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Calculus of functors, operad formality, and rational homology of embedding spaces

by

Gregory Arone
University of Virginia
Charlottesville, VA, U.S.A.

Pascal Lambrechts
Université Catholique de Louvain
Louvain-la-Neuve, Belgium

Ismar Volić
Wellesley College
Wellesley, MA, U.S.A.

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1. Introduction

Let M be a smooth manifold of dimension m. M may be non-compact, but we always assume that M is the interior of a compact manifold with boundary. Let V be a Euclidean space. Let $\mathrm{Emb}(M,V)$ be the space of smooth embeddings of M into V. For technical reasons, rather than study $\mathrm{Emb}(M,V)$ directly, we will focus on the space

$$\overline{\mathrm{Emb}}(M,V) := \mathrm{hofiber}(\mathrm{Emb}(M,V) \to \mathrm{Imm}(M,V)),$$

where $\mathrm{Imm}(M,V)$ denotes the space of immersions of M into V. Note that the definition requires that we fix an embedding (or at minimum an immersion) $\alpha: M \hookrightarrow V$, to act as a basepoint. Most of the time we will work with the suspension spectrum $\Sigma^\infty \overline{\mathrm{Emb}}(M,V)_+$, and our results are really about the rationalization of this spectrum,

$$\Sigma^\infty_{\mathbf{Q}} \overline{\mathrm{Emb}}(M,V)_+ \simeq \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,V)_+.$$

In other words, our results are about the rational homology of $\overline{\mathrm{Emb}}(M,V)$.

Our framework is provided by the Goodwillie–Weiss calculus of functors. One of the main features of calculus of functors is that with a functor it associates a tower of fibrations, analogous to the Taylor series of a function. The functor $\mathrm{Emb}(M,V)$ is a functor of two variables, and accordingly one may do "Taylor expansion" in at least two ways: in either the variable M or the variable V (or both). Since the two variables of $\mathrm{Emb}(M,V)$ are of rather different nature (for example, one is contravariant and the other one is covariant), there are two versions of calculus needed for dealing with them—*embedding calculus* (for the variable M) and *orthogonal calculus* (for the variable V).

Embedding calculus [23], [11] is designed for studying contravariant isotopy functors on manifolds, such as $F(M) = \mathrm{Emb}(M,V)$. With such a functor F, embedding calculus associates a tower of fibrations under F:

$$F(-) \longrightarrow (T_\infty F(-) \to \ldots \to T_k F(-) \to T_{k-1} F(-) \to \ldots \to T_1 F(-)). \tag{1}$$

Here

$$T_k F(U) := \operatorname*{holim}_{\{U' \in \mathcal{O}_k(M) : U' \subset U\}} F(U'),$$

where $\mathcal{O}_k(M)$ is the category of open subsets of M that are homeomorphic to the disjoint union of at most k open balls.

$T_\infty F$ is defined to be the homotopy inverse limit of $T_k F$. When circumstances are favorable, the natural map $F(M) \to T_\infty F(M)$ is a homotopy equivalence, and then one says that the embedding tower converges. There is a deep and important convergence result, due to Goodwillie and Klein (unpublished, see [9]), for the functor

$$F(M) = \mathrm{Emb}(M, N),$$

where N is a fixed manifold. We will state it now, being an important fact in the background, but we will not really use it in this paper.

THEOREM 1.1. (Goodwillie–Klein, [9]) *The Taylor tower* (*as defined above*) *of the embedding functor* $\mathrm{Emb}(M,N)$ (*or* $\overline{\mathrm{Emb}}(M,N)$) *converges if* $\dim N - \dim M \geqslant 3$.

We will only need a much weaker convergence result, whose proof is accordingly easier. The "weak convergence theorem" says that the above Taylor tower converges if $2\dim M + 2 < \dim N$ and a proof can be found in the remark after Corollary 4.2.4 in [10]. The weak convergence result also holds for $\mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,N)_+$ by the main result of [24].

Let us have a closer look at the functor $U \mapsto \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(U,V)_+$. If U is homeomorphic to a disjoint union of finitely many open balls, say $U \cong k_U \times D^m$, then $\overline{\mathrm{Emb}}(U,V)$ is homotopy equivalent to the configuration space $\mathrm{C}(k_U, V)$ of k_U-tuples of distinct points in V or, equivalently, the space of k_U-tuples of disjoint balls inside the unit ball of V, which we denote by $\mathrm{B}(k_U, V)$. Abusing notation slightly, we can write that

$$T_k \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,V)_+ := \operatorname*{holim}_{U \in \mathcal{O}_k(M)} \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(U,V)_+ \simeq \operatorname*{holim}_{U \in \mathcal{O}_k(M)} \mathbf{HQ} \wedge \mathrm{B}(k_U, V)_+. \quad (2)$$

The right-hand side in the above formula is not really well-defined, because $\mathrm{B}(k_U, V)$ is not a functor on $\mathcal{O}_k(M)$, but it gives the right idea. The formula tells us that under favorable circumstances (e.g., if $2\dim M + 2 < \dim V$), the spectrum $\mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,V)_+$ can be written as a homotopy inverse limit of spectra of the form $\mathbf{HQ} \wedge \mathrm{B}(k_U, V)_+$. It is obvious that the maps in the diagram are closely related to the structure map in the little balls operad. Therefore, information about the rational homotopy type of the little balls operad may yield information about the homotopy type of spaces of embeddings. The key fact about the little balls operad that we want to use is the theorem of Kontsevich [14, Theorem 2, §3.2], asserting that this operad is formal. Let $\mathrm{B}(\bullet, V) = \{\mathrm{B}(n, V)\}_{n \geqslant 0}$ be our notation for the operads of little balls inside the unit ball of V.

THEOREM 1.2. (Kontsevich, [14]) *The little balls operad is formal over the reals. Explicitly, there is a chain of quasi-isomorphisms of operads of chain complexes connecting the operads* $\mathrm{C}_*(\mathrm{B}(\bullet, V))\otimes\mathbf{R}$ *and* $\mathrm{H}_*(\mathrm{B}(\bullet, V);\mathbf{R})$.

The formality theorem was announced by Kontsevich in [14], and an outline of a proof was given there. However, not all the steps of the proof are given in [14] in as much detail as some readers might perhaps wish. Because of this, the second and third authors decided to write another paper [16], whose primary purpose is to spell out the details of the proof of the formality theorem, following Kontsevich's outline. The paper [16] also has a second purpose, which is to prove a slight strengthening of the formality theorem, which we call "a relative version" of the formality theorem (Theorem 6.1 in this paper). We will give a sketch of the proof of the relative version in §6. Using the relative version of formality, together with some abstract homotopy theory, we deduce our first theorem (see Theorem 7.2 for a precise statement).

THEOREM 1.3. *Suppose that the basepoint embedding* $\alpha: M\hookrightarrow V$ *factors through a vector subspace* $W\subset V$ *such that* $\dim V\geqslant 2\dim W+1$. *Then the contravariant functor from* $\widetilde{\mathcal{O}}^s_k(M)$ *to chain complexes*

$$U \longmapsto \mathrm{C}_*(\overline{\mathrm{Emb}}(U,V))\otimes\mathbf{R}$$

is formal. This means that there is a chain of weak equivalences, natural in U:

$$\mathrm{C}_*(\overline{\mathrm{Emb}}(U,V))\otimes\mathbf{R}\simeq \mathrm{H}_*(\overline{\mathrm{Emb}}(U,V);\mathbf{R}).$$

Here, $\widetilde{\mathcal{O}}^s_k(M)$ is a suitable variation of the category $\mathcal{O}_k(M)$ (where k can be arbitrarily large). We will now give the rough idea of the proof. The category $\widetilde{\mathcal{O}}^s_k(M)$ is a subcategory of $\widetilde{\mathcal{O}}^s_k(W)$, so it is enough to prove that $\mathrm{C}_*(\overline{\mathrm{Emb}}(U,V))\otimes\mathbf{R}$ is formal as a functor on $\widetilde{\mathcal{O}}^s_k(W)$ (this is Theorem 7.2 in this paper). The category $\widetilde{\mathcal{O}}^s_k(W)$ is a category of balls in W, and so it is related to the little balls operad $\mathrm{B}(\bullet, W)$. Therefore, the category of (contravariant) functors from $\widetilde{\mathcal{O}}^s_k(W)$ to (real) chain complexes is closely related to the category of (right) modules over the chains on little balls operad $\mathrm{C}_*(\mathrm{B}(\bullet, W))\otimes\mathbf{R}$. The space $\overline{\mathrm{Emb}}(U,V)$, where U is the union of n balls, is equivalent to the nth space in the V-balls operad, $\mathrm{B}(n, V)$. We will show that the formality of the functor $\mathrm{C}_*(\overline{\mathrm{Emb}}(U,V))\otimes\mathbf{R}$ follows from the formality of the operad $\mathrm{C}_*(\mathrm{B}(\bullet, V))\otimes\mathbf{R}$ *as a module over the operad* $\mathrm{C}_*(\mathrm{B}(\bullet, W))\otimes\mathbf{R}$. The last formality statement follows from the relative formality theorem.

Remark 1.4. In several places, our argument relies on the following simple observation: the formality of a discrete (i.e., unenriched) diagram of chain complexes is equivalent to the splitting up to homotopy of the diagram as a direct sum of diagrams concentrated in a single homological degree (Proposition 3.3). This is convenient, because this kind of splitting is a homotopy invariant property, that is preserved by various Quillen equivalences between diagram categories that we need to consider, while the property of being formal cannot, in general, be transferred across a Quillen equivalence. On the other hand, for enriched diagrams, this observation is not true. We can see it in the example of modules over operads (which are a special case of enriched functors). The homology of the little balls operad is formal, but it does not split, as a module over itself, into a direct sum of modules concentrated in a single homological dimension. However, if we consider the operad $\mathrm{H}_*(\mathrm{B}(\bullet, V))$ as a module over the operad $\mathrm{H}_*(\mathrm{B}(\bullet, W))$, where W is a proper subspace of V, then it does split as a direct sum, for the silly reason that in this case the action of $\mathrm{H}_*(\mathrm{B}(\bullet, W))$ factors through an action of $\mathrm{H}_0(\mathrm{B}(\bullet, W))$, and so it is essentially a discrete action. This follows from the elementary fact that a proper linear inclusion of vector spaces $W \hookrightarrow V$ induces a null-homotopic map of spaces of little balls $\mathrm{B}(n, W) \to \mathrm{B}(n, V)$. Using this observation, together with the relative formality theorem, we are able to conclude the highly non-obvious statement that if $2 \dim W < \dim V$ then the operad $\mathrm{C}_*(\mathrm{B}(\bullet, V)) \otimes \mathbf{R}$ splits, as a module over $\mathrm{C}_*(\mathrm{B}(\bullet, W)) \otimes \mathbf{R}$, into a direct sum of modules that are homologically concentrated in a single dimension. This splitting is then used to prove an analogous splitting (and therefore the formality) of the functor $\mathrm{C}_*(\overline{\mathrm{Emb}}(U, V)) \otimes \mathbf{R}$. It is for this reason that we have to assume that M lies in a proper linear subspace of V, and to utilize a relative formality theorem.

A formality theorem similar to Theorem 1.3 was used in [15] for showing the collapse (at E^2) of a certain spectral sequence associated with the *embedding* tower for spaces of *knot embeddings*. However, to obtain a collapsing result for a spectral sequence of more general embedding spaces, we need, curiously enough, to turn to Weiss' *orthogonal* calculus (the standard reference is [22], and a brief overview can be found in §8). This is a calculus of covariant functors from the category of vector spaces and linear isometric inclusions to topological spaces (or spectra). With such a functor G, orthogonal calculus associates a tower of fibrations of functors $P_n G(V)$, where $P_n G$ is the nth Taylor polynomial of G in the orthogonal sense. Let $D_n G(V)$ denote the nth homogeneous layer in the orthogonal Taylor tower, namely the fiber of the map $P_n G(V) \to P_{n-1} G(V)$.

The functor that we care about is, of course, $G(V) = \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M, V)_+$, where M is fixed. We will use the notation $P_n \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M, V)_+$ and $D_n \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M, V)_+$ to denote its Taylor approximations and homogeneous layers in the sense of orthogonal calculus. It turns out that Theorem 1.3 implies that, under the same condition on the codimension, the orthogonal tower of $\mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M, V)_+$ splits as a product of its layers. The following is our main theorem (Theorem 10.10 in this paper).

THEOREM 1.5. *Under the assumptions of Theorem* 1.3, *there is a homotopy equivalence*, *natural with respect to embeddings in the M-variable* (*note that we do not claim that the splitting is natural in V*)

$$P_n\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+ \simeq \prod_{i=0}^{n} D_i\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+.$$

The following corollary is just a reformulation of the theorem.

COROLLARY 1.6. *Under the assumptions of Theorems* 1.3 *and* 1.5, *the spectral sequence for* $\mathrm{H}_*(\overline{\mathrm{Emb}}(M,V);\mathbf{Q})$ *that arises from the Taylor tower* (*in the sense of orthogonal calculus*) *of* $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ *collapses at* E^1.

Here is a sketch of the proof of Theorem 1.5. Embedding calculus tells us, roughly speaking, that $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ can be written as a homotopy limit of a diagram of spectra of the from $\mathbf{HQ}\wedge\mathrm{B}(k,V)_+$. Since there is a Quillen equivalence between the categories of rational spectra and rational chain complexes, we may pass to a diagram of rational chain complexes of the form $\mathrm{C}_*(\mathrm{B}(k,V))\otimes\mathbf{Q}$, whose homotopy limit is $\mathrm{C}_*(\overline{\mathrm{Emb}}(M,V))\otimes\mathbf{Q}$. On the other hand, Theorem 1.3 tells us that this diagram of chain complexes is formal when tensored with $\mathbf{R}$. One concludes that $\mathrm{C}_*(\overline{\mathrm{Emb}}(M,V))\otimes\mathbf{R}$ splits as the product of inverse limits of layers in the Postnikov towers of $\mathrm{C}_*(\mathrm{B}(k,V))\otimes\mathbf{R}$. It turns out that in our case tensoring with $\mathbf{R}$ commutes with taking the homotopy limit, and so it follows that there must be a similar splitting for $\mathrm{C}_*(\overline{\mathrm{Emb}}(M,V))\otimes\mathbf{Q}$ and therefore for $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$. On the other hand, it turns out that for functors of the form $\mathbf{HQ}\wedge\mathrm{B}(k,V)_+$, the Postnikov tower coincides, up to regrading, with the orthogonal tower, and therefore $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ splits as the product of inverse limits of layers in the orthogonal tower of rationally stabilized configuration spaces. But, taking the nth layer in the orthogonal tower is an operation that commutes (in our case) with homotopy inverse limits (unlike the operation of taking the nth layer of the Postnikov tower), and therefore $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ splits as the product of layers of its orthogonal tower.

Remark 1.7. In the case of knot embeddings, the spectral sequence associated with the orthogonal tower seems to coincide with the famous spectral sequence constructed by Vassiliev, since the latter also collapses, and the initial terms are isomorphic. We hope to come back to this in future papers.

In §11 we write an explicit description of $D_n\Sigma^\infty\overline{\mathrm{Emb}}(M,V)_+$, in terms of certain spaces of partitions (which can also be described as spaces of rooted trees) attached to M. This section is an announcement; detailed proofs appear in [2]. We do note the following consequence of our description of the layers: The homotopy groups of the layers depend only on the stable homotopy type of M and similarly the rational homotopy groups of the layers depend only on the rational homology type of M (Corollary 11.3). Combining this with Theorem 1.5, we obtain the following theorem (Theorem 11.6 in this paper).

THEOREM 1.8. *Under the assumptions of Theorem* 1.5, *the rational homology groups of the space* $\overline{\mathrm{Emb}}(M,V)$ *are determined by the rational homology type of* M. *More precisely*, *suppose that* M_1, M_2 *and* V *satisfy the assumptions of Theorem* 1.5, *and suppose that there is a zig-zag of maps*, *each inducing an isomorphism in rational homology*, *connecting* M_1 *and* M_2. *Then there is an isomorphism*

$$\mathrm{H}_*(\overline{\mathrm{Emb}}(M_1,V);\mathbf{Q}) \cong \mathrm{H}_*(\overline{\mathrm{Emb}}(M_2,V);\mathbf{Q}).$$

In view of this result, one may wonder whether the rational *homotopy* groups, or maybe even the rational homotopy *type* (rather than just rational homology) of $\overline{\mathrm{Emb}}(M,V)$ could be an invariant of the rational homotopy type of M (in high enough codimension). One could derive further hope from the fact that the little balls operad is not only formal, but also coformal. We would like to propose the following conjecture.

CONJECTURE 1.9. *Under the assumptions of Theorem* 1.3, *the rational homotopy spectral sequence for* $\pi_*(\overline{\mathrm{Emb}}(M,V))\otimes\mathbf{Q}$ *that arises from the Taylor tower* (*in the sense of orthogonal calculus*) *of* $\overline{\mathrm{Emb}}(M,V)$ *collapses at* E^1.

A statement essentially equivalent to this conjecture is proved in [3] in the special case of spaces of long knots in dimension $\geqslant 4$. In this case, the space of embeddings is an H-space (in fact, a double loop space), and this gives one enough control over the homotopy type of the space to force the desired conclusion.

A general point that we are trying to make with this paper is this: while embedding calculus is important, and is in some ways easier to understand than orthogonal calculus, the Taylor tower in the sense of orthogonal calculus is also interesting and is worthy of a further study. We hope that §11 will convince the reader that the layers of the orthogonal tower, while not exactly simple, are interesting, and it may be possible to do calculations with them. We intend to come back to this in the future.

1.1. A section by section outline

In §2 we review background material and fix terminology on spaces, spectra and chain complexes. In §3 we define the notion of formality of diagram chain complexes. The main result of this section is the following simple but useful observation: the stable formality of a diagram can be interpreted as the splitting of its Postnikov tower.

Our next goal is to exploit Kontsevich's formality of the little balls operads and deduce some formality results of diagrams of embedding spaces. In order to do that we first review, in §4, enriched categories, their modules and the associated homotopy theory. In §5 we review classical operads and their modules and give an alternative viewpoint on those in terms of enriched categories. This will be useful for the study of the homotopy theory of modules over an operad. We then digress in §6 to prove a relative version of Kontsevich's formality of the little balls operads that we need for our applications. In §7 we deduce the formality of a certain diagram of real-valued chains on embedding spaces.

In §8 we digress again to give a review of embedding calculus and orthogonal calculus, and record some generalities on how these two brands of calculus may interact. In §9 we use the formality of a diagram of chains on embedding spaces established in §7 to show that the stages in the embedding tower of $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ split in a certain way, but *not* as the product of the layers in the embedding tower. In §10 we reinterpret this splitting once again, to prove our main theorem: Under a certain codimension hypothesis, the orthogonal tower of $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ splits as the product of its layers. In §11 we sketch a description of the layers in the orthogonal tower, and deduce that the rational homology of the space of embeddings (modulo immersions) of a manifold into a high-dimensional vector space is determined by the rational homology type of the manifold.

1.2. Acknowledgments

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2. Spaces, spectra, and chain complexes

Let us introduce the basic categories that we will work with.

- Top will stand for the category of compactly generated spaces (we choose *compactly generated* to make it a *closed* monoidal category, see §4). If X is a space, we denote by X_+ the based space obtained by adjoining a disjoint basepoint.
- Spectra will be the category of (-1)-connected spectra. We denote by $\mathbf{HQ}$ the Eilenberg–MacLane spectrum such that $\pi_0(\mathbf{HQ})=\mathbf{Q}$. A *rational spectrum* is a module spectrum over $\mathbf{HQ}$. For a space X, $\Sigma^\infty X_+$ stands for the suspension spectrum of X, and $\mathbf{HQ}\wedge X_+$ denotes the stable rationalization of X. It is well known that there is a *rational* equivalence $\mathbf{HQ}\wedge X_+\simeq\Sigma^\infty X_+$.
- $\mathcal{V}$ will denote the category of rational vector spaces (or $\mathbf{Q}$-vector spaces), and $\mathcal{V}^{\Delta^{\mathrm{op}}}$ the category of simplicial $\mathbf{Q}$-vector spaces.
- $\mathrm{Ch}_{\mathbf{Q}}$ and $\mathrm{Ch}_{\mathbf{R}}$ will denote the category of non-negatively graded rational and real chain complexes, respectively. We will sometimes use Ch to denote either one of these two categories.

Most of the above categories have a *Quillen model structure*, which means that one can apply the techniques of homotopy theory to them. A good introduction to closed model categories is [8], a good reference is [13]. There are slight variations in the literature as to the precise definition of model structure. We use the definition given in [13]. In particular, we assume the existence of functorial fibrant and cofibrant replacements. The category for which we will use the model structure most heavily is the category of chain complexes. Thus we remind the reader that the category of chain complexes over a field has a model structure where weak equivalences are quasi-isomorphisms, fibrations are chain maps that are surjective in positive degrees, and cofibrations are (since all modules are projective) chain maps that are injective in all degrees [8, Theorem 7.2]. We will also need the fact that the category of rational spectra is a Quillen model category and is Quillen equivalent to the category $\mathrm{Ch}_{\mathbf{Q}}$. For a proof of this (in fact, of a more general statement, involving the category of module spectra over a general Eilenberg–MacLane commutative ring-spectrum) see, for example, [18].

We now define some basic functors between the various categories in which we want to do homotopy theory.

2.0.1. Homology

We think of homology as a functor from chain complexes to chain complexes. Thus if C is a chain complex, then $\mathrm{H}_*(C)$ is the chain complex whose chain groups are the homology groups of C, and whose differentials are zero. Moreover, we define $\mathrm{H}_n(C)$ to be the chain complex having the nth homology group of C in degree n and zero in all other degrees. Thus, H_n is a functor from Ch to Ch as well. Notice that there are obvious isomorphisms of functors

$$\mathrm{H}_* \cong \bigoplus_{n=0}^{\infty} \mathrm{H}_n \cong \prod_{n=0}^{\infty} \mathrm{H}_n\,.$$

2.0.2. The normalized chains functor

To get from spaces to chain complexes, we will use the *normalized singular chains* functor $\mathrm{C}_*\colon \mathrm{Top}\to\mathrm{Ch}$, defined by

$$\mathrm{C}_*(X) = \mathrm{N}(\mathbf{Q}[\mathcal{S}_\bullet(X)]).$$

Here $\mathcal{S}_\bullet(X)$ is the simplicial set of singular simplices of X, $\mathbf{Q}[\mathcal{S}_\bullet(X)]$ is the simplicial $\mathbf{Q}$-vector space generated by $\mathcal{S}_\bullet(X)$, and $\mathrm{N}\colon \mathcal{V}^{\Delta^{\mathrm{op}}}\to\mathrm{Ch}$ is the normalized chains functor as defined for example in [21, Chapter 8].

2.1. Postnikov sections

We will need to use Postnikov towers in the categories of chain complexes, and spectra. We now review the construction of Postnikov towers in the category of chain complexes. For an integer n and a chain complex (C,d), let $d(C_{n+1})$ be the n-dimensional boundaries in C. We define the *n-th Postnikov section* of C, denoted $(\mathrm{Po}_n(C),d')$, as follows:

$$(\mathrm{Po}_n(C))_i = \begin{cases} C_i, & \text{if } i \leqslant n, \\ d(C_{n+1}), & \text{if } i = n+1, \\ 0, & \text{if } i > n+1. \end{cases}$$

The differential d' is defined to be d in degrees $\leqslant n$, and the obvious inclusion

$$d(C_{n+1}) \hookrightarrow C_n$$

in degree $n+1$. It is easy to see that Po_n defines a functor from Ch to Ch. Moreover, $\mathrm{H}_i(\mathrm{Po}_n(C)) \cong \mathrm{H}_i(C)$ for $i \leqslant n$ and $\mathrm{H}_i(\mathrm{Po}_n(C))=0$ for $i>n$.

For each n, there is a natural fibration (i.e., a degreewise surjection)

$$\pi_n : \mathrm{Po}_n(C) \twoheadrightarrow \mathrm{Po}_{n-1}(C)$$

defined as follows: π_n is the identity in all degrees except $n+1$ and n; in degree $n+1$ it is the zero homomorphism; and in degree n it is the obvious surjective map $d: C_n \to d(C_n)$. Since π_n is a fibration, $\ker(\pi_n)$ can serve as the model for its homotopy fiber. Clearly, $\ker(\pi_n)$ is a chain complex concentrated in dimensions n and $n+1$. The homology of the kernel is concentrated in dimension n, and in this dimension it equals the homology of the original complex C. A similar formula defines a natural map $\varrho_n : C \to \mathrm{Po}_n(C)$, and we have $\pi_n \varrho_n = \varrho_{n-1}$. Note that ϱ_n, like π_{n+1}, is an isomorphism (on chain level) in degrees $\leqslant n$.

2.2. Diagrams

Let $\mathcal{A}$ be a small category and let $\mathcal{E}$ be a category. An *$\mathcal{A}$-diagram in $\mathcal{E}$* is just a functor $F: \mathcal{A} \to \mathcal{E}$. In this paper a diagram can be a functor which is either covariant or contravariant. A *morphism* of $\mathcal{A}$-diagrams is a natural transformation between two functors. Such a morphism is called a *weak equivalence* if it is a weak equivalence objectwise, for a given notion of weak equivalence in the category $\mathcal{E}$. In practice, we will only consider diagrams of spaces, chain complexes or spectra.

2.3. Homotopy limits

We will make heavy use of homotopy limits of diagrams in Spectra and in Ch. Homotopy limits of diagrams in a general model category are treated in [13, Chapter 19]. Generally, when we take the homotopy limit of a diagram, we assume that all the objects in the diagram are fibrant and cofibrant—this will ensure "correct" homotopical behavior in all cases. Since most of our homotopy limits will be taken in the category of chain complexes over $\mathbf{Q}$ or $\mathbf{R}$, in which all objects are fibrant and cofibrant, this is a moot point in many cases. The only other category in which we will take homotopy limits is the category of rational spectra, in which case we generally assume that we have taken fibrant-cofibrant replacement of all objects, whenever necessary.

It follows from the results in [13, §19.4] that if R and L are the right and left adjoint in a Quillen equivalence, then both R and L commute with homotopy limits up to a zig-zag of natural weak equivalences. In particular, this enables us to shuttle back and forth between homotopy limits of diagrams of rational spectra and diagrams of rational chain complexes.

3. Formality and homogeneous splitting of diagrams

The notion of formality was first introduced by Sullivan in the context of rational homotopy theory [20], [7]. Roughly speaking a chain complex (possibly with additional structure) is called formal if it is weakly equivalent to its homology. In this paper we will only use the notion of formality of diagrams of chain complexes (over $\mathbf{Q}$ and over $\mathbf{R}$).

Definition 3.1. Let $\mathcal{A}$ be a small category. An $\mathcal{A}$-diagram of chain complexes

$$F\colon \mathcal{A} \longrightarrow \mathrm{Ch},$$

is *formal* if there is a chain of weak equivalences $F\simeq \mathrm{H}_* \circ F$.

Formality of chain complexes has a convenient interpretation as the splitting of the Postnikov tower.

Definition 3.2. Let $\mathcal{A}$ be a small category. We say that an $\mathcal{A}$-diagram of chain complexes, $F\colon \mathcal{A}\to\mathrm{Ch}$, *splits homogeneously* if there exist $\mathcal{A}$-diagrams $\{F_n\}_{n=0}^{\infty}$ of chain complexes such that $F\simeq\bigoplus_{n=0}^{\infty} F_n$ and $\mathrm{H}_*(F_n)=\mathrm{H}_n(F_n)$ (i.e., F_n is homologically concentrated in degree n).

PROPOSITION 3.3. *Let $\mathcal{A}$ be a small category. An $\mathcal{A}$-diagram of chain complexes is formal if and only if it splits homogeneously.*

Proof. Let F be an $\mathcal{A}$-diagram of chain complexes.

In one direction, if F is formal then $F \simeq \mathrm{H}_*(F)$. Since $\mathrm{H}_* = \bigoplus_{n=0}^{\infty} \mathrm{H}_n$, we get the homogeneous splitting $F \simeq \bigoplus_{n=0}^{\infty} \mathrm{H}_n(F)$.

In the other direction, suppose that $F \simeq \bigoplus_{n=0}^{\infty} F_n$ with $\mathrm{H}_*(F_n) = \mathrm{H}_n(F_n) = \mathrm{H}_n(F)$. Recall the definition of Postnikov sections of chain complexes from §2. Then

$$\ker(\mathrm{Po}_n(F_n) \xrightarrow{\pi_n} \mathrm{Po}_{n-1}(F_n))$$

is concentrated in degrees n and $n+1$ and its homology is exactly $\mathrm{H}_n(F)$. Thus we have a chain of quasi-isomorphisms

$$\begin{aligned} F_n \xrightarrow{\simeq} \mathrm{Po}_n(F_n) \xleftarrow{\simeq} & \ker(\mathrm{Po}_n(F_n) \to \mathrm{Po}_{n-1}(F_n)) \\ & \xrightarrow{\simeq} \mathrm{H}_n(\ker(\mathrm{Po}_n(F_n) \to \mathrm{Po}_{n-1}(F_n))) \cong \mathrm{H}_n(F), \end{aligned}$$

and so $F \simeq \bigoplus_{n=0}^{\infty} \mathrm{H}_n(F) = \mathrm{H}_*(F)$. $\square$

Remark 3.4. Note that above we have proved the following (elementary) statement: Suppose that F and G are two $\mathcal{A}$-diagrams of chain complexes such that both F and G are homologically concentrated in degree n and such that there is an isomorphism of diagrams $\mathrm{H}_n(F) \cong \mathrm{H}_n(G)$. Then there is a chain of weak equivalences, $F \simeq G$. Using the Quillen equivalence between rational spectra and rational chain complexes, one can prove the analogous statement for diagrams of rational Eilenberg–MacLane spectra: If F and G are two $\mathcal{A}$-diagrams of rational Eilenberg–MacLane spectra concentrated in degree n, and if there is an isomorphism of diagrams $\pi_n(F) \cong \pi_n(G)$, then there is a chain of weak equivalences $F \simeq G$. Actually this is true more generally, for any diagrams of Eilenberg–MacLane spectra concentrated in one degree, but we will not need that fact.

Remark 3.5. Let F be a diagram with values in Ch. There is a tower of fibrations converging to $\operatorname{holim} F$ whose nth stage is $\operatorname{holim} \mathrm{Po}_n F$. We call it the lim-*Postnikov tower*. Of course, this tower does not usually coincide with the Postnikov tower of $\operatorname{holim} F$. Since $\mathrm{H}_* \cong \prod_{n=0}^{\infty} \mathrm{H}_n$, and homotopy limits commute with products, it follows immediately that if F is a formal diagram then the lim-Postnikov tower of $\operatorname{holim} F$ splits as a product, namely

$$\operatorname{holim} F \simeq \prod_{n=0}^{\infty} \operatorname{holim} \mathrm{H}_n \circ F.$$

The proof of the following result is also straightforward.

LEMMA 3.6. *Let $\lambda: \mathcal{A} \to \mathcal{A}'$ be a functor between small categories and let F be an $\mathcal{A}'$-diagram of chain complexes. If the $\mathcal{A}'$-diagram F is formal, then so is the $\mathcal{A}$-diagram $\lambda^*(F) := F \circ \lambda$.*

4. Enriched categories and their modules

We now briefly recall some definitions and facts about symmetric monoidal categories, enriched categories, Quillen model structures, etc. The standard reference for symmetric monoidal categories and enriched categories is [4, Chapter 6]. We will also need some results of Schwede and Shipley on the homotopy theory of enriched categories developed in [19], especially §6, from which we also borrow some of our notation and terminology.

4.1. Monoidal model categories and enriched categories

A *closed symmetric monoidal category* is a triple $(\mathcal{C}, \otimes, \mathbf{1})$ such that $\otimes$ and $\mathbf{1}$ endow the category $\mathcal{C}$ with a symmetric monoidal structure, and such that, for each object Y, the endofunctor $-\otimes Y: \mathcal{C} \to \mathcal{C}$, $X \mapsto X \otimes Y$ admits a right adjoint denoted by

$$\mathcal{C}(Y, -): Z \longmapsto \mathcal{C}(Y, Z).$$

It is customary to think of $\mathcal{C}(Y, Z)$ as an "internal mapping object". Throughout this section, $\mathcal{C}$ stands for a closed symmetric monoidal category.

A *monoidal model category* is a closed symmetric monoidal category equipped with a compatible Quillen model structure (see [19, Definition 3.1] for a precise definition).

The only examples of monoidal model categories that we will consider in this paper are the following:

(1) The category $(\mathrm{Top}, \times, *)$ of compactly generated topological spaces with cartesian product;

(2) The category $(\mathrm{Ch}, \otimes, \mathbf{K})$ of non-negatively graded chain complexes over $\mathbf{K}$ (where $\mathbf{K}$ is $\mathbf{Q}$ or $\mathbf{R}$), with tensor product.

The *internal* hom *functor* in the category Ch is defined as follows. Let Y_* and Z_* be chain complexes. Then $\mathrm{Ch}(Y_*, Z_*)$ is the chain complex that in positive degrees $p>0$ is defined by

$$\mathrm{Ch}_p(Y_*, Z_*) = \prod_{n=0}^{\infty} \hom(Y_n, Z_{n+p}),$$

while in degree zero, we have

$$\mathrm{Ch}_0(Y_*, Z_*) = \{\text{chain homomorphisms from } Y_* \text{ to } Z_*\}.$$

The differential in $\mathrm{Ch}(Y_*, Z_*)$ is determined by the formula

$$D(\{f_n\}) = \{d_Z f_n - (-1)^p f_{n-1} d_Y\}, \quad \text{for } f_n \in \hom(Y_n, Z_{n+p}).$$

4.2. Enriched categories

A category $\mathcal{O}$ *enriched over* $\mathcal{C}$, or a $\mathcal{C}$*-category*, consists of a class I (representing the *objects* of $\mathcal{O}$), and, for any objects $i,j,k\in I$, a $\mathcal{C}$-object $\mathcal{O}(i,j)$ (representing the morphisms from i to j in $\mathcal{O}$) and $\mathcal{C}$-morphisms

$$\mathcal{O}(i,j)\otimes\mathcal{O}(j,k)\longrightarrow\mathcal{O}(i,k)\quad\text{and}\quad\mathbf{1}\longrightarrow\mathcal{O}(i,i)$$

(representing the composition of morphisms in $\mathcal{O}$ and the identity morphism on i). These structure morphisms are required to be associative and unital in the evident sense. Notice that a closed symmetric monoidal cateory $\mathcal{C}$ is enriched over itself since $\mathcal{C}(Y,Z)$ is an object of $\mathcal{C}$. Following [19], we use the term $\mathcal{C}I$*-category* to signify a category enriched over $\mathcal{C}$, whose set of objects is I.

Let $\mathcal{O}$ be a $\mathcal{C}I$-category and $\mathcal{R}$ be a category enriched over $\mathcal{C}$. A (covariant) *functor enriched over* $\mathcal{C}$, or $\mathcal{C}$*-functor* from $\mathcal{O}$ to $\mathcal{R}$,

$$M:\mathcal{O}\longrightarrow\mathcal{R},$$

consists of an $\mathcal{R}$-object $M(i)$ for every $i\in I$, and of morphisms in $\mathcal{C}$,

$$M(i,j):\mathcal{O}(i,j)\longrightarrow\mathcal{R}(M(i),M(j)),$$

for every $i,j\in I$, that are associative and unital. There is an analogous notion of a *contravariant* $\mathcal{C}$*-functor*.

A *natural transformation enriched over* $\mathcal{C}$, $\Phi: M\to M'$, between two $\mathcal{C}$-functors

$$M,M':\mathcal{O}\longrightarrow\mathcal{R}$$

consists of $\mathcal{C}$-morphisms

$$\Phi_i:\mathbf{1}\longrightarrow\mathcal{R}(M(i),M'(i))$$

for every object i of $\mathcal{O}$, that satisfy the obvious commutativity conditions for a natural transformation (see [4, Definition 6.2.4]). Notice that if $\mathcal{R}=\mathcal{C}$ then a morphism

$$\Phi_i:\mathbf{1}\longrightarrow\mathcal{C}(M(i),M'(i))$$

is the same as the adjoint morphism $\Phi(i): M(i)\to M'(i)$ in $\mathcal{C}$.

For fixed $\mathcal{C}$ and I, we consider the collection of $\mathcal{C}I$-categories as a category in its own right. A morphism of $\mathcal{C}I$-categories is an enriched functor which is the *identity* on the set of objects.

Suppose now that $\mathcal{C}$ is a monoidal model category. In particular, $\mathcal{C}$ is equipped with a notion of weak equivalence. Then we say that a morphism $\Psi:\mathcal{O}\to\mathcal{R}$ of $\mathcal{C}I$-categories is a *weak equivalence* if it is a weak equivalence pointwise, i.e., if the map $\mathcal{O}(i,j)\to\mathcal{R}(i,j)$ is a weak equivalence in $\mathcal{C}$ for all $i,j\in I$.

4.3. Homotopy theory of right modules over enriched categories

For a $\mathcal{C}I$-category $\mathcal{O}$, a (*right*) $\mathcal{O}$-*module* is a contravariant $\mathcal{C}$-functor from $\mathcal{O}$ to $\mathcal{C}$. Explicitly an $\mathcal{O}$-module M consists of objects $M(i)$ in $\mathcal{C}$ for $i\in I$ and (since $\mathcal{C}$ is a *closed* monoidal category and since it is enriched over itself) of $\mathcal{C}$-morphisms

$$M(j)\otimes\mathcal{O}(i,j)\longrightarrow M(i)$$

which are associative and unital. A *morphism* of $\mathcal{O}$-modules, $\Phi: M\to M'$, is an enriched natural transformation, i.e., a collection of $\mathcal{C}$-morphisms $\Phi(i): M(i)\to M'(i)$ satisfying the usual naturality requirements. Such a morphism of $\mathcal{O}$-module is a *weak equivalence* if each $\Phi(i)$ is a weak equivalence in $\mathcal{C}$. We denote by $\mathcal{M}od\text{-}\mathcal{O}$ the category of right $\mathcal{O}$-modules and natural transformations.

Let $\Psi:\mathcal{O}\to\mathcal{R}$ be a morphism of $\mathcal{C}I$-categories. Clearly, Ψ induces a *restriction of scalars* functor on module categories

$$\Psi^*: \mathcal{M}od\text{-}\mathcal{R}\longrightarrow\mathcal{M}od\text{-}\mathcal{O},$$
$$M\longmapsto M\circ\Psi.$$

As explained in [19, p. 323], the functor Ψ^* has a left adjoint functor Ψ_*, also denoted

$$-\otimes_{\mathcal{O}}\mathcal{R}$$

(one can think of Ψ_* as the left Kan extension). Schwede and Shipley [19, Theorem 6.1] prove that under some technical hypotheses on $\mathcal{C}$, the category $\mathcal{M}od\text{-}\mathcal{O}$ has a Quillen model structure, and moreover, if Ψ is a weak equivalence of $\mathcal{C}I$-categories, then the pair (Ψ^*,Ψ_*) induces a Quillen equivalence of module categories.

We will need this result in the case $\mathcal{C}=\mathrm{Ch}$. In keeping with our notation, we use $\mathrm{Ch}I$-categories to denote categories enriched over chain complexes, with object set I. Note that the category of modules over a $\mathrm{Ch}I$-category admits coproducts (i.e. direct sums).

THEOREM 4.1. (Schwede–Shipley, [19])

(1) *Let $\mathcal{O}$ be a* $\mathrm{Ch}I$*-category. Then $\mathcal{M}od\text{-}\mathcal{O}$ has a cofibrantly generated Quillen model structure, with fibrations and weak equivalences defined objectwise.*

(2) *Let $\Psi:\mathcal{O}\to\mathcal{R}$ be a weak equivalence of* $\mathrm{Ch}I$*-categories. Then (Ψ^*,Ψ_*) induce a Quillen equivalence of the associated module categories.*

Proof. General conditions on $\mathcal{C}$ that guarantee the result are given in [19, Theorem 6.1]. It is straightforward to check that the conditions are satisfied by the category of chain complexes (the authors of [19] verify them for various categories of spectra, and the verification for chain complexes is strictly easier). □

Let $\mathcal{O}$ and $\mathcal{R}$ be $\mathcal{C}I$-categories and let M and N be right modules over $\mathcal{O}$ and $\mathcal{R}$, respectively. A morphism of pairs $(\mathcal{O}, M)\to(\mathcal{R}, N)$ consists of a morphism of $\mathcal{C}I$-categories $\Psi: \mathcal{O}\to\mathcal{R}$ and a morphism of $\mathcal{O}$-modules $\Phi: M\to\Psi^*(N)$. The corresponding category of pairs $(\mathcal{O}, M)$ is called the *$\mathcal{C}I$-module category.*

A morphism (Ψ, Φ) in a $\mathcal{C}I$-module is called a *weak equivalence* if both Ψ and Φ are weak equivalences. Two objects of a $\mathcal{C}I$-module are called *weakly equivalent* if they are linked by a chain of weak equivalences, pointing in either direction.

In our study of the formality of the little balls operad, we will consider certain splittings of $\mathcal{O}$-modules into direct sums. The following homotopy invariance property of such a splitting will be important.

PROPOSITION 4.2. *Let $(\mathcal{O}, M)$ and $(\mathcal{O}', M')$ be weakly equivalent* $\mathrm{Ch}I$*-modules. If M is weakly equivalent as an $\mathcal{O}$-module to a direct sum $\bigoplus_n M_n$, then M' is weakly equivalent as an $\mathcal{O}'$-module to a direct sum $\bigoplus_n M_n'$ such that $(\mathcal{O}, M_n)$ is weakly equivalent to $(\mathcal{O}', M_n')$ for each n.*

Proof. It is enough to prove that for a direct weak equivalence

$$(\Psi, \Phi): (\mathcal{O}, M) \xrightarrow{\simeq} (\mathcal{R}, N),$$

M splits as a direct sum if and only if N splits in a compatible way.

In one direction, suppose that $N\simeq\bigoplus_n N_n$ as $\mathcal{R}$-modules. It is clear that the restriction of the scalars functor Ψ^* preserves direct sums and weak equivalences (quasi-isomorphisms). Therefore $\Psi^*(N)\simeq\bigoplus_n \Psi^*(N_n)$. Since by hypothesis M is weakly equivalent to $\Psi^*(N)$, we have the required splitting of M.

In the other direction suppose that the $\mathcal{O}$-module M is weakly equivalent to $\bigoplus_n M_n$. We can assume that each M_n is cofibrant, hence so is $\bigoplus_n M_n$. Moreover $\Psi^*(N)$ is fibrant because every $\mathcal{O}$-module is. Therefore, since M is weakly equivalent to $\Psi^*(N)$, there exists a direct weak equivalence $\gamma: \bigoplus_n M_n \xrightarrow{\simeq} \Psi^*(N)$. Since (Ψ^*, Ψ_*) is a Quillen equivalence, the weak equivalence γ induces an adjoint weak equivalence

$$\gamma^\flat: \Psi_*\left(\bigoplus_n M_n\right) \xrightarrow{\simeq} N.$$

As a left adjoint, Ψ_* commutes with coproducts, therefore we get the splitting

$$\bigoplus_n \Psi_*(M_n) \xrightarrow{\simeq} N.$$

Moreover, we have a weak equivalence $M_n \xrightarrow{\simeq} \Psi^*\Psi_*(M_n)$, because it is the adjoint of the identity map on $\Psi_*(M_n)$, M_n is cofibrant, and (Ψ^*, Ψ_*) is a Quillen equivalence. Thus this splitting of N is compatible with the given splitting of M. □

4.4. Lax monoidal functors, enriched categories, and their modules

Let $\mathcal{C}$ and $\mathcal{D}$ be two symmetric monoidal categories. A *lax symmetric monoidal functor* $F:\mathcal{C}\to\mathcal{D}$ is a (non-enriched) functor, together with morphisms

$$\mathbf{1}_{\mathcal{D}} \longrightarrow F(\mathbf{1}_{\mathcal{C}}) \quad \text{and} \quad F(X)\otimes F(Y) \longrightarrow F(X\otimes Y),$$

natural in $X,Y\in\mathcal{C}$, that satisfy the obvious unit, associativity, and symmetry relations. In this paper, we will sometimes use "monoidal" to mean "lax symmetric monoidal", as this is the only notion of monoidality that we will consider.

Such a lax symmetric monoidal functor F induces a functor (which we will still denote by F) from $\mathcal{C}I$-categories to $\mathcal{D}I$-categories. Explicitly if $\mathcal{O}$ is a $\mathcal{C}I$-category then $F(\mathcal{O})$ is the $\mathcal{D}$-category whose set of objects is I and morphisms are

$$(F(\mathcal{O}))(i,j) := F(\mathcal{O}(i,j)).$$

Moreover, F induces a functor from $\mathcal{M}od$-$\mathcal{O}$ to $\mathcal{M}od$-$F(\mathcal{O})$. We will denote this functor by F as well.

The main examples that we will consider are those from §2.0.1 and §2.0.2, and their composites:

(1) Homology: $\mathrm{H}_*:(\mathrm{Ch},\otimes,\mathbf{K})\to(\mathrm{Ch},\otimes,\mathbf{K})$;

(2) Normalized singular chains: $\mathrm{C}_*:(\mathrm{Top},\times,*)\to(\mathrm{Ch},\otimes,\mathbf{K})$, $X\mapsto \mathrm{C}_*(X)$.

The fact that the normalized chains functor is lax monoidal, and equivalent to the unnormalized chains functor, is explained in [19, §2]. As is customary, we often abbreviate the composite $\mathrm{H}_*\circ\mathrm{C}_*$ as H_*.

Recall that we also use the functor $\mathrm{H}_n:(\mathrm{Ch},\otimes,\mathbf{K})\to(\mathrm{Ch},\otimes,\mathbf{K})$, where $\mathrm{H}_n(C,d)$ is seen as a chain complex concentrated in degree n. The functor H_n is not monoidal for $n>0$. However, H_0 is monoidal.

Thus, if $\mathcal{B}$ is a small $\mathrm{Top}I$-category then $\mathrm{C}_*(\mathcal{B})$ and $\mathrm{H}_*(\mathcal{B})$ are $\mathrm{Ch}I$-categories. Also if $B:\mathcal{B}\to\mathrm{Top}$ is a $\mathcal{B}$-module then $\mathrm{C}_*(B)$ is a $\mathrm{C}_*(\mathcal{B})$-module and $\mathrm{H}_*(B)$ is an $\mathrm{H}_*(\mathcal{B})$-module. We also have the $\mathrm{Ch}I$-category $\mathrm{H}_0(\mathcal{B})$.

4.5. Discretization of enriched categories

When we want to emphasize that a category is *not* enriched (or, equivalently, enriched over Set), we will use the term *discrete category*. When we speak of an $\mathcal{A}$-diagram we always assume that $\mathcal{A}$ is a discrete category.

Let $\mathcal{C}$ be a closed symmetric monoidal category. Consider the forgetful functor

$$\begin{aligned}\phi:\mathcal{C} &\longrightarrow \mathrm{Set},\\ C &\longmapsto \hom_{\mathcal{C}}(\mathbf{1},C).\end{aligned}$$

It is immediate from the definitions that ϕ is a monoidal functor. Therefore, it induces a functor from categories enriched over $\mathcal{C}$ to discrete categories. We will call this induced functor the *discretization* functor. Let $\mathcal{O}$ be a category enriched over $\mathcal{C}$. The discretization of $\mathcal{O}$ will be denoted $\mathcal{O}^\delta$. It has the same objects as $\mathcal{O}$, and its sets of morphisms are given by the discretization of morphisms in $\mathcal{O}$. For example, Top can be either the Top-enriched category or the associated discrete category. For Ch, the set of morphisms between two chain complexes X_* and Y_* in the discretization of Ch is the set of cycles of degree 0 in the chain complex $\mathrm{Ch}(X_*, Y_*)$, i.e. the set of chain maps. It is easy to see that if $\mathcal{C}$ is a closed symmetric monoidal category, then the discretization of $\mathcal{C}$ is the same as $\mathcal{C}$, considered as a discrete category. We will not use special notation to distinguish between $\mathcal{C}$ and its underlying discrete category.

Let $M: \mathcal{O} \to \mathcal{R}$ be a $\mathcal{C}$-functor between two $\mathcal{C}$-categories. The *underlying discrete functor* is the functor $M^\delta: \mathcal{O}^\delta \to \mathcal{R}^\delta$ induced in the obvious way from M. More precisely, if i is an object of $\mathcal{O}$ then $M^\delta(i) = M(i)$. If j is another object and $f \in \mathcal{O}^\delta(i,j)$, that is $f: \mathbf{1} \to \mathcal{O}(i,j)$, then $M^\delta(f) \in \mathcal{R}^\delta(M^\delta(i), M^\delta(j))$ is defined as the composite

$$\mathbf{1} \xrightarrow{f} \mathcal{O}(i,j) \xrightarrow{M(i,j)} \mathcal{R}(M(i), M(j)).$$

Similarly if $\Phi: M \to M'$ is an enriched natural transformation between enriched functors, we have an induced discrete natural transformation $\Phi^\delta: M^\delta \to (M')^\delta$. In particular, an $\mathcal{O}$-module M induces an $\mathcal{O}^\delta$-diagram M^δ in $\mathcal{C}$ and a morphism of $\mathcal{O}$-modules induces a morphism of $\mathcal{O}^\delta$-diagrams.

Let $F: \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor, let $\mathcal{O}$ be a $\mathcal{C}I$-category, and let $M: \mathcal{O} \to \mathcal{C}$ be an $\mathcal{O}$-module. As explained before, we have an induced $\mathcal{D}I$-category $F(\mathcal{O})$, and an $F(\mathcal{O})$-module $F(M)$. We may compare $\mathcal{O}^\delta$ and $F(\mathcal{O})^\delta$ by means of a functor

$$F^\delta_{\mathcal{O}}: \mathcal{O}^\delta \longrightarrow F(\mathcal{O})^\delta$$

which is the identity on objects and if $f: \mathbf{1}_{\mathcal{C}} \to \mathcal{O}(i,j)$ is a morphism in $\mathcal{O}^\delta$, then $F^\delta_{\mathcal{O}}(f)$ is the composite $\mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{F(f)} F(\mathcal{O}(i,j))$.

It is straightforward to verify the following two properties of discretization.

LEMMA 4.3. *Let $F: \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor, let $\mathcal{O}$ be a $\mathcal{C}I$-category and let M be an $\mathcal{O}$-module. The following diagram of discrete functors commutes*

$$\begin{tikzcd} \mathcal{O}^\delta \arrow[r, "M^\delta"] \arrow[d, "F^\delta_{\mathcal{O}}"'] & \mathcal{C} \arrow[d, "F"] \\ F(\mathcal{O})^\delta \arrow[r, "F(M)^\delta"] & \mathcal{D}. \end{tikzcd}$$

LEMMA 4.4. *Let $\mathcal{C}$ be a monoidal model category and let $\mathcal{O}$ be a $\mathcal{C}I$-category. If $\Phi\colon M \xrightarrow{\simeq} M'$ is a weak equivalence of $\mathcal{O}$-modules then $\Phi^\delta\colon M^\delta \xrightarrow{\simeq} (M')^\delta$ is a weak equivalence of $\mathcal{O}^\delta$-diagrams.*

5. Operads and associated enriched categories

We will first recall the notions of operads, right modules over operads, and weak equivalences of operads. We will then describe the enriched category associated with an operad. Finally, we will treat the central example of the little balls operad. The enriched category viewpoint will help us to deduce (in §7) the formality of certain topological functors from the formality of the little balls operad.

5.1. Operads and right modules

Among the many references for operads, a recent one that covers them from a viewpoint similar to ours is Ching's paper [5]. However, there is one important difference between our setting and Ching's: He only considers operads without the zero-th term, while we consider operads with one. Briefly, an *operad* in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, or a $\mathcal{C}$-*operad*, is a symmetric sequence $O(\bullet)=\{O(n)\}_{n=0}^\infty$ of objects of $\mathcal{C}$, equipped with structure maps

$$O(n)\otimes O(m_1)\otimes \ldots \otimes O(m_n) \longrightarrow O(m_1+\ldots+m_n) \quad \text{and} \quad \mathbf{1} \longrightarrow O(1),$$

satisfying certain associativity, unit, and symmetry axioms. There is an obvious notion of a *morphism of operads*.

When $\mathcal{C}$ is a monoidal model category, we say that a morphism $f\colon O(\bullet)\to R(\bullet)$ of $\mathcal{C}$-operads is a *weak equivalence* if $f(n)$ is a weak equivalence in $\mathcal{C}$ for each natural number n. If $f\colon O(\bullet)\to R(\bullet)$ and $f'\colon O'(\bullet)\to R'(\bullet)$ are morphisms of operads, a *morphism of arrows* from f to f' is a pair $(o\colon O(\bullet)\to O'(\bullet), r\colon R(\bullet)\to R'(\bullet))$ of morphisms of operads such that the obvious square diagrams commute. Such a pair (o, r) is called a *weak equivalence* if both o and r are weak equivalences.

A *right module* over a $\mathcal{C}$-operad $O(\bullet)$ is a symmetric sequence $M(\bullet)=\{M(n)\}_{n=0}^\infty$ of objects of $\mathcal{C}$, equipped with structure morphisms

$$M(n)\otimes O(m_1)\otimes \ldots \otimes O(m_n) \longrightarrow M(m_1+\ldots+m_n)$$

satisfying certain obvious associativity, unit, and symmetry axioms (see [5] for details). Notice that a morphism of operads $f\colon O(\bullet)\to R(\bullet)$ endows $R(\bullet)$ with the structure of a right $O(\bullet)$-module.

5.2. Enriched category associated with an operad

Fix a closed symmetric monoidal category $\mathcal{C}$ that admits finite coproducts. Recall from §4.2 that a $\mathcal{C}\mathbf{N}$-category is a category enriched over $\mathcal{C}$ whose set of objects is $\mathbf{N}$. The $\mathcal{C}\mathbf{N}$-*category associated with the* $\mathcal{C}$-*operad* $O(\bullet)$ is the category $\mathcal{O}$ defined by

$$\mathcal{O}(m,n) = \coprod_{\alpha:\underline{m}\longrightarrow\underline{n}} O(\alpha^{-1}(1))\otimes...\otimes O(\alpha^{-1}(n)),$$

where the coproduct is taken over set maps

$$\alpha: \underline{m} := \{1,...,m\} \longrightarrow \underline{n} := \{1,...,n\}$$

and $O(\alpha^{-1}(j))=O(m_j)$, where m_j is the cardinality of $\alpha^{-1}(j)$. Composition of morphisms is prescribed by operad structure maps in $O(\bullet)$. In particular, $\mathcal{O}(m,1)=O(m)$.

Let $O(\bullet)$ be a $\mathcal{C}$-operad and let $\mathcal{O}$ be the associated $\mathcal{C}\mathbf{N}$-category. A right module (in the sense of operads) $M(\bullet)$ over $O(\bullet)$ gives rise to a right $\mathcal{O}$-module (in the sense of §4)

$$\begin{aligned} M(-)\colon \mathcal{O} &\longrightarrow \mathcal{C}, \\ n &\longmapsto M(n), \end{aligned}$$

where $M(-)$ is defined on morphisms by the $\mathcal{C}$-morphisms

$$M(m,n)\colon \mathcal{O}(m,n) \longrightarrow \mathcal{C}(M(n),M(m))$$

obtained by adjunction from the structure maps

$$M(n)\otimes\mathcal{O}(m,n) = \coprod_{\alpha:\underline{m}\longrightarrow\underline{n}} M(n)\otimes O(\alpha^{-1}(1))\otimes...\otimes O(\alpha^{-1}(n)) \longrightarrow M(m).$$

If $f\colon O(\bullet)\to R(\bullet)$ is a morphism of operads, then we have an associated right $\mathcal{O}$-module $R(-)\colon\mathcal{O}\to\mathcal{C}$.

It is obvious that if $O(\bullet)$ and $O'(\bullet)$ are weakly equivalent, objectwise cofibrant, operads over a monoidal model category $\mathcal{C}$ then the associated $\mathcal{C}\mathbf{N}$-categories $\mathcal{O}$ and $\mathcal{O}'$ are weakly equivalent. Also, if $f\colon O(\bullet)\to R(\bullet)$ and $f'\colon O'(\bullet)\to R'(\bullet)$ are weakly equivalent morphisms of operads, then the pair $(\mathcal{O}, R(-))$ is weakly equivalent, in the category of $\mathcal{C}\mathbf{N}$-modules, to the pair $(\mathcal{O}', R'(-))$.

Let $F\colon\mathcal{C}\to\mathcal{D}$ be a lax symmetric monoidal functor, and suppose that $O(\bullet)$ is an operad in $\mathcal{C}$. Let $\mathcal{O}$ be the $\mathcal{C}\mathbf{N}$-category associated with $O(\bullet)$. Then $F(O(\bullet))$ is an operad in $\mathcal{D}$, and $F(\mathcal{O})$ is a $\mathcal{D}\mathbf{N}$-category. It is easy to see that there is a natural morphism from the $\mathcal{D}\mathbf{N}$-category associated with the $\mathcal{D}$-operad $F(O(\bullet))$ to $F(\mathcal{O})$. This morphism is not an isomorphism, unless F is strictly monoidal and also takes coproducts to coproducts, but in all cases that we consider, it will be a weak equivalence. Similarly, if $f: O(\bullet) \to R(\bullet)$ is a morphism of operads in $\mathcal{C}$ and if $R(-)$ is the right $\mathcal{O}$-module associated with the $O(\bullet)$-module $R(\bullet)$, then $F(R(-))$ has a natural structure of an $F(\mathcal{O})$-module, extending the structure of an $F(O(\bullet))$-module possessed by $F(R(\bullet))$.

5.3. The standard little balls operad

The most important operad for our purposes is what we will call the standard balls operad. Let V be a Euclidean space. By a *standard ball* in V we mean a subset of V, that is obtained from the open unit ball by dilation and translation. The operad of standard balls will be denoted by $\mathrm{B}(\bullet, V)$. It is the well-known operad in $(\mathrm{Top}, \times, *)$, consisting of the topological spaces

$$\mathrm{B}(n, V) = \{n\text{-tuples of disjoint standard balls inside the unit ball of } V\}$$

with the structure maps given by composition of inclusions after suitable dilations and translations.

The Top$\mathbf{N}$-category associated with the standard balls operad $\mathrm{B}(\bullet, V)$ will be denoted by $\mathcal{B}(V)$. An object of $\mathcal{B}(V)$ is a non-negative integer n which can be thought of as an abstract (i.e., not embedded) disjoint union of n copies of the unit ball in V. The space of morphisms $\mathcal{B}(V)(m,n)$ is the space of embeddings of m unit balls into n unit balls, which on each ball are obtained by dilations and translations.

Let $j: W \hookrightarrow V$ be a linear isometric inclusion of Euclidean spaces. Such a map induces a morphism of operads

$$j: \mathrm{B}(\bullet, W) \longrightarrow \mathrm{B}(\bullet, V),$$

where a ball centered at $w \in W$ is sent to the ball of the same radius centered at $j(w)$.

Hence $\mathrm{B}(\bullet, V)$ is a right module over $\mathrm{B}(\bullet, W)$, and we get a right $\mathcal{B}(W)$-module

$$\begin{aligned} \mathrm{B}(-, V): \mathcal{B}(W) &\longrightarrow \mathrm{Top}, \\ n &\longmapsto \mathrm{B}(n, V). \end{aligned}$$

We can apply lax monoidal functors to the above setting. For example, $\mathrm{C}_*(\mathrm{B}(\bullet, W))$ and $\mathrm{H}_*(\mathrm{B}(\bullet, W))$ are operads in $(\mathrm{Ch}, \otimes, \mathbf{K})$. Hence we get Ch$\mathbf{N}$-categories $\mathrm{C}_*(\mathcal{B}(W))$ and $\mathrm{H}_*(\mathcal{B}(W))$, a right $\mathrm{C}_*(\mathcal{B}(W))$-module $\mathrm{C}_*(\mathrm{B}(-, V))$, and a right $\mathrm{H}_*(\mathcal{B}(W))$-module $\mathrm{H}_*(\mathrm{B}(-, V))$.

We will also consider the discrete categories $\mathcal{B}(W)^\delta$ and $\mathrm{C}_*(\mathcal{B}(W))^\delta$ obtained by the discretization process from $\mathcal{B}(W)$ and $\mathrm{C}_*(\mathcal{B}(W))$, respectively. Note that

$$\mathrm{C}_*(\mathcal{B}(W))^\delta = \mathbf{K}[\mathcal{B}(W)^\delta].$$

6. Formality and splitting of the little balls operad

In this section, all chain complexes and homology groups are taken with coefficients in $\mathbf{R}$. A deep theorem of Kontsevich (Theorem 1.2 of the introduction and Theorem 2 of [14]) asserts that the standard balls operad is formal over the reals. We will need a slight strengthening of this result. Throughout this section, let $j\colon W\hookrightarrow V$ be, as usual, a linear isometric inclusion of Euclidean spaces. Recall the little balls operad and the associated enriched categories and modules as in §5.3. Here is the version of Kontsevich's theorem we need.

THEOREM 6.1. (Relative formality) *If* $\dim V > 2\dim W$ *then the morphism of chain operads*

$$\mathrm{C}_*(j)\colon \mathrm{C}_*(\mathrm{B}(\bullet, W))\otimes\mathbf{R} \longrightarrow \mathrm{C}_*(\mathrm{B}(\bullet, V))\otimes\mathbf{R}$$

is weakly equivalent to the morphism

$$\mathrm{H}_*(j)\colon \mathrm{H}_*(\mathrm{B}(\bullet, W);\mathbf{R}) \longrightarrow \mathrm{H}_*(\mathrm{B}(\bullet, V);\mathbf{R}).$$

Sketch of the proof. A detailed proof will appear in [16]. Here we give a sketch based on the proof of absolute formality given in [14, Theorem 2], and we follow that paper's notation. Denote by $\mathrm{FM}_d(n)$ the Fulton–MacPherson compactification of the configuration space of n points in $\mathbf{R}^d$. This defines an operad $\mathrm{FM}_d(\bullet)$ which is homotopy equivalent to the little balls operad $B(\bullet,\mathbf{R}^d)$. Kontsevich constructs a quasi-isomorphism

$$\Psi\colon \mathsf{SemiAlgChain}_*(\mathrm{FM}_d(n)) \xrightarrow{\simeq} \mathsf{Graphs}_d(n)\widehat{\otimes}\mathbf{R},$$

where $\mathsf{SemiAlgChain}_*$ is a chain complex of semi-algebraic chains naturally quasi-isomorphic to singular chains and $\mathsf{Graphs}_d(n)\widehat{\otimes}\mathbf{R}$ is the chain complex dual, over $\mathbf{R}$, to the chain complex of admissible graphs as defined in [14, Definitions 13 and 15]. For a semi-algebraic chain ξ on $\mathrm{FM}_d(n)$, the map Ψ is defined by

$$\Psi(\xi)(\Gamma) = \langle\omega_\Gamma,\xi\rangle,\quad \text{for any admissible graph } \Gamma,$$

where ω_Γ is the associated differential form defined in [14, Definition 14].

Let $j_*\colon \mathrm{FM}_{\dim W}(n)\to\mathrm{FM}_{\dim V}(n)$ be the map induced by the inclusion of Euclidean spaces j. Notice that $H_i(j_*)=0$ for $i>0$. Define $\varepsilon\colon \mathsf{Graphs}_{\dim W}(n)\to\mathsf{Graphs}_{\dim V}(n)$ to be the dual of the map that sends graphs with at least one edge to zero, and the graph without edges to itself. We need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{SemiAlgChain}_*(\mathrm{FM}_{\dim W}(n)) \arrow[r, "\simeq"] \arrow[d, "j_*"] & \mathsf{Graphs}_{\dim W}(n)\widehat{\otimes}\mathbf{R} \arrow[r, "\simeq"] \arrow[d, "\varepsilon"] & \mathrm{H}_*(\mathrm{FM}_{\dim W}(n)) \arrow[d, "\mathrm{H}(j_*)"] \\
\mathsf{SemiAlgChain}_*(\mathrm{FM}_{\dim V}(n)) \arrow[r, "\simeq"] & \mathsf{Graphs}_{\dim V}(n)\widehat{\otimes}\mathbf{R} \arrow[r, "\simeq"] & \mathrm{H}_*(\mathrm{FM}_{\dim V}(n)).
\end{tikzcd}$$

The commutativity of the right-hand square is clear. For the left-hand square it suffices to check that for any admissible graph of positive degree Γ and for any non-zero semi-algebraic chain $\xi \in \mathsf{SemiAlgChain}_*(\mathrm{FM}_{\dim W}(n))$ we have $\langle \omega_\Gamma, j_*(\xi) \rangle = 0$.

The first n vertices $1, ..., n$ of Γ are called *external* and the other are called *internal*. If every external vertex of Γ is connected to an edge, then, using the fact that internal vertices are at least trivalent, we obtain that the form ω_Γ on $\mathrm{FM}_{\dim V}(n)$ is of degree $\geqslant \frac{1}{2} n(\dim V - 1)$. Since $\dim V > 2 \dim W$, we get that $\deg(\omega_\Gamma) > \dim \mathrm{FM}_{\dim W}(n)$. Therefore $\deg(\omega_\Gamma) > \deg(j_*(\xi))$ and $\langle \omega_\Gamma, j_*(\xi) \rangle = 0$.

If Γ has an isolated external vertex, then $\langle \omega_\Gamma, j_*(\xi) \rangle = \langle \omega_\Gamma, j_*(\xi') \rangle$, where ξ' is a chain in $\mathsf{SemiAlgChain}_*(\mathrm{FM}_{\dim W}(m))$ with $m < n$ and the proof proceeds by induction. □

We remark once again that the formality theorem is for chain complexes over $\mathbf{R}$, not over $\mathbf{Q}$. We do not know if the little balls operad is formal over the rational numbers, but we do think it is an interesting question. We note that a general result about descent of formality from $\mathbf{R}$ to $\mathbf{Q}$ was proved in [12], for operads *without a term in degree zero*. The proof does not seem to be easily adaptable to operads with a zero term.

To deduce the formality of certain diagrams more directly related to spaces of embeddings, we first reformulate relative formality in terms of homogeneous splittings in the spirit of Proposition 3.3. With this in mind, we introduce the following enrichment of Definition 3.2.

Definition 6.2. Let $\mathcal{O}$ be a $\mathrm{Ch}I$-category. We say that an $\mathcal{O}$-module $M: \mathcal{O} \to \mathrm{Ch}$ *splits homogeneously* if there exists a sequence $\{M_n\}_{n=0}^\infty$ of $\mathcal{O}$-modules such that $M \simeq \bigoplus_{n=0}^\infty M_n$ and $\mathrm{H}_*(M_n) = \mathrm{H}_n(M_n)$.

Our first example (a trivial one) of such a homogeneous splitting of modules is given by the following lemma.

Lemma 6.3. *If* $\dim V > \dim W$ *then the* $\mathrm{H}_*(\mathcal{B}(W))$*-module* $\mathrm{H}_*(\mathrm{B}(-, V))$ *splits homogenously.*

Proof. Notice that $\mathrm{H}_0(\mathcal{B}(W))$ is also a $\mathrm{Ch}\mathbf{N}$-category. Since our chain complexes are non-negatively graded and with a zero differential, we have an obvious inclusion functor

$$i: \mathrm{H}_0(\mathcal{B}(W)) \hookrightarrow \mathrm{H}_*(\mathcal{B}(W))$$

and a projection functor

$$\Phi: \mathrm{H}_*(\mathcal{B}(W)) \longrightarrow \mathrm{H}_0(\mathcal{B}(W))$$

between $\mathrm{Ch}\mathbf{N}$-categories, where $\Phi \circ i$ is the identity. Therefore, an $\mathrm{H}_*(\mathcal{B}(W))$-module admits a structure of an $\mathrm{H}_0(\mathcal{B}(W))$-module via i. Since $\mathrm{H}_0(\mathcal{B}(W))$ is a category of chain complexes concentrated in degree 0 and $\mathrm{H}_*(\mathrm{B}(-,V))$ has no differentials, it is clear that we have a splitting of $\mathrm{H}_0(\mathcal{B}(W))$-modules

$$\mathrm{H}_*(\mathrm{B}(-,V)) \cong \bigoplus_{n=0}^{\infty} \mathrm{H}_n(\mathrm{B}(-,V)). \tag{3}$$

Moreover, since $\dim W < \dim V$ the morphisms

$$\mathrm{H}_*(\mathrm{B}(n,W)) \longrightarrow \mathrm{H}_*(\mathrm{B}(n,V))$$

are zero in positive degrees. Hence the $\mathrm{H}_*(\mathcal{B}(W))$-module structure on $\mathrm{H}_*(\mathrm{B}(-,V))$ factors through the above-mentioned $\mathrm{H}_0(\mathcal{B}(W))$-module structure via Φ. Therefore, the splitting (3) is a splitting of $\mathrm{H}_*(\mathcal{B}(W))$-modules. □

Using Lemma 6.3 and the relative formality theorem, we obtain the following highly non-trivial splitting.

LEMMA 6.4. *If* $\dim V > 2\dim W$ *then the* $\mathrm{C}_*(\mathcal{B}(W))$*-module* $\mathrm{C}_*(\mathrm{B}(-,V))$ *splits homogenously.*

Proof. We deduce from Theorem 6.1 that the $\mathrm{Ch}\mathbf{N}$-module categories

$$(\mathrm{C}_*(\mathcal{B}(W)), \mathrm{C}_*(\mathrm{B}(-,V))) \quad \text{and} \quad (\mathrm{H}_*(\mathcal{B}(W)), \mathrm{H}_*(\mathrm{B}(-,V)))$$

are equivalent. By Lemma 6.3, the latter splits homogeneously, hence, by Proposition 4.2, the same is true of the former. □

Recall from §4.5 that the enriched category $\mathcal{B}(W)$ has an underlying discrete category $\mathcal{B}(W)^\delta$ and that the $\mathcal{B}(W)$-module $\mathrm{B}(-,V)$ induces a $\mathcal{B}(W)^\delta$-diagram $\mathrm{B}(-,V)^\delta$.

PROPOSITION 6.5. *If* $\dim V > 2\dim W$ *then the* $\mathcal{B}(W)^\delta$*-diagram*

$$\mathrm{C}_*(\mathrm{B}(-,V))^\delta : \mathcal{B}(W)^\delta \longrightarrow \mathrm{Ch}_{\mathbf{R}}$$

is formal.

Proof. By Lemma 4.3, the following diagram of discrete functors commutes:

$$\begin{tikzcd} \mathcal{B}(W)^\delta \arrow[r, "\mathrm{B}(-,V)^\delta"] \arrow[d, "(\mathrm{C}_*)^\delta_{\mathcal{B}(W)}"'] & \mathrm{Top} \arrow[d, "\mathrm{C}_*"] \\ \mathrm{C}_*(\mathcal{B}(W))^\delta \arrow[r, "\mathrm{C}_*(\mathrm{B}(V,-))^\delta"] & \mathrm{Ch}_{\mathbf{R}}, \end{tikzcd}$$

where the discretization $\mathrm{C}_*(\mathcal{B}(W))^\delta$ is nothing else than the linearization of $\mathcal{B}(W)^\delta$, that is the category whose morphisms are formal $\mathbf{R}$-linear combinations of morphism in $\mathcal{B}(W)$ and the objects are the same as in $\mathcal{B}(W)$. We want to prove that the $\mathcal{B}(W)^\delta$-diagram $\mathrm{C}_*(\mathrm{B}(-,V)^\delta)$ is formal. By the commutativity of the square above and Lemma 3.6, it is enough to prove that the $\mathrm{C}_*(\mathcal{B}(W))^\delta$-diagram $\mathrm{C}_*(\mathrm{B}(V,-))^\delta$ is formal. By Lemma 6.4, the $\mathrm{C}_*(\mathcal{B}(W))$-module $\mathrm{C}_*(\mathrm{B}(-,V))$ splits homogeneously. By Lemma 4.4, we deduce that the $\mathrm{C}_*(\mathcal{B}(W))^\delta$-diagram $\mathrm{C}_*(\mathrm{B}(-,V))^\delta$ splits homogeneously, which implies, by Proposition 3.3, the formality of that diagram. □

7. Formality of a certain diagram arising from embedding calculus

In this section, all chain complexes are still taken over the real numbers. As before, fix a linear isometric inclusion of Euclidean vector spaces $j\colon W\hookrightarrow V$. Let $\mathcal{O}(W)$ be the poset of open subsets of W. As explained in the introduction, we have two contravariant functors

$$\mathrm{Emb}(-,V), \mathrm{Imm}(-,V)\colon \mathcal{O}(W) \longrightarrow \mathrm{Top}\,.$$

Moreover, the fixed embedding $j\colon W\hookrightarrow V$ can serve as a basepoint, so we can consider the homotopy fiber of the inclusion $\mathrm{Emb}(-,V)\to\mathrm{Imm}(-,V)$, which we denote by

$$\overline{\mathrm{Emb}}(-,V)\colon \mathcal{O}(W) \longrightarrow \mathrm{Top}\,.$$

Our goal in this section is to compare a certain variation of this functor with the functor

$$\mathrm{B}(-,V)^\delta\colon \mathcal{B}(W)^\delta \longrightarrow \mathrm{Top}$$

and to deduce, in Theorem 7.2, the stable formality of certain diagrams of embedding spaces. In order to do this we first introduce a subcategory $\mathcal{O}^s(W)$ of $\mathcal{O}(W)$ and a category $\widetilde{\mathcal{O}}^s(W)$ which will serve as a turning table between $\mathcal{O}^s(W)$ and $\mathcal{B}(W)^\delta$.

To describe $\mathcal{O}^s(W)$, recall that a *standard ball* in W is an open ball in the metric space W, i.e. it is obtained in a unique way by a dilation and translation of the unit ball in W. The category $\mathcal{O}^s(W)$ is the full subcategory of $\mathcal{O}(W)$ whose objects are finite unions of disjoint standard balls.

The category $\widetilde{\mathcal{O}}^s(W)$ is a kind of covering of $\mathcal{O}^s(W)$. Recall that the object $m\in\mathbf{N}$ of $\mathcal{B}(W)$ can be thought of as an abstract disjoint union of m copies of the unit ball of W. An object of $\widetilde{\mathcal{O}}^s(W)$ is then an embedding $\phi\colon m\hookrightarrow W$ such that the restriction of ϕ to each unit ball amounts to a dilation and translation. In other words, an object (ϕ,m) of $\widetilde{\mathcal{O}}^s(W)$ is the same as an ordered m-tuple of disjoint standard balls in W. The union of these m standard balls is an object of $\mathcal{O}^s(W)$ that we denote by $\phi(m)$, as the image of the embedding ϕ. By definition, there is a morphism in $\widetilde{\mathcal{O}}^s(W)$ between two objects (ϕ, m) and (ψ, n) if and only if $\phi(n)\subset\psi(m)$, and such a morphism is unique.

We define functors

$$\mathcal{B}(W)^\delta \xleftarrow{\lambda} \widetilde{\mathcal{O}}^s(W) \xrightarrow{\pi} \mathcal{O}^s(W).$$

Here π is defined on objects by $\pi(\phi, m)=\phi(m)$ and is defined on morphisms by sending a morphism $\alpha\colon(\phi_1, m_1)\to(\phi_2, m)$ to the inclusion $\phi_1(m_1)\hookrightarrow\phi_2(m_2)$, and this functor is easily seen to be an equivalence of categories. The functor λ is defined on objects by $\lambda(\phi, m)=m$, and is defined on morphisms using the fact that any two standard balls in W can be canonically identified by a unique transformation that is a combination of dilation and translation.

We would like to compare the following two composed functors

$$\overline{\mathrm{Emb}}(\pi(-), V))\colon \widetilde{\mathcal{O}}^s(W) \xrightarrow{\pi} \mathcal{O}^s(W) \xrightarrow{\overline{\mathrm{Emb}}(-,V)} \mathrm{Top},$$

$$\mathrm{B}(\lambda(-), V)^\delta\colon \widetilde{\mathcal{O}}^s(W) \xrightarrow{\lambda} \mathcal{B}(W)^\delta \xrightarrow{\mathrm{B}(-,V)^\delta} \mathrm{Top}\,.$$

Proposition 7.1. *The $\widetilde{\mathcal{O}}^s(W)$-diagrams $\mathrm{B}(\lambda(-), V)^\delta$ and $\overline{\mathrm{Emb}}(\pi(-), V)$ are weakly equivalent.*

Proof. Define subspaces

$$\mathrm{AffEmb}(\phi(n), V)\subset\mathrm{Emb}(\phi(n), V) \quad \text{and} \quad \mathrm{AffImm}(\phi(n), V)\subset\mathrm{Imm}(\phi(n), V)$$

to be the spaces of embeddings and immersions, respectively, that are affine on each ball. It is well known that the above inclusion maps are homotopy equivalences. We may define $\overline{\mathrm{AffEmb}}(\phi(n), V)$ to be the homotopy fiber of the map

$$\mathrm{AffEmb}(\phi(n), V) \longrightarrow \mathrm{AffImm}(\phi(n), V).$$

Thus there is a natural homotopy equivalence

$$\overline{\mathrm{AffEmb}}(\phi(n), V) \xrightarrow{\simeq} \overline{\mathrm{Emb}}(\phi(n), V).$$

Define $\widetilde{\mathrm{Inj}}(W, V)$ as the space of injective linear maps from W to V, quotiented out by the multiplicative group of positive reals, i.e. defined up to dilation. Then there is a natural homotopy equivalence

$$\mathrm{AffImm}(\phi(n), V) \xleftarrow{\simeq} \widetilde{\mathrm{Inj}}(W, V)^n$$

obtained by differentiating the immersion at each component of $\phi(n)$. Now consider the map

$$\mathrm{AffEmb}(\phi(n), V) \longrightarrow \widetilde{\mathrm{Inj}}(W, V)^n.$$

We denote the homotopy fiber of this map by $F(n,\phi)$, and we obtain a natural equivalence

$$\overline{\mathrm{AffEmb}}(\phi(n),V) \xrightarrow{\simeq} F(n,\phi).$$

Finally since the composite map

$$\mathrm{B}(n,V) \hookrightarrow \mathrm{AffEmb}(\phi(n),V) \longrightarrow \widetilde{\mathrm{Inj}}(W,V)^n$$

is the constant map into the basepoint, there is a natural map $\mathrm{B}(n,V) \to F(n,\phi)$. It is easy to see that the map is an equivalence. To summarize, we have constructed the following chain of natural weak equivalences

$$\overline{\mathrm{Emb}}(\phi(n),V) \xleftarrow{\simeq} \overline{\mathrm{AffEmb}}(\phi(n),V) \xrightarrow{\simeq} F(n,\phi) \xleftarrow{\simeq} \mathrm{B}(n,V). \qquad \square$$

We are ready to prove the main result of this section.

THEOREM 7.2. *If* $\dim V > 2\dim W$ *then the* $\widetilde{\mathcal{O}}^s(W)$*-diagram* $\mathrm{C}_*(\overline{\mathrm{Emb}}(\pi(-),V))$ *is stably formal.*

Proof. By Proposition 6.5 and Lemma 3.6, the diagram $\mathrm{C}_*(\mathrm{B}(\lambda(-),V))^\delta$ is stably formal. Proposition 7.1 implies the theorem. □

8. More generalities on calculus of functors

In this section we digress to review in a little more detail the basics of embedding and orthogonal calculus. We will also record some general observations about bifunctors to which both brands of calculus apply. The standard references are [23] and [22].

8.1. Embedding calculus

Let M be a smooth manifold (for convenience, we assume that M is the interior of a compact manifold with boundary). Let $\mathcal{O}(M)$ be the poset of open subsets of M and let $\mathcal{O}_k(M)$ be the subposet consisting of open subsets homeomorphic to a union of at most k open balls. Embedding calculus is concerned with the study of contravariant isotopy functors from F to a Quillen model category (Weiss only considers functors into the category of spaces, and, implicitly, spectra, but much of the theory works just as well in the more general setting of model categories). Following [23, p. 5], we say that a contravariant functor is *good* if it converts isotopy equivalences to weak equivalences and filtered unions to homotopy limits. Polynomial functors are defined in terms of certain cubical diagrams, similarly to the way they are defined in Goodwillie's homotopy calculus. Recall that a cubical diagram of spaces is called *strongly co-cartesian* if each of its 2-dimensional faces is a homotopy pushout square. A contravariant functor F on $\mathcal{O}(M)$ is *polynomial* of *degree* k if it takes strongly co-cartesian $(k+1)$-dimensional cubical diagrams of opens subsets of M to homotopy cartesian cubical diagrams (homotopy cartesian cubical diagram is synonymous with homotopy pullback cubical diagram). Good functors can be approximated by the stages of the tower defined by

$$T_k F(U) = \operatorname*{holim}_{\{U' \in \mathcal{O}_k(M): U' \subset U\}} F(U').$$

It turns out that $T_k F$ is polynomial of degree k, and moreover there is a natural map $F \to T_k F$ which in some sense is the best possible approximation of F by a polynomial functor of degree k. More precisely, the map $F \to T_k F$ can be characterized as the essentially unique map from F to a polynomial functor of degree k that induces a weak equivalence when evaluated on an object of $\mathcal{O}_k(M)$. In the terminology of [23], $T_k F$ is the kth Taylor polynomial of F. F is said to be *homogeneous of degree* k if it is polynomial of degree k and $T_{k-1}F$ is equivalent to the trivial functor. For each k, there is a natural map $T_k F \to T_{k-1} F$, compatible with the maps $F \to T_k F$ and $F \to T_{k-1} F$. Its homotopy fiber is a homogeneous functor of degree k, and it is called the *k-th layer* of the tower. It plays the role of the kth term in the Taylor series of a function. For functors with values in pointed spaces, there is a useful general formula for the kth layer in terms of spaces of sections of a certain bundle $p: E \to \binom{M}{k}$ over the space $\binom{M}{k}$ of unordered k-tuples of distinct points in M. The fiber of p at a point $\underline{m} = \{m_1, ..., m_k\}$ is $\widehat{F(m)}$, which is defined to be the total fiber of the k-dimensional cube $S \mapsto F(N(S))$, where S ranges over subsets of $\underline{m}$ and $N(S)$ stands for a "small tubular neighborhood" of S in M, i.e., a disjoint union of open balls in M. The fibration p has a preferred section. See [23], especially §8 and §9, for more details and a proof of the following proposition.

PROPOSITION 8.1. (Weiss) *The homotopy fiber of the map $T_k F \to T_{k-1} F$ is equivalent to the space of sections of the fibration p above, which agree with the preferred section in a neighborhood of the fat diagonal in M^k.*

We denote this space of restricted sections by

$$\Gamma_c\left(\binom{M}{k}, \widehat{F(k)}\right).$$

Even though $T_k F$ is defined as the homotopy limit of an infinite category, for most moral and practical purposes it behaves as if it was the homotopy limit of a *very small category* (i.e., a category whose simplicial nerve has finitely many non-degenerate simplices). This is so because of the following proposition.

PROPOSITION 8.2. *There is a very small subcategory $\mathcal{C}$ of $\mathcal{O}_k(M)$ such that restriction from $\mathcal{O}_k(M)$ to $\mathcal{C}$ induces an equivalence on homotopy limits of all good functors.*

Proof. It is not difficult to show, using handlebody decomposition and induction (the argument is essentially contained in the proof of Theorem 5.1 of [23]), that one can find a finite collection $\{U_1, ..., U_N\}$ of open subsets of M such that all their possible intersections are objects of $\mathcal{O}_k(M)$ and

$$M^k = \bigcup_{i=1}^{N} U_i^k.$$

This is equivalent to saying that the sets U_i cover M in what Weiss calls the Grothendieck topology $\mathcal{J}_k$. By [23, Theorem 5.2], polynomial functors of degree k are homotopy sheaves with respect to $\mathcal{J}_k$. In practice, this means the following. Let $\mathcal{C}$ be the subposet of $\mathcal{O}_k(M)$ given by the sets U_i and all their possible intersections (clearly, $\mathcal{C}$ is a very small category). Let G be a polynomial functor of degree k. Then the following canonical map is a homotopy equivalence:

$$G(M) \longrightarrow \operatorname*{holim}_{U \in \mathcal{C}} G(U).$$

We conclude that for a good functor F, there is the following zig-zag of weak equivalences:

$$\operatorname*{holim}_{U \in \mathcal{C}} F(U) \xrightarrow{\simeq} \operatorname*{holim}_{U \in \mathcal{C}} T_k F(U) \xleftarrow{\simeq} T_k F(M).$$

Here the left map is a weak equivalence because the map $F \to T_k F$ is a weak equivalence on objects of $\mathcal{O}_k(M)$, and all objects of $\mathcal{C}$ are objects of $\mathcal{O}_k(M)$. The right map is an equivalence because $T_k F$ is a polynomial functor of degree k, in view of the discussion above. $\square$

The important consequence of the proposition is that $T_k F$ commutes, up to a zig-zag of weak equivalences, with filtered homotopy colimits of functors. In the same spirit, we have the following proposition.

PROPOSITION 8.3. *Let $F: \mathcal{O}_k(M) \to \mathrm{Ch}_{\mathbf{Q}}$ be a good functor into rational chain complexes. Then the natural map*

$$(T_k F(M)) \otimes \mathbf{R} \longrightarrow T_k(F \otimes \mathbf{R})(M)$$

is a weak equivalence.

Proof. Tensoring with $\mathbf{R}$ obviously commutes up to homotopy with very small homotopy limits, and so the claim follows from Proposition 8.2. $\square$

8.2. Orthogonal calculus

The basic reference for orthogonal calculus is [22]. Let $\mathcal{J}$ be the topological category of Euclidean spaces and linear isometric inclusions. Orthogonal calculus is concerned with the study of *continuous* functors from $\mathcal{J}$ to a model category enriched over Top_*. We will only consider functors into Top_*, Spectra and closely related categories. Like embedding calculus, orthogonal calculus comes equipped with a notion of a polynomial functor, and with a construction that associates with a functor G a tower of approximating functors P_nG such that P_nG is, in a suitable sense, the best possible approximation of G by a polynomial functor of degree n. P_n is defined as a certain filtered homotopy colimit of compact homotopy limits. For each n, there is a natural map $P_nG\to P_{n-1}G$ and its fiber (again called the *n-th layer*) is denoted by D_nG. D_nG is a homogeneous functor, in the sense that it is polynomial of degree n and $P_{n-1}D_nG\simeq *$. The following characterization of homogeneous functors is proved in [22].

THEOREM 8.4. (Weiss) *Every homogeneous functor of degree n from vector spaces to spectra is equivalent to a functor of the form*

$$(C_n\wedge S^{nV})_{h\mathrm{O}(n)},$$

where C_n is a spectrum with an action of the orthogonal group $\mathrm{O}(n)$, S^{nV} is the one-point compactification of the vector space $\mathbf{R}^n\otimes V$, and the subscript $h\mathrm{O}(n)$ denotes homotopy orbits.

It follows, in particular, that given a (spectrum-valued) functor G to which orthogonal calculus applies, D_nG has the form described in the theorem, with some spectrum C_n. The spectrum C_n is called the *n-th derivative* of G. There is a useful description of the derivatives of G as stabilizations of certain types of iterated cross-effects of G.

Let G_1 and G_2 be two functors to which orthogonal calculus applies. Let $\alpha: G_1\to G_2$ be a natural transformation. Very much in the spirit of Goodwillie's homotopy calculus, we say that G_1 and G_2 *agree to n-th order via α* if the map $\alpha(V): G_1(V)\to G_2(V)$ is $((n+1)\dim V+c)$-connected, where c is a possibly negative constant, independent of V. Using the description of derivatives in terms of cross-effects, it is easy to prove the following proposition.

PROPOSITION 8.5. *Suppose that G_1 and G_2 agree to n-th order via a natural transformation $\alpha: G_1\to G_2$. Then α induces an equivalence on the first n derivatives, and therefore an equivalence on n-th Taylor polynomials*

$$P_n\alpha: P_nG_1\xrightarrow{\simeq}P_nG_2.$$

8.3. Bifunctors

In this paper we consider bifunctors

$$E: \mathcal{O}(M)^{\mathrm{op}} \times \mathcal{J} \longrightarrow \mathrm{Top}/\,\mathrm{Spectra}$$

such that the adjoint contravariant functor $\mathcal{O}(M) \to \mathrm{Funct}(\mathcal{J}, \mathrm{Top}/\,\mathrm{Spectra})$ is good (in the evident sense) and the adjoint functor $\mathcal{J} \to \mathrm{Funct}(\mathcal{O}(M)^{\mathrm{op}}, \mathrm{Top}/\,\mathrm{Spectra})$ is continuous. We may apply both embedding calculus and orthogonal calculus to such a bifunctor. Thus, by $P_n E(M, V)$ we mean the functor obtained from E by considering it as a functor of V, (with M being a "parameter") and taking the nth Taylor polynomial in the orthogonal sense. Similarly, $T_k E(M, V)$ is the functor obtained by taking the kth Taylor polynomial in the sense of embedding calculus.

We will need a result about the interchangeability of order of applying the differential operators P_n and T_k. The operator T_k is constructed using a homotopy limit, while P_n is constructed using a homotopy limit (over a compact topological category) and a filtered homotopy colimit. It follows that there is a natural transformation

$$P_n T_k E(M, V) \longrightarrow T_k P_n E(M, V)$$

and a similar natural transformation, where P_n is replaced by D_n.

LEMMA 8.6. *Let E be a bifunctor as above. For all n and k the natural map*

$$P_n T_k E(M, V) \xrightarrow{\simeq} T_k P_n E(M, V)$$

is an equivalence. There is a similar equivalence where P_n is replaced by D_n.

Proof. By Proposition 8.2, T_k can be presented as a very small homotopy limit. Therefore, it commutes up to homotopy with homotopy limits and filtered homotopy colimits. P_n is constructed using homotopy limits and filtered homotopy colimits. Therefore, T_k and P_n commute. □

9. Formality and the embedding tower

In this section we assume that $\alpha: M \hookrightarrow W$ is an inclusion of an *open subset* into a Euclidean space W. From our point of view, there is no loss of generality in this assumption, because if M is an embedded manifold in W, we can replace M with an open tubular neighborhood, without changing the homotopy type of $\overline{\mathrm{Emb}}(M, V)$. As usual, we fix an isometric inclusion $j: W \hookrightarrow V$ of Euclidean vector spaces.

Recall that we defined the functor

$$\overline{\mathrm{Emb}}(-,V)\colon \mathcal{O}(M) \longrightarrow \mathrm{Top}\,.$$

The stable rationalization $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(-,V)_+$ of $\overline{\mathrm{Emb}}(-,V)$ admits a Taylor tower (in this section, Taylor towers are taken in the sense of embedding calculus). Our goal is to give, in Theorem 9.2, a splitting of the kth stage of this tower. The splitting is *not* as a product of the layers in the embedding towers. Rather, we will see in the next section that the splitting is as a product of the layers in the orthogonal tower.

Recall the poset $\mathcal{O}^s(W)$ of finite unions of standard balls in W from §7. Let $\mathcal{O}^s(M)$ be the full subcategory of $\mathcal{O}^s(W)$ consisting of the objects which are subsets of M. For a natural number k, we define $\mathcal{O}^s_k(M)$ as the full subcategory of $\mathcal{O}^s(M)$ consisting of disjoint unions of at most k standard balls in M.

Proposition 9.1. *Let M be an open submanifold of a vector space W and let $F\colon \mathcal{O}(M)\to\mathrm{Top}$ be a good functor. The restriction map*

$$T_kF(M) := \operatorname*{holim}_{U\in\mathcal{O}_k(M)} F(U) \longrightarrow \operatorname*{holim}_{U\in\mathcal{O}^s_k(M)} F(U),$$

induced by the inclusion of categories $\mathcal{O}^s_k(M)\to\mathcal{O}_k(M)$, is a homotopy equivalence.

Proof. Define $T^s_kF(M) := \operatorname*{holim}_{U\in\mathcal{O}^s_k(M)} F(U)$. There are projection maps

$$T^s_kF(M) \longrightarrow T^s_{k-1}F(M)$$

induced by the inclusion of categories $\mathcal{O}^s_{k-1}(M)\to\mathcal{O}^s_k(M)$, and the map $T_kF\to T^s_kF$ extends to a map of towers. One can adapt the methods of [23] to analyze the functors T^s_kF. In particular, it is not hard to show, using the same methods as in [23], that our map induces a homotopy equivalence from the homotopy fibers of the map $T_kF\to T_{k-1}F$ to the homotopy fibers of the map $T^s_kF\to T^s_{k-1}F$, for all k. Our assertion follows by induction on k. □

Recall the category $\widetilde{\mathcal{O}}^s(W)$ defined in §7. Let $\widetilde{\mathcal{O}}^s(M)$ be the full subcategory of $\widetilde{\mathcal{O}}^s(W)$ consisting of objects (ϕ,m) such that $\phi(m)$ is a subset of M. Define also $\widetilde{\mathcal{O}}^s_k(M)$ to be the full subcategory of $\widetilde{\mathcal{O}}^s(W)$ consisting of objects (ϕ,m) such that m is at most k.

Recall the functor $\pi\colon \widetilde{\mathcal{O}}^s(W)\to\mathcal{O}^s(W)$, $(\phi,m)\mapsto\phi(m)$, defined in §7. It is clear that this functor restricts to a functor $\pi\colon\widetilde{\mathcal{O}}^s_k(M)\to\mathcal{O}^s_k(M)$. Moreover it is an equivalence of categories, therefore pullbacks along π induce weak equivalences between homotopy limits.

We can now prove the main result of this section. Recall from §7 the functor

$$\mathrm{B}(\lambda(-),V)\colon \widetilde{\mathcal{O}}^s(W) \longrightarrow \mathrm{Top},$$

which by abuse of notation we denote by $(\phi,m)\mapsto B(m,V)$.

THEOREM 9.2. *Let $W \subset V$ be an inclusion of Euclidean vector spaces, let M be an open submanifold of W, and let k be a natural number. If $\dim V > 2\dim W$ then there is an equivalence of spectra*

$$\begin{aligned} T_k \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,V)_+ &\simeq \prod_{i=0}^{\infty} T_k \|\mathrm{H}_i(\overline{\mathrm{Emb}}(M,V))\| \\ &\simeq \prod_{i=0}^{\infty} \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}^s_k(M)} \|\mathrm{H}_i(\overline{\mathrm{Emb}}(\pi(\phi,m),V))\|, \end{aligned}$$

where $\|\mathrm{H}_i(X)\|$ is the Eilenberg–MacLane spectrum that has the i-th rational homology of X in degree i.

Proof. By Proposition 9.1 and since π is an equivalence of categories, we have

$$T_k \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,V)_+ \simeq \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}^s_k(M)} \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(\pi(\phi,m),V))_+.$$

By Proposition 7.1, the functors $\overline{\mathrm{Emb}}(\pi(\phi,m),V)$ and $\mathrm{B}(\lambda(\phi,m),V) = \mathrm{B}(m,V)$ are weakly equivalent, as functors on $\widetilde{\mathcal{O}}^s_k(W)$. It follows that their restrictions to $\widetilde{\mathcal{O}}^s_k(M)$ are weakly equivalent, and so

$$T_k \mathbf{HQ} \wedge \overline{\mathrm{Emb}}(M,V)_+ \simeq \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}^s_k(M)} \mathbf{HQ} \wedge \mathrm{B}(m,V)_+.$$

Using the Quillen equivalence between rational spectra and rational chain complexes, and the fact that homotopy limits are preserved by Quillen equivalences, we conclude that there is a weak equivalence (or more precisely a zig-zag of weak equivalences) in $\mathrm{Ch}_{\mathbf{Q}}$:

$$T_k \mathrm{C}_*(\overline{\mathrm{Emb}}(M,V)) \simeq \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}^s_k(M)} \mathrm{C}_*(\mathrm{B}(m,V)).$$

On the other hand, by Proposition 6.5 and Lemma 3.6, the functor $m \mapsto \mathrm{C}_*(\mathrm{B}(m,V)) \otimes \mathbf{R}$ from $\widetilde{\mathcal{O}}^s_k(M)$ to $\mathrm{Ch}_{\mathbf{R}}$ is formal. By Remark 3.5, we get that

$$\operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}^s_k(M)} \mathrm{C}_*(\overline{\mathrm{Emb}}(\pi(\phi,m),V)) \otimes \mathbf{R} \simeq \prod_{i=0}^{\infty} \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}^s_k(M)} \mathrm{H}_i(\mathrm{B}(m,V);\mathbf{R}).$$

Recall that $\mathrm{B}(m,V)$ is equivalent to the space of configurations of m points in V and it only has homology in dimensions at most $(m-1)(\dim V - 1)$. Since $m \leqslant k$, the product on the right-hand side of the above formula is in fact finite (more precisely, it is non-zero only for $i = 0, \dim V - 1, 2(\dim V - 1), \dots, (k-1)(\dim V - 1)$). Therefore, we may think of the product as a direct sum, and so tensoring with $\mathbf{R}$ commutes with product in the displayed formulas below. By Proposition 8.3, we know that tensoring with $\mathbf{R}$ commutes, in our case, with holim, and so we obtain the weak equivalence

$$T_k \mathrm{C}_*(\overline{\mathrm{Emb}}(M,V)) \otimes \mathbf{R} \simeq \left(\prod_{i=0}^{\infty} \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}_k^s(M)} \mathrm{H}_i(\mathrm{B}(m,V);\mathbf{Q}) \right) \otimes \mathbf{R}.$$

It is well known (and is easy to prove using calculus of functors) that spaces such as $\overline{\mathrm{Emb}}(M,V)$ are homologically of finite type, therefore all chain complexes involved are homologically of finite type. Two rational chain complexes of homologically finite type which are quasi-isomorphic after tensoring with $\mathbf{R}$ are, necessarily, quasi-isomorphic over $\mathbf{Q}$. Therefore, we have a weak equivalence in $\mathrm{Ch}_{\mathbf{Q}}$:

$$T_k \mathrm{C}_*(\overline{\mathrm{Emb}}(M,V)) \simeq \prod_{i=0}^{\infty} \operatorname*{holim}_{(\phi,m)\in \widetilde{\mathcal{O}}_k^s(M)} \mathrm{H}_i(\mathrm{B}(m,V);\mathbf{Q}).$$

The desired result follows by using, once again, Proposition 7.1 and the equivalence between $\mathrm{Ch}_{\mathbf{Q}}$ and rational spectra. □

10. Formality and the splitting of the orthogonal tower

In this section we show that Theorem 9.2, which is about the splitting of a certain lim-Postnikov tower, can be reinterpreted as the splitting of the orthogonal tower of $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$. Thus in this section we mainly focus on the functoriality of

$$\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$$

in V and, accordingly, terms like "Taylor polynomials", "derivatives", etc. are always used in the context of orthogonal calculus.([1])

As we have seen, embedding calculus tells us, roughly speaking, that

$$\Sigma^\infty \overline{\mathrm{Emb}}(M,V)_+$$

can be written as a homotopy inverse limit of spectra of the form $\Sigma^\infty \mathrm{C}(k,V)_+$, where $\mathrm{C}(k,V) := \mathrm{Emb}(\{1,...,k\},V)$ is the space of configurations of k points in V. A good place to start is therefore to understand the orthogonal Taylor tower of $V \mapsto \Sigma^\infty \mathrm{C}(k,V)_+$. The only thing that we will need in this section is the following simple fact (we will only use a rationalized version of it, but it is true integrally).

([1]) We are committing a slight abuse of notation here, because the definition of $\overline{\mathrm{Emb}}(M,V)$ depends on choosing a fixed embedding $M \hookrightarrow W$, and therefore $\overline{\mathrm{Emb}}(M,V)$ is only defined for vector spaces containing W. One way around this problem would be to work with the functor $V \mapsto \overline{\mathrm{Emb}}(M, W\oplus V)$. To avoid introducing ever messier notation, we chose to ignore this issue, as it does not affect our arguments in the slightest.

PROPOSITION 10.1. *The functor $V \mapsto \Sigma^\infty C(k,V)_+$ is polynomial of degree $k-1$. For $0 \leqslant i \leqslant k-1$, the i-th layer in the orthogonal tower of this functor,*(2) $D_i \Sigma^\infty C(k,V)_+$, *is equivalent to a wedge of spheres of dimension $i(\dim V - 1)$.*

This proposition is an immediate consequence of Proposition 10.3 below, and its rational version is restated more precisely as Corollary 10.5.

We now digress to do a detailed calculation of the derivatives of $\Sigma^\infty \mathrm{C}(k,V)_+$. First, we need some definitions.

Definition 10.2. Let S be a finite set. A *partition* Λ of S is an equivalence relation on S. Let $P(S)$ be the poset of all partitions of S, ordered by refinement (the finer the bigger). We say that a partition Λ is *irreducible* if each component of Λ has at least two elements.

The poset $P(S)$ has both an initial and a final object. Therefore, its geometric realization, which we denote by $|P(S)|$, is a contractible simplicial complex. One standard way to construct a non-trivial homotopy type out of $P(S)$ is to consider the subposet $P_0(S)$, obtained from $P(S)$ by removing both the initial and the final object. We are going to construct a variation of $|P_0(S)|$ as follows. First, consider the following subcomplex of $|P(S)|$, which we denote by $\partial|P(S)|$. $\partial|P(S)|$ is built of those simplices of the nerve of $P(S)$ that do not contain the morphism from the initial object to the final object as a 1-dimensional face. It is an easy exercise to show that $\partial|P(S)|$ is homeomorphic to the unreduced suspension of $|P_0(S)|$. Let T_S be the quotient space $|P(S)|/\partial|P(S)|$. Since $|P(S)|$ is contractible, it follows that T_S is equivalent to the suspension of $\partial|P(S)|$, and therefore to the double suspension of $|P_0(S)|$.

If $S = \{1, ..., n\}$, we denote $P(S)$ by $P(n)$ and T_S by T_n. There is a well-known equivalence (see, e.g., [17, Theorem 4.109], where the analogous statement is proved for $|P_0(S)|$, which in [17] is called the Folkman complex of the braid arrangement):

$$T_n \simeq \bigvee_{(n-1)!} S^{n-1}.$$

Now let Λ be a partition of $S = \{1, ..., n\}$, and let $P(\Lambda)$ be the poset of all refinements of Λ. Again, $P(\Lambda)$ is a poset with both an initial and a final object. Define $\partial|P(\Lambda)|$ in the same way as before, and let

$$T_\Lambda = |P(\Lambda)|/\partial|P(\Lambda)|.$$

It is not hard to see that if Λ is a partition with components $(\lambda_1, ..., \lambda_j)$, then there is an isomorphism of posets

$$P(\Lambda) \cong P(\lambda_1) \times ... \times P(\lambda_j)$$

(2) Note that we are not speaking of the derivative of this functor.

and therefore a homeomorphism

$$T_\Lambda \cong T_{\lambda_1} \wedge \ldots \wedge T_{\lambda_j}.$$

In particular, T_Λ is equivalent to a wedge of spheres of dimension $n-j$. We call this number the *excess* of Λ and denote it by $e(\Lambda)$.

PROPOSITION 10.3. *For $i>0$, the i-th layer of $\Sigma^\infty C(k,V)_+$ is equivalent to*

$$D_i \Sigma^\infty C(k,V)_+ \simeq \bigvee_{\{\Lambda \in P(k): e(\Lambda)=i\}} \mathrm{Map}_*(T_\Lambda, \Sigma^\infty S^{iV}),$$

where the wedge sum is over the set of partitions of k of excess i.

Proof. Denote the *fat diagonal* of kV by

$$\Delta^k V := \{(v_1, ..., v_k) \in kV : v_i = v_j \text{ for some } i \neq j\}.$$

The *smashed-fat-diagonal* of S^{kV} is

$$\Delta^k S^V := \left\{x_1 \wedge \ldots \wedge x_k \in \textstyle\bigwedge_{i=1}^k S^V = S^{kV} : x_i = x_j \text{ for some } i \neq j\right\}.$$

Thus

$$\mathrm{C}(k,V) = kV \setminus \Delta^k V = ((kV) \cup \{\infty\}) \setminus ((\Delta^k V) \cup \{\infty\}) = S^{kV} \setminus \Delta^k S^V.$$

Recall that for a subpolyhedron in a sphere, $j: K \hookrightarrow S^n$, Spanier–Whitehead duality gives a weak equivalence of spectra

$$\Sigma^\infty (S^n \setminus K)_+ \simeq \mathrm{Map}_*(S^n/K, \Sigma^\infty S^n)$$

which is natural with respect to inclusions $L \subset K$ and commutes with suspensions. In our case, Spanier–Whitehead duality gives an equivalence

$$\Sigma^\infty \mathrm{C}(k,V)_+ \simeq \mathrm{Map}_*(S^{kV}/\Delta^k S^V, \Sigma^\infty S^{kV})$$

which is natural with respect to linear isometric injections. The right-hand side is equivalent to the homotopy fiber of the map

$$\mathrm{Map}_*(S^{kV}, \Sigma^\infty S^{kV}) \longrightarrow \mathrm{Map}_*(\Delta^k S^V, \Sigma^\infty S^{kV}).$$

Since $\mathrm{Map}_*(S^{kV}, \Sigma^\infty S^{kV}) \simeq \Sigma^\infty S^0$ is a constant functor, it has no layers of degree greater than zero. Therefore, for $i>0$,

$$D_i \Sigma^\infty \mathrm{C}(k,V)_+ \simeq \Omega(D_i \,\mathrm{Map}_*(\Delta^k S^V, \Sigma^\infty S^{kV})).$$

It is not hard to see (see [1, Lemma 2.2] for a proof) that $\Delta^k S^V$ can be "filtered" by excess. More precisely, there is a sequence of spaces

$$*=\Delta^k_0 S^V \longrightarrow \Delta^k_1 S^V \longrightarrow \Delta^k_2 S^V \longrightarrow \dots \longrightarrow \Delta^k_{k-1} S^V = \Delta^k S^V$$

such that the homotopy cofiber of the map $\Delta^k_{i-1} S^V \to \Delta^k_i S^V$ is equivalent to

$$\bigvee_{\{\Lambda\in P(k):e(\Lambda)=i\}} K_\Lambda \wedge S^{(k-i)V},$$

where K_Λ is a de-suspension of T_Λ. It follows that $\mathrm{Map}_*(\Delta^k S^V, \Sigma^\infty S^{kV})$ can be decomposed into a finite tower of fibrations

$$\mathrm{Map}_*(\Delta^k S^V, \Sigma^\infty S^{kV}) = X_{k-1} \longrightarrow X_{k-2} \longrightarrow \dots \longrightarrow X_1,$$

where the homotopy fiber of the map $X_i \to X_{i-1}$ is equivalent to

$$\prod_{\{\Lambda\in P(k):e(\Lambda)=i\}} \mathrm{Map}_*(K_\Lambda, \Sigma^\infty S^{iV}).$$

Since this is obviously a homogeneous functor of degree i, it follows that X_i is the ith Taylor polynomial of $\mathrm{Map}_*(\Delta^k S^V, \Sigma^\infty S^{kV})$. The proposition follows. $\square$

Remark 10.4. Proposition 10.3 is closely related to the homology calculations done by Cohen and Taylor in [6].

Rationalizing, we obtain the following corollary.

Corollary 10.5. *Each layer in the orthogonal tower of the functor*

$$V \longmapsto \mathbf{HQ} \wedge C(k,V)_+$$

is an Eilenberg–Maclane spectrum. More precisely,

$$D_i(\mathbf{HQ}\wedge C(k,V)_+) \simeq \begin{cases} \|\mathrm{H}_{i(\dim V-1)}(C(k,V))\|, & \text{if } i \leqslant k-1, \\ *, & \text{otherwise,} \end{cases}$$

where

$$\|\mathrm{H}_{i(\dim V-1)}(C(k,V))\|$$

is the Eilenberg–MacLane spectrum that has the $i(\dim V-1)$-th rational homology of $C(k,V)$ in degree $i(\dim V-1)$.

Therefore, this orthogonal tower coincides, up to indexing, with the Postnikov tower, i.e.

$$P_n(\mathbf{HQ}\wedge C(k,V)_+) \simeq \mathrm{Po}_{d(n)}(\mathbf{HQ}\wedge C(k,V)_+),$$

where $d(n)$ is any number satisfying $n(\dim V-1) \leqslant d(n) < (n+1)(\dim V-1)$.

Proof. The computation of the layers is an immediate application of the previous proposition. Set $X=\mathbf{HQ}\wedge \mathrm{C}(k,V)_+$ and consider the following commutative square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Po}_d(X) \arrow[d] \\ P_n(X) \arrow[r] & \mathrm{Po}_d\, P_n(X). \end{tikzcd}$$

A study of the homotopy groups of the layers shows that the bottom and the right maps are weak equivalences when d is the prescribed range. □

Let $\Sigma^\infty\widehat{\mathrm{C}(k,V)}_+$ be the total homotopy fiber of the k-dimensional cubical diagram which sends a subset $S\subset\{1,...,k\}$ to $\Sigma^\infty\mathrm{C}(S,V)_+$ (where $\mathrm{C}(S,V)=\mathrm{Emb}(S,V)$), and where the maps are the obvious restriction maps. $\Sigma^\infty\widehat{\mathrm{C}(k,V)}_+$ is a functor of V, and so we may ask about the homogeneous layers of this functor. We have the following variation of Proposition 10.3.

PROPOSITION 10.6. *Let $P^{\mathrm{irr}}(k)$ be the set of* irreducible *partitions of k (i.e., partitions without singletons). For $i\geqslant 0$, the i-th layer of $\Sigma^\infty\widehat{C(k,V)}_+$ is equivalent to*

$$D_i\Sigma^\infty\widehat{C(k,V)}_+\simeq \bigvee_{\{\Lambda\in P^{\mathrm{irr}}(k):e(\Lambda)=i\}} \mathrm{Map}_*(T_\Lambda,\Sigma^\infty S^{iV}),$$

where the wedge sum is over the set of irreducible partitions of k of excess i.

Proof. It is clear by inspection that

$$D_0\Sigma^\infty\widehat{\mathrm{C}(k,V)}_+\simeq *$$

so we may assume that $i>0$. Proposition 10.3 can be rephrased as saying that for $i>0$, the ith layer of $\Sigma^\infty\mathrm{C}(S,V)_+$ is equivalent to

$$\prod_{\{\Lambda\in P(S):e(\Lambda)=i\}} \mathrm{Map}_*(T_\Lambda,\Sigma^\infty S^{iV}),$$

where the product is over the set of partitions of S of excess i. Let $S_1\hookrightarrow S_2$ be an inclusion, and consider the corresponding projection of configuration spaces $\mathrm{C}(S_2,V)\longrightarrow\mathrm{C}(S_1,V)$. It is not hard to show, by inspecting the proof of Proposition 10.3, that the corresponding map of ith layers

$$D_i\Sigma^\infty\mathrm{C}(S_2,V)_+\longrightarrow D_i\Sigma^\infty\mathrm{C}(S_1,V)_+$$

corresponds to the projection

$$\prod_{\{\Lambda\in P(S_2):e(\Lambda)=i\}} \mathrm{Map}_*(T_\Lambda,\Sigma^\infty S^{iV})\longrightarrow \prod_{\{\Lambda\in P(S_1):e(\Lambda)=i\}} \mathrm{Map}_*(T_\Lambda,\Sigma^\infty S^{iV})$$

associated with the obvious inclusion of posets $P(S_1)\hookrightarrow P(S_2)$. The proposition follows by elementary combinatorics. □

Corollary 10.7. *Suppose that* $k=2l-1$ *or* $k=2l$. *Then*

$$D_i\Sigma^\infty\widehat{C(k,V)}_+\simeq * \quad \textit{for}\ \ i<l.$$

Proof. According to the preceding proposition, $D_i\Sigma^\infty\widehat{\mathrm{C}(k,V)}_+$ is a wedge sum indexed by irreducible partitions of k of excess i. It is clear from the definition of excess that the lowest possible excess of an irreducible partition of k is attained by the irreducible partition of k with the maximal number of components. It is easy to see that the irreducible partitions of k with the maximal number of components are partitions of type

$$\underbrace{2\text{-}\ \dots\ \text{-}2}_{l-2}\text{-}3$$

if $k=2l-1$, and partitions of type

$$\underbrace{2\text{-}\ \dots\ \text{-}2}_{l}$$

if $k=2l$. In both of these cases, the excess of the partition is l. It follows that the wedge sum in the statement of Proposition 10.6 is empty for $i<l$, and so the left-hand side is contractible in this case. □

Corollary 10.8. *Suppose that* $k=2l-1$ *or* $k=2l$. *Then* $\Sigma^\infty\widehat{C(k,V)}_+$ *is*

$$(l(\dim V-1)-1)\textit{-connected.}$$

Proof. By the preceding corollary, the smallest i for which $D_i\Sigma^\infty\widehat{\mathrm{C}(k,V)}_+$ is nontrivial is l. By Proposition 10.6, this layer is equivalent to a (stable) wedge of spheres of dimension $l(\dim V-1)$, and so it is $(l(\dim V-1)-1)$-connected. Clearly, higher layers are more highly connected. Since the Taylor tower of $\Sigma^\infty\mathrm{C}(k,V)_+$ converges, the statement follows. □

We will also need the following proposition.

Proposition 10.9. *For every* n *there exists a large enough* k *such that the natural map*

$$P_n\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+\xrightarrow{\simeq}\operatorname*{holim}_{U\in\mathcal{O}_k(M)}P_n\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(U,V)_+$$

is an equivalence. The same holds if P_n *is replaced by* D_n.

Proof. We will only prove the P_n version. The target of the map is

$$T_kP_n\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+.$$

Applying Lemma 8.6 to the functor $E(M,V)=\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$, it is enough to prove that for a large enough k the map

$$P_n\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+ \longrightarrow P_nT_k\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$$

is an equivalence. Consider again the formula for the kth layer in the embedding tower

$$\Gamma_c\left(\binom{M}{k}, \mathbf{HQ}\widehat{\wedge\mathrm{C}(k},V)_+\right).$$

It follows from Corollary 10.8 that the spectrum $\mathbf{HQ}\widehat{\wedge\mathrm{C}(k},V)_+$ is roughly $\frac{1}{2}k\dim V$-connected. Thus, $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ and $T_k\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ agree to order roughly $\frac{1}{2}k$ (in the sense defined in §8). Hence, by Proposition 8.5, the map

$$\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+ \longrightarrow T_k\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$$

induces an equivalence on P_n, for roughly $n\leqslant\frac{1}{2}k$. □

We are now ready to state and prove our main theorem.

Theorem 10.10. *Under the assumptions of Theorem* 9.2, *the orthogonal tower of the functor* $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ *splits. In other words*, *there is an equivalence*

$$P_n\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+ \simeq \prod_{i=0}^{n} D_i\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+.$$

Proof. By Lemma 8.6 and Proposition 10.9, and using the model for

$$T_k\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$$

given in Theorem 9.2, it is enough to show that

$$P_n\Big(\operatorname*{holim}_{(m,\phi)\in\widetilde{\mathcal{O}}_k^s(M)}\mathbf{HQ}\wedge\mathrm{B}(m,V)_+\Big)\simeq\prod_{i=0}^{n}\operatorname*{holim}_{(m,\phi)\in\widetilde{\mathcal{O}}_k^s(M)}D_i(\mathbf{HQ}\wedge\mathrm{B}(m,V)_+).$$

By Corollary 10.5, the Taylor tower of $\mathbf{HQ}\wedge\mathrm{B}(m,V)_+$ coincides, up to regrading, with the Postnikov tower. By the proof of Theorem 9.2, the homotopy limit $\operatorname{holim}\mathbf{HQ}\wedge\mathrm{B}(m,V)_+$ splits as a product of the homotopy limits of the layers in the Postnikov towers. Since diagrams of layers in the Postnikov towers and diagrams of layers in the orthogonal towers are diagrams of Eilenberg–MacLane spectra that are equivalent on homotopy groups, they are equivalent diagrams (as per Remark 3.4). It follows that $\operatorname{holim}\mathbf{HQ}\wedge\mathrm{B}(n,V)_+$ splits as a product of the homotopy limits of the layers in the orthogonal towers. □

It is easy to see that the splitting is natural with respect to inclusions of submanifolds of M, but notice that we do not claim that the splitting is natural in V.

11. The layers of the orthogonal tower

In this section we explicitly describe the layers (in the sense of orthogonal calculus) of the Taylor tower of $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)$ as the twisted cohomology of certain spaces of partitions attached to M. We will try to give a "plausibility argument" for our formulas, but a detailed proof will appear in [2].

We encountered partition posets in §10. Here, however, we need to consider a different category of partitions. If Λ is a partition of S, we call S the support of Λ. When we need to emphasize that S is the support of Λ, we use the notation $S(\Lambda)$. Also, we denote by $c(\Lambda)$ the set of components of Λ. Then Λ can be represented by a surjection $S(\Lambda)\twoheadrightarrow c(\Lambda)$. Let C_Λ be the mapping cylinder of this surjection. Then $S(\Lambda)\subset C_\Lambda$. In the previous section we defined the excess of Λ to be $e(\Lambda):=|S(\Lambda)|-|c(\Lambda)|$. It is easy to see that

$$e(\Lambda)=\mathrm{rank}(\mathrm{H}_1(C_\Lambda,S(\Lambda)).$$

Let Λ_1 and Λ_2 be partitions of S_1 and S_2, respectively. A "pre-morphism" $\alpha\colon\Lambda_1\to\Lambda_2$ is defined to be a surjection (which we denote with the same letter) $\alpha\colon S_1\twoheadrightarrow S_2$ such that Λ_2 is the equivalence relation generated by $\alpha(\Lambda_1)$. It is easy to see that such a pre-morphism induces a map of pairs $(C_{\Lambda_1},S(\Lambda_1))\to(C_{\Lambda_2},S(\Lambda_2))$, and therefore a homomorphism

$$\alpha_*\colon\mathrm{H}_1(C_{\Lambda_1},S(\Lambda_1))\longrightarrow\mathrm{H}_1(C_{\Lambda_2},S(\Lambda_2)).$$

We say that α is a *morphism* if α_* is an isomorphism. In particular, there can only be a morphism between partitions of equal excess. Roughly speaking, morphisms are allowed to fuse components together, but are not allowed to bring together two elements in the same component.

For $k\geqslant 1$, let $\mathcal{E}_k$ be the category of irreducible partitions (recall that Λ is irreducible if none of the components of Λ is a singleton) of excess k, with morphisms as defined above. Notice that if Λ is irreducible of excess k then the size of the support of Λ must be between $k+1$ and $2k$.

Next we define two functors on $\mathcal{E}_k$—one covariant and one contravariant. Recall from the previous section that $P(\Lambda)$ is the poset of refinements of Λ. A morphism $\Lambda\to\Lambda'$ induces a map of posets $P(\Lambda)\to P(\Lambda')$. It is not difficult to check that this map takes boundary into boundary, and therefore it induces a map $T_\Lambda\to T_{\Lambda'}$. This construction gives rise to a functor $\mathcal{E}_k\to\mathrm{Top}$, given on objects by

$$\Lambda\longmapsto T_\Lambda.$$

In fact, to conform with the classification of homogeneous functors in orthogonal calculus, we would like to induce up T_Λ to make a space with an action of the orthogonal group $\mathrm{O}(k)$. Let

$$\widetilde{T}_\Lambda := \mathrm{Iso}(\mathbf{R}^k, \mathrm{H}_1(T(\Lambda), S(\Lambda); \mathbf{R}))_+ \wedge T_\Lambda,$$

where $\mathrm{Iso}(V, W)$ is the space of linear isomorphisms from V to W (thus $\mathrm{Iso}(V, W)$ is abstractly homeomorphic to the general linear group if V and W are isomorphic, and is empty otherwise). In this way we get a covariant functor from $\mathcal{E}_k$ to spaces with an action of $\mathrm{O}(k)$. When we want to emphasize the functoriality of this construction, we denote this functor by $\widetilde{T}_{(-)}$.

We now construct another functor on $\mathcal{E}_k$, this time a contravariant one. To begin with, there is an obvious contravariant functor, defined on objects by

$$\Lambda \longmapsto M^{S(\Lambda)}.$$

Let $f: S(\Lambda) \to M$ be an element of $M^{S(\Lambda)}$. The image of f is a finite subset of M, and $f(\Lambda)$ is a partition of $f(S(\Lambda))$. Clearly, f defines a pre-morphism $\Lambda \mapsto f(\Lambda)$. We say that f is a *good* element of $M^{S(\Lambda)}$ if the pre-morphism $\Lambda \mapsto f(\Lambda)$ is in fact a morphism. Otherwise, we say that f is a bad element of $M^{S(\Lambda)}$.

Example 11.1. Let Λ be the partition $(1,2)(3,4)$. Let f be a map $f: \{1,2,3,4\} \to M$. If f is injective, then f is good. If $f(2) = f(3)$, but otherwise f is injective, then f is good. If $f(1) = f(2)$, then f is bad. In general, if f is not injective on some component of Λ, then f is bad, but the converse is not true. In our example, if $f(1) = f(3)$ and $f(2) = f(4)$ then f is bad, even though it may be injective on each component.

Let $\Delta^\Lambda(M)$ be the subspace of $M^{S(\Lambda)}$ consisting of all the bad elements. For example, if Λ is the partition with one component, then $\Delta^\Lambda(M)$ is the usual fat diagonal. It is not hard to see that the contravariant functor $\Lambda \mapsto M^{S(\Lambda)}$ passes to a contravariant functor from $\mathcal{E}_k$ to spaces given on objects by

$$\Lambda \longmapsto M^{S(\Lambda)} / \Delta^\Lambda(M).$$

Let $M^{[\Lambda]} := M^{S(\Lambda)} / \Delta^\Lambda(M)$. This is a contravariant functor on $\mathcal{E}_k$. Again, when we want to emphasize that this is a functor, we denote this construction by $M^{[-]}$.

Given a covariant functor and a contravariant functor on $\mathcal{E}_k$, we may consider the "tensor product" (also known as coend, which in our case is equivalent to the homotopy coend)

$$\widetilde{T}_{(-)} \otimes_{\mathcal{E}_k} M^{[-]},$$

which is a space with an action of $\mathrm{O}(k)$.

THEOREM 11.2. *The i-th layer of the orthogonal calculus tower of* $\Sigma^\infty \overline{\mathrm{Emb}}(M,V)_+$ *is equivalent to*

$$\mathrm{Map}_*(\widetilde{T}_{(-)} \otimes_{\mathcal{E}_i} M^{[-]}, \Sigma^\infty S^{Vi})^{\mathrm{O}(i)}.$$

Idea of proof. Embedding calculus suggests that it is almost enough to prove the theorem in the case of M homeomorphic to a finite disjoint union of balls. In this case $\overline{\mathrm{Emb}}(M,V)$ is equivalent to the configuration space $\mathrm{C}(k,V)$. It is not hard to show that then the formula in the statement of the current theorem is equivalent to the formula given by Proposition 10.3. The current theorem restates the formula of Proposition 10.3 in a way that is well defined and natural for all manifolds M. □

It follows that the kth layer of $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)_+$ is given by the same formula as in the theorem, with Σ^∞ replaced by $\mathbf{HQ}\wedge$.

COROLLARY 11.3. *Suppose that M_1 and M_2 are related by a zig-zag of map inducing an isomorphism in homology. Then, for each n, the n-th layers of the orthogonal towers of the two functors*

$$V \longmapsto \Sigma^\infty \overline{\mathrm{Emb}}(M_i,V)_+, \quad i=1,2,$$

are homotopy equivalent. Similarly, if the maps in the aforementioned zig-zag induce isomorphisms in rational homology then the layers of the orthogonal towers of

$$V \longmapsto \mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M_i,V)_+$$

are equivalent.

Proof. It is not hard to show that $\widetilde{T}_{(-)} \otimes_{\mathcal{E}_k} M^{[-]}$ is equivalent to a finite CW complex with a free action (in the pointed sense) of $\mathrm{O}(k)$. Since the action is free, the fixed points construction in the formula for the layers in the orthogonal tower can be replaced by the homotopy fixed points construction. Thus, the kth layer in the orthogonal tower of $\Sigma^\infty \overline{\mathrm{Emb}}(M,V)_+$ is equivalent to

$$\mathrm{Map}_*(\widetilde{T}_{(-)} \otimes_{\mathcal{E}_k} M^{[-]}, \Sigma^\infty S^{Vk})^{h\,\mathrm{O}(k)}.$$

It is easy to see that this is a functor that takes homology equivalences in M to homotopy equivalences. For the rational case, notice that $\mathrm{Map}_*(\widetilde{T}_{(-)} \otimes_{\mathcal{E}_k} M^{[-]}, \mathbf{HQ}\wedge S^{Vk})^{h\,\mathrm{O}(k)}$ is a functor of M that takes rational homology equivalences to homotopy equivalences. □

Some remarks are perhaps in order.

Remark 11.4. It may be helpful to note that the space $\widetilde{T}_{(-)}\otimes_{\mathcal{E}_k} M^{[-]}$ can be filtered by the size of support of Λ (that is, by the number of points in M involved). This leads to a decomposition of the kth layer in the orthogonal tower of $\Sigma^\infty\overline{\mathrm{Emb}}(M,V)$ as a finite tower of fibrations, with k terms, indexed $k+1\leqslant i\leqslant 2k$, corresponding to the number of points in M. This is the embedding tower of the kth layer of the orthogonal tower. For example, the second layer of the orthogonal tower fits into the following diagram, where $\Delta^{2,2}M$ is the singular set of the action of $\Sigma_2\wr\Sigma_2$ on M^4, the left row is a fibration sequence, and the square is a homotopy pullback:

$$\begin{tikzcd}
\mathrm{Map}_*((M^4/\Delta^4M)\wedge T_2\wedge T_2, \Sigma^\infty S^{2V})^{\Sigma_2\wr\Sigma_2} \arrow[d] & \\
D_2\Sigma^\infty\overline{\mathrm{Emb}}(M,V) \arrow[r] \arrow[d] & \mathrm{Map}_*((M^4/\Delta^{2,2}M)\wedge T_2^{\wedge 2}, \Sigma^\infty S^{2V})^{\Sigma_2\wr\Sigma_2} \arrow[d] \\
\mathrm{Map}_*((M^3/\Delta^3M)\wedge T_3, \Sigma^\infty S^{2V})^{\Sigma_3} \arrow[r] & \mathrm{Map}_*((M^3/\Delta^3M)\wedge T_2^{\wedge 2}, \Sigma^\infty S^{2V})^{\Sigma_2}.
\end{tikzcd}$$

Remark 11.5. To relate this to something "classical", note that the top layer of the embedding tower of the kth layer of the orthogonal tower is

$$\mathrm{Map}_*((M^{2k}/\Delta^{2k}M)\wedge T_2^{\wedge k}, \Sigma^\infty S^{kV})^{\Sigma_2\wr\Sigma_k}.$$

This is the space of "chord diagrams" on M, familiar from knot theory. In fact, in the case of M being a circle (or an interval, in which case one considers embeddings fixed near the boundary), it is known from [15] that the Vassiliev homology spectral sequence, which also converges to the space of knots, collapses at E^1. Thus the orthogonal tower spectral sequence for $\mathbf{HQ}\wedge\overline{\mathrm{Emb}}(M,V)$ must coincide with Vassiliev's. It is not hard to verify directly that the two E^1 terms are isomorphic (up to regrading).

Finally, we deduce the rational homology invariance of $\overline{\mathrm{Emb}}(M,V)$ from our main theorem and Corollary 11.3.

THEOREM 11.6. *Let M and M' be two manifolds such that there is a zig-zag of maps, each inducing an isomorphism in rational homology, connecting M and M'. If*

$$\dim V > 2\max\{\mathrm{ED}(M), \mathrm{ED}(M')\},$$

then $\overline{\mathrm{Emb}}(M,V)$ and $\overline{\mathrm{Emb}}(M',V)$ have the same rational homology groups.

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Gregory Arone
Department of Mathematics
University of Virginia
Charlottesville, VA 22903
U.S.A.
zga2m@virginia.edu

Pascal Lambrechts
Institut Mathématique
Université Catholique de Louvain
2 Chemin du Cyclotron
BE-1348 Louvain-la-Neuve
Belgium
pascal.lambrechts@uclouvain.be

Ismar Volić
Department of Mathematics
Wellesley College
Wellesley, MA 02481
U.S.A.
ivolic@wellesley.edu

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