{so}(2,3)$, and that there exists a copy of $\mathfrak{so}(1,3)$ in $\mathfrak{hol}(\mathcal{M},a)$ that is also locally transitive. Since these Lie algebras have dimensions 10 and 6 and since, on the other hand, there are transitive subalgebras of $\mathfrak{hol}(\mathcal{M},o)$ of dimension 5, these two Lie algebras do not satisfy the minimality condition (VI). In the following we consider both cases separately.

$\mathfrak{s}\cong\mathfrak{so}(2,3)$. The only Lie subalgebras $\mathfrak{h}$ in the normal real form $\mathfrak{so}(2,3)$ with $\mathrm{codim}_{\mathfrak{s}}\,\mathfrak{h}\leqslant 3$ are the 3-codimensional maximal parabolic subalgebras. We need to take a closer look at the structure of these subalgebras. There are, up to isomorphisms of $\mathfrak{s}$, only two such parabolic subalgebras: If $\Pi(\mathfrak{a})=\{\alpha,\beta\}$ is a basis of the root system $\Phi(\mathfrak{a})$ ($\mathfrak{a}$ a split Cartan subalgebra, α long and β short), then

$$\mathfrak{h}_1 := (\mathfrak{a}\oplus\mathfrak{s}_\alpha\oplus\mathfrak{s}_{-\alpha})\oplus\mathfrak{s}_{-\beta}\oplus\mathfrak{s}_{-\alpha-\beta}\oplus\mathfrak{s}_{-\alpha-2\beta}=\mathfrak{h}_1^{\mathrm{r}}\ltimes\mathfrak{h}_1^{-\mathrm{n}},$$
$$\mathfrak{h}_2 := (\mathfrak{a}\oplus\mathfrak{s}_\beta\oplus\mathfrak{s}_{-\beta})\oplus\mathfrak{s}_{-\alpha}\oplus\mathfrak{s}_{-\alpha-\beta}\oplus\mathfrak{s}_{-\alpha-2\beta}=\mathfrak{h}_2^{\mathrm{r}}\ltimes\mathfrak{h}_2^{-\mathrm{n}}$$

are representatives of the corresponding isomorphy classes. The only instance where the filtration $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}\subset\mathfrak{s}_{\mathfrak{H}}\subset\mathfrak{s}$ could be nontrivial (recall that $\mathfrak{s}_o$ and $\mathfrak{s}_{\mathfrak{F}}$ are subalgebras, $\mathfrak{s}_{\mathfrak{H}}$ is an $\mathrm{ad}(\mathfrak{s}_{\mathfrak{F}})$-stable subspace) arises when $\mathfrak{s}_o\subset\mathfrak{s}_{\mathfrak{F}}=\mathfrak{h}_j\subset\mathfrak{s}_{\mathfrak{H}}\subset\mathfrak{s}$ for $j=1,2$. The possibility $\mathfrak{h}_2=\mathfrak{s}_{\mathfrak{F}}$ cannot occur since the adjoint representation of $\mathfrak{h}_2^{\mathrm{r}}$ on $\mathfrak{h}_2^{\mathrm{n}}\cong\mathfrak{s}/\mathfrak{s}_{\mathfrak{F}}$ is irreducible, contradicting the existence of a 2-dimensional $\mathrm{ad}(\mathfrak{s}_{\mathfrak{F}})$-stable subspace $\mathfrak{s}_{\mathfrak{H}}/\mathfrak{s}_{\mathfrak{F}}$. The possibility that $\mathfrak{s}_{\mathfrak{F}}=\mathfrak{h}_1\supset\mathfrak{s}_o$ remains. From now on, $\mathfrak{h}:=\mathfrak{h}_1$ and we analyze the various possibilities for $\dim\mathfrak{s}_{\mathfrak{F}}-\dim\mathfrak{s}_o\in\{0,1,2\}$. The equation $\mathfrak{s}_{\mathfrak{F}}=\mathfrak{s}_o$ contradicts condition $(\mathsf{VI})_2$, since in that case the proper subalgebra $(\mathfrak{h}^{\mathrm{n}}\oplus\mathfrak{s}')\ltimes\mathrm{rad}(\mathfrak{g})$ would be transitive on (M,o). The case when $\mathfrak{s}_o$ is of codimension 1 in $\mathfrak{s}_{\mathfrak{F}}=\mathfrak{h}$ can also be ruled out: either $\mathfrak{h}^{-\mathrm{n}}\subset\mathfrak{s}_o$, and then $(\mathfrak{h}^{\mathrm{opp}}\oplus\mathfrak{s}')\ltimes\mathrm{rad}(\mathfrak{g})$ would be a transitive proper subalgebra of $\mathfrak{g}$, or the intersection of $\mathfrak{s}_o$ with $\mathfrak{h}^{-\mathrm{n}}=(\mathfrak{s}_{-\beta}\oplus\mathfrak{s}_{-\beta-\alpha})\oplus\mathfrak{s}_{-\alpha-2\beta}$ is a 2-dimensional subalgebra. In such a case the image $\pi(\mathfrak{s}_o)$ of the projection $\pi\colon\mathfrak{h}=\mathfrak{h}^{-\mathrm{n}}\rtimes\mathfrak{h}^{\mathrm{r}}\to\mathfrak{h}^{\mathrm{r}}$ coincides with $\mathfrak{h}^{\mathrm{r}}$. But this also leads to a contradiction: neither the intersection $\mathfrak{s}_o\cap\mathfrak{h}^{-\mathrm{n}}$ can coincide with $\mathfrak{s}_{-\beta}\oplus\mathfrak{s}_{-\beta-\alpha}$ (since it is not a subalgebra), nor $(\mathfrak{s}_{-\beta}\oplus\mathfrak{s}_{-\beta-\alpha})\cap\mathfrak{s}_o$ can be 1-dimensional (since $\mathfrak{h}^{\mathrm{r}}$ acts irreducibly on $(\mathfrak{s}_{-\beta}\oplus\mathfrak{s}_{-\beta-\alpha})$). Finally we are left with the case $\dim\mathfrak{s}_{\mathfrak{F}}-\dim\mathfrak{s}_o=2$, that is, $\mathfrak{s}$ is transitive on (M,o). Thus $\mathfrak{g}=\mathfrak{s}$ by assumption (VI). But then $\dim\mathfrak{g}_o=5$ and there always exist proper subalgebras $\mathfrak{g}'\subset\mathfrak{g}$ with $\mathfrak{g}'+\mathfrak{g}_0=\mathfrak{g}$, a contradiction to condition $(\mathsf{VI})_2$.

$\mathfrak{s}\cong\mathfrak{so}(1,3)\cong\mathfrak{sl}(2,\mathbb{C})$. We work out this case by investigating various possibilities for $\dim\mathfrak{s}_o$.

• $\dim\mathfrak{s}_o\geqslant3$. We claim that then there exists a solvable subalgebra $\mathfrak{r}\subset\mathfrak{s}$ such that $\mathfrak{s}_o+\mathfrak{r}=\mathfrak{s}$. The case $\dim\mathfrak{s}_o\geqslant4$ is easily settled as all 4-dimensional subalgebras in $\mathfrak{s}$ are maximal, i.e., they are Borel subalgebras of $\mathfrak{sl}(2,\mathbb{C})$ and consequently have nilpotent complementary subalgebras. There do not exist (real) subalgebras of $\mathfrak{sl}(2,\mathbb{C})$ of dimension 5. If $\dim\mathfrak{s}_o=3$ then $\mathfrak{s}$ is either semisimple or solvable. In the semisimple case we work with an explicit matrix realization $\mathfrak{s}\subset\mathbb{C}^{2\times2}$: either $\mathfrak{s}_o\cong\mathfrak{su}(2)$ or $\mathfrak{s}_o\cong\mathfrak{su}(1,1)$, i.e.,

$$\mathfrak{s}_o\cong\left\{\begin{pmatrix} it & \varepsilon\bar{z}\\ z & -it\end{pmatrix} : t\in\mathbb{R}\text{ and }z\in\mathbb{C}\right\}\quad\text{for }\varepsilon=1\text{ or }\varepsilon=-1.$$

In both cases the upper triangular Borel subalgebra $\mathfrak{b}^+\subset\mathfrak{sl}(2,\mathbb{C})\subset\mathbb{C}^{2\times2}$ forms a linear complement of $\mathfrak{s}_o$. This cannot happen, since then condition $(\mathsf{VI})_2$ would be violated.

If $\mathfrak{s}_o$ is solvable then $\mathfrak{s}_o$ is contained as a certain 1-codimensional (real) subalgebra in a (complex) Borel subalgebra $\mathfrak{b}=\mathfrak{t}\ltimes\mathfrak{b}^{\mathrm{nil}}$ of $\mathfrak{s}\cong\mathfrak{sl}(2,\mathbb{C})$. We claim that $\mathfrak{s}_o\supset\mathfrak{b}^{\mathrm{nil}}$: otherwise (that is, if $\dim_{\mathbb{R}}\mathfrak{s}_o\cap\mathfrak{b}^{\mathrm{nil}}=1$) we would have $\pi(\mathfrak{s}_o)=\mathfrak{t}$, where $\pi\colon\mathfrak{b}\to\mathfrak{t}$ is the projection homomorphism, and consequently $\mathfrak{s}_o$ would contain a certain complex Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{b}$ which acts $\mathbb{R}$-irreducibly on $\mathfrak{b}^{\mathrm{nil}}$.

However, from our claim it follows that the opposite Borel subalgebra is a complementary subspace of $\mathfrak{s}_o$ in $\mathfrak{s}$. The case '$\dim\mathfrak{s}_o=3$' is now completely ruled out.

• $\dim\mathfrak{s}_o=2$. It follows that $\mathfrak{s}_o$ is solvable and either complex or totally real. Since it is immediate that every complex subalgebra in $\mathfrak{sl}(2,\mathbb{C})$ has a complementary subalgebra (a simple check), it remains only to deal with the totally real case, i.e., with $\mathfrak{s}_o$ being a real form of a Borel subalgebra $\mathfrak{b}\subset\mathfrak{sl}(2,\mathbb{C})$. Let $\tau\colon\mathfrak{b}\to\mathfrak{b}$ be the conjugation with $\mathfrak{b}^{\tau}=\mathfrak{s}_o$. It is well-known that there exists a τ-stable Cartan subalgebra $\mathfrak{t}\subset\mathfrak{b}$, and consequently we have the τ-stable decomposition $\mathfrak{b}=\mathfrak{t}\ltimes[\mathfrak{b},\mathfrak{b}]=:\mathfrak{t}\ltimes\mathfrak{n}$. Select $\mathsf{h}\in\mathfrak{t}$ and $\mathsf{e}\in\mathfrak{n}$ such that $[\mathsf{h},\mathsf{e}]=2\mathsf{e}$. By construction $\tau(\mathsf{h})=a\cdot\mathsf{h}$ and $\tau(\mathsf{e})=b\cdot\mathsf{e}$ for suitable $a,b\in\mathbb{C}$. Since τ is an automorphism, $a=1$. As τ is an involution, $|b|=1$. This shows that, up to a conjugation, we may assume that $\mathfrak{b}$ is the upper-triangular Borel subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ and the real forms $\mathfrak{s}_o\subset\mathfrak{b}^+$ have the realization

$$\mathfrak{s}_o\cong\left\{\begin{pmatrix} t & sc\\ 0 & -t\end{pmatrix} : t,s\in\mathbb{R}\right\}\quad\text{for some } c\in\mathbb{C}^*.$$

For every $c\in\mathbb{C}^*$ at least one of the Borel subalgebras

$$\mathbb{C}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\oplus\mathbb{C}\begin{pmatrix} 1 & -1\\ 1 & -1\end{pmatrix}\quad\text{or}\quad\mathbb{C}\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}\oplus\mathbb{C}\begin{pmatrix} 1 & i\\ i & -1\end{pmatrix}$$

is then complementary to $\mathfrak{s}_o$ in $\mathfrak{sl}(2,\mathbb{C})$.

• $\dim\mathfrak{s}_o=1$, that is, $\mathfrak{s}$ itself is locally transitive and thus $\mathfrak{g}=\mathfrak{s}$ by the minimality condition $(\mathsf{VI})_2$. Consider the following matrix realization:

$$\mathfrak{l}=\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})\subset\mathbb{C}^{2\times2}\times\mathbb{C}^{2\times2},\quad \sigma(\mathsf{x},\mathsf{y})=(\bar{\mathsf{y}},\bar{\mathsf{x}}),\quad \mathfrak{g}=\{(\mathsf{x},\bar{\mathsf{x}}):\mathsf{x}\in\mathfrak{sl}(2,\mathbb{C})\}\subset\mathfrak{l}.$$

The 3-dimensional complex subalgebra $\mathfrak{q}\subset\mathfrak{l}$ is either simple or solvable.

$\mathfrak{q}$ *simple.* Then $\mathfrak{q}\cong\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{q}$ is either one of the two factors of $\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{q}$ is conjugate to $\mathfrak{g}$ in $\mathfrak{l}$. The first case can be ruled out immediately since then $\mathfrak{q}\cap\mathfrak{g}=0=\mathfrak{g}_o$ which is absurd. In the second case, all simple subalgebras $\mathfrak{q}\subset\mathfrak{l}$ which are not ideals are conjugate to each other. We may select the particular subalgebra

$$\mathfrak{q}=\{(\mathsf{x},-\mathsf{x}^t):\mathsf{x}\in\mathfrak{sl}(2,\mathbb{C})\}\subset\mathfrak{l},$$

where x^t is the transpose of x. The corresponding Lie subgroup

$$Q = \{(\mathsf{q}, (\mathsf{q}^{-1})^t) : \mathsf{q} \in \mathrm{SL}(2,\mathbb{C})\} \subset L := \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$$

is closed, and the map $(\mathsf{x},\mathsf{y}) \mapsto \mathsf{x}\cdot\mathsf{y}^t$ identifies the quotient with the affine quadric

$$\mathrm{SL}(2,\mathbb{C}) \subset \mathbb{C}^{2\times 2}$$

on which $S{:=}\mathrm{SL}(2,\mathbb{C})$ (considered as a real Lie group) acts by the holomorphic transformations $(\mathsf{s},\mathsf{z}) \mapsto \mathsf{s}\mathsf{z}\bar{\mathsf{s}}^t$. Hence, in this situation every CR-germ associated with a CR-algebra $(\mathfrak{s},\mathfrak{q})$ is globalizable. It is well known (see, for instance [16]) that the hypersurface S-orbits in L/Q are either Levi nondegenerate or locally CR-equivalent to the tube $\mathcal{M}$ over the light cone. However, $\dim \mathfrak{g}{=}6$ and in this case $(\mathfrak{g},\mathfrak{q})$ satisfies (I)–(V), $(\mathrm{VI})_1$ and $(\mathrm{VI})_2$, but not (VI).

$\mathfrak{q}$ *solvable.* We will show that also this case contradicts the fundamental assumption in §9.8. If $\mathfrak{l}_o{=}\mathbb{C}(\mathsf{t},\bar{\mathsf{t}})$ is ad-semisimple (and without loss of generality $\mathsf{t}{=}\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right)$ for some $t{\in}\mathbb{C}^*$) then $\mathfrak{l}_o$ is a regular torus in $\mathfrak{l}$. Consequently, the centralizer $C_{\mathfrak{l}}(\mathfrak{l}_o){=:}\mathfrak{t}_1\times\mathfrak{t}_2$ is a σ-stable Cartan subalgebra. Either $C_{\mathfrak{l}}(\mathfrak{l}_o){\subset}\mathfrak{q}$ (but then $\mathfrak{q}{+}\sigma\mathfrak{q}$ is of codimension 2 in $\mathfrak{l}$) or $C_{\mathfrak{l}}(\mathfrak{l}_o){\cap}\mathfrak{q}{=}\mathfrak{l}_o$. In the latter case, denote by $\mathfrak{l}^{\pm\alpha_1}$ and $\mathfrak{l}^{\pm\alpha_2}$ the root spaces with respect to $\mathfrak{t}_1\times\mathfrak{t}_2$. A direct check shows that $\mathfrak{q}$ is the direct sum of $\mathfrak{l}_o$ and two other root spaces (also if $\bar{\mathsf{t}}{=}\mathsf{t}$). Since $\mathfrak{q}$ is solvable, there are four possibilities of choosing such pairs of root spaces. In all four cases either $\mathfrak{q}{+}\sigma\mathfrak{q}$ is too small or $\mathfrak{f}{=}\mathfrak{l}_o$, that is, the corresponding CR-germ is Levi nondegenerate.

It remains to discuss the case when $\mathfrak{l}_o{=}\mathbb{C}(\mathsf{n},\bar{\mathsf{n}})$ is ad-nilpotent. Since $\mathfrak{q}$ is solvable, it is contained in a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{l}$. Consequently, $\mathfrak{l}_o{\subset}\mathfrak{b}^{\mathrm{nil}}{=}C_{\mathfrak{l}}(\mathfrak{l}_o){\cong}\mathbb{C}\mathsf{n}\times\mathbb{C}\bar{\mathsf{n}}$. Let $\pi\colon \mathfrak{b}{\to}\mathfrak{b}/\mathfrak{b}^{\mathrm{nil}}$ be the canonical projection. The image $\pi(\mathfrak{q})$ cannot be surjective, since there is no 3-dimensional subalgebra $\mathbb{C}(\mathsf{n},\bar{\mathsf{n}}){\subset}\mathfrak{q}{\subset}\mathfrak{b}$ with this property. Only the possibility $\mathfrak{b}^{\mathrm{nil}}{\subset}\mathfrak{q}$ remains, but then $\mathfrak{q}{\cap}\sigma\mathfrak{q}{\supset}\mathfrak{b}^{\mathrm{nil}}$, which is too big.

Summarizing, we have for the CR-algebra $(\mathfrak{g},\mathfrak{q})$ satisfying the assumption in §9.8 that $\mathfrak{g}^{\mathrm{ss}}$ must be a finite direct sum of copies of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$. In the next section we direct our attention to factors of type $\mathfrak{sl}(2,\mathbb{R})$.

12. The case $\mathfrak{s}{\cong}\mathfrak{sl}(2,\mathbb{R})$

In this section we continue the proof of Theorem II and consider only CR-algebras $(\mathfrak{g},\mathfrak{q})$ subject to the assumption in §9.8, for which the simple factor $\mathfrak{s}$ of $\mathfrak{g}^{\mathrm{ss}}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$; compare (10.1). As already proved, the remaining simple factors in $\mathfrak{s}'$ (if there are any) are isomorphic to $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{R})$.

From $\mathfrak{hol}(\mathcal{M},o)\cong\mathfrak{so}(2,3)$ for the light cone tube $\mathcal{M}$ it is clear that $\mathfrak{hol}(\mathcal{M},o)$ contains copies of $\mathfrak{so}(2,2)\cong\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R})$. Fixing a subalgebra $\mathfrak{r}\subset\mathfrak{sl}(2,\mathbb{R})$ (which is necessarily nonabelian), the 5-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{r}$ can be embedded into $\mathfrak{hol}(\mathcal{M},o)$, and this can be done in such a way that the image is a transitive subalgebra. Therefore, the simple factor $\mathfrak{sl}(2,\mathbb{R})$ of $\mathfrak{g}^{\mathrm{ss}}$ cannot be avoided in the classification proof. However, we will show that this factor only occurs in the instance described above, that is, in connection with $\mathcal{M}$.

Our first result is that in $\mathfrak{g}^{\mathrm{ss}}$ the simple factor $\mathfrak{sl}(2,\mathbb{R})$ can occur at most once. Here and in the following we repeatedly use the basic fact that each proper subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ has a solvable complementary subalgebra. Furthermore, throughout this subsection we denote by $\pi\colon\mathfrak{g}=(\mathfrak{s}\times\mathfrak{s}')\ltimes\mathrm{rad}(\mathfrak{g})\to\mathfrak{s}$ the canonical projection. We analyze various possibilities for the image of the isotropy subalgebra $\mathfrak{g}_o$ under π.

If $\pi(\mathfrak{g}_o)\neq0$ then there exists a solvable complement $\mathfrak{r}\subset\mathfrak{s}$ to $\pi(\mathfrak{g}_o)$, and $(\mathfrak{r}\times\mathfrak{s}')\ltimes\mathrm{rad}(\mathfrak{g})$ is a proper transitive subalgebra of $\mathfrak{g}$ violating $(\mathsf{VI})_2$.

If $\pi(\mathfrak{g}_o)=0$ then $\mathfrak{g}_o\subset\mathfrak{s}'\ltimes\mathrm{rad}(\mathfrak{g})$ and is there of codimension 2. It follows that there is no factor $\mathfrak{h}\cong\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{s}'$: Otherwise, counting dimensions we would have $\dim\mathfrak{h}\cap\mathfrak{s}_o\geqslant1$, which implies that there is a proper transitive subalgebra of $\mathfrak{g}$, violating $(\mathsf{VI})_2$. This proves that $\mathfrak{s}'$ is a product of factors which all are isomorphic to $\mathfrak{su}(2)$. Finally, we show that $\mathfrak{s}'$ consists of at most *one* such simple factor: Indeed, suppose that $\mathfrak{s}'=\mathfrak{h}\times\mathfrak{s}''$ for some ideal $\mathfrak{h}\cong\mathfrak{su}(2)$ of $\mathfrak{s}'$. Denote by $\pi_{\mathfrak{h}}\colon(\mathfrak{s}\times\mathfrak{h}\times\mathfrak{s}'')\ltimes\mathrm{rad}(\mathfrak{g})\to\mathfrak{h}$ the canonical projection. Then $\pi_{\mathfrak{h}}(\mathfrak{g}_o)$ has codimension $\leqslant2$ in $\mathfrak{h}$. Since $\mathfrak{su}(2)$ does not have a subalgebra of dimension 2, the dimension of $\pi_{\mathfrak{h}}(\mathfrak{g}_o)$ can only be 1 or 3. But dimension 3 violates $(\mathsf{VI})_2$ since then $\mathfrak{g}=\mathfrak{g}_o+\ker(\pi_{\mathfrak{h}})$. The case $\dim\pi_{\mathfrak{h}}(\mathfrak{g}_o)=1$ remains. Then $\pi_{\mathfrak{h}}(\mathfrak{g}_o)$ is a torus in $\mathfrak{h}$ and $\pi(\mathfrak{g}_o)=\mathfrak{h}\cap\mathfrak{g}_o$. Since $\mathfrak{g}':=\mathfrak{s}\times\mathfrak{h}$ is a transitive subalgebra of $\mathfrak{g}$, we must have $\mathfrak{g}'=\mathfrak{g}$ by $(\mathsf{VI})_2$. But this is not possible, as we show in the following result.

Lemma 12.1. $\mathfrak{s}\cong\mathfrak{sl}(2,\mathbb{R})$ *implies that* $\mathfrak{g}=\mathfrak{s}\ltimes\mathrm{rad}(\mathfrak{g})$ *and* $\mathfrak{g}_o\subset\mathrm{rad}(\mathfrak{g})$.

Proof. By the above discussion, we only have to rule out the case $\mathfrak{g}=\mathfrak{s}\times\mathfrak{s}'$ with $\mathfrak{g}_o\subset\mathfrak{s}'=\mathfrak{su}(2)$. The key point here is that the isotropy subalgebra $\mathfrak{g}_o$ is toral in $\mathfrak{su}(2)$ and that there exists a unique $\mathrm{ad}(\mathfrak{g}_o)$-stable subspace $\mathfrak{p}_*\subset\mathfrak{su}(2)$ on which the adjoint representation of $\mathfrak{g}_o$ is irreducible. Let $\pi_{\mathfrak{su}}\colon\mathfrak{s}\oplus\mathfrak{su}(2)\to\mathfrak{su}(2)$ be the projection onto the second factor. Either $\pi_{\mathfrak{su}}(\mathfrak{F})=\mathfrak{su}(2)$ or $\pi_{\mathfrak{su}}(\mathfrak{F})=\mathfrak{g}_o$. In the first case, we actually have $\mathfrak{F}=\mathfrak{su}(2)$ as $[\mathfrak{g}_o,\mathfrak{F}]\subset\mathfrak{F}$ and $[\mathfrak{s},\mathfrak{g}_o]=0$. But this cannot be true: Independently of what exactly $\mathfrak{H}=W\oplus\mathfrak{su}(2)$ would be, we always would have $[\mathfrak{H},\mathfrak{F}]\subset\mathfrak{F}$. But this violates the nondegeneracy condition (V).

The case $\pi_{\mathfrak{su}}(\mathfrak{F})=\mathfrak{g}_o$ remains, that is, $\mathfrak{F}=\mathfrak{b}\oplus\mathfrak{g}_o$, where $\mathfrak{b}=\mathfrak{F}\cap\mathfrak{s}\subset\mathfrak{s}$ is a 2-dimensional real subalgebra. By a dimensional argument, $\dim\mathfrak{H}\cap\mathfrak{su}(2)\geqslant2$. But this intersection must be $\mathrm{ad}(\mathfrak{g}_o)$-stable, which implies that $\mathfrak{H}=\mathfrak{b}\oplus\mathfrak{su}(2)$, i.e., $\mathfrak{H}$ is a subalgebra of $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{su}(2)$. Clearly, this violates condition (III). $\square$

In the next lemma we show that the semidirect product $\mathfrak{s}\ltimes\mathrm{rad}(\mathfrak{g})$ in Lemma 12.1 actually is a direct product $\mathfrak{s}\times\mathrm{rad}(\mathfrak{g})$ and that the radical has dimension 2, i.e. $\mathfrak{g}$ has dimension 5. Recall the obvious fact that, up to isomorphism, there exist precisely two Lie algebras of dimension 2, the abelian Lie algebra $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$ and a nonabelian one.

LEMMA 12.2. $\mathfrak{s}\cong\mathfrak{sl}(2,\mathbb{R})$ *implies that* $\mathfrak{g}=\mathfrak{s}\times\mathfrak{r}$ *with* $\mathfrak{r}:=\mathrm{rad}(\mathfrak{g})$ *being nonabelian and of dimension* 2.

Proof. Let $\pi\colon\mathfrak{g}\to\mathfrak{s}$ be the canonical projection. We use the same symbol also for the complex extension $\mathfrak{l}\to\mathfrak{s}^{\mathbb{C}}$. Because of (I) and $\mathfrak{g}_o\subset\mathfrak{r}$, the image $\pi(\mathfrak{F})$ has dimension $\leqslant 2$.

- $\dim\pi(\mathfrak{F})=0$ implies $\mathfrak{F}=\mathfrak{r}$, that is, $\mathfrak{F}$ is an ideal in $\mathfrak{g}$. But this violates (V).
- $\dim\pi(\mathfrak{F})=1$ implies $\pi(\mathfrak{f})=\pi(\sigma\mathfrak{f})$ and $\dim\pi(\mathfrak{f})=1$. Consequently, $\pi(\mathfrak{q})$ and $\pi(\sigma\mathfrak{q})$ are 2-dimensional (Borel) subalgebras which generate $\mathfrak{sl}(2,\mathbb{C})$ as a linear space (otherwise (M,o) would be Levi flat). It follows that $\pi^{-1}(\pi(\sigma\mathfrak{q}))\cap\mathfrak{q}=\mathfrak{f}$. But this implies that $[\mathfrak{f},\sigma\mathfrak{q}]\subset\mathfrak{f}+\sigma\mathfrak{q}$, a contradiction to (V).
- $\dim\pi(\mathfrak{F})=2$. Note that $\pi(\mathfrak{f})$ and $\pi(\sigma\mathfrak{f})$ are 1-dimensional, since $\mathfrak{l}_o\subset\mathfrak{r}^{\mathbb{C}}$. Since $\pi(\mathfrak{f})+\pi(\sigma\mathfrak{f})$ is a 2-dimensional subalgebra, $\mathfrak{t}:=\pi(\mathfrak{f})$ and $\mathfrak{t}':=\pi(\sigma\mathfrak{f})=\sigma\mathfrak{t}$ are tori, as follows with the elementary structure theory of $\mathfrak{sl}(2,\mathbb{C})$. The case $\pi(\mathfrak{f})=\pi(\mathfrak{q})$ can be excluded, since otherwise we would have $\pi(\mathfrak{q}+\sigma\mathfrak{q})=\pi(\mathfrak{f}+\sigma\mathfrak{f})$ which is a subalgebra, a contradiction to (III). Hence, the only possibility remaining is $\pi(\mathfrak{f})\neq\pi(\mathfrak{q})$. This implies that $\mathfrak{q}\cap\mathrm{rad}=\sigma\mathfrak{q}\cap\mathrm{rad}=\mathfrak{l}_o$. Furthermore, $\mathfrak{l}_o$ is an ideal in $\mathfrak{q}$ and in $\sigma\mathfrak{q}$, and, since $[\mathfrak{q},\sigma\mathfrak{q}]=\mathfrak{l}$, even an ideal of $\mathfrak{l}$. By the effectivity assumption $(\mathrm{VI})_1$, this implies that $\mathfrak{g}_o=0$, i.e., $\mathfrak{g}$ has dimension 5 and thus

$$\mathfrak{g}\cong\mathfrak{sl}(2,\mathbb{R})\ltimes_\varrho\mathbb{R}^2 \quad\text{or}\quad \mathfrak{g}\cong\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{r},\quad \text{with } \dim\mathfrak{r}=2, \tag{12.3}$$

where $\varrho\colon\mathfrak{sl}(2,\mathbb{R})\to\mathrm{End}(\mathbb{R}^2)$ is the canonical inclusion.

CLAIM 12.4. *The first case in* (12.3), *that is* $\mathfrak{g}\cong\mathfrak{sl}(2,\mathbb{R})\ltimes_\varrho\mathbb{R}^2$, *cannot occur.*

Proof. Let $\pi\colon\mathfrak{l}\to\mathfrak{sl}(2,\mathbb{C})$ be the canonical projection and $\sigma\colon\mathfrak{l}\to\mathfrak{l}$ be the complex conjugation defining the real form $\mathfrak{g}$ of $\mathfrak{l}$. The possibility $\pi(\mathfrak{q})=0$ $(=\pi(\sigma\mathfrak{q}))$ can be excluded, since then $\mathfrak{q}=\mathrm{rad}(\mathfrak{l})=\sigma\mathfrak{q}$, violating (III). Also $\dim\pi(\mathfrak{q})=1$ can be ruled out: Then $\pi(\mathfrak{q})\neq\pi(\mathfrak{f})$, since otherwise $\pi(\mathfrak{q}+\sigma\mathfrak{q})=\pi(\mathfrak{f}+\sigma\mathfrak{f})$ would be a subalgebra, violating (III). Hence, $\mathfrak{f}=\mathfrak{q}\cap\mathrm{rad}(\mathfrak{l})$. But this contradicts (V), since then $[\mathfrak{f},\sigma\mathfrak{q}]\subset\mathrm{rad}(\mathfrak{l})=\mathfrak{f}\oplus\sigma\mathfrak{f}$.

The most involved case is $\dim\pi(\mathfrak{q})=2$ (that is, $\mathfrak{q}\cap\mathrm{rad}(\mathfrak{l})=0$). Then $\pi(\mathfrak{q})=\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{sl}(2,\mathbb{C})$. To rule out also this case we need some preparations. First realize $\mathfrak{l}$ as

$$\mathfrak{sl}(2,\mathbb{C})\ltimes_\varrho\mathbb{C}^2=\{(X,w):X\in\mathfrak{sl}(\mathbb{C}^2),w\in\mathbb{C}^2\},\ \text{ with }\ [(X,w),(Y,u)]=([X,Y],Xu-Yw)$$

and complex conjugation σ given by $(X,w)\mapsto(\overline{X},\overline{w})$. Since (M,o) is not Levi flat (compare (III)), we necessarily have $\mathfrak{b}+\sigma\mathfrak{b}=\mathfrak{sl}(2,\mathbb{C})$, and the 1-dimensional intersection $\mathfrak{b}\cap\sigma\mathfrak{b}$ is a toral subalgebra $\mathfrak{t}\subset\mathfrak{sl}(2,\mathbb{C})$ (we use here the well-known fact that the [σ-stable] intersection of any two Borel subalgebras contains a [σ-stable] Cartan subalgebra). In the next two paragraphs we recall some elementary facts from the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ which we need to complete our proof.

12.5. σ-adapted $\mathfrak{sl}_2$-triples

Let $\mathsf{h}\in\mathfrak{t}\subset\mathfrak{sl}(2,\mathbb{C})$ be the element for which $[\mathsf{h},E]=2E$ and $[\mathsf{h},F]=-2F$ for every $E\in\mathfrak{b}^{\mathrm{nil}}$ and $F\in\sigma(\mathfrak{b}^{\mathrm{nil}})=(\sigma\mathfrak{b})^{\mathrm{nil}}$. Since σ interchanges the eigenspaces of $\mathrm{ad}(\mathsf{h})$, we have $\sigma(\mathsf{h})=-\mathsf{h}$. There are crucial technical points here: We claim that there exists $\mathsf{e}\in\mathfrak{b}^{\mathrm{nil}}$ such that $[\mathsf{e},\sigma(\mathsf{e})]=\mathsf{h}$, i.e., $(\mathsf{e},\mathsf{h},\mathsf{f})$, with $\mathsf{f}:=\sigma(\mathsf{e})$, is an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}(2,\mathbb{C})$ (i.e., $[\mathsf{h},\mathsf{e}]=2\mathsf{e}$, $[\mathsf{h},\mathsf{f}]=-2\mathsf{f}$ and $[\mathsf{e},\mathsf{f}]=\mathsf{h}$).

Construction. We have to be careful here about signs: Since σ defines a *noncompact* real form of $\mathfrak{sl}(2,\mathbb{C})$, the nondegenerate Hermitian 2-form $\varkappa(\cdot\,,\sigma(\cdot))$ has precisely one negative eigenvalue, where $\varkappa$ denotes the Killing form. As $\varkappa(\mathsf{h},\sigma(\mathsf{h}))=\varkappa(\mathsf{h},-\mathsf{h})<0$, it follows that $\varkappa(E,\sigma(E))>0$ and consequently $[\lambda E,\sigma(\lambda E)]=\mathsf{h}$ for an appropriately chosen $\lambda\in\mathbb{C}^*$ (keeping in mind the general formula $[E,F]=\varkappa(E,F)H$, where H, the coroot, is a positive multiple of h). Define then $\mathsf{e}:=\lambda E$ and $\mathsf{f}:=\sigma(\mathsf{e})$. Each $\mathfrak{sl}_2$-triple $(\mathsf{e}',\mathsf{h}',\mathsf{f}')$, with $\mathsf{e}',\mathsf{h}',\mathsf{f}'\in\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(2,\mathbb{C})$ and $\sigma(\mathsf{e}')=\mathsf{f}'$, is called σ-*adapted* in the sequel.

12.6. Complexifying $\mathfrak{sl}(2,\mathbb{R})\ltimes_\varrho\mathbb{R}^2$, we briefly discuss how the 2-dimensional abelian radical $\mathbb{C}^2$, considered as an $\mathfrak{sl}(2,\mathbb{C})$-module, is related to the real structure. Let a σ-adapted $\mathfrak{sl}_2$-triple $(\mathsf{e},\mathsf{h},\mathsf{f})$ be given. Let $\mathbb{C}^2=V_+\oplus V_-$ be the decomposition into (±1) h-eigenspaces. They are interchanged by σ. We claim that there exists a $v_+\in V_+$ with $v_-:=\sigma(\mathsf{e})v_+=\sigma(v_+)\neq0$: Indeed, choose $w_+\in V_+\setminus\{0\}$ arbitrarily. Then there is a $c\in\mathbb{C}^*$ with $\mathsf{f}w_+=c\sigma(w_+)$, and a direct check shows that $|c|=1$. Choose a $b\in\mathbb{C}$ with $b^2=\bar{c}$ and put $v_+:=b\cdot w_+$.

We use the linear basis v_+,v_- of the radical $\mathrm{rad}(\mathfrak{l})\cong\mathbb{C}^2$ in the following computations.

We now resume the proof of Claim 12.4. For short, write $\mathfrak{r}^{\mathbb{C}}:=\mathrm{rad}(\mathfrak{l})$ for the abelian radical. Since $\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}=0$, we can write $\mathfrak{q}=\mathbb{C}\cdot(\mathsf{e},w_1)\oplus\mathbb{C}\cdot(\mathsf{h},w_2)$ for suitable $w_1,w_2\in\mathbb{C}^2$. Clearly, the w_j's are not arbitrary: Write $w_1=\lambda_1v_++\mu_1v_-$ and $w_2=\lambda_2v_++\mu_2v_-$. Note that $w_2 \neq -\bar{w}_2$ (otherwise $\mathfrak{q} \cap \sigma\mathfrak{q} \neq 0$), i.e., $\lambda_2 \neq -\bar{\mu}_2$. The fact that $\mathfrak{q}$ is a subalgebra imposes one more condition on the coefficients λ_j and μ_j: A simple computation shows that $\mu_1=0$ and $\mu_2=-\lambda_1$, i.e., for each $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \bar{\mu}$ we have the two 2-dimensional complex subalgebras $\mathfrak{q}=\mathfrak{q}_{\lambda,\mu}$ and $\sigma\mathfrak{q}$:

$$\mathfrak{q} = \mathbb{C}\cdot(\mathsf{e}, \lambda v_+) \oplus \mathbb{C}\cdot(\mathsf{h}, \mu v_+ - \lambda v_-) \quad \text{and} \quad \sigma\mathfrak{q} = \mathbb{C}\cdot(\mathsf{f}, \bar{\lambda} v_-) \oplus \mathbb{C}\cdot(-\mathsf{h}, -\bar{\lambda} v_+ + \bar{\mu} v_-).$$

Recall the definition $\mathfrak{f}=\{u \in \mathfrak{q} : [u, \sigma\mathfrak{q}] \subset \mathfrak{q}+\sigma\mathfrak{q}\}$ from §9.5 (iv). A straightforward calculation shows that in all cases $\mathfrak{f}=0$ holds. This contradicts (IV) and proves Claim 12.4. □

We proceed with the proof of Lemma 12.2 by restricting the radical of $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}$ further.

Claim 12.7. *The Lie algebra $\mathfrak{g}$ cannot be isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}^2$.*

Proof. Assume on the contrary that $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}^2$ and denote by $\pi_1 : \mathfrak{l} \to \mathfrak{sl}(2,\mathbb{C})$ and $\pi_2 : \mathfrak{l} \to \mathbb{C}^2$ the canonical projections. As in the proof of the previous claim, the cases $\dim \mathfrak{q} \cap \operatorname{rad}(\mathfrak{l}) \geqslant 1$ can be ruled out immediately. We therefore only have to exclude the case $\mathfrak{q} \cap \mathbb{C}^2=0$. In this case, let $\mathfrak{b}$ be the 2-dimensional π_1-image in $\mathfrak{sl}(2,\mathbb{C})$. It follows that $\sigma(\mathfrak{b}) \neq \mathfrak{b}$ since otherwise $\pi_{\mathfrak{s}}(\mathfrak{q}+\sigma\mathfrak{q}) \subset \mathfrak{b}$ would contradict (III). Consequently, there exist $\mathsf{e} \in \mathfrak{b}$, $\mathsf{h} \in \mathfrak{b} \cap \sigma\mathfrak{b}$ and $\mathsf{f}=\sigma(\mathsf{e}) \in \sigma(\mathfrak{b})=\pi(\sigma\mathfrak{q})$ with the properties described in §12.5. The fact that $\mathfrak{q}$ and $\sigma\mathfrak{q}$ are subalgebras and that $\pi(\mathfrak{q})=\mathbb{C}\mathsf{h} \oplus \mathbb{C}\mathsf{e}$ determines $\mathfrak{q}$ and $\sigma\mathfrak{q}$ as follows: $\mathfrak{q}=\mathbb{C}\cdot(\mathsf{h}, w) \oplus \mathbb{C}\cdot(\mathsf{e}, 0)$ and $\sigma\mathfrak{q}=\mathbb{C}\cdot(\mathsf{h}, -\bar{w}) \oplus \mathbb{C}\cdot(\mathsf{f}, 0)$ for a $w \in \mathbb{C}^2$.

We may further assume that w and $\bar{w}$ are linearly independent, otherwise

$$\pi_2(\mathfrak{q}+\sigma\mathfrak{q}) \neq \mathbb{C}^2$$

would contradict (III). As a consequence, $(\mathsf{h}, 0) \notin \mathfrak{q}+\sigma\mathfrak{q}$. The definition of $\mathfrak{f}$ then shows that $\mathfrak{f}=\mathbb{C}\cdot(\mathsf{h}, w)$. On the other hand, for this $\mathfrak{f}$ we have $[\mathfrak{f}, \sigma\mathfrak{q}] \subset \mathfrak{f} \oplus \sigma\mathfrak{q}$, in contradiction to (V). This proves Claim 12.7. □

The proof of Lemma 12.2 is now complete. □

So far we know that under the assumption $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{R})$, necessarily $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}$, where $\mathfrak{r}$ is the 2-dimensional nonabelian Lie algebra. To determine the full CR-algebra $(\mathfrak{g}, \mathfrak{q})$, we still have to find out how the complex subalgebra $\mathfrak{q} \subset \mathfrak{l}$ sits inside $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{r}^{\mathbb{C}}$.

Claim 12.8. *Up to a CR-algebra automorphism of $(\mathfrak{g}, \mathfrak{q})$, the subalgebra $\mathfrak{q} \subset \mathfrak{l}$ is obtained in the following way. Fix a linear basis x, z of $\mathfrak{r}^{\mathbb{C}}$, with $[\mathsf{x}, \mathsf{z}]=\mathsf{z}$, and a σ-adapted $\mathfrak{sl}_2$-triple $(\mathsf{e}, \mathsf{h}, \mathsf{f})$, with $\mathsf{e}, \mathsf{h}, \mathsf{f} \in \mathfrak{sl}(2,\mathbb{C})$ (see §12.5 for the definition). Then*

$$\mathfrak{q} = \mathbb{C}(\mathsf{h}, 2\mathsf{x}+\mu\mathsf{z}) \oplus \mathbb{C}(\mathsf{e}, \nu\mathsf{z})$$

for suitable $\mu, \nu \in \mathbb{C}$ with $\operatorname{Im}\mu=\pm 2$ and $|\nu|=1$.

Proof. The case $\mathfrak{q}=\mathfrak{r}^{\mathbb{C}}$ can clearly be excluded. If the intersection $\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}$ would be 1-dimensional, then $\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}\oplus\sigma\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}=\mathfrak{r}^{\mathbb{C}}$ (as the sum $\mathfrak{q}+\sigma\mathfrak{q}$ must be direct). Since $\mathfrak{f}$ must also be 1-dimensional, we have $\mathfrak{f}=\mathfrak{r}^{\mathbb{C}}\cap\mathfrak{q}$. But then $[\mathfrak{f},\sigma\mathfrak{q}]\subset\mathfrak{r}^{\mathbb{C}}=\mathfrak{f}\oplus\sigma\mathfrak{f}$, violating condition (V).

The case $\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}=0$ remains, that is, $\pi_{\mathfrak{s}}(\mathfrak{q})$ is a Borel subalgebra $\mathbb{C}\mathsf{e}\oplus\mathbb{C}\mathsf{h}$, and

$$\pi(\sigma\mathfrak{q})=\mathbb{C}\mathsf{h}\oplus\mathbb{C}\mathsf{f},$$

where $\mathsf{h},\mathsf{e},\mathsf{f}\in\mathfrak{sl}(2,\mathbb{C})$ are chosen as explained in §12.5. Since $\mathfrak{q}\subset\mathfrak{s}^{\mathbb{C}}\times\mathfrak{r}^{\mathbb{C}}$ is a Lie sub*algebra*, it necessarily has the following form:

$$\begin{gathered}\mathfrak{q}=\mathbb{C}\cdot(\mathsf{h},\lambda\mathsf{x}+\mu\mathsf{z})\oplus\mathbb{C}\cdot(\mathsf{e},\nu\mathsf{z})\quad\text{with}\quad\sigma\mathfrak{q}=\mathbb{C}\cdot(-\mathsf{h},\bar\lambda\mathsf{x}+\bar\mu\mathsf{z})\oplus\mathbb{C}\cdot(\mathsf{f},\bar\nu\mathsf{z})\\ \text{and}\quad\lambda,\mu,\nu\in\mathbb{C}\quad\text{satisfying}\quad\lambda=2\text{ if }\nu\neq0.\end{gathered}\tag{12.9}$$

Case $\nu=0$. Then $\lambda\mathsf{x}+\mu\mathsf{z}$ and $\bar\lambda\mathsf{x}+\bar\mu\mathsf{z}$ must be linearly independent in $\mathfrak{r}^{\mathbb{C}}$, that is, $\lambda\bar\mu\neq\bar\lambda\mu$ (otherwise $\pi_{\mathfrak{r}}(\mathfrak{q}+\sigma\mathfrak{q})\neq\mathfrak{r}^{\mathbb{C}}$). Observe that $(\mathsf{h},0)\notin\mathfrak{q}+\sigma\mathfrak{q}$ and $\lambda\neq0$. A direct verification shows that $\mathfrak{f}=\mathbb{C}\cdot(\mathsf{h},\lambda\mathsf{x}+\mu\mathsf{z})$ if $\lambda\in i\mathbb{R}$ and $\mathfrak{f}=0$ otherwise. But then $\mathfrak{f}\oplus\sigma\mathfrak{q}$ is a subalgebra, in contradiction to (V).

Case $\nu\neq0$. Possibly after replacing z by $|\nu|\mathsf{z}$, we may assume that $|\nu|=1$ in (12.9). Employing the definition of $\mathfrak{f}$ in §9.5 (iv), a simple calculation shows that

$$\mathfrak{f}=\begin{cases}\mathbb{C}(\mathsf{h}\mp2i\bar\nu\mathsf{e},2\mathsf{x}+(\operatorname{Re}\mu)\mathsf{z}), & \text{if }\operatorname{Im}\mu=\pm2,\\ 0, & \text{otherwise,}\end{cases}\tag{12.10}$$

that is, $\mathfrak{f}$ is 1-dimensional only if $\operatorname{Im}\mu=\pm2$. This proves the claim. □

LEMMA 12.11. *For the CR-algebra $(\mathfrak{g},\mathfrak{q})$ in Claim 12.8 the associated CR-germ (M,o) is CR-equivalent to $(\mathcal{M},a)$, where $\mathcal{M}$ is the tube over the future light cone.*

Proof. We start by giving a particular representing manifold for the associated germ; compare [16, Example 6.6]. Let μ and ν be the constants occurring in Claim 12.8 and consider the affine quadric $Z:=\mathrm{SL}(2,\mathbb{C})\subset\mathbb{C}^{2\times2}$, on which the group $\hat L:=\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ acts holomorphically by $z\mapsto gzh^{-1}$ for all $(g,h)\in\hat L$. Then $\widehat G:=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ is a real form of $\hat L$. Via

$$\begin{aligned}&\mathsf{x}:=\frac12\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \mathsf{z}:=\frac12\begin{pmatrix}1&-1\\1&-1\end{pmatrix} &&\in\mathfrak{r}\\ &\mathsf{e}:=\frac{\nu}{2}\begin{pmatrix}1&i\\i&-1\end{pmatrix},\quad \mathsf{h}:=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \mathsf{f}:=\sigma(\mathsf{e}) &&\in\mathfrak{s}^{\mathbb{C}}\end{aligned}\tag{12.12}$$

we consider $\mathfrak{l}=\mathfrak{s}^{\mathbb{C}}\times\mathfrak{r}^{\mathbb{C}}$ as a complex subalgebra of $\hat{\mathfrak{l}}=\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{g}$ as a subalgebra of $\hat{\mathfrak{g}}$.

We recall some basic facts (compare [16]). The polynomial function

$$\psi(z):=\det(z+\bar{z})-2$$

on Z is invariant under the action of the group $\widehat{G}$. Furthermore, the nonsingular part M of the algebraic subset $S{:=}\psi^{-1}(\{\pm 2\})$ is a (nonconnected) hypersurface in Z, locally CR-equivalent to $\mathcal{M}$. On the other hand, the singular part of S is a totally real submanifold of Z. Consider the point

$$o:=\frac{e^{-\pi i/4}}{4}\begin{pmatrix} 4+\mu & -\mu \\ i\mu & i(4-\mu) \end{pmatrix}\in Z,$$

which actually is in S because $\psi(o){=}\operatorname{Im}\mu$. A direct computation shows that the isotropy subalgebra $\mathfrak{l}_o$ of $\mathfrak{l}\subset\hat{\mathfrak{l}}$ at $o\in Z$ is $\mathbb{C}(\mathsf{h}, 2\mathsf{x}+\mu\mathsf{z})\oplus\mathbb{C}(\mathsf{e}, \nu\mathsf{z})$ and that the isotropy subalgebra of $\mathfrak{g}\subset\hat{\mathfrak{g}}$ at o is trivial. This implies that $a\in M$ and also that the CR-algebra $(\mathfrak{g},\mathfrak{q})$ is associated with the germ (M,o). Since M is locally CR-equivalent to the tube $\mathcal{M}$ over the future light cone, the proof for Lemma 12.11 is complete. □

13. The case $\mathfrak{s}\cong\mathfrak{su}(2)$

The status of our proof so far can be summarized as follows. For the CR-algebra $(\mathfrak{g},\mathfrak{q})$ satisfying the assumption in §9.8 and $\mathfrak{g}^{\mathrm{ss}}\neq 0$, the simple ideal $\mathfrak{s}$ of $\mathfrak{g}^{\mathrm{ss}}$ can only be isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$, Furthermore, the case $\mathfrak{s}\cong\mathfrak{sl}(2,\mathbb{R})$ only occurs if the associated CR-germ is equivalent to $(\mathcal{M},o)$, with $\mathcal{M}$ being the light cone tube, and then $\mathfrak{g}\cong\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{r}$ with nonabelian 2-dimensional $\mathfrak{r}$. Consequently, only the situation needs to be investigated when $\mathfrak{g}^{\mathrm{ss}}$ only contains simple ideals isomorphic to $\mathfrak{su}(2)$. We assume this throughout this section and state our main lemma.

LEMMA 13.1. *There is no simple factor of* $\mathfrak{g}^{\mathrm{ss}}$ *isomorphic to* $\mathfrak{su}(2)$.

Proof. The proof will be subdivided into several claims. Note that contrary to $\mathfrak{sl}(2,\mathbb{R})$, the only nontrivial proper subalgebras of $\mathfrak{su}(2)$ are 1-dimensional tori. As before, let $\mathfrak{s}$ be a fixed simple factor of $\mathfrak{g}^{\mathrm{ss}}$ and denote by $\pi\colon\mathfrak{g}\to\mathfrak{s}$ the canonical projection. Then, the image $\pi(\mathfrak{g}_o)$ must be a proper subalgebra, since otherwise $\mathfrak{g}'{=}\pi^{-1}(0)$ would violate $(\mathsf{VI})_2$. Our first observation is the following statement.

CLAIM 13.2. *For* $\mathfrak{g}$ *and* $\mathfrak{r}{:=}\operatorname{rad}(\mathfrak{g})$ *only the following cases may occur*:

$$\mathfrak{g}\cong\mathfrak{su}(2)\ltimes\mathfrak{r},\quad \mathfrak{g}\cong\mathfrak{su}(2)\times\mathfrak{su}(2)\quad\text{or}$$

$$\mathfrak{g}\cong(\mathfrak{su}(2)\times\mathfrak{su}(2))\ltimes\mathfrak{r}\quad \textit{with } \mathfrak{r}\neq 0 \textit{ and } \mathfrak{g}_o=(\mathbb{R}\mathsf{t}_1\times\mathbb{R}\mathsf{t}_2)\ltimes(\mathfrak{g}_o\cap\mathfrak{r}) \textit{ for } \mathsf{t}_j\in\mathfrak{su}(2).$$

Proof. Write $\mathfrak{g}=(\mathfrak{s}_1\oplus\mathfrak{s}_2\oplus\mathfrak{s}'')\ltimes\mathfrak{r}$, where $\mathfrak{s}_1,\mathfrak{s}_2\cong\mathfrak{su}(2)$, $\mathfrak{s}''$ is the complementary ideal in $\mathfrak{g}^{\mathrm{ss}}$, and π_1 and π_2 denote the projections onto $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively. Only one of the following possibilities can occur:

- $\pi_1(\mathfrak{g}_o)=0$. Then $\mathfrak{s}_1\oplus\mathfrak{s}_2$ is already a transitive subalgebra of $\mathfrak{g}$ and $\mathfrak{s}''=\mathfrak{r}=0$.
- $\pi_1(\mathfrak{g}_o)\neq 0\neq\pi_2(\mathfrak{g}_o)$. Then either $\mathfrak{g}=\mathfrak{s}_1\oplus\mathfrak{s}_2$ or

$$\mathfrak{g}_o\cap(\mathfrak{s}_1\oplus\mathfrak{s}_2)=(\mathfrak{g}_o\cap\mathfrak{s}_1)\oplus(\mathfrak{g}_o\cap\mathfrak{s}_2)=\mathbb{R}\mathfrak{t}_1\oplus\mathbb{R}\mathfrak{t}_2$$

for suitable nonzero $\mathfrak{t}_1,\mathfrak{t}_2\in\mathfrak{su}(2)$. In the latter case $\mathfrak{g}_o\cap(\mathfrak{s}''\ltimes\mathfrak{r})$ is of codimension 1 in $\mathfrak{s}''\ltimes\mathfrak{r}$. But then we conclude that $\mathfrak{s}''=0$ as there are no 1-codimensional subalgebras in $\mathfrak{su}(2)$. □

For the proof of Lemma 13.1, we only need to investigate the above three types of $\mathfrak{g}$. We repeatedly use the fact that each 1- or 2-dimensional representation of $\mathfrak{su}(2)\cong\mathfrak{so}(3)$ is trivial and that each toral subalgebra $\mathfrak{t}\subset\mathfrak{su}(2)$ acts irreducibly on $\mathfrak{su}(2)/\mathfrak{t}$.

Claim 13.3. *$\mathfrak{g}\cong(\mathfrak{su}(2)\times\mathfrak{su}(2))\ltimes\mathfrak{r}$ implies that $\mathfrak{r}=0$.*

Proof. Let π_1 and π_2 be the projections onto the first and the second simple factor, respectively. As observed in the preceding claim, $\mathfrak{g}_o=(\mathfrak{t}_1\times\mathfrak{t}_2)\ltimes\mathfrak{r}_o$ with $\mathfrak{r}_o:=\mathfrak{g}_o\cap\mathfrak{r}$ and 1-dimensional toral subalgebras $\mathfrak{t}_j$. Recall that we have the $\mathfrak{g}_o$-stable filtration $\mathfrak{g}\supset\mathfrak{H}\supset\mathfrak{F}\supset\mathfrak{g}_o$. At least one of the images $\pi_j(\mathfrak{F})$ coincides with $\mathfrak{s}_j$, say for $j=2$. Since $\mathfrak{t}_j$ acts irreducibly on $\mathfrak{s}_j/\mathfrak{t}_j$, it follows that $\mathfrak{F}=\mathfrak{s}_2\ltimes\mathfrak{r}_o$. Then, since $\mathfrak{H}\cap\mathfrak{s}$ is at least 2-dimensional, the irreducibility of the action of $\mathfrak{t}_1$ implies that $\mathfrak{H}=\mathfrak{s}\times\mathfrak{s}_2\ltimes\mathfrak{r}_o$. But this would imply that $[\mathfrak{H},\mathfrak{H}]\subset\mathfrak{H}$, contradicting (III). □

Claim 13.4. $\mathfrak{g}\ncong\mathfrak{su}(2)\times\mathfrak{su}(2)$.

Proof. Suppose on the contrary that $\mathfrak{g}=\mathfrak{su}(2)\times\mathfrak{su}(2)$. Since $\mathfrak{F}\subset\mathfrak{g}$ is a subalgebra of dimension 3 and there is no solvable subalgebra of this dimension, necessarily $\mathfrak{F}\cong\mathfrak{su}(2)$. Consequently, either $\mathfrak{F}$ is one of the simple factors of $\mathfrak{g}$ or $\mathfrak{F}$ is the graph of an automorphism of $\mathfrak{su}(2)$. Both possibilities lead to a contradiction: in the first case $\mathfrak{F}$ is an ideal in $\mathfrak{g}$, contradicting (V), and in the second case $\mathfrak{F}$ acts irreducibly on $\mathfrak{g}/\mathfrak{F}$, violating the existence of the $\mathrm{ad}(\mathfrak{F})$-stable proper subspace $\mathfrak{H}/\mathfrak{F}\subset\mathfrak{g}/\mathfrak{F}$. □

The remaining case $\mathfrak{g}=\mathfrak{su}(2)\ltimes\mathfrak{r}$ with $\mathfrak{r}:=\mathrm{rad}(\mathfrak{g})$ is the most involved. In this situation the image of $\mathfrak{g}_o$ under the canonical projection $\pi:\mathfrak{g}\to\mathfrak{su}(2)$ is of dimension $\leqslant 1$. As usual, we denote the canonical projection $\mathfrak{l}\to\mathfrak{sl}(2,\mathbb{C})$ by the same symbol π. We investigate the various possibilities for the π-images of the Lie subalgebras $\mathfrak{l}_o\subset\mathfrak{f}\subset\mathfrak{q}$ defined in §9.4 (i)–(iii) and §9.5 (iv).

- $\pi(\mathfrak{l}_o)=\pi(\mathfrak{f})=\pi(\mathfrak{q})$. Since $\pi(\mathfrak{l}_o)$ is a σ-stable torus, $\pi(\mathfrak{q}+\sigma\mathfrak{q})=\pi(\mathfrak{l}_o)$ would follow. Counting dimensions, this cannot happen.

• $\pi(\mathfrak{l}_o)\subsetneq\pi(\mathfrak{f})=\pi(\mathfrak{q})$. Here, we have to rule out the two possibilities $\dim\pi(\mathfrak{l}_o)\leqslant 1$.

If $\dim\pi(\mathfrak{l}_o)=0$ then $\pi(\mathfrak{F})$ is a 1-dimensional toral subalgebra in $\mathfrak{su}(2)$, and consequently $\pi(\mathfrak{f})=\pi(\sigma\mathfrak{f})=\pi(\mathfrak{q})=\pi(\sigma\mathfrak{q})$ is 1-dimensional. This contradicts the fact that $\mathfrak{q}+\sigma\mathfrak{q}$ is a hyperplane in $\mathfrak{l}$. The case when $\pi(\mathfrak{l}_o)$ is 1-dimensional remains. Then $\pi(\mathfrak{F})=\mathfrak{s}$, and possibly after replacing $\mathfrak{s}\cong\mathfrak{su}(2)$ by the Levi factor contained in $\mathfrak{F}$, we may assume that $\mathfrak{s}\subset\mathfrak{F}$. For dimensional reasons, $\mathfrak{r}_o:=\mathfrak{g}_o\cap\mathfrak{r}=\mathfrak{F}\cap\mathfrak{r}$. Further, $\mathfrak{r}_o$ is an ideal in $\mathfrak{F}$ and we have $\mathfrak{F}=\mathfrak{s}\ltimes\mathfrak{r}_o$ as well as $\mathfrak{H}=\mathfrak{s}\ltimes\mathfrak{r}_{\mathfrak{H}}$ with $\mathfrak{r}_{\mathfrak{H}}:=\mathfrak{H}\cap\mathfrak{r}$. As $\dim\mathfrak{r}_{\mathfrak{H}}-\dim\mathfrak{r}_o=2$, the ad-representation of $\mathfrak{s}$ on $\mathfrak{r}_{\mathfrak{H}}/\mathfrak{r}_o$ is trivial, that is, $[\mathfrak{s},\mathfrak{r}_{\mathfrak{H}}]\subset\mathfrak{r}_o$. Since $\mathfrak{f}=\mathfrak{l}_o\oplus\mathbb{C}\mathsf{y}$ for some $\mathsf{y}\in\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(2,\mathbb{C})$ and $\sigma\mathfrak{q}=\sigma\mathfrak{f}\oplus\mathbb{C}\bar{\mathsf{x}}$ for some $\bar{\mathsf{x}}\in\mathfrak{r}_{\mathfrak{H}}^{\mathbb{C}}$, we would have

$$[\mathfrak{f},\sigma\mathfrak{q}]=[\mathfrak{l}_o\oplus\mathbb{C}\mathsf{y},\sigma\mathfrak{f}\oplus\mathbb{C}\bar{\mathsf{x}}]\subset\mathfrak{f}+\sigma\mathfrak{f},$$

in contradiction to (V).

• $\pi(\mathfrak{l}_o)=\pi(\mathfrak{f})\subsetneq\pi(\mathfrak{q})$. The possibility $0=\pi(\mathfrak{f})$ can be excluded since otherwise $\mathfrak{F}=\mathfrak{r}$, i.e. $\mathfrak{f}+\sigma\mathfrak{f}$ would be an ideal in $\mathfrak{l}$, contradicting condition (V). Next, we deal with the case when $\pi(\mathfrak{l}_o)=\pi(\mathfrak{f})=\pi(\sigma\mathfrak{f})$ is a (1-dimensional) toral subalgebra. By assumption $\mathfrak{b}:=\pi(\mathfrak{q})$ is then 2-dimensional, that is, a Borel subalgebra, which implies that $\mathfrak{b}+\sigma\mathfrak{b}=\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(2,\mathbb{C})$, or equivalently $\pi(\mathfrak{H})=\mathfrak{s}$. Counting dimensions, $\mathfrak{r}_{\mathfrak{F}}:=\mathfrak{F}\cap\mathfrak{r}=\mathfrak{H}\cap\mathfrak{r}$. Further, $[\mathfrak{r}_{\mathfrak{F}},\mathfrak{H}]\subset\mathfrak{r}\cap\mathfrak{H}=\mathfrak{r}_{\mathfrak{F}}$ and since, due to condition (III), $\mathfrak{H}$ generates $\mathfrak{g}$ as a Lie algebra, we deduce that $\mathfrak{r}_{\mathfrak{F}}$ is an ideal in $\mathfrak{g}$. From $\pi(\mathfrak{l}_o)=\pi(\mathfrak{f})$ follows that $\mathfrak{f}=\mathfrak{l}_o\oplus\mathbb{C}\mathsf{r}$ with $\mathsf{r}\in\mathfrak{r}_{\mathfrak{F}}^{\mathbb{C}}$. But this implies that $[\mathfrak{f},\sigma\mathfrak{q}]=[\mathfrak{l}_0\oplus\mathbb{C}\mathsf{r},\sigma\mathfrak{q}]\subset\sigma\mathfrak{q}+\mathfrak{r}_{\mathfrak{F}}^{\mathbb{C}}\subset\sigma\mathfrak{q}\oplus\mathfrak{f}$, violating (V).

One last possibility remains for the flag $\mathfrak{l}_o\subset\mathfrak{f}\subset\mathfrak{q}$ in $\mathfrak{l}$, which we consider in the next claim.

Claim 13.5. *If* $\pi(\mathfrak{l}_o)\subsetneq\pi(\mathfrak{f})\subsetneq\pi(\mathfrak{q})$ *then* $\mathfrak{l}_o=0$ *and* $\mathfrak{g}\cong\mathfrak{su}(2)\times\mathfrak{r}$ *with* $\mathfrak{r}:=\mathrm{rad}(\mathfrak{g})$ *of dimension* 2.

Proof. The properness of the inclusions implies that $\mathfrak{l}_o\cap\mathfrak{r}^{\mathbb{C}}=\mathfrak{f}\cap\mathfrak{r}^{\mathbb{C}}=\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}=\sigma\mathfrak{q}\cap\mathfrak{r}^{\mathbb{C}}$. Since $\mathfrak{q}$ and $\sigma\mathfrak{q}$ generate $\mathfrak{l}$ as a Lie algebra, it follows that $\mathfrak{l}_o\cap\mathfrak{r}^{\mathbb{C}}$ is an ideal in $\mathfrak{l}$, or equivalently $\mathfrak{g}_o\cap\mathfrak{r}\lhd\mathfrak{g}$. By the effectivity assumption $(\mathsf{VI})_1$, we have $\mathfrak{g}_o\cap\mathfrak{r}=0$. Consequently, as $\dim\pi(\mathfrak{l}_o)\leqslant 1$, the same estimate holds for $\dim\mathfrak{l}_o$. Next, we show that the case $1=\dim_{\mathbb{R}}\pi(\mathfrak{g}_o)$ $(=\dim\pi(\mathfrak{l}_o)=\dim\mathfrak{l}_o)$ cannot happen. Assume on the contrary that $\pi(\mathfrak{g}_o)=1$. Then $\pi(\mathfrak{F})=\mathfrak{s}$ and, possibly after replacing the Levi factor $\mathfrak{s}\subset\mathfrak{g}$ by a conjugate one, we may assume that $\mathfrak{s}\subset\mathfrak{F}$. Counting dimensions yields then $\mathfrak{F}=\mathfrak{s}\cong\mathfrak{su}(2)$. Since $[\mathfrak{F},\mathfrak{H}]\subset\mathfrak{H}$ by (II), we have a representation of $\mathfrak{s}$ on $\mathfrak{H}/\mathfrak{F}$. But this yields the contradiction as $\dim\mathfrak{H}/\mathfrak{F}=2$ (compare (I)) implies that this representation is trivial, that is, $[\mathfrak{s},\mathfrak{H}]=[\mathfrak{F},\mathfrak{H}]\subset\mathfrak{F}$, violating (V).

We have proved that the only possibility for $\mathfrak{g}$ is $\mathfrak{su}(2)\ltimes\mathfrak{r}$ with a 2-dimensional radical $\mathfrak{r}$. Since $\mathfrak{su}(2)$ can act only trivially on such an $\mathfrak{r}$, the above semidirect product is in fact direct. This proves the claim. ∎

Claim 13.6. *The case* $\mathfrak{g}\cong\mathfrak{su}(2)\times\mathfrak{r}$, $\dim\mathfrak{r}=2$, *cannot occur either.*

Proof. We write $\pi=\pi_{\mathfrak{s}}$ and $\pi_{\mathfrak{r}}$ for the projections onto $\mathfrak{s}^{\mathbb{C}}\cong\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{r}^{\mathbb{C}}$, respectively. The proof analyses various possibilities for $\mathfrak{f}\subset\mathfrak{q}$.

If $\pi_{\mathfrak{s}}(\mathfrak{F})=0$ then $\mathfrak{F}=\mathfrak{r}$ would be an ideal in $\mathfrak{g}$ contradicting (V). The case when $\pi_{\mathfrak{s}}(\mathfrak{F})$, as well as $\pi_{\mathfrak{s}}(\mathfrak{f})$, is 1-dimensional remains. In that situation one necessarily has $\pi_{\mathfrak{s}}(\mathfrak{q})\neq\pi_{\mathfrak{s}}(\mathfrak{f})$, and $\pi(\mathfrak{q})$ is a Borel subalgebra in $\mathfrak{sl}(2,\mathbb{C})$. Further, $\pi_{\mathfrak{s}}(\mathfrak{q})+\sigma(\pi_{\mathfrak{s}}(\mathfrak{b}))=\mathfrak{sl}(2,\mathbb{C})$ and $\pi_{\mathfrak{s}}(\mathfrak{q})\cap\pi_{\mathfrak{s}}(\sigma\mathfrak{b})$ is a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$. Similar to the situation considered in §12.5, we can also here choose a *standard triple* $\mathsf{e},\mathsf{f},\mathsf{h}\in\mathfrak{s}^{\mathbb{C}}$, with $\pi_{\mathfrak{s}}(\mathfrak{q})=\mathbb{C}\mathsf{h}\oplus\mathbb{C}\mathsf{e}$ and $\sigma(\mathsf{h})=-\mathsf{h}$, but now $\sigma(\mathsf{e})=-\mathsf{f}$ for the Cartan involution σ. The radical $\mathfrak{r}$ cannot be abelian: otherwise, exactly as in the proof of Claim 12.7, we obtain a contradiction. In the remaining nonabelian case, we proceed as in the proof of Claim 12.8 by investigating the various positions of $\mathfrak{q}$ in $\mathfrak{l}$. The cases $\dim\mathfrak{r}^{\mathbb{C}}\cap\mathfrak{q}>0$ are easily ruled out (the same argument as in the proof of Claim 12.8, following (12.9)). Hence, we may assume that $\mathfrak{q}$ and $\sigma\mathfrak{q}$ are given by the formula (12.9), except that now $\sigma\mathfrak{q}=\mathbb{C}\cdot(-\mathsf{h},\bar{\lambda}\mathsf{x}+\bar{\mu}\mathsf{z})\oplus\mathbb{C}\cdot(-\mathsf{f},\bar{\nu}\mathsf{z})$, that is, the sign in front of f has changed. This slight difference is precisely the reason why in the case $\nu\neq0$, contrary to (12.10), the CR-germ (M,o) associated with $(\mathfrak{g},\mathfrak{q})$ would be Levi nondegenerate, as shown by a simple computation. This contradicts our fundamental assumption and concludes the proof of the claim. □

The proof of Lemma 13.1 is now complete. □

14. Reduction to the case where $\mathfrak{g}$ is solvable and of dimension 5

Striking the balance for the proof of Theorem II obtained so far, we have shown the following:

Let (M,o) *be an arbitrary locally homogeneous* 2*-nondegenerate CR-germ of dimension* 5 *and let* $(\mathfrak{g},\mathfrak{q})$ *be an associated CR-algebra. If the Lie algebra* $\mathfrak{g}$ *is not solvable then* M *is locally CR-equivalent at* $o\in M$ *to the tube* $\mathcal{M}$ *over the future light cone.*

For the rest of the proof we therefore assume that every CR-algebra $(\mathfrak{g},\mathfrak{q})$ under consideration satisfies the fundamental assumption in §9.8 and that $\mathfrak{g}$ is solvable. The following is the main result of this section.

Lemma 14.1. *The solvable Lie algebra* $\mathfrak{g}$ *has dimension* 5, *i.e.*, $\mathfrak{g}_o=0$.

Proof. The proof will be subdivided into several steps. Recall that by definition the *nilradical* $\mathfrak{g}^{\mathrm{nil}}$ of $\mathfrak{g}$ is the maximal nilpotent ideal in $\mathfrak{g}$. It is well known that $\mathfrak{g}^{\mathrm{nil}}$ contains the commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$, and each element $\xi\in\mathfrak{g}^{\mathrm{nil}}$ is ad-nilpotent in $\mathfrak{g}$. Similarly, we denote the (complex) nilradical of $\mathfrak{l}$ by $\mathfrak{l}^{\mathrm{nil}}$. We retain the notation from §9.4 and §9.5.

CLAIM 14.2. $\mathfrak{g}_o \subset \mathfrak{g}^{\mathrm{nil}}$.

Proof. Since $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}^{\mathrm{nil}}$, the quotient $\mathfrak{g}/\mathfrak{g}^{\mathrm{nil}}$ is abelian. Let $\pi\colon\mathfrak{g}\to\mathfrak{g}/\mathfrak{g}^{\mathrm{nil}}$ be the projection. Assume that $\mathfrak{g}_o\not\subset\mathfrak{g}^{\mathrm{nil}}$, i.e., $\pi(\mathfrak{g}_o)\neq 0$. Select a subalgebra $\mathfrak{r}\subset\mathfrak{g}/\mathfrak{g}^{\mathrm{nil}}$ (possibly 0) which is a complement of $\pi(\mathfrak{g}_o)$. But then $\mathfrak{g}':=\pi^{-1}(\mathfrak{r})$ would violate $(\mathsf{VI})_2$. Consequently, we necessarily have $\pi(\mathfrak{g}_o)=0$, that is, $\mathfrak{g}_o\subset\mathfrak{g}^{\mathrm{nil}}$ as claimed. □

14.3. Recall that for the CR-algebra $(\mathfrak{g},\mathfrak{q})$ under consideration there exist flags $\mathfrak{l}_o\subset\mathfrak{f}\subset\mathfrak{q}$ and $\mathfrak{l}_o\subset\sigma\mathfrak{f}\subset\sigma\mathfrak{q}$ of subalgebras in $\mathfrak{l}$, as defined in §9.4. For the subsequent considerations, we select the following elements in $\mathfrak{l}$: $\mathsf{y}\in\mathfrak{f}\setminus\mathfrak{l}_o$ and $\mathsf{x}\in\mathfrak{q}\setminus\mathfrak{f}$, and write $\bar{\mathsf{x}}:=\sigma(\mathsf{x})$ and $\bar{\mathsf{y}}:=\sigma(\mathsf{y})$, for short. Then

$$\mathfrak{f}=\mathbb{C}\mathsf{y}\oplus\mathfrak{l}_o,\quad \mathfrak{q}=\mathbb{C}\mathsf{x}\oplus\mathbb{C}\mathsf{y}\oplus\mathfrak{l}_o,\quad \sigma\mathfrak{f}=\mathbb{C}\bar{\mathsf{y}}\oplus\mathfrak{l}_o\quad\text{and}\quad \sigma\mathfrak{q}=\mathbb{C}\bar{\mathsf{x}}\oplus\mathbb{C}\bar{\mathsf{y}}\oplus\mathfrak{l}_o.$$

The inclusion $\mathfrak{l}_o\subset\mathfrak{l}^{\mathrm{nil}}$ is guaranteed by Claim 14.2. Hence, $\mathfrak{l}_o$ acts by ad-nilpotent endomorphisms on $\mathfrak{l}$. In particular, $[\mathfrak{l}_o,\mathfrak{f}]\subset\mathfrak{l}_o$ and $[\mathfrak{l}_o,\mathfrak{q}]\subset\mathfrak{f}$. The condition (III) means that $[\mathsf{x},\bar{\mathsf{x}}]\notin\mathfrak{q}+\sigma\mathfrak{q}$, and (V) is equivalent to $[\bar{\mathsf{x}},\mathsf{y}]\notin\mathfrak{f}+\sigma\mathfrak{q}$.

Let $\mathfrak{z}\subset\mathfrak{l}^{\mathrm{nil}}$ be the center of the nilradical (which is nontrivial if $\mathfrak{l}\neq 0$), to which we refer as the *nilcenter* of $\mathfrak{l}$. As for every characteristic ideal, we have $\sigma(\mathfrak{l}^{\mathrm{nil}})=\mathfrak{l}^{\mathrm{nil}}$ and $\sigma(\mathfrak{z})=\mathfrak{z}$.

CLAIM 14.4. (i) *The nilcenter* $\mathfrak{z}$ *of* $\mathfrak{l}$ *is not contained in* $\mathfrak{q}+\sigma\mathfrak{q}$.
(ii) $\mathfrak{l}_o=0$ *if* $\dim\mathfrak{z}\geqslant 2$.

Proof. (i) Assume that (i) is not true and let $\mathsf{x},\mathsf{y},\bar{\mathsf{x}},\bar{\mathsf{y}}\in\mathfrak{l}$ be as in §14.3. Let

$$\zeta:=a\mathsf{x}+\bar{a}\bar{\mathsf{x}}+b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o,$$

with $\gamma_o\in\mathfrak{g}_o$ and $a,b\in\mathbb{C}$, be an arbitrary element in $\mathfrak{z}^\sigma$. Then $[\bar{\mathsf{x}},\zeta]\in\mathfrak{z}\subset\mathfrak{q}+\sigma\mathfrak{q}$, since $\mathfrak{z}$ is an ideal in $\mathfrak{l}$. On the other hand,

$$[\bar{\mathsf{x}},a\mathsf{x}+\bar{a}\bar{\mathsf{x}}+b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o]=a[\bar{\mathsf{x}},\mathsf{x}]+[\bar{\mathsf{x}},b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o]\equiv a[\bar{\mathsf{x}},\mathsf{x}]\equiv 0\mod\mathfrak{q}+\sigma\mathfrak{q},$$

which implies that $a=0$. This shows that $\mathfrak{z}\subset\mathfrak{f}+\sigma\mathfrak{f}$. Given then $\zeta=b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o\in\mathfrak{z}^\sigma$, the inclusion $[\bar{\mathsf{x}},\zeta]\subset\mathfrak{z}$ holds, since $\mathfrak{z}$ is an ideal. On the other hand,

$$[\bar{\mathsf{x}},b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o]=b[\bar{\mathsf{x}},\mathsf{y}]+[\bar{\mathsf{x}},\bar{b}\bar{\mathsf{y}}+\gamma_o]\equiv b[\bar{\mathsf{x}},\mathsf{y}]\equiv 0\mod\mathfrak{f}+\sigma\mathfrak{q}.$$

Hence, $b=0$ as a consequence of the above equation and (V). But this cannot be true, since then $\mathfrak{z}\subset\mathfrak{l}_o$ would be a nontrivial ideal of $\mathfrak{l}$, violating $(\mathsf{VI})_1$.

(ii) Assume that $\dim\mathfrak{z}\geqslant 2$, that is, $\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})\neq 0$. Recall that $\mathfrak{l}_o\subset\mathfrak{l}^{\mathrm{nil}}$, by Claim 14.2, and consequently $[\mathsf{y},\mathfrak{l}_o]\subset\mathfrak{l}_o$ and $[\bar{\mathsf{y}},\mathfrak{l}_o]\subset\mathfrak{l}_o$. Let an arbitrary

$$\zeta:=a\mathsf{x}+\bar{a}\bar{\mathsf{x}}+b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o\in\mathfrak{z}^\sigma\cap(\mathfrak{q}+\sigma\mathfrak{q})$$

be given. Computing its bracket with $\mathfrak{l}_o$, we obtain

$$[\zeta,\mathfrak{l}_o]=a[\mathsf{x},\mathfrak{l}_o]+\bar{a}[\bar{\mathsf{x}},\mathfrak{l}_o]\equiv 0 \mod \mathfrak{l}_o.$$

Either $a=0$ for all such ζ, and then $\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}+\sigma\mathfrak{f})$, or $a\neq 0$, and then

$$[\mathsf{x},\mathfrak{l}_o]\subset\mathfrak{l}_o\supset[\bar{\mathsf{x}},\mathfrak{l}_o].$$

In the latter case, it follows that $\mathfrak{l}_o$ is an ideal in $\mathfrak{l}$ (due to Lemma 9.7 (III), x, y, $\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$ and $\mathfrak{l}_o$ generate $\mathfrak{l}$ as a Lie algebra), hence $\mathfrak{l}_o=0$ as claimed. The other possibility would be $\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}+\sigma\mathfrak{f})$ and we show that this cannot happen. Given an arbitrary $\zeta=b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o\in\mathfrak{z}^\sigma\cap(\mathfrak{f}+\sigma\mathfrak{f})$, note that

$$[\bar{\mathsf{x}},\zeta]\in\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}+\sigma\mathfrak{f}).$$

More explicitly, $[\bar{\mathsf{x}},b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o]=b[\bar{\mathsf{x}},\mathsf{y}]+[\bar{\mathsf{x}},\bar{b}\bar{\mathsf{y}}+\gamma_o]\in b[\bar{\mathsf{x}},\mathsf{y}]+\sigma\mathfrak{q}$. But then the above equation together with (V) would imply $b=0$, i.e., $\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})=\mathfrak{z}\cap\mathfrak{l}_o\neq 0$. This is absurd, as

$$[\mathfrak{q}+\sigma\mathfrak{q},\mathfrak{z}\cap\mathfrak{l}_o]\subset\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})=\mathfrak{z}\cap\mathfrak{l}_o\neq 0,$$

i.e., since $\mathfrak{q}+\sigma\mathfrak{q}$ generates $\mathfrak{l}$ as a Lie algebra, $\mathfrak{z}\cap\mathfrak{l}_o$ would be a nontrivial ideal in $\mathfrak{l}$. ☐

The case when the nilcenter is 1-dimensional remains.

CLAIM 14.5. *Suppose* $\dim\mathfrak{z}=1$. *Then* $\mathfrak{l}=\mathfrak{z}\oplus(\mathfrak{q}+\sigma\mathfrak{q})$ *and* $\mathfrak{g}_o=\mathfrak{l}_o=0$.

Proof. Since $\mathfrak{z}\not\subset\mathfrak{q}+\sigma\mathfrak{q}$ by Claim 14.4, the sum $\mathfrak{z}+(\mathfrak{q}+\sigma\mathfrak{q})$ is direct. Recall that the recursively defined subspaces $C_0(\mathfrak{l}^{\mathrm{nil}}):=0$ and $C_k:=C_k(\mathfrak{l}^{\mathrm{nil}}):=\{u\in\mathfrak{l}^{\mathrm{nil}}:[u,\mathfrak{l}^{\mathrm{nil}}]\subset C_{k-1}(\mathfrak{l}^{\mathrm{nil}})\}$ for every $k>0$, form the ascending central series of $\mathfrak{l}^{\mathrm{nil}}$. Clearly, $\mathfrak{z}=C_1$ and $\sigma(C_k)=C_k$ for all k. Either $\mathfrak{z}=C_1=C_2=\mathfrak{l}^{\mathrm{nil}}$, and consequently $\mathfrak{l}_o=0$ (due to $(\mathsf{VI})_1$, $\mathfrak{l}_o$ must be a proper subalgebra of the 1-dimensional algebra $\mathfrak{l}^{\mathrm{nil}}$), or $C_1\neq C_2$. In the latter case $C_2\cap(\mathfrak{q}+\sigma\mathfrak{q})\neq 0$. Let $\eta=a\mathsf{x}+\bar{a}\bar{\mathsf{x}}+b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o\in C_2^\sigma\cap(\mathfrak{q}+\sigma\mathfrak{q})$ be arbitrary. Since $[\eta,\mathfrak{l}_o]\subset\mathfrak{z}\cap(\mathfrak{q}+\sigma\mathfrak{q})=0$, we have

$$[\eta,\mathfrak{l}_o]\equiv a[\mathsf{x},\mathfrak{l}_o]+\bar{a}[\bar{\mathsf{x}},\mathfrak{l}_o]\equiv 0 \mod \mathfrak{l}_o.$$

If $a\neq 0$ then $[\mathsf{x},\mathfrak{l}_o],[\bar{\mathsf{x}},\mathfrak{l}_o]\subset\mathfrak{l}_o$ and $\mathfrak{l}_o$ is an ideal in $\mathfrak{l}$, that is, $\mathfrak{l}_o=0$ by $(\mathsf{VI})_1$. If $a=0$ for every choice of $\eta\in C_2^\sigma\cap(\mathfrak{q}+\sigma\mathfrak{q})$ as above, then $C_2\cap(\mathfrak{q}+\sigma\mathfrak{q})=C_2\cap(\mathfrak{f}+\sigma\mathfrak{f})$. This possibility can be ruled out as follows. For a nonzero $\eta{=}b\mathsf{y}{+}\bar{b}\bar{\mathsf{y}}{+}\gamma_o{\in}C_2^\sigma{\cap}(\mathfrak{f}{+}\sigma\mathfrak{f})$ we have $[\eta,\mathsf{x}]{\in}(\mathfrak{q}{+}\sigma\mathfrak{q}){\cap}C_2{=}(\mathfrak{f}{+}\sigma\mathfrak{f}){\cap}C_2$, that is,

$$[b\mathsf{y}+\bar{b}\bar{\mathsf{y}}+\gamma_o,\mathsf{x}]=\bar{b}[\bar{\mathsf{y}},\mathsf{x}]+([b\mathsf{y}+\gamma_o,\mathsf{x}])\equiv 0 \mod \mathfrak{f}+\sigma\mathfrak{f},$$

which is only possible if $\bar{b}{=}0$. This would imply that $(\mathfrak{q}{+}\sigma\mathfrak{q}){\cap}C_2{=}\mathfrak{l}_o{\cap}C_2$. On the other hand, the identity $(\mathfrak{q}{+}\sigma\mathfrak{q}){\cap}\, C_2{=}\mathfrak{l}_o{\cap}C_2$ implies that $\mathfrak{l}_o{\cap}C_2$ is an ideal of $\mathfrak{l}$. The effectivity of $(\mathfrak{g},\mathfrak{q})$ then forces $\mathfrak{l}_o{\cap}C_2{=}0$, contradicting $(\mathfrak{q}{+}\sigma\mathfrak{q}){\cap}C_2{\neq}0$. This completes the proof of Claim 14.5. □

The proof of Lemma 14.1 is now complete. □

15. The existence of a 3-dimensional abelian ideal in $\mathfrak{g}$ suffices

The proof of Theorem II has brought us to the point where we may and do henceforth assume that the Lie algebra $\mathfrak{g}$ in the CR-algebra $(\mathfrak{g},\mathfrak{q})$ satisfying the assumption in §9.8 is solvable. The subalgebra $\mathfrak{q}{\subset}\mathfrak{l}{=}\mathfrak{g}{\oplus}i\mathfrak{g}$ then necessarily is of complex dimension 2.

The Main lemma 15.14 of this section states that the CR-germ associated with $(\mathfrak{g},\mathfrak{q})$ is represented by the tube $F+i\mathbb{R}^3$ over an affinely homogeneous surface $F{\subset}\mathbb{R}^3$ if $\mathfrak{g}$ is isomorphic to a semidirect product $\mathfrak{h}{\ltimes}\mathfrak{r}$, with $\mathfrak{h}$ being a 2-dimensional Lie subalgebra and $\mathfrak{r}{\cong}\mathbb{R}^3$ being an abelian ideal. Once this main lemma is proved, our proof of the classification theorem will be complete as soon as we can show that every 5-dimensional solvable Lie algebra $\mathfrak{g}$ occurring in $(\mathfrak{g},\mathfrak{q})$ indeed is isomorphic to a semidirect product as above. This will be achieved in the final §16. In this section we only prove the partial result that if $\mathfrak{q}$ is abelian, then also the commutator $[\mathfrak{g},\mathfrak{g}]$ is abelian and 3-dimensional. Moreover, there exists an abelian subalgebra $\mathfrak{h}{\subset}\mathfrak{g}$ such that $\mathfrak{g}{=}\mathfrak{h}{\ltimes}[\mathfrak{g},\mathfrak{g}]$.

Since there is no general structure theory for solvable Lie algebras, we develop ad hoc methods and describe the structure constants in $\mathfrak{l}{=}\mathfrak{g}^{\mathbb{C}}$ with respect to a particularly chosen basis. Every CR-algebra $(\mathfrak{g},\mathfrak{q})$ under consideration gives rise to the 1-dimensional subalgebras $\mathfrak{f}$ and $\sigma\mathfrak{f}$ of the 2-dimensional subalgebras $\mathfrak{q}$ and $\sigma\mathfrak{q}$, respectively. We construct a basis of $\mathfrak{l}$ which reflects the conditions (I)–(V) and investigate the various possibilities for the values of the corresponding structure constants. Select a nonzero $\mathsf{z}{\in}\mathfrak{z}^{-\sigma}\backslash(\mathfrak{q}{+}\sigma\mathfrak{q})$ (this is possible due to Claim 14.5) and an $\mathsf{x}{\in}\mathfrak{q}\backslash\mathfrak{f}$ such that for $\bar{\mathsf{x}}{:=}\sigma\mathsf{x}$ the congruence $[\mathsf{x},\bar{\mathsf{x}}]{\equiv}\mathsf{z}$ mod $\mathfrak{q}{+}\sigma\mathfrak{q}$ holds (this is possible due to (III)). By (V), it is further possible to select $\mathsf{y}{\in}\mathfrak{f}\backslash\mathfrak{l}_o$ such that for $\bar{\mathsf{y}}{:=}\sigma\mathsf{y}$ the structure equations of $\mathfrak{l}$ are of the following form (in particular, the coefficient in front of $\bar{\mathsf{x}}$ in the second equation is $b_2=1$):

$$\begin{aligned}
[\mathsf{x},\bar{\mathsf{x}}] &= \mathsf{z}+a_1\mathsf{x}-\bar{a}_1\bar{\mathsf{x}}+\ a_2\mathsf{y}-\bar{a}_2\bar{\mathsf{y}}, \\
[\mathsf{x},\bar{\mathsf{y}}] &= \qquad b_1\mathsf{x}+\quad \bar{\mathsf{x}}+\ b_3\mathsf{y}+b_4\bar{\mathsf{y}}, \\
[\mathsf{y},\bar{\mathsf{y}}] &= \qquad\qquad\qquad\quad c\mathsf{y}-\ \bar{c}\bar{\mathsf{y}}, \\
[\mathsf{y},\mathsf{x}] &= \qquad d_1\mathsf{x}\qquad\ +d_2\mathsf{y}, \\
[\mathsf{z},\eta] &\in \mathfrak{z}(\mathfrak{l}^{\mathrm{nil}}) \qquad\qquad\qquad\qquad \text{for every } \eta\in\mathfrak{l}.
\end{aligned} \tag{15.1}$$

The brackets $[\mathsf{y},\bar{\mathsf{x}}]$ and $[\bar{\mathsf{y}},\bar{\mathsf{x}}]$ are completely determined by the above five equations, due to the fact that $\sigma\colon\mathfrak{l}\to\mathfrak{l}$ is an antilinear Lie algebra automorphism. Of course, not for all values of the constants the above identities give rise to a Lie algebra. In fact the above structure constants $a_1, ..., d_2$, are subject to further constraints, imposed by (1) the Jacobi identity, (2) our assumption that $\mathfrak{l}$ is solvable and, (3) our assumption $\mathsf{z}\in\mathfrak{z}(\mathfrak{l}^{\mathrm{nil}})$.

Conversely, let a 5-dimensional solvable complex Lie algebra

$$\mathfrak{l}=\mathbb{C}\mathsf{z}\oplus\mathbb{C}\mathsf{x}\oplus\mathbb{C}\bar{\mathsf{x}}\oplus\mathbb{C}\mathsf{y}\oplus\mathbb{C}\bar{\mathsf{y}}$$

be given with structure equations as in (15.1) (together with $[\bar{\mathsf{x}},\mathsf{y}]=\mathsf{x}+\bar{b}_1\bar{\mathsf{x}}+\bar{b}_4\mathsf{y}+\bar{b}_3\bar{\mathsf{y}}$ and $[\bar{\mathsf{y}},\bar{\mathsf{x}}]=\bar{d}_1\bar{\mathsf{x}}+\bar{d}_2\bar{\mathsf{y}}$) for certain $a_1, .., d_2\in\mathbb{C}$ and $\mathsf{z}\in\mathfrak{z}^{-\sigma}$, where $\mathfrak{z}$ is the nilcenter of $\mathfrak{l}$. Define

$$\mathfrak{g}:=\mathbb{R}i\mathsf{z}\oplus\mathbb{R}(\mathsf{x}+\bar{\mathsf{x}})\oplus\mathbb{R}(i\mathsf{x}-i\bar{\mathsf{x}})\oplus\mathbb{R}(\mathsf{y}+\bar{\mathsf{y}})\oplus\mathbb{R}(i\mathsf{y}-i\bar{\mathsf{y}}) \quad\text{and}\quad \mathfrak{q}:=\mathbb{C}\mathsf{x}\oplus\mathbb{C}\mathsf{y}.$$

Then the CR-algebra $(\mathfrak{g},\mathfrak{q})$ satisfies the fundamental assumption in §9.8.

As already mentioned, besides the geometrically motivated conditions in Lemma 9.7, which are already incorporated in (15.1), further conditions affect the particular values of the structure constants, for example those given by the Jacobi identity. We use

$$[[\xi_1,\xi_2,\xi_3]]:=[[\xi_1,\xi_2],\xi_3]+[[\xi_2,\xi_3],\xi_1]+[[\xi_3,\xi_1],\xi_2]$$

as shorthand. The identity $[[\mathsf{x},\mathsf{y},\bar{\mathsf{y}}]]=0$ implies that

$$|c|=1,\quad d_1=\bar{c}-\bar{b}_1 \quad\text{and}\quad \bar{c}b_1\in\mathbb{R}. \tag{15.2}$$

Remark 15.3. From $c\neq0$, that is, $[\mathsf{y},\bar{\mathsf{y}}]\neq0$, it follows that the solvable subalgebra $\mathfrak{f}+\sigma\mathfrak{f}$, and in turn $\mathfrak{l}$, cannot be nilpotent.

Keeping in mind the identities (15.2), it is possible to readjust the basis $\mathsf{x}, ..., \mathsf{z}$ of $\mathfrak{l}$ as follows. Write $c=f^2$ for the coefficient c in the third equation of (15.1) and replace x by $f\mathsf{x}$, as well as y by $c\mathsf{y}$. After this replacement, the structure equations (15.1) keep their form, only c changes to $c=1$ and b_1 becomes real.

Notational agreement. For the rest of this section we use α, β, γ and δ to denote real numbers, while a_j, b_j, c_j and d_j stand for complex numbers. In particular, we write $\beta_1 := b_1$ to underline that the structure constant b_1 is real.

The Jacobi identity $[[\mathsf{x},\mathsf{y},\bar{\mathsf{y}}]]=0$ implies more relations between the coefficients in the structure equations (15.1):

$$c=1, \quad \beta_1 := b_1 \in \mathbb{R}, \quad d_1 = 1-\beta_1, \quad \bar{b}_3 - d_2 = b_4(\beta_1 - 1), \quad -\beta_1(b_3+d_2) = b_4 - \bar{b}_4. \quad (15.4)$$

15.5. Perfect basis

Summarizing, there exists a basis $\mathsf{z},\mathsf{x},\bar{\mathsf{x}},\mathsf{y},\bar{\mathsf{y}}$ ($\bar{\mathsf{x}}:=\sigma(\mathsf{x})$ and $\bar{\mathsf{y}}:=\sigma(\mathsf{y})$) of $\mathfrak{l}=\mathfrak{g}^{\mathbb{C}}$ with $\mathsf{z}\in\mathfrak{z}^{-\sigma}\setminus(\mathfrak{q}\oplus\sigma\mathfrak{q})$, $\mathfrak{q}=\mathbb{C}\mathsf{x}\oplus\mathbb{C}\mathsf{y}$ and $\mathfrak{f}=\mathbb{C}\mathsf{y}$, such that the corresponding structure constants in (15.1) satisfy (15.4). We call each such basis *perfect*.

There are still more constraints for the structure constants given by the Jacobi identity for other triples of elements and also by the fact that $\mathfrak{z}$ is in the center of $\mathfrak{l}^{\mathrm{nil}}$. Later on, we characterize these additional conditions more explicitly. For now, we elaborate the particular structure of the nilcenter in the next result.

LEMMA 15.6. *The nilcenter* $\mathfrak{z}\subset\mathfrak{g}$ *has dimension* 1 *or* 3.

Proof. We closely analyze the conditions in Lemma 9.7 in order to get the dimension estimates. It is clear that $1\leqslant\dim\mathfrak{z}\leqslant4$; see Remark 15.3. Assume that $\dim\mathfrak{z}=4$. Then $\dim\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})=3$ follows by Lemma 14.4. But since $\mathfrak{f}+\sigma\mathfrak{f}$ is not abelian, that is,

$$\dim\mathfrak{z}\cap(\mathfrak{f}\oplus\sigma\mathfrak{f})=1,$$

there exist elements in $\mathfrak{z}$ of the form $\mathsf{x}+\eta_1$ and $\bar{\mathsf{x}}+\eta_2$ with $\eta_j\in\mathfrak{f}+\sigma\mathfrak{f}$. This leads to a contradiction: indeed, on the one hand $[\mathsf{x}+\eta_1,\bar{\mathsf{x}}+\eta_2]=0$, and on the other hand (V) implies that $[\mathsf{x}+\eta_1,\bar{\mathsf{x}}+\eta_2]\equiv[\mathsf{x},\bar{\mathsf{x}}]\not\equiv0 \bmod \mathfrak{q}+\sigma\mathfrak{q}$. Hence, we have proved that $1\leqslant\dim\mathfrak{z}\leqslant3$.

The main difficulty is to rule out the possibility $\dim\mathfrak{z}=2$. Select a perfect basis in $\mathfrak{l}$ as described in §15.5, keeping in mind (15.1) and (15.4). We have to deal with two subcases.

CLAIM 15.7. *If* $[\mathsf{x},\mathsf{y}]\neq0$, *i.e.*, $\mathfrak{q}$ *is not abelian*, *then* $\dim\mathfrak{z}=1$.

Proof. We first show that $\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}\oplus\sigma\mathfrak{f})$. Select an arbitrary

$$\mathsf{z}' := \lambda\mathsf{x}+\bar{\lambda}\bar{\mathsf{x}}+\mu\mathsf{y}+\bar{\mu}\bar{\mathsf{y}}\in\mathfrak{z}^{\sigma}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q}).$$

We have to investigate the two possibilities $\beta_1\neq1$ and $\beta_1=1$. In the first case we get

$$[\mathsf{z}',(1-\beta_1)\mathsf{x}+d_2\mathsf{y}]\equiv\bar{\lambda}(1-\beta_1)[\bar{\mathsf{x}},\mathsf{x}]\equiv0 \mod \mathfrak{q}+\sigma\mathfrak{q},$$

since $[\mathsf{y},\mathsf{x}]\in\mathfrak{l}^{\mathrm{nil}}$. Therefore, $\lambda=0$ by the condition (III), i.e., $\mathsf{z}'\in\mathfrak{f}+\sigma\mathfrak{f}$. If $\beta_1=1$, i.e., $d_1=0$, then, by our assumption, $d_2\neq0$, and in turn $\mathsf{y}\in\mathfrak{l}^{\mathrm{nil}}$. Hence, $[\mathsf{z}',\mathsf{y}]\equiv\bar\lambda[\bar{\mathsf{x}},\mathsf{y}]\equiv0 \bmod \mathfrak{f}+\sigma\mathfrak{f}$, which, together with (V), also forces $\lambda=0$. This proves that $\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}\oplus\sigma\mathfrak{f})$.

We claim that this identity can only hold if both sides vanish, i.e., $\dim\mathfrak{z}=1$ by Lemma 14.4 (i). Assuming on the contrary that $\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})\neq0$, then on the one hand there exists $\mathsf{z}'=\mu\mathsf{y}+\bar\mu\bar{\mathsf{y}}\in\mathfrak{z}\cap(\mathfrak{f}\oplus\sigma\mathfrak{f})$ with $\mu\neq0$. On the other hand,

$$[\mathsf{x},\mathsf{z}']=[\mathsf{x},\mu\mathsf{y}+\bar\mu\bar{\mathsf{y}}]\in\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}\oplus\sigma\mathfrak{f}),$$

which in view of (V) is only possible if $\mu=0$. This shows that $\dim\mathfrak{z}=1$. □

It remains to rule out the second subcase.

CLAIM 15.8. *If* $[\mathsf{y},\mathsf{x}]=0$, *i.e.*, $\mathfrak{q}$ *is abelian*, *then* $[\mathfrak{l},\mathfrak{l}]$ *is a* 3-*dimensional abelian ideal. Further*, *if* $\dim\mathfrak{z}\geqslant2$ *then* $[\mathfrak{l},\mathfrak{l}]=\mathfrak{l}^{\mathrm{nil}}$, *and consequently* $\mathfrak{z}=[\mathfrak{l},\mathfrak{l}]$ *is* 3-*dimensional.*

Proof. This is the most involved case. The assumption $[\mathsf{y},\mathsf{x}]=0$, that is, $d_1=d_2=0$, implies $\beta_1=1$, $b_3=0$ and $b_4:=\beta_4\in\mathbb{R}$, see the table below:

$$\begin{aligned}
[\mathsf{x},\bar{\mathsf{x}}]&=\mathsf{z}+a_1\mathsf{x}-\bar a_1\bar{\mathsf{x}}+a_2\mathsf{y}-\bar a_2\bar{\mathsf{y}},\\
[\mathsf{x},\bar{\mathsf{y}}]&=\mathsf{x}+\bar{\mathsf{x}}+\beta_4\bar{\mathsf{y}},\\
[\mathsf{y},\bar{\mathsf{y}}]&=\mathsf{y}-\bar{\mathsf{y}},\\
[\mathsf{y},\mathsf{x}]&=0,\\
[\mathsf{z},\eta]&=c_\eta\mathsf{z}+\mathsf{q}_\eta, \qquad \eta\in\mathfrak{l},\ \mathsf{q}_\eta\in\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q}).
\end{aligned}\tag{15.9}$$

We need to analyze the relations between the structure constants in more detail. Let

$$\mathsf{q}_\mathsf{x}=z_1\mathsf{x}+z_2\bar{\mathsf{x}}+z_3\mathsf{y}+z_4\bar{\mathsf{y}} \quad\text{and}\quad \mathsf{q}_\mathsf{y}=w_1\mathsf{x}+w_2\bar{\mathsf{x}}+w_3\mathsf{y}+w_4\bar{\mathsf{y}}.$$

The fact that the q_η's commute with all elements of $[\mathfrak{l},\mathfrak{l}]\subset\mathfrak{l}^{\mathrm{nil}}$ (and in particular with $\mathsf{y}-\bar{\mathsf{y}}$) implies that $z_1=z_2$, $w_1=w_2$, $z_4=z_1\beta_4-z_3$ and $w_4=w_1\beta_4-w_3$. The Jacobi identity $[[\mathsf{x},\bar{\mathsf{x}},\mathsf{y}]]=0$ yields

$$w_j=0,\quad c_\mathsf{y}=1,\ \text{i.e. } [\mathsf{z},\mathsf{y}]=[\mathsf{z},\bar{\mathsf{y}}]=\mathsf{z},\quad a_1=i\alpha_1\quad\text{and}\quad a_2=\alpha_2+\tfrac12 i\alpha_1\beta_4 \tag{15.10}$$

for some $\alpha_1,\alpha_2\in\mathbb{R}$, and $[[\bar{\mathsf{x}},\mathsf{y},\mathsf{z}]]=0$ implies that

$$\operatorname{Im} z_1=0,\quad \operatorname{Re} z_3=\tfrac12 z_1\beta_4\quad\text{and}\quad \operatorname{Re} c_\mathsf{x}=-\tfrac12\beta_4. \tag{15.11}$$

Summarizing,

$$[\mathfrak{l},\mathfrak{l}]=\mathbb{C}\mathsf{z}\oplus\mathbb{C}\big(\mathsf{x}+\bar{\mathsf{x}}+\tfrac12\beta_4\mathsf{y}+\tfrac12\beta_4\bar{\mathsf{y}}\big)\oplus\mathbb{C}(\mathsf{y}-\bar{\mathsf{y}}), \tag{15.12}$$

and, using the above relations between the structure constants, a simple check shows that this ideal is abelian. Moreover, the subset

$$\mathfrak{h} := \mathbb{R}i\big(\mathsf{x}-\bar{\mathsf{x}}+\tfrac{1}{2}\beta_4\mathsf{y}-\tfrac{1}{2}\beta_4\bar{\mathsf{y}}\big)\oplus\mathbb{R}(\mathsf{y}+\bar{\mathsf{y}}) \tag{15.13}$$

is an abelian subalgebra of $\mathfrak{g}$ (and $\mathfrak{g}=\mathfrak{h}\oplus[\mathfrak{g},\mathfrak{g}]$; in fact $\mathfrak{g}=\mathfrak{h}\ltimes[\mathfrak{g},\mathfrak{g}]$).

It should be noted that the nilcenter $\mathfrak{z}$ may be 1-dimensional, and then $\mathfrak{l}^{\mathrm{nil}}$ properly contains $[\mathfrak{l},\mathfrak{l}]$.

We next show that the case $\dim\mathfrak{z}=2$ does not occur. Assume on the contrary that $\dim\mathfrak{z}=2$. Then $\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})$ is nonzero. Let $\mathsf{z}':=\lambda\mathsf{x}+\bar{\lambda}\bar{\mathsf{x}}+\mu\mathsf{y}+\bar{\mu}\bar{\mathsf{y}}\in\mathfrak{z}^\sigma\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})$ be arbitrary. Either, for all such z' the coefficient λ is 0, that is, $\mathfrak{z}\cap(\mathfrak{q}\oplus\sigma\mathfrak{q})=\mathfrak{z}\cap(\mathfrak{f}\oplus\sigma\mathfrak{f})$, and then this case can be ruled out by a similar argument as in the proof of the above claim. Or, there exists z' with $\lambda\neq0$. In such a situation the identity $[\mathsf{z}',\mathsf{y}-\bar{\mathsf{y}}]=0$ implies that $\lambda\in\mathbb{R}^*$ and, without loss of generality, we may suppose that $\mathsf{z}'=\mathsf{x}+\bar{\mathsf{x}}+z_3\mathsf{y}+\bar{z}_3\bar{\mathsf{y}}$ for some $z_3\in\mathbb{C}$. The condition $[\mathsf{z},\mathsf{z}']=0$ gives $\operatorname{Re} z_3=\frac{1}{2}\beta_4$. But then

$$\mathsf{z}'=\big(\mathsf{x}+\bar{\mathsf{x}}+\tfrac{1}{2}\beta_4\cdot(\mathsf{y}+\bar{\mathsf{y}})\big)+i\operatorname{Im} z_3\cdot(\mathsf{y}-\bar{\mathsf{y}})\in[\mathfrak{l},\mathfrak{l}],$$

compare (15.12). We claim that, in the situation under consideration, $\mathfrak{l}^{\mathrm{nil}}=[\mathfrak{l},\mathfrak{l}]$. To prove this, we simply compute the ad-action of $\mathsf{y}+\bar{\mathsf{y}}$ and $\mathsf{x}-\bar{\mathsf{x}}$ on $[\mathfrak{l},\mathfrak{l}]$. Since $[\mathsf{x}-\bar{\mathsf{x}},\mathsf{z}']\in\mathbb{C}\mathsf{z}\oplus\mathbb{C}\mathsf{z}'$, the relation $\beta_4^2+4(\operatorname{Im} z_3)^2=4\alpha_1\operatorname{Im} z_3+4\alpha_2$ must also be fulfilled. Once again, a simple computation yields

$$\operatorname{ad}(\mathsf{y}+\bar{\mathsf{y}})|_{[\mathfrak{l},\mathfrak{l}]}=-2\cdot\mathrm{id}\quad\text{and}\quad[\mathsf{x}-\bar{\mathsf{x}},\mathsf{z}']=2\mathsf{z}+2i(\alpha_1-\operatorname{Im} z_3)\mathsf{z}'.$$

The above identities show that for every $\mathsf{v}:=u_1\cdot(\mathsf{x}-\bar{\mathsf{x}})+u_2(\mathsf{y}+\bar{\mathsf{y}})+u_3\eta$, $u_j\in\mathbb{C}$, $\eta\in[\mathfrak{l},\mathfrak{l}]$, the condition $[\mathsf{v},\mathsf{z}']=0$ implies that $u_1=u_2=0$, i.e., the centralizer $C_{\mathfrak{l}}(\mathsf{z}')$ coincides with $[\mathfrak{l},\mathfrak{l}]$. This proves $[\mathfrak{l},\mathfrak{l}]=\mathfrak{l}^{\mathrm{nil}}=C_{\mathfrak{l}}(\mathsf{z}')$. But this is absurd, since then the nilcenter $\mathfrak{z}$ would coincide with the 3-dimensional abelian ideal $[\mathfrak{l},\mathfrak{l}]$, contrary to our assumption $\dim\mathfrak{z}=2$.

Finally, we need to investigate the case $\dim\mathfrak{z}=3$. We claim that $\mathfrak{z}=[\mathfrak{l},\mathfrak{l}]=\mathfrak{l}^{\mathrm{nil}}$. To see this, it is enough to show that $\mathfrak{l}^{\mathrm{nil}}$ is 3-dimensional, as, due to (15.12), $[\mathfrak{l},\mathfrak{l}]$ is 3-dimensional too. As already mentioned (see the sentence following (15.2)), $\mathfrak{l}^{\mathrm{nil}}$ can be at most 4-dimensional. But the 4-dimensional case can be excluded, otherwise $\mathfrak{l}^{\mathrm{nil}}=\mathfrak{z}\oplus\mathbb{C}\mathsf{n}$ for $\mathsf{n}\in\mathfrak{l}^{\mathrm{nil}}\setminus[\mathfrak{l},\mathfrak{l}]$, which would imply that $\mathfrak{l}^{\mathrm{nil}}$ is abelian. Hence, the nilradical is 3-dimensional. This proves Claim 15.8. □

The proof of Lemma 15.6 is now complete. □

The next statement is one of the key points in our classification of 5-dimensional 2-nondegenerate homogeneous CR-germs. Before stating it, we first fix some notation. Given a vector space V, write $\mathfrak{aff}(V)$ for the Lie algebra consisting of affine maps of V. This Lie algebra (as well as the corresponding Lie group $\mathrm{Aff}(V)$) has the natural semidirect product structure: $\mathfrak{aff}(V)=V\rtimes\mathfrak{gl}(V)$ (with $\mathfrak{gl}(V)=\{X\in\mathfrak{aff}(V):X(0)=0\}$). Let $\pi:\mathfrak{aff}(V)\to\mathfrak{gl}(V)$ be the projection homomorphism. We use similar notation on the Lie group level and write, for instance, $\pi:\mathrm{Aff}(V)=V\rtimes\mathrm{GL}(V)\to\mathrm{GL}(V)$ for the corresponding group homomorphism. Sometimes, we simply write $\psi^{\mathrm{lin}}:=\pi(\psi)$ for the linear part of an element in $\mathfrak{aff}(V)$ or $\mathrm{Aff}(V)$.

Main lemma 15.14. *Let $(\mathfrak{g},\mathfrak{q})$ be a CR-algebra satisfying the fundamental assumption in §9.8 and let $\mathfrak{g}$ be solvable and of dimension 5. Suppose that there exists a 3-dimensional abelian ideal $\mathfrak{v}\subset\mathfrak{g}$ and a 2-dimensional subalgebra $\mathfrak{h}\subset\mathfrak{g}$ with $\mathfrak{h}\cap\mathfrak{v}=\{0\}$. Then, the associated CR-germ (M,o) is locally CR-equivalent to a tube $F\times i\mathbb{R}^3\subset\mathbb{C}^3$, where $F\subset\mathbb{R}^3$ is an affinely homogeneous surface.*

Proof. The proof is divided into several steps which give more precise (but also more technical) information concerning the structure of the Lie groups corresponding to $\mathfrak{g}$, $\mathfrak{l}$ and $\mathfrak{q}$, and a realization of the CR-germ (M,o).

Claim 15.15. *The adjoint representation $\mathrm{ad}:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{v})$ is faithful. Consequently, identifying $\mathfrak{h}$ with the subalgebra $\mathrm{ad}(\mathfrak{h})\subset\mathfrak{gl}(\mathfrak{v})$, the Lie algebras $\mathfrak{g}$ and $\mathfrak{l}$ can be realized as Lie subalgebras of affine transformations:*

$$\mathfrak{g}=\mathfrak{v}\rtimes\mathfrak{h}=\mathfrak{v}\rtimes\mathrm{ad}(\mathfrak{h})\subset\mathfrak{aff}(\mathfrak{v})\cong\mathfrak{aff}(\mathbb{R}^3)\quad\textit{and}\quad\mathfrak{l}=\mathfrak{v}^{\mathbb{C}}\rtimes\mathfrak{h}^{\mathbb{C}}\subset\mathfrak{aff}(\mathfrak{v}^{\mathbb{C}})\cong\mathfrak{aff}(\mathbb{C}^3).$$

Proof. Let $\mathfrak{n}\subset\mathfrak{h}$ be the kernel of the adjoint representation $\mathrm{ad}:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{v})$. The case $\dim\mathfrak{n}=1$ can be excluded, otherwise $\mathfrak{n}\oplus\mathfrak{v}=\mathfrak{g}^{\mathrm{nil}}=\mathfrak{z}$ would be 4-dimensional, contradicting Lemma 15.6. The case $\dim\mathfrak{n}=2$ can be also excluded, otherwise $\mathfrak{g}=\mathfrak{v}\times\mathfrak{h}$ would be abelian or contain a 4-dimensional abelian nilradical which in both cases would contradict conditions (III)–(V). □

Write $V\cong\mathbb{R}^3$ for a vector group with Lie algebra $\mathfrak{v}$ and $E:=V^{\mathbb{C}}$ for its complexification. Let $H_{\mathrm{GL}}\subset\mathrm{GL}(V)$ and $H_{\mathrm{GL}}^{\mathbb{C}}\subset\mathrm{GL}(E)$ be the Lie subgroups corresponding to the Lie algebras $\mathrm{ad}(\mathfrak{h})$ and $\mathrm{ad}(\mathfrak{h}^{\mathbb{C}})$, respectively. Since $\mathrm{GL}(V)\cong\mathrm{GL}(3,\mathbb{R})$ contains no compact torus of dimension $\geqslant 2$, each subgroup, in particular H_{GL}, is closed. This is in general not true for the complex subgroup $H_{\mathrm{GL}}^{\mathbb{C}}$. Let $H^{\mathbb{C}}$ be the simply connected Lie group with Lie algebra $\mathfrak{h}^{\mathbb{C}}$, $\mathrm{pr}:H^{\mathbb{C}}\to H_{\mathrm{GL}}^{\mathbb{C}}\subset\mathrm{GL}(\mathfrak{v}^{\mathbb{C}})$ the homomorphism induced by $\mathrm{ad}:\mathfrak{h}^{\mathbb{C}}\to\mathrm{ad}(\mathfrak{h}^{\mathbb{C}})\subset\mathfrak{gl}(\mathfrak{v}^{\mathbb{C}})$ and $L:=V^{\mathbb{C}}\rtimes H^{\mathbb{C}}$. For simplicity, for each $h\in H^{\mathbb{C}}$ we also write $h^{\mathrm{lin}}\subset\mathrm{GL}(E)$ instead of $\pi(h)$. Let $G=V\rtimes H\subset L$ be the real form. Since every 2-dimensional Lie algebra is solvable, we deduce that also $\mathfrak{l}=\mathfrak{v}^{\mathbb{C}}\rtimes\mathfrak{h}^{\mathbb{C}}$ (as well as $\mathfrak{g}$, L and G) is solvable.

Claim 15.16. *Let $Q\subset L$ be the subgroup corresponding to the Lie subalgebra $\mathfrak{q}\subset\mathfrak{l}$. Then Q is closed and $Q\cap V^{\mathbb{C}}=\{e\}$. Hence $L=V^{\mathbb{C}}\rtimes Q$ is a semidirect product.*

Proof. Let $\pi\colon\mathfrak{l}\to\mathfrak{h}^{\mathbb{C}}$ be the projection homomorphism. Our first observation is that $\pi(\mathfrak{q})=\mathfrak{h}^{\mathbb{C}}$. The case $\pi(\mathfrak{q})=0$ can clearly be excluded, as in such a situation $\mathfrak{q}\subset\mathfrak{v}^{\mathbb{C}}$, and in turn $\mathfrak{q}+\sigma\mathfrak{q}\subset\mathfrak{v}^{\mathbb{C}}$, which is absurd.

The possibility $\dim\pi(\mathfrak{q})=1$ can be ruled out as follows. We may assume that $\pi(\mathfrak{q}\oplus\sigma\mathfrak{q})=\mathfrak{h}^{\mathbb{C}}$ (otherwise $\mathfrak{q}\oplus\sigma\mathfrak{q}$ would be a subalgebra). Either, $\pi(\mathfrak{f})=0$, that is, $\mathfrak{q}\cap\mathfrak{v}^{\mathbb{C}}=\mathfrak{f}$, and in turn $(\mathfrak{q}\oplus\sigma\mathfrak{q})\cap\mathfrak{v}^{\mathbb{C}}=\mathfrak{f}\oplus\sigma\mathfrak{f}$: this leads to a contradiction, since then $[\mathfrak{F},\mathfrak{H}]\subset\mathfrak{H}\cap\mathfrak{v}=\mathfrak{F}$, violating ($\mathsf{V}$). Or, $\pi(\mathfrak{f})=\pi(\mathfrak{q})$. But then $\mathfrak{q}=\mathfrak{f}\oplus\mathbb{C}\mathsf{x}$ with a nonzero $\mathsf{x}\in\mathfrak{q}\cap\mathfrak{v}^{\mathbb{C}}$, and in turn $[\mathfrak{q},\sigma\mathfrak{q}]\subset\mathfrak{q}\oplus\sigma\mathfrak{q}$, contradicting ($\mathsf{III}$). Summarizing, $\pi(\mathfrak{q})=\mathfrak{h}^{\mathbb{C}}$.

On the group level, since L is simply connected and solvable, every connected subgroup is closed. The restriction of π to Q induces a surjective homomorphism $Q\to H^{\mathbb{C}}$. Since both groups are 2-dimensional, this homomorphism is a covering. Our assumption that $H^{\mathbb{C}}$ is simply connected finally implies that $\pi|_Q\colon Q\to H^{\mathbb{C}}$ is an isomorphism. In particular $Q\cap V^{\mathbb{C}}=Q\cap\ker\pi=\{e\}$. □

CLAIM 15.17. *With respect to the identification $Z:=L/Q=V^{\mathbb{C}}\cong\mathbb{C}^3$ the real form G acts on L/Q by affine transformations and $V\subset G$ by translations.*

Proof. The existence of the decomposition $L=V^{\mathbb{C}}\rtimes Q$ implies that there are well-defined functions $v\colon L\to V^{\mathbb{C}}$ and $q\colon L\to Q$ such that $l=v(l)\cdot q(l)$ for every $l\in L$. Let

$$g=w\cdot h\in V\rtimes H=G$$

(with $w\in V$ and $h\in H$) be arbitrary. Then, for any $z\in V^{\mathbb{C}}$ we have

$$g\cdot zQ=w\cdot h\cdot zQ=w\cdot v(h)\cdot q(h)\cdot zQ=w\cdot v(h)\cdot(q(h)\cdot z\cdot q(h)^{-1})Q$$

and $q(h)\cdot z\cdot q(h)^{-1}=v(h)^{-1}h\cdot z\cdot h^{-1}v(h)=h^{\mathrm{lin}}(z)$. Hence, with respect to the identification $V^{\mathbb{C}}=L/Q$ (induced by the inclusion $V^{\mathbb{C}}\hookrightarrow L$ such that 0 corresponds to the point $eQ\in L/Q$), the action of G can be written as follows:

$$g\cdot z=h^{\mathrm{lin}}(z)+v(h)+w,\qquad\text{with } g=w\cdot h\in L \text{ and } z\in V^{\mathbb{C}}. \tag{15.18}$$

In particular, the subgroup $V\subset G$ acts by translations $z\mapsto z+w$. □

Consequently,

$$M:=G\cdot 0=VH\cdot 0=V+F\subset V\oplus iV$$

with $F:=M\cap iV$. It should be noted, however, that in general $F:=(G\cdot 0)\cap iV\neq H\cdot 0$. Nevertheless, as we shortly will see, F is affinely homogeneous under a slightly different subgroup of $\mathrm{Aff}(iV)$. Clearly, multiplying a tube manifold $F+iV\subset V\oplus iV$, $F\subset V$, by the imaginary unit i, we get the CR-equivalent realization $V+iF=V+F'$ with $F'=iF\subset iV$. The latter form of a tube manifold is more suitable in our particular setup, and we keep this notation until the end of the proof of the main lemma.

Claim 15.19. *Retaining the previous notation, there exists a subgroup*

$$B\subset iV\rtimes \mathrm{GL}(iV)=\mathrm{Aff}(iV)$$

such that $F:=(G\cdot 0)\cap iV=B\cdot 0$.

Proof. Let $\mathrm{pr}^i\colon V\oplus iV\to iV$ be the linear projection. A glance at (15.18) shows that $F=\mathrm{pr}^i\{v(h):h\in H\}$. In order to determine $v(h)$ more explicitly, we need to analyze the position of Q in $V^{\mathbb{C}}\rtimes H^{\mathbb{C}}$ in greater detail. Since $\mathfrak{h}$ is 2-dimensional, there exists a basis $\mathsf{s},\mathsf{n}\in\mathfrak{h}$ such that $[\mathsf{s},\mathsf{n}]=\varepsilon\mathsf{n}$ with $\varepsilon\in\{0,1\}$. Recall that the projection map $\pi\colon\mathfrak{q}\to\mathfrak{h}^{\mathbb{C}}$ is an isomorphism. Consequently, there exist $\mathsf{w}_\mathsf{s},\mathsf{w}_\mathsf{n}\in\mathfrak{v}^{\mathbb{C}}=V^{\mathbb{C}}$ such that the elements $\mathsf{w}_\mathsf{s}+\mathsf{s}$ and $\mathsf{w}_\mathsf{n}+\mathsf{n}$ in $\mathfrak{l}=V^{\mathbb{C}}\oplus\mathfrak{h}^{\mathbb{C}}$ generate $\mathfrak{q}$. Then, since $\sigma\mathfrak{q}=\mathbb{C}(\overline{\mathsf{w}}_\mathsf{s}+\mathsf{s})\oplus\mathbb{C}(\overline{\mathsf{w}}_\mathsf{n}+\mathsf{n})$, we must have $\mathsf{w}_\mathsf{s}\neq\overline{\mathsf{w}}_\mathsf{s}$ and $\mathsf{w}_\mathsf{n}\neq\overline{\mathsf{w}}_\mathsf{n}$ (otherwise $\mathfrak{q}\cap\sigma\mathfrak{q}\neq 0$). Let $\exp\colon\mathfrak{l}\to L$ and $\mathrm{Exp}\colon\mathrm{ad}(\mathfrak{h}^{\mathbb{C}})\to\mathrm{GL}(V^{\mathbb{C}})$ be the exponential maps (that is, $\mathrm{Exp}(\mathrm{ad}(\mathsf{v}))=\pi(\exp(\mathsf{v}))$, with π as in the paragraph preceding the Main lemma 15.14). Furthermore, let Ψ be the entire function defined by

$$\Psi(z)=\frac{e^z-1}{z}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}.$$

Then, for $\mathsf{S}^t:=\Psi(t\,\mathrm{ad}(\mathsf{s}))$ and $\mathsf{N}^u:=\Psi(u\,\mathrm{ad}(\mathsf{n}))$, a simple computation shows that

$$\begin{aligned} H&=\{\exp(t\mathsf{s})\cdot\exp(u\mathsf{n}):t,u\in\mathbb{R}\},\\ Q&=\{\exp t(\mathsf{w}_\mathsf{s}+\mathsf{s})\cdot\exp u(\mathsf{w}_\mathsf{n}+\mathsf{n}):t,u\in\mathbb{C}\}\\ &=\{\mathsf{S}^t(t\mathsf{w}_\mathsf{s})\cdot\exp(t\mathsf{s})\cdot\mathsf{N}^u(u\mathsf{w}_\mathsf{n})\cdot\exp(u\mathsf{n}):t,u\in\mathbb{C}\}\\ &=\{(\mathsf{S}^t(t\mathsf{w}_\mathsf{s})\cdot\mathrm{Exp}(t\,\mathrm{ad}(\mathsf{s}))(\mathsf{N}^u(u\mathsf{w}_\mathsf{n})))\cdot\exp(t\mathsf{s})\cdot\exp(u\mathsf{n}):t,u\in\mathbb{C}\}\subset V^{\mathbb{C}}\cdot H^{\mathbb{C}}=L. \end{aligned}\tag{15.20}$$

The explicit form of $v(h)$ (compare the proof of Claim 15.17) can be read off the last line in (15.20):

$$v(h)=v(\exp(t\mathsf{s})\exp(u\mathsf{n}))=(\mathsf{S}^t(t\mathsf{w}_\mathsf{s}))^{-1}\cdot(\mathrm{Exp}(t\,\mathrm{ad}(\mathsf{s}))(\mathsf{N}^u(u\mathsf{w}_\mathsf{n})))^{-1}.$$

Since $\mathrm{ad}(\mathsf{s})$, N^u and S^t are real operators, it follows, for $h=\exp(t\mathsf{s})\cdot\exp(u\mathsf{n})$ as before, that

$$\begin{aligned} \mathrm{pr}^i(v(h))&=(\mathsf{S}^t(t\mathsf{w}_\mathsf{s}))^{-1}\cdot(\mathrm{Exp}(t\,\mathrm{ad}(\mathsf{s}))(\mathsf{N}^u(u\mathsf{w}_\mathsf{n})))^{-1}\\ &=\exp(t(-\mathsf{w}_\mathsf{s}^i+\mathsf{s}))\cdot\exp(u(-\mathsf{w}_\mathsf{n}^i+\mathsf{n}))\cdot 0\subset iV. \end{aligned}\tag{15.21}$$

(Using additive notation, $\mathrm{pr}^i(v(h))=-\mathsf{S}^t(t\mathsf{w}_\mathsf{s}^i)-\mathrm{Exp}(t\,\mathrm{ad}(\mathsf{s}))(\mathsf{N}^u(u\mathsf{w}_\mathsf{n}^i))\subset iV$.) Define

$$\mathfrak{b}:=\mathbb{R}(-\mathsf{w}_\mathsf{s}^i+\mathsf{s})\oplus\mathbb{R}(-\mathsf{w}_\mathsf{n}^i+\mathsf{n})\subset\mathfrak{l}=V^{\mathbb{C}}\rtimes\mathfrak{h}^{\mathbb{C}}$$

and check that this is a Lie algebra. Then $B:=\exp(\mathbb{R}(-\mathsf{w}_\mathsf{s}^i+\mathsf{s}))\cdot\exp(\mathbb{R}(-\mathsf{w}_\mathsf{n}^i+\mathsf{n}))$ is the subgroup of L with Lie algebra $\mathfrak{b}$, and (15.21) shows that $F=\mathrm{pr}^i\{v(h):h\in H\}=B\cdot 0$. This finishes the proof of the claim. □

The proof of the Main lemma 15.14 is now complete. □

16. The final steps

Our final step toward the complete classification of all 5-dimensional 2-nondegenerate and homogeneous CR-germs is to deduce that each 5-dimensional solvable Lie algebra $\mathfrak{g}$ which occurs in a CR-algebra $(\mathfrak{g}, \mathfrak{q})$ subject to the assumption in §9.8 also satisfies the assumptions of the preceding Main lemma 15.14.

LEMMA 16.1. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be a CR-algebra satisfying the assumption in* §9.8 *and suppose* $\mathfrak{g}$ *is solvable and of dimension* 5. *Then there exists a semidirect product decomposition* $\mathfrak{g}=\mathfrak{v}\rtimes\mathfrak{h}$ *with a* 3*-dimensional abelian ideal* $\mathfrak{v}\subset\mathfrak{g}$ *and a* 2*-dimensional subalgebra* $\mathfrak{h}$.

Proof. We have already observed in Lemma 15.6 that the nilcenter $\mathfrak{z}$ has dimension 1 or 3. If $\dim\mathfrak{z}=3$ then, due to Claim 15.7, we can apply Lemma 15.8. Consequently, we can choose $\mathfrak{v}=[\mathfrak{g}, \mathfrak{g}]$ (compare (15.12)) and $\mathfrak{h}$ as defined in (15.13).

The situation $\dim\mathfrak{z}=1$ requires some more elaborate work. We classify all CR-algebras $(\mathfrak{g}, \mathfrak{q})$ under consideration in terms of the corresponding structure equations with respect to some perfect basis $\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}$ of $\mathfrak{l}$ (see §15.5). Given $(\mathfrak{g}, \mathfrak{q})$, let the corresponding structure equations be as in (15.1), taking into account (15.4). To handle the various sets of relations between the structure constants, we divide the class of CR-algebras under consideration into the subclasses A, B and C, see below.

Case A. $\beta_1\neq\pm1$. In this situation it is possible to assume that $a_1=0$ (simply replace x by $\mathsf{x}+\lambda\mathsf{y}$ with $\lambda=u+iv$ defined by $u:=(\operatorname{Re} a_1)/(1-\beta_1)$ and $v:=(\operatorname{Im} a_1)/(1+\beta_1)$). The structure equations then read

$$\begin{aligned}
[\mathsf{x}, \bar{\mathsf{x}}] &= \mathsf{z} + a_2\mathsf{y} - \bar{a}_2\bar{\mathsf{y}}, \\
[\mathsf{x}, \bar{\mathsf{y}}] &= \beta_1\mathsf{x} + \bar{\mathsf{x}} + b_3\mathsf{y} + b_4\bar{\mathsf{y}}, \\
[\mathsf{y}, \bar{\mathsf{y}}] &= \mathsf{y} - \bar{\mathsf{y}}, \\
[\mathsf{y}, \mathsf{x}] &= (1-\beta_1)\mathsf{x} + d_2\mathsf{y}, \qquad d_2 = \bar{b}_3+(1-\beta_1)b_4, \\
[\mathsf{z}, \eta] &= c_\eta\mathsf{z} \qquad \text{for every } \eta\in\mathfrak{l},
\end{aligned} \tag{16.2}$$

and we now work out more constraints imposed on the constants. An explicit evaluation of the Jacobi identity $[[\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}]]=0$ yields $c_{\mathsf{y}}=c_{\bar{\mathsf{y}}}=2\beta_1-1$. Furthermore, we obtain the equations $\bar{b}_3(1+\beta_1)=b_4(\beta_1^2-\beta_2)$ and $\bar{b}_3(1+\beta_1)=(b_4-\bar{b}_4)(\beta_1-1)$ which imply $b_4(1-\beta_1)=\bar{b}_4$.

In order to investigate most conveniently additional relations between the structure constants, we deal separately with the following three subcases:

$$\text{(AI)}\ \ b_4\in i\mathbb{R}^* \text{ and } \beta_1=2, \quad \text{(AII)}\ \ b_4\in\mathbb{R}^* \text{ and } \beta_1=0, \quad \text{(AIII)}\ \ b_4=0.$$

(AI) Put $\beta_4:=-ib_4$. In this particular situation the identity $[[\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}]]=0$ implies that $b_3=-\frac{2}{3}i\beta_4$ and $a_2=-\frac{2}{9}\beta_4^2$, and $[[\mathsf{x}, \bar{\mathsf{y}}, \mathsf{z}]]=0$ implies that $c_{\mathsf{x}}=-i\beta_4$. There are no more conditions imposed by the Jacobi identity and the structure equations of $\mathfrak{l}$ are now

$$\begin{aligned}
[\mathsf{x},\bar{\mathsf{x}}] &= \mathsf{z} - 2\gamma^2\mathsf{y} + 2\gamma^2\bar{\mathsf{y}},\\
[\mathsf{x},\bar{\mathsf{y}}] &= 2\mathsf{x} + \bar{\mathsf{x}} - 2i\gamma\mathsf{y} + 3i\gamma\bar{\mathsf{y}},\\
[\mathsf{y},\bar{\mathsf{y}}] &= \mathsf{y} - \bar{\mathsf{y}},\\
[\mathsf{y},\mathsf{x}] &= -\mathsf{x} - i\gamma\mathsf{y},\\
[\mathsf{z},\mathsf{x}] &= -3i\gamma\mathsf{z},\\
[\mathsf{z},\mathsf{y}] &= -3\mathsf{z},
\end{aligned}\tag{16.3}$$

where $\gamma:=\frac{1}{3}\beta_4\in\mathbb{R}^*$. Keeping in mind Main lemma 15.14, the structure of $\mathfrak{l}$ and $\mathfrak{g}$ and the position of $\mathfrak{q}$ is determined by (16.3). This can be seen more clearly by decomposing $\mathfrak{g}$ and $\mathfrak{l}$ into the eigenspaces of $\operatorname{ad}(\mathsf{s})$, where $\mathsf{s}:=-\frac{1}{2}(\mathsf{y}+\bar{\mathsf{y}})$. Define the elements n, v_1, v_2 and v_3 from $\mathfrak{g}$ by

$$\begin{aligned}
\mathsf{n} &:= i\mathsf{x} - i\bar{\mathsf{x}} - \gamma\mathsf{y} - \gamma\bar{\mathsf{y}},\\
\mathsf{v}_1 &:= \tfrac{1}{2}i\mathsf{y} - \tfrac{1}{2}i\bar{\mathsf{y}},\\
\mathsf{v}_2 &:= \mathsf{x} + \bar{\mathsf{x}} - 2i\gamma\mathsf{y} + 2i\gamma\bar{\mathsf{y}},\\
\mathsf{v}_3 &:= 2i\mathsf{z}.
\end{aligned}$$

It is clear that s, n, v_1, v_2 and v_3 form a basis of $\mathfrak{g}$. The bracket relations are

$$[\mathsf{s},\mathsf{v}_k]=k\mathsf{v}_k,\quad [\mathsf{s},\mathsf{n}]=\mathsf{n},\quad [\mathsf{n},\mathsf{v}_1]=\mathsf{v}_2,\quad [\mathsf{n},\mathsf{v}_2]=\mathsf{v}_3\quad\text{and}\quad [\mathsf{n},\mathsf{v}_3]=0.$$

Further, $\mathfrak{v}:=\mathbb{R}\mathsf{v}_1\oplus\mathbb{R}\mathsf{v}_2\oplus\mathbb{R}\mathsf{v}_3\cong\mathbb{R}^3$ is an abelian ideal in $\mathfrak{g}$ with $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}^{\mathrm{nil}}=\mathbb{R}\mathsf{n}\oplus\mathfrak{v}$. Hence, $\mathfrak{g}$ has the structure of the semidirect product $\mathfrak{v}\rtimes\mathfrak{h}$ with $\mathfrak{h}=\mathbb{R}\mathsf{s}\oplus\mathbb{R}\mathsf{n}$ and the Main lemma 15.14 applies.

Remark. A direct verification shows that for every $\gamma\in\mathbb{R}^*$, $(\mathfrak{g},\mathfrak{q})$ in (16.3) is associated with Example 8.5.

We show that in the next case the Lie algebra cannot be solvable, hence, this case can be discarded.

(AII) Write $\beta_4:=b_4$. The Jacobi identity implies that $a_2=0=b_3$ and $d_2=\beta_4$, and (15.9) reads

$$\begin{aligned}
[\mathsf{x},\bar{\mathsf{x}}] &= \mathsf{z},\\
[\mathsf{x},\bar{\mathsf{y}}] &= \bar{\mathsf{x}} + \beta_4\bar{\mathsf{y}},\\
[\mathsf{y},\bar{\mathsf{y}}] &= \mathsf{y} - \bar{\mathsf{y}},\\
[\mathsf{y},\mathsf{x}] &= \mathsf{x} + \beta_4\mathsf{y},\\
[\mathsf{z},\mathsf{x}] &= \beta_4\mathsf{z},\\
[\mathsf{z},\mathsf{y}] &= -\mathsf{z}.
\end{aligned}\tag{16.4}$$

But then the linear span of the vectors

$$\begin{aligned}
\mathsf{e}^+ &:= \mathsf{y}-\bar{\mathsf{y}},\\
\mathsf{h} &:= -\frac{1}{\beta_4}(\mathsf{x}+\bar{\mathsf{x}}+(\beta_4-1)\mathsf{y}+(\beta_4+1)\bar{\mathsf{y}}),\\
\mathsf{e}^- &:= \frac{1}{4\beta_4^2}(2\mathsf{z}+(2-2\beta_4)\mathsf{x}+(2+2\beta_4)\bar{\mathsf{x}}-(1-\beta_4)^2\mathsf{y}+(1+\beta_4)^2\bar{\mathsf{y}})
\end{aligned}$$

is a copy of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{l}$, that is, $\mathfrak{l}$ is not solvable.

(AIII) The condition $b_4=0$ together with the Jacobi identity $[[\mathsf{x},\bar{\mathsf{x}},\mathsf{y}]]=0$ implies that $a_2=b_3=0$ and $c_{\mathsf{y}}=2\beta_1-1$. Since $1-\beta_1\neq0$, the identity $[\mathsf{z},[\mathsf{y},\mathsf{x}]]=0$ implies that $[\mathsf{z},\mathsf{x}]=0$, see the table

$$\begin{aligned}
[\mathsf{x},\bar{\mathsf{x}}] &= \mathsf{z},\\
[\mathsf{x},\bar{\mathsf{y}}] &= \beta_1\mathsf{x}+\bar{\mathsf{x}},\\
[\mathsf{y},\bar{\mathsf{y}}] &= \mathsf{y}-\bar{\mathsf{y}},\\
[\mathsf{y},\mathsf{x}] &= (1-\beta_1)\mathsf{x}+\beta_4\mathsf{y},\\
[\mathsf{z},\mathsf{x}] &= 0,\\
[\mathsf{z},\mathsf{y}] &= -\mathsf{z}.
\end{aligned}\tag{16.5}$$

Select the following basis of $\mathfrak{g}$:

$$\mathsf{n}:=\frac{1}{2}i(\mathsf{x}-\bar{\mathsf{x}}),\quad \mathsf{s}:=\frac{1}{2-2\beta_1}(\mathsf{y}+\bar{\mathsf{y}}),\quad \mathsf{v}_1:=i\mathsf{y}-i\bar{\mathsf{y}},\quad \mathsf{v}_2:=\mathsf{x}+\bar{\mathsf{x}},\quad \mathsf{v}_3:=i\mathsf{z}.$$

One checks immediately that $\mathfrak{v}:=\mathbb{R}\mathsf{v}_1\oplus\mathbb{R}\mathsf{v}_2\oplus\mathbb{R}\mathsf{v}_3$ is an abelian ideal and $\mathfrak{h}:=\mathbb{R}\mathsf{s}\oplus\mathbb{R}\mathsf{n}$ is a subalgebra with $[\mathsf{s},\mathsf{n}]=\mathsf{n}$. Further,

$$[\mathsf{s},\mathsf{v}_j]=\frac{2-j+(j-1)\beta_1}{\beta_1-1}\mathsf{v}_j \text{ for } j=1,2,3 \quad\text{and}\quad [\mathsf{n},\mathsf{v}_1]=\mathsf{v}_2,\ [\mathsf{n},\mathsf{v}_2]=\mathsf{v}_3,\ [\mathsf{n},\mathsf{v}_3]=0.$$

Hence, $\mathfrak{g}=\mathfrak{v}\rtimes\mathfrak{h}$ as claimed, and the Main lemma 15.14 applies. Also in this case, for all $\beta_1\neq\pm1$ the CR-algebra $(\mathfrak{g},\mathfrak{q})$ is associated with Example 8.5.

It remains to discuss the cases $\beta_1=\pm1$.

Case B. $\beta_1=1$. Plugging β_1 into (15.4) gives $d_2=\bar{b}_3$ and $d_1=0$. A direct check shows that $[[\mathsf{x},\bar{\mathsf{x}},\mathsf{y}]]=0$ implies that $0=b_3=d_2$, i.e., $[\mathsf{x},\mathsf{y}]=0$. Lemma 15.8 then gives that $\mathfrak{g}$ is isomorphic to the semidirect product $\mathfrak{v}\rtimes\mathfrak{h}$, with $\mathfrak{v}=[\mathfrak{g},\mathfrak{g}]$, and the abelian subalgebra $\mathfrak{h}$ as defined in (15.13).

Case C. $\beta_1=-1$. We proceed as in the preceding cases. Starting from the structure equations (15.1) with respect to some perfect basis $\mathsf{x},\bar{\mathsf{x}},\mathsf{y},\bar{\mathsf{y}},\mathsf{z}$, we first evaluate (15.4) for this particular value of β_1. We get $d_1=2$ and $d_2=\bar{b}_3+2b_4$. Next, the Jacobi identity

$[[\mathsf{x},\bar{\mathsf{x}},\mathsf{y}]]=0$ implies $b_4:=\beta_4\in\mathbb{R}$ and $a_1=\beta_4$. Further, $[[\mathsf{x},\mathsf{y},\bar{\mathsf{y}}]]=0$ implies $b_3=-\beta_4+i\gamma$, $\gamma\in\mathbb{R}$. Next, $[[\mathsf{x},\bar{\mathsf{x}},\mathsf{y}]]=0$ determines the value of a_2:

$$a_2=\tfrac{1}{4}(3\beta_4^2+\gamma^2)-\tfrac{1}{2}i\beta_4\gamma.$$

Finally, $[[\mathsf{x},\mathsf{y},\mathsf{z}]]=0$ implies $[\mathsf{z},\mathsf{x}]=\left(\frac{3}{2}\beta_4-\frac{3}{2}i\gamma\right)\mathsf{z}$, see the diagram

$$\begin{array}{lllll}
[\mathsf{x},\bar{\mathsf{x}}]= & \mathsf{z}+\beta_4\mathsf{x}-\beta_4\bar{\mathsf{x}}+a_2\mathsf{y}-\bar{a}_2\bar{\mathsf{y}}, & & a_2=\frac{1}{4}(3\beta_4^2+\gamma^2)-\frac{1}{2}i\gamma\beta_4, \\
[\mathsf{x},\bar{\mathsf{y}}]= & -\mathsf{x}+\bar{\mathsf{x}}+b_3\mathsf{y}+\beta_4\bar{\mathsf{y}}, & & b_3=-\beta_4+i\gamma, \\
[\mathsf{y},\bar{\mathsf{y}}]= & \mathsf{y}-\bar{\mathsf{y}}, \\
[\mathsf{y},\mathsf{x}]= & 2\mathsf{x}-b_4\mathsf{y}, \\
[\mathsf{z},\mathsf{x}]= & \left(\frac{3}{2}\beta_4-\frac{3}{2}i\gamma\right)\mathsf{z}, \\
[\mathsf{z},\mathsf{y}]= & -3\mathsf{z}.
\end{array} \tag{16.6}$$

Select the following basis of $\mathfrak{g}$:

$$\begin{aligned}
\mathsf{n}&:=2i\mathsf{x}-2i\bar{\mathsf{x}}-ib_3\mathsf{y}+i\bar{b}_3\bar{\mathsf{y}},\\
\mathsf{s}&:=-\tfrac{1}{4}\mathsf{y}-\tfrac{1}{4}\bar{\mathsf{y}},\\
\mathsf{v}_1&:=\tfrac{1}{2}i\mathsf{y}-\tfrac{1}{2}i\bar{\mathsf{y}},\\
\mathsf{v}_2&:=2\mathsf{x}+2\bar{\mathsf{x}}-b_3\mathsf{y}-\bar{b}_3\bar{\mathsf{y}},\\
\mathsf{v}_3&:=4i\mathsf{z}.
\end{aligned}$$

A direct computation (using (16.6)) shows that $\mathfrak{v}:=\mathbb{R}\mathsf{v}_1\oplus\mathbb{R}\mathsf{v}_2\oplus\mathbb{R}\mathsf{v}_3$ is an abelian ideal and

$$[\mathsf{s},\mathsf{n}]=\mathsf{n},\quad [\mathsf{s},\mathsf{v}_1]=-\tfrac{1}{2}\mathsf{v}_1,\quad [\mathsf{s},\mathsf{v}_2]=\tfrac{1}{2}\mathsf{v}_2,\quad [\mathsf{s},\mathsf{v}_3]=\tfrac{3}{2}\mathsf{v}_3,$$

$$[\mathsf{n},\mathsf{v}_1]=\mathsf{v}_2,\quad [\mathsf{n},\mathsf{v}_2]=\mathsf{v}_3\quad\text{and}\quad [\mathsf{n},\mathsf{v}_3]=0.$$

Again, this shows that $\mathfrak{g}$ is isomorphic to the semidirect product $\mathfrak{v}\rtimes\mathfrak{h}$ with $\mathfrak{h}=\mathbb{R}\mathsf{n}\oplus\mathbb{R}\mathsf{s}$. Actually, an explicit realization of the corresponding CR-manifold M along the lines of proof of the Main lemma shows that $(\mathfrak{g},\mathfrak{q})$ is associated with the tube over the future light cone. □

We close by stating that the Main lemma 15.14 together with Lemma 16.1 finishes the proof of the classification theorem.

References

[1] Andreotti, A. & Fredricks, G. A., Embeddability of real analytic Cauchy–Riemann manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 6 (1979), 285–304.

[2] Andreotti, A. & Hill, C. D., Complex characteristic coordinates and tangential Cauchy–Riemann equations. *Ann. Scuola Norm. Sup. Pisa*, 26 (1972), 299–324.

[3] Baouendi, M. S., Ebenfelt, P. & Rothschild, L. P., *Real Submanifolds in Complex Space and their Mappings*. Princeton Mathematical Series, 47. Princeton University Press, Princeton, NJ, 1999.

[4] Baouendi, M. S., Huang, X. & Rothschild, L. P., Regularity of CR mappings between algebraic hypersurfaces. *Invent. Math.*, 125 (1996), 13–36.

[5] Beloshapka, V. K., Symmetries of real hypersurfaces of a three-dimensional complex space. *Mat. Zametki*, 78 (2005), 171–179 (Russian); English translation in *Math. Notes*, 78 (2005), 156–163.

[6] Boggess, A., *CR Manifolds and the Tangential Cauchy–Riemann Complex*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991.

[7] Bourbaki, N., *Éléments de mathématique.* XXVI. *Groupes et algèbres de Lie. Chapitre* 1–9. Hermann, Paris, 1960.

[8] Cartan, É., Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes. *Ann. Mat. Pura Appl.*, 11 (1933), 17–90.

[9] Chern, S. S. & Moser, J. K., Real hypersurfaces in complex manifolds. *Acta Math.*, 133 (1974), 219–271.

[10] Dadok, J. & Yang, P., Automorphisms of tube domains and spherical hypersurfaces. *Amer. J. Math.*, 107 (1985), 999–1013.

[11] Doubrov, B., Komrakov, B. & Rabinovich, M., Homogeneous surfaces in the three-dimensional affine geometry, in *Geometry and Topology of Submanifolds*, VIII (Brussels, 1995/Nordfjordeid, 1995), pp. 168–178. World Sci. Publ., River Edge, NJ, 1996.

[12] Eastwood, M. & Ezhov, V., On affine normal forms and a classification of homogeneous surfaces in affine three-space. *Geom. Dedicata*, 77 (1999), 11–69.

[13] Ebenfelt, P., Normal forms and biholomorphic equivalence of real hypersurfaces in $\mathbf{C}^3$. *Indiana Univ. Math. J.*, 47 (1998), 311–366.

[14] — Uniformly Levi degenerate CR manifolds: the 5-dimensional case. *Duke Math. J.*, 110 (2001), 37–80. Correction in *Duke Math. J.*, 131 (2006), 589–591.

[15] Fels, G., Locally homogeneous finitely nondegenerate CR-manifolds. *Math. Res. Lett.*, 14 (2007), 893–922.

[16] Fels, G. & Kaup, W., CR-manifolds of dimension 5: a Lie algebra approach. *J. Reine Angew. Math.*, 604 (2007), 47–71.

[17] Gaussier, H. & Merker, J., A new example of a uniformly Levi degenerate hypersurface in $\mathbf{C}^3$. *Ark. Mat.*, 41 (2003), 85–94. Correction in *Ark. Mat.*, 45 (2007), 269–271.

[18] Hartshorne, R., *Algebraic Geometry.* Graduate Texts in Mathematics, 52. Springer, New York, 1977.

[19] Hörmander, L., *An Introduction to Complex Analysis in Several Variables.* North-Holland Mathematical Library, 7. North-Holland, Amsterdam, 1990.

[20] Isaev, A. V. & Mishchenko, M. A., Classification of spherical tube hypersurfaces that have one minus in the Levi signature form. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52 (1988), 1123–1153, 1327 (Russian); English translation in *Math. USSR–Izv.*, 33 (1989), 441–472.

[21] Kaup, W., Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen. *Invent. Math.*, 3 (1967), 43–70.

[22] — On Levi-degenerate homogeneous CR-manifolds, in *Contemporary Geometry and Topology and Related Topics* (8th International Workshop on Differential Geometry and its Applications, Cluj-Napoca, 2007), pp. 187–199. Cluj University Press, 2008.

[23] Kaup, W. & Zaitsev, D., On local CR-transformation of Levi-degenerate group orbits in compact Hermitian symmetric spaces. *J. Eur. Math. Soc.* (*JEMS*), 8 (2006), 465–490.

[24] Knapp, A. W., *Lie Groups beyond an Introduction*. Progress in Mathematics, 140. Birkhäuser, Boston, MA, 2002.

[25] Loboda, A. V., Homogeneous real hypersurfaces in $\mathbb{C}^3$ with two-dimensional isotropy groups. *Tr. Mat. Inst. Steklova*, 235 (2001), 114–142 (Russian); English translation in *Proc. Steklov Inst. Math.*, 235 (2001), 107–135.

[26] — Homogeneous nondegenerate surfaces in $\mathbb{C}^3$ with two-dimensional isotropy groups. *Funktsional. Anal. i Prilozhen.*, 36 (2002), 80–83 (Russian); English translation in *Funct. Anal. Appl.*, 36 (2002), 151–153.

[27] — On the determination of a homogeneous strictly pseudoconvex hypersurface from the coefficients of its normal equation. *Mat. Zametki*, 73 (2003), 453–456 (Russian); English translation in *Math. Notes*, 73 (2003), 419–423.

[28] Medori, C. & Nacinovich, M., Algebras of infinitesimal CR automorphisms. *J. Algebra*, 287 (2005), 234–274.

[29] Palais, R. S., A global formulation of the Lie theory of transformation groups. *Mem. Amer. Math. Soc.*, 22 (1957).

[30] Sakai, T., *Riemannian Geometry*. Shokabo, Tokyo, 1992 (Japanese); English translation in Transl. Math. Monogr., 149. Amer. Math. Soc., Providence, RI, 1996.

[31] Sergeev, A. G. & Vladimirov, V. S., Complex analysis in the future tube, in *Current Problems in Mathematics*, Fundamental Directions, 8, pp. 191–266, 274. Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1985 (Russian); English translation in *Several Complex Variable* II, Encyclopaedia Math. Sci., 8, pp. 179–253, Springer, Berlin–Heidelberg, 1994.

[32] Tanaka, N., On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables. *J. Math. Soc. Japan*, 14 (1962), 397–429.

[33] Zaitsev, D., On different notions of homogeneity for CR-manifolds. *Asian J. Math.*, 11 (2007), 331–340.

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