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The scaling limit of loop-erased random walk in three dimensions

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1. Introduction

Loop-erased random walk (LERW) is a model for a random simple path, created by taking a simple random walk and, whenever the random walk hits its path, removing the resulting loop and continuing. See §1.5 for a precise definition. It is strongly related to the uniform spanning tree (UST), a random spanning tree of a graph G selected uniformly between all spanning trees of G: the path in the UST between two points is distributed like a LERW between them [P91], and further, the entire UST can be generated using repeated use of LERW by Wilson's algorithm [W96]. Both models, and the connections between them, are interesting on a general graph, but we shall be most interested in lattices on $\mathbf{R}^d$ and open subsets thereof, in which case these models arise naturally in statistical mechanics in conjunction with the Potts model.

Of all the non-Gaussian models in statistical mechanics, LERW is probably the most tractable. Above five dimensions, it can be analyzed using the non-intersections of simple random walk directly [L91, Chapter 7] giving an easy proof that the scaling limit is Brownian motion. In four dimensions a logarithmic correction is required [L95], and that too has been proved with no use of the difficult technique of lace expansion (see [BS85] and [HvdHS03] for lace expansion). Borrowing a term from physics, we might say that the *upper critical dimension* for this model is 4. In two dimensions, LERW is conformally invariant in the limit as the lattice becomes finer and finer. This allowed physicists to make precise conjectures about fractal dimensions, critical exponents and winding numbers [D92], [M92]. Rigorously, three different approaches proved fruitful: the connection to random domino tilings [K00a], [K00b], the connection to SLE [LSW04a], and the approach that we will pursue in this paper, [K]. In fact, SLE was discovered [S00] in the context of LERW.

Attempts to understand LERW in dimension three focused mainly on the number of steps it takes to reach the distance r. Physicists conjecture that it is $\approx r^{\xi}$ and made numerical experiments to show that $\xi=1.62\pm0.01$ [GB90]. Rigorously the existence of ξ has not been proved (so we must talk about upper and lower exponents $\underline{\xi}\leqslant\overline{\xi}$), and the best estimates known are $1<\underline{\xi}\leqslant\overline{\xi}\leqslant\frac{5}{3}$ [L99]. LERW has no natural continuum equivalent in dimensions smaller than four—Brownian motion has a dense set of loops and therefore it is not clear how to remove them in chronological order. In two dimensions the scaling limit is radial SLE 2, but it is not clear if this can be interpreted as a "Brownian motion with loops removed". For example, take a coupling of Brownian motion and SLE 2 which is the scaling limit of the couple $(R, \mathrm{LE}(R))$—it has not been proved that this limit exists, but for the purpose of the discussion we may assume that it does or alternatively take a subsequential limit. It is not known whether in that coupling the SLE 2 path is a function of the Brownian path (I was informed of this question by O. Schramm).

In this paper we shall show that LERW has a scaling limit in three dimensions. More precisely we shall show the following theorem.

THEOREM 1. *Let $\mathcal{D}\subset\mathbf{R}^d$, $d=2,3$, be a polyhedron and let $a\in\mathcal{D}$. Let $\mathbf{P}_n$ be the distribution of the loop-erasure of a random walk on $\mathcal{D}\cap 2^{-n}\mathbf{Z}^3$ starting from a and stopped when hitting $\partial\mathcal{D}$. Then $\mathbf{P}_n$ converge in the space $\mathcal{M}(\mathcal{H}(\overline{\mathcal{D}}))$.*

Here $\mathcal{H}(\mathcal{X})$ is the space of compact subsets of $\mathcal{X}$ with the Hausdorff metric, and $\mathcal{M}(\mathcal{X})$ is the space of measures on $\mathcal{X}$ with the topology of weak convergence (this, and some other notation is explained in §1.4). In general, the choice of topologies above is not canonical. For example, [LSW04a] shows the existence of a scaling limit for LERW in two dimensions replacing $\mathcal{H}$ above with the somewhat stronger topology of "minimal distance after optimal change of variables". However, for our techniques the Hausdorff metric is the natural choice. I believe that the tools that will be developed here can be used for a number of convergence questions for LERW (e.g. the existence of ξ, the existence of the scaling limit on more general domains, and in stronger topologies, universality and so on). However, as this paper is long as it is, I chose to show only the simplest consequences: that the limit exists and is invariant under dilations and rotations.

Since we are interested in scaling limits, it might be useful to quickly review known results of this type. The archetypical example is of course the Donsker invariance principle [RW94, p. 16] stating that the scaling limit of simple random walk is Brownian motion in any dimension. As already remarked, in two dimensions the scaling limit of LERW is radial SLE 2, and a good deal of other discrete models have been shown to converge to SLE: critical percolation on the triangular lattice converges to chordal SLE 6 [S01], [SW01] or [BR06, Chapter 7], the Peano curve of the UST converges to chordal SLE 8 [LSW04a], and the harmonic explorer converges to SLE 4 [SS]. The interface of the critical Ising model is about to join this family too [S]. The case of the self-avoiding walk demonstrates nicely the difficulties involved: it has been proved that if the limit exists and is conformally invariant, it would be chordal SLE $\frac{8}{3}$ [LSW04b], but the existence of the limit is still open. A similar situation is facing the Laplacian-b

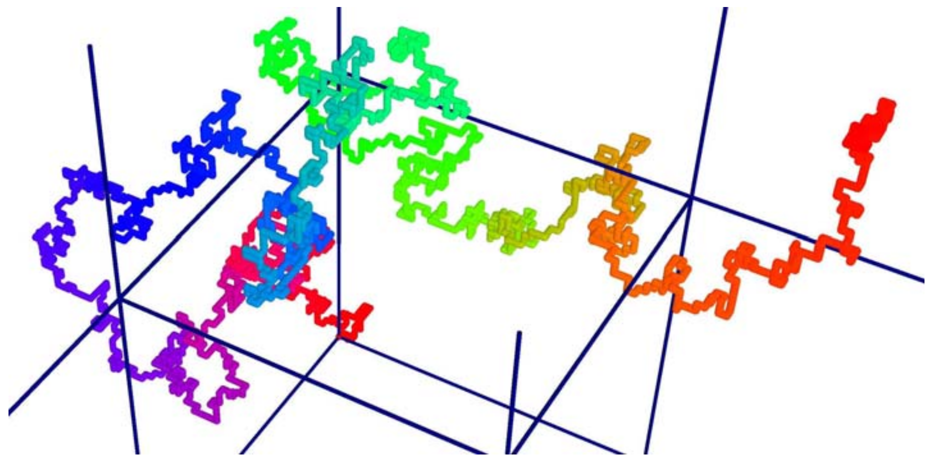

Figure 1. A loop-erased random walk in three dimensions. Color or brightness change along the path.

random walk [L]. In high dimensions lace expansion has been used to show that the scaling limit of the self-avoiding walk is Brownian motion [BS85], [HS92], and that the scaling limit of percolation, oriented percolation and lattice trees is integrated super-Brownian excursion [NY95], [DS98], [S99], [vdHS03]. In intermediate dimensions much less is known. The discrete Gaussian free field converges to the continuum Gaussian free field and the Richardson model was shown to have a limit shape from subadditivity arguments in any dimension [R73], and I cannot resist citing the beautiful work on branched polymers in dimension three [BI03]. But these examples are the exception, not the rule.

1.1. Sketch of the proof

The core of the argument is very similar to that of [K], so let us recall the arguments there. Let R be a random walk on a "three-dimensional graph" starting from a and stopped on the boundary of some domain $\mathcal{D}$. Let $B\subset\mathcal{D}$ be some (small) ball. We write

$$\mathrm{LE}(R)=\gamma_1\cup\gamma_2\cup\gamma_3,$$

where γ_1 is the portion of $\mathrm{LE}(R)$ until the first time when $\mathrm{LE}(R)$ hits B. Notice that this is *not* the same as the loop-erasure of a random walk stopped on ∂B! Then γ_2 is the portion of $\mathrm{LE}(R)$ until the last time when $\mathrm{LE}(R)$ is inside B, and γ_3 is the reminder (the precise form of this division is in the proof of Lemma 5.8, p. 129). Tracing the process of loop-erasure in $\mathcal{D}$, one sees that γ_1 does not depend on anything that happens inside B: when one knows all entry and exit points of R from B, and all the trajectories that R does outside B, one can calculate γ_1. In particular, if we compare random walks R^1 and

R^2 on graphs G^1 and G^2, where $G^1\setminus B=G^2\setminus B$, and inside B we have some estimate of the sort

$$p^1(v)\simeq p^2(v), \tag{1}$$

where $p^i(v)$ is the probability that a random walk on G^i exits B at a particular vertex v, then we should have that

$$\gamma_1^1\simeq\gamma_1^2.$$

This argument and the precise meaning of "$\simeq$" are contained in Lemma 5.7. To make this argument work for γ_3, we have to use the symmetry of loop-erased random walk.

Now, if γ_2 is large, then we are in the situation that was coined in [S00] a "quasi-loop", namely two close points such that the path passes through both but goes a long way in-between. In two dimensions it is possible to show that LE(R) has few quasi-loops, by tracing the process that creates them and seeing that it necessitates that a random walk that starts quite close to a loop-erased random walk will avoid hitting it for a long while. This, however, contradicts the discrete Beurling projection principle (see [K87]) that states that a random walk starting near any path has a high probability to intersect it.(1) See [S00, Lemma 2.1] or [K, Lemma 18]. Unfortunately this argument no longer holds in three dimensions. A random walk starting, say, at a distance n from a straight line, has a reasonable probability to intersect it only after extending to a distance of n^2, and even after n^2 the probability to not intersecting the line only decreases logarithmically. In other words, in three dimensions not all paths are "hittable", and we have to show specifically that a loop-erased random walk is, using facts about its structure. This will be done in §4 and we shall show that the probability that a random walk starting near a loop-erased random walk will avoid hitting it decreases like a power law. This will allow us to repeat the above argumentation in three dimensions, show that there are no quasi-loops and hence that LERW is similar on our G^1 and G^2.

The proof that a loop-erased random walk is hittable is based on searching for (local) cut times. A cut time for a random walk R is a time t such that $R[0,t]\cap R[t+1,\infty[=\varnothing$. The number of cut times is connected to the non-intersection exponent: by considering the parts of R up to t and from t on as two random walks, and reversing the first part, we see that it is important to estimate the probability that two random walks of length t will not intersect. This is $\approx t^{-2\xi}$ [L96b], where ξ is the famous non-intersection exponent of Brownian motion. See §3.2 for a description of this topic. Heuristically speaking, a set is "hittable" if its Hausdorff dimension is >1, and the set of cut times has dimension $2-\xi$, so the argument terminates by the well-known fact that $\xi<1$ in three dimensions [BL90b].

(1) Kesten's theorem is stronger, and claims that the minimum probability is achieved, up to a constant, for a straight line, in which case it can be estimated directly. However we will not need this level of accuracy.

Once the fact that LE(R) has no quasi-loops on either G^1 or G^2 is established, we get that it is similar on these two graphs. Hence we can show that LE(R) is similar on $\mathbf{Z}^3$ and $2\mathbf{Z}^3$ by interpolating between them: dissecting into a grid of cubes of intermediate size, and at each step changing one cube from $\mathbf{Z}^3$ to $2\mathbf{Z}^3$. Hence all of the above discussion actually referred to graphs formed by cutting and gluing together cubes of these two graphs.

Here are some corresponding reading recommendations:

• If you are familiar with [K], the most interesting part for you would probably be the proof that there are no quasi-loops. Read the definition of a Euclidean net (§2.2, p. 43), the definition of an isotropic graph (§3.3, p. 62) and the statements of Theorems 2 and 3 (pp. 73 and 77, respectively) and jump directly to §4 (p. 85).

• If you are unfamiliar with [K], the most interesting part for you would probably be the proof core sketched above. Read the definitions of a Euclidean net and an isotropic graph as above; and the definition of a quasi-loop and the statement of Theorem 4 (p. 85). Then jump directly to §5 (p. 118) or even to §5.2 (p. 127).

• If you are the kind of person who prefers explicit examples to generalizations, start with §6 (p. 141) and read a few examples of isotropic graphs and isotropic interpolations. Then you can read the rest of the paper keeping in mind that an "isotropic graph" is really cubes of $\mathbf{Z}^3$ and $2\mathbf{Z}^3$ (and other variations) cut and sewn together so that a random walk would behave like a Brownian motion. §6.1 also contains the proof of the invariance of the scaling limit to dilations and rotations.

• §2 and §3 are mostly recommended for students and non-specialists. §2 consists mostly of citing well-known connections between rough isometries, heat kernel decay, Harnack inequality and similar topics. In §3 we are forced to replicate the results of Lawler [L96b] in our settings. Roughly we show that a relatively simple estimate of hitting probabilities allows us to couple a random walk and a Brownian motion and then Lawler's argument goes through almost unchanged, giving that on the graphs that interest us random walk has many cut times.

Finally I wish to point out how this paper improves over [K] in the two-dimensional case. The use of a computer to calculate precise estimates for the harmonic potential on $\mathbf{Z}^2$ [KS04] and on "hybrid graphs" has been made completely unnecessary by the use of electrical conductance techniques (see Lemma 2.3). The use of "nice rectangles" to ensure that hitting probabilities are comparable was replaced by a multi-scale application of Harnack's inequality together with a coupling argument (and in particular we use spheres throughout rather than rectangles). See Lemmas 3.5–3.6 and 5.1–5.5. Here the representation is of comparable length, but is possibly less cumbersome. Finally, the proof here of the final limit process is much shorter and simpler.

1.2. About the settings

The usual settings for these problems is that of a lattice in $\mathbf{R}^d$. However, as explained above, the proof has to cut and sew together different graphs, and even if these graphs were to be grids, the *intermediate objects* that we must handle would not be. Hence we need to understand random walks on graphs which are "similar" to $\mathbf{Z}^d$. It turns out that there are two important levels of similarity, which correspond to "metric" and "conformal" properties.

Much effort has gone into understanding which properties of random walks are related to the metric structure only, or, more formally, are satisfied by any graph *roughly isometric* to $\mathbf{Z}^d$ (see definition and background at p. 41). However, one cannot expect LERW on a graph roughly isometric to $\mathbf{Z}^d$ to converge to a limit independent of the graph. Indeed, the scaling limit of the random walk on the graph $\mathbf{Z}^2\times 2\mathbf{Z}$ is not a Brownian motion but a stretched version of it. These are not identical—indeed, even their hitting distribution on, say, a sphere, differ, which implies that the scaling limit of LERW on $\mathbf{Z}^3$ and $\mathbf{Z}^2\times 2\mathbf{Z}$ also differ. Hence we need some condition to ensure that locally the graph is not stretched in any direction. In other words, we need to preserve the conformal structure of $\mathbf{R}^d$.

Properties related to the conformal structure are less well understood. The "invariance principle", that is the fact that a random walk converges to a Brownian motion, which is a conformal property, has been researched intensively, but, as it seems, only in different contexts than here. Hence we will use a definition of *isotropic graph* which is, to the best of my knowledge, new. These graphs will satisfy the invariance principle (this is more or less a tautology) and they preserve many properties which are not preserved by the metric structure alone, such as escape probabilities from a line, the non-intersection exponent, and so on.

Our definition of an isotropic graph (see §3) is definitely not the most general imaginable. There are at least two important examples which fall out of its scope. The first is a conformal map of a grid—for example the graph $\mathbf{Z}^2/\{(a,b)\sim(-a,-b)\}$ embedded into $\mathbf{C}$ via the map $(a,b)\mapsto(a+ib)^2$ (here i stands for the imaginary unit). The second is random graphs, such as the Delaunay triangulation of a Poisson process or the infinite cluster of super-critical percolation. These graphs are not even roughly isometric to $\mathbf{R}^2$ and yet are "isotropic" in some heuristic sense. For example, the percolation cluster is isotropic in the sense that it satisfies the invariance principle, see [Ko85], [DFGW89], [SS04], [BB07], [MP07]. I conjecture that the results here extend to these graphs, but will not complicate the paper by considering them.

We shall prove Theorem 1 (and other results) in both the two- and three-dimensional cases. While the three-dimensional case is the more interesting one, the two-dimensional proof is not quite a subset of known results: it is proved for multiply connected domains and for graphs more general than grids. However, at points the presentation of specifically two-dimensional issues will be sketchy.

1.3. Acknowledgements

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1.4. Preliminaries

A *weighted graph* is a couple (G,ω), where G is a set and $\omega: G\times G\to[0,\infty[$ is such that $\omega(v,w)=\omega(w,v)$. We shall often call (G,ω) simply G and use ω only in the places it is needed. For $v\in G$ the *neighbors* of v are the vertices w such that $\omega(v,w)>0$. We denote by $v\sim w$ the neighborhood relation. We shall always assume that the number of neighbors of every vertex is bounded and that the graph has *bounded weights*, i.e.

$$\sup_{v,w}\omega(v,w)<\infty \quad\text{and}\quad \inf_{v\sim w}\omega(v,w)>0.$$

We do not assume that $\omega(v,v)=0$, i.e. we allow self-loops.

A *directed graph* is a graph where $\omega(v,w)$ might be different from $\omega(w,v)$. We will only use directed graphs once, in §2.5. Unless specifically marked "directed", everything below should be assumed to hold for undirected graphs only.

For a subset $X\subset G$ we denote by ∂X the external boundary, namely all vertices of $G\setminus X$ with a neighbor in X. When this is not clear from the context, we shall denote $\partial_G X$ for the graph boundary and ∂_{cont} for the boundary of subsets of $\mathbf{R}^d$ in the usual sense. We write $\overline{X}=X\cup\partial X$.

A *path* in a graph is a function γ from $\{1,...,n\}$ to G such that $\gamma(i)$ and $\gamma(i+1)$ are neighbors; n is the *length* of the path, denoted by $\operatorname{len}\gamma$. If γ_1 and γ_2 are two paths and $\gamma_1(\operatorname{len}\gamma_1)$ is a neighbor of $\gamma_2(1)$ (in which case we call γ_1 and γ_2 *concatenatable*) we shall define $\gamma_1 \cup \gamma_2$ to be the path of length $\operatorname{len}\gamma_1 + \operatorname{len}\gamma_2$ obtained by concatenating them. It will be convenient to regard $\varnothing$ as a path of length 0 and define $\gamma\cup\varnothing=\varnothing\cup\gamma=\gamma$. The notation $\gamma[a,b]$ will be a short for the path γ' of length $b-a+1$ defined by

$$\gamma'(i)=\gamma(a+i-1),$$

and also for the set $\{\gamma(t):t\in[a,b]\}$ (there will rarely be a need to differentiate between a path and its image). The same holds for other types of segments (open, half-open). We say that γ is *between* $\gamma(1)$ and $\gamma(\operatorname{len}\gamma)$, and call G *connected* if there exists a path between any two vertices.

A random walk on a weighted graph is a process R in discrete time such that $R(t+1)$ depends only on $R(t)$ and

$$\mathbf{P}(R(t+1)=w \mid R(t)=v)=\frac{\omega(v,w)}{\omega(v)}, \quad \text{where } \omega(v):=\sum_w \omega(v,w). \tag{2}$$

Here $\mathbf{P}$ denotes probability. We shall denote the expectation by $\mathbf{E}$. When we shall need to specify the starting point, we shall do so using $\mathbf{P}^v$ for the probability when $R(0)=v$, and similarly $\mathbf{E}^v$. When we shall need to specify the graph, we shall do so using $\mathbf{P}^v_G$, and so on. Occasionally (as in the statement of Theorem 1) we will have a graph with an embedding in $\mathbf{R}^d$ and the "starting point" would be an $a\in\mathbf{R}^d$. In this case we mean by $\mathbf{P}^a_G$ a random walk on G starting from the point of G closest to a (if more than one such point exists, choose one, say by lexicographic order).

For a subset $X\subset G$ and a random walk R, we denote by $T(X)$ the *hitting time* of X, i.e.

$$T(X):=\begin{cases}\infty, & \text{if the set is never hit,}\\ \min\{t\geqslant 1: R(t)\in X\}, & \text{otherwise.}\end{cases}$$

If $X=\{x\}$ we shall write $T(x)$ as short for $T(\{x\})$. If d is some metric on G and if $X=\partial B(v,r)$, where $B(v,r)$ is a ball around v of radius r in the metric d, we will write for short $T_{v,r}:=T(\partial B(v,r))$ (and assume that the metric is clear from the context). Note that even if we start from v, $T(v)$ is non-trivial since hitting is defined only for $t\geqslant 1$.

Sometimes we will have a few independent walks denoted by $R^1, R^2, \dots$. In this case the corresponding stopping times will be denoted by $T^i(X)$ and $T^i_{v,r}$. Similarly we shall use $\mathbf{P}^{1,v,2,w}$ when we want to denote that R^1 started from v, while R^2 started from w.

The strong Markov property says that for any stopping time T the random walk after T behaves like a regular random walk. We shall often use it, say for an event E that depends only on what happened after T, in the form $\mathbf{P}(E)=\mathbf{E}\mathbf{P}^{R(T)}(E)$. Here $\mathbf{E}$ denotes expectation over the value of $R(T)$.

Random walk is reversible, which means that the probabilities to traverse a given path in one direction and in the opposite direction are equal up to the ratio of ω at the beginning and at the end. In particular, we can sum over all paths of length t and get

$$\omega(v)\mathbf{P}^v(R(t)=w)=\omega(w)\mathbf{P}^w(R(t)=v) \quad \text{for all } t,\ v \text{ and } w. \tag{3}$$

A similar argument shows that if $v,w\in A\subset G$ then

$$\omega(v)\mathbf{P}^v(T(A)<\infty \text{ and } R(T(A))=w)=\omega(w)\mathbf{P}^w(T(A)<\infty \text{ and } R(T(A))=v). \tag{4}$$

For a function $f\colon G\to\mathbf{R}$ (or to any linear space over $\mathbf{R}$) we define the (discrete) Laplacian of f, Δf, by

$$(\Delta f)(v)=-f(v)+\sum_{w\sim v}\frac{\omega(v,w)}{\omega(v)}f(w).$$

A function f such that Δf is zero will be called (*discretely*) *harmonic*. If Δf is zero on a set $A\subset G$ we shall call f *harmonic on* A. Harmonic functions satisfy the maximum principle, i.e. a function harmonic on a finite set A attains its maximum in $\bar{A}$ on the boundary ∂A. Harmonic functions are related to random walks by the following simple and well known fact: if f is harmonic on a finite set A and $v\in A$ then

$$f(v)=\mathbf{E}^v(f(R(T(\partial A)))). \tag{5}$$

For two sets A and B in a metric space (X,d), we define their *distance* by

$$d(A,B):=\inf_{\substack{a\in A\\ b\in B}} d(a,b).$$

If $x\in X$ we write $d(x,A)$ as a shortening for $d(\{x\},A)$. The *Hausdorff distance* between A and B is defined by

$$d_{\mathrm{Haus}}(A,B):=\max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\}.$$

d_{Haus} makes the collection of all closed subsets of X into a metric space, which we will denote by $\mathcal{H}(X)$. The *diameter* of a set is defined by

$$\operatorname{diam} A:=\sup_{a,b\in A} d(a,b).$$

If the metric space has an addition structure, we will use the notation $A+B$ for the *Minkowski sum* of the sets A and B, i.e.

$$A+B:=\{a+b : a\in A \text{ and } b\in B\}.$$

In particular, if B is a ball centered at 0, then

$$A+B(0,r)=\{x:d(x,A)<r\}.$$

We will sometimes abuse notation by denoting the right-hand side by $A+B(r)$ even when the metric space has no addition structure.

A *domain* is a non-empty bounded open connected subset of $\mathbf{R}^d$. A *polyhedron* is a domain whose boundary is composed of a collection of non-degenerate linear polyhedra of dimension $d-1$. In particular, we do not require that the boundary of the polyhedron is connected. For simplicity, however, we will not allow slits. Let $G=(V,\omega)$ be a graph with $V\subset\mathbf{R}^d$. Then for a domain $\mathcal{D}$ we denote by $G\cap\mathcal{D}$ the graph whose vertex set is $V\cap\mathcal{D}$ and whose weight function ω' is

$$\omega'(x,y)=\begin{cases}\omega(x,y), & \text{when the linear segment } [x,y]\subset\mathcal{D},\\ 0, & \text{otherwise.}\end{cases}$$

We will never intersect a graph with a subset of $\mathbf{R}^d$ which is not open. In particular, when we examine G with the metric induced by the embedding into $\mathbf{R}^d$, the notation $\partial B(v,r)$ relates to the *graph-boundary* of $G\cap B(v,r)$, i.e. of the ball in the $\mathbf{R}^d$ metric, and not to the intersection of G with the $\mathbf{R}^d$-sphere $\partial_{\mathrm{cont}}B(v,r)$.

By C and c, we denote absolute constants which may be different from place to place. C will usually denote constants which are "large enough" and c "small enough". We shall number $(c_1,c_2,\dots)$ only constants to which we will need to refer to again. Sometimes we shall also write $C(\cdot)$ and $c(\cdot)$ for a constant which is not properly absolute—it depends on some parameters—but is best thought of as absolute. This notation implicitly means that I cannot think of any applications where the parameters are not themselves constants. Again, $C(G)$ could change from place to place, and we shall number only those that we shall need to refer to in the future. If, say, C_{187} depends on some parameters, we shall only note this once, and from that place on refer to it as simply C_{187}, not $C_{187}(\alpha,\tau,\mathcal{H})$.

As usual we denote by $\lfloor x\rfloor$ the largest integer $\leqslant x$ and by $\lceil x\rceil$ the smallest integer $\geqslant x$.

1.5. Loop-erasure

For a finite path $\gamma:\{1,\dots,n\}\to G$ we define its *loop-erasure*, $\mathrm{LE}(\gamma)$, which is a simple path in G, by the consecutive removal of loops from γ. Formally,

$$\begin{aligned}\mathrm{LE}(\gamma)(1)&:=\gamma(1)\\ \mathrm{LE}(\gamma)(i+1)&:=\gamma(j_i+1)\quad j_i:=\max\{j:\gamma(j)=\mathrm{LE}(\gamma)(i)\},\end{aligned}\tag{6}$$

which is defined for all i such that $j_{i-1}<n$.

Lemma 1.1. *Let $b^0, b^1 \in B \subset G$. Let R^i be a random walk starting at b^i, stopped at B and conditioned to hit b^{1-i}. Then* $\mathrm{LE}(R^0)$ *has the same distribution as the reversal of* $\mathrm{LE}(R^1)$.

This is well known. See, e.g., [K, Lemma 2].

The following lemma was discovered with Omer Angel. To the best of our knowledge, it has never been published before.

Lemma 1.2. *Let G be a finite weighted graph and let $v, w \in G$. Let R be a random walk on G starting from v and let T_n be the n-th time R is at w. Then*

$$\mathrm{LE}(R[0,T_1]) \sim \mathrm{LE}(R[0,T_n]) \quad \textit{for all } n=2,3,\ldots.$$

The notation $\sim$ here stands, as usual, for "having the same distribution".

Proof. Let γ be any path starting from v not containing w and let $k=1,\ldots,n$. Let $l=\operatorname{len}\gamma$. Let $X_{\gamma,k}$ be the event that $\gamma=\mathrm{LE}(R[0,T_n])[1,l]$ and that $j_l\in[T_{n-k},T_{n-k+1}]$, where j_l is from the definition of LE above and where we consider T_0 to be 0. Let x be any neighbor of $\gamma(l)$. We have that x is the next element of $\mathrm{LE}(R[0,T_n])$ if and only if $R(j_l+1)=x$. Denote this event by N_x.

Conditioning by $X_{\gamma,k}$, we get that $R[j_l,T_n]$ is a random walk on G starting from $\gamma(l)$ and conditioned not to hit γ before hitting w for k times. Therefore

$$\mathbf{P}(N_x \mid X_{\gamma,k}) = \mathbf{P}^{\gamma(l)}(R(1)=x \mid T_k < T(\gamma)).$$

The point of the lemma is that the right-hand side is independent of k—after T_1 it is no longer possible to know anything about the value of $R(1)$. Therefore

$$\mathbf{P}(N_x \mid X_{\gamma,k}) = \mathbf{P}(N_x \mid X_{\gamma,1}),$$

and summing over k we get (letting $X_\gamma=\bigcup_{k=1}^n X_{\gamma,k}$)

$$\mathbf{P}(N_x \mid X_\gamma) = \sum_{k=1}^n \frac{\mathbf{P}(N_x\cap X_{\gamma,k})}{\mathbf{P}(X_\gamma)} = \sum_{k=1}^n \frac{\mathbf{P}(N_x|X_{\gamma,1})\mathbf{P}(X_{\gamma,k})}{\mathbf{P}(X_\gamma)} = \mathbf{P}(N_x \mid X_{\gamma,1}).$$

This last term is equal to the probability that $\mathrm{LE}(R[0,T_1])$ will, if conditioned to start from γ, have x as its next vertex. Indeed, this is the well-known "Laplacian random walk" representation of loop-erased random walk; see [L80], [L87]. □

Lemma 1.3. *Lemma* 1.2 *also holds when the random walk is conditioned not to hit a given subset $B\subset G$, $w\notin B$. In a formula,*

$$\mathrm{LE}(R[0,T_1]) \mid R[1,T_1]\cap B=\varnothing \sim \mathrm{LE}(R[0,T_n]) \mid R[1,T_n]\cap B=\varnothing \quad \textit{for all } n=2,3,\ldots.$$

The proof is identical to that of the previous lemma, except that now the random walk is conditioned not to hit $B\cup\gamma$ instead of just γ.

2. Euclidean nets

2.1. Background on rough isometries

Let X and Y be two metric spaces. A function $f: X \to Y$ is called a *rough morphism* if

$$d(f(x), f(y)) \leqslant Cd(x,y)+C$$

for some C which depends on f. A *rough identity* is a function $f: X \to X$ satisfying

$$d(f(x), x) \leqslant C$$

for some C which depends on f. Notice that f need neither to be one-to-one nor onto! X and Y will be called *roughly isometric* if there exist rough morphisms $f: X \to Y$ and $g: Y \to X$ such that both $f \circ g$ and $g \circ f$ are rough identities. In this case we call both f and g *rough isometries*. This term was introduced by Kanai in [Ka85], though in more restricted settings it already appeared in [G81]. There are various equivalent definitions in the literature, but I prefer the above "categorical" one. A rough isometry completely ignores all local structure, and in fact $\mathbf{R}^d$ is roughly isometric to $\mathbf{Z}^d$ and more generally, any manifold is roughly isometric to any net inside it.

To talk about rough isometries of graphs, we need to introduce a metric. Let therefore G be a weighted connected graph. Define

$$\delta(v,w) := \min_{\gamma: v \to w} \operatorname{len} \gamma,$$

where the minimum is taken over all paths γ from v to w. Clearly this makes G into a metric space.

The "Euclidean nets" we are going to define in the next section are graphs roughly isometric to $\mathbf{R}^d$. Whether properties of random walks are preserved under rough isometries is in general not obvious. In some cases (e.g. transience) this requires an equivalent representation as a geometric property. In others (e.g. Harnack's inequality) it is actually unknown. Let us therefore state here some of the connections between random walks and the geometry of the graph that we will use.

Definition. We say that a graph G satisfies the *volume doubling property* if there exists a constant C such that for any $v \in G$ and any $r \geqslant 1$, $\omega(B(v,2r)) \leqslant C\omega(B(v,r))$, where $\omega(A) := \sum_{x\in A} \omega(x)$, and ω is from equation (2).

Definition. We say that a graph G satisfies the *weak Poincaré inequality* if there exists a constant C such that for any function $f: G \to \mathbf{R}$, any $v \in G$ and any integer r,

$$\sum_{w\in B(v,r)} \omega(w)|f(w)-\bar{f}|^2 \leqslant Cr^2 \sum_{w\sim x\in B(v,2r)} \omega(w,x)|f(w)-f(x)|^2,$$

where

$$\bar{f} := \frac{1}{\omega(B(v,r))} \sum_{x\in B(v,r)} \omega(x) f(x).$$

The inequality is called "weak" because the sum on the right-hand side is over a ball of radius $2r$. The regular Poincaré inequality is defined with the sum over the ball of radius r. However, under the assumption of the volume doubling property, these properties are equivalent; see [J86, §5] (the settings there are a little different but the proof carries through literally the same). In fact, the equivalence (under volume doubling) of the weak Poincaré inequality under different constants >1 replacing the "2" in the radius of the ball is much easier and the only thing that we will use: this easily implies that the combination of the volume doubling property and the weak Poincaré inequality is invariant under rough isometries.

Another common variation on this inequality is an L^1 version, i.e.

$$\sum \omega |f(w)-\bar{f}| \leqslant Cr \sum \omega |f(w)-f(x)|.$$

The L^1 version is stronger—indeed, since the L^∞ version $|f(w)-\bar{f}|\leqslant 2r\max|f(w)-f(x)|$ is obviously always true, the L^2 version follows from the L^1 version by interpolation.

Definition. We say that a graph G satisfies the *elliptic Harnack inequality* if there exists a constant C such that for any $v\in G$ and $r\geqslant 1$ and any function f harmonic and positive on $B(v,2r)$ one has

$$\max\{f(x): x\in B(v,r)\} \leqslant C\min\{f(x): x\in B(v,r)\}. \tag{7}$$

Definition. We say that a graph G satisfies the *parabolic Harnack inequality* if there exists a constant C such that for any $v\in G$ and $r\geqslant 1$ and any positive function f on $B(v,2r)\times[0,4r^2]$ satisfying

$$f(\cdot,t+1)-f(\cdot,t) = \Delta f(\cdot,t) \tag{8}$$

one has

$$\max\{f(x,t):(x,t)\in B(v,r)\times[r^2,2r^2]\} \leqslant C\min\{f(x,t):(x,t)\in B(v,r)\times[3r^2,4r^2]\}. \tag{9}$$

Clearly, the parabolic Harnack inequality is stronger than the elliptic one. A difficulty in applying this fact is as follows: if the graph is bipartite (say $\mathbf{Z}^d$) then the parabolic Harnack inequality cannot hold—for example, $f(x,t)=\mathbf{P}(R(t)=x)$ satisfies (8) but the right-hand side of (9) is 0 for any $r>1$, since $f(x,t)=0$ whenever $t+\sum_i x_i$ is odd.

However, adding self-loops will allow the graph to satisfy the parabolic Harnack inequality without changing the set of harmonic functions at all. After adding self-loops it is not at all easy to construct examples of graphs satisfying the elliptic Harnack inequality without satisfying the parabolic one. See [BB99] and [GSC05] for some constructions (see also [HSC01]).

THEOREM. (Delmotte) *Let G be an infinite connected graph and assume that there exists some constant c (the "laziness constant") such that $\omega(v,v)>c\sum_w \omega(v,w)$ for all $v\in G$. Then the following are equivalent*:

(i) *G satisfies the volume doubling property and the weak Poincaré inequality*;

(ii) *G satisfies the parabolic Harnack inequality*;

(iii) *the random walk on G satisfies upper and lower Gaussian estimates, namely*

$$\frac{c}{\omega(B(v,\sqrt{t}\,))}e^{-C\delta(v,w)^2/t} \leqslant \mathbf{P}^v(R(t)=w) \leqslant \frac{C}{\omega(B(v,\sqrt{t}\,))}e^{-c\delta(v,w)^2/t}$$

for all $\delta(v,w)\leqslant t$.

Further, for any two clauses, all constants in the first one depend only on the constants in the second one and on the laziness constant.

See [D99]. One of the important consequences of this theorem is that the parabolic Harnack inequality is invariant under rough isometries: as already remarked, the combination of the volume doubling property and the Poincaré inequality is invariant under rough isometries. For the elliptic Harnack inequality, the question of its invariance is still open.

2.2. Definition

A d-dimensional *Euclidean net* is a graph (G,ω) such that $G\subset\mathbf{R}^d$ and

(i) G has bounded weight;

(ii) the inclusion $\iota: G\to\mathbf{R}^d$ is a rough isometry between (G,δ) and $\mathbf{R}^d$;

(iii) $\inf\{|v-w|: v\neq w\in G\}>0$.

We shall mostly be interested in the $\mathbf{R}^d$ distance on G, which we will denote by $d(v,w)$ or $|v-w|$. Likewise, the notation $B(v,r)$ will relate to a ball in the $\mathbf{R}^d$ distance, while a ball in the metric δ will be denoted by B_δ.

2.3. Harnack's inequality

LEMMA 2.1. *A Euclidean net satisfies the elliptic Harnack inequality* (7) *for r sufficiently large.*

Two comments should be made on the formulation of the lemma. The first is that in the definition of the elliptic Harnack inequality (7) we mean balls in the $\mathbf{R}^d$ metric and not in the graph metric. The second is about the constant in (7). We implicitly assume that the constant $C=C(G)$ depends only on the following parameters:

(i) the bounds for ω;

(ii) the constants of the rough isometries ι and f between G and $\mathbf{R}^d$;

(iii) the constants of the rough identities $\iota\circ f$ and $f\circ\iota$;

(iv) the lower bound for $|v-w|$;

(v) d.

We call the aggregation of these parameters the *Euclidean net structure constants*. Whenever we use the notation $C(G)$ we mean a constant depending only on these parameters. Similarly, constants implicit in the o, O and $\approx$ notation are not universal but may depend on the structure constants of the Euclidean net. The phrase "sufficiently large" means "larger than a constant depending on the Euclidean structure constants only".

In §5 we shall apply results obtained up to that point to families of graphs with uniformly bounded structure constants, hence it is important that C does not depend on other properties of G.

Let us stress once again that all the results of §2 are known "folk" results. In particular, Lemma 2.1 for the graph metric is well known. It is the slightly non-standard setting of a Euclidean net that forces us to supply a proof.

Proof. Construct an auxiliary graph G^* with the same vertex set as G and with the weights defined by $\omega_{G^*}(v,v)=\omega_G(v,v)+\omega_G(v)$. In other words, the random walk on G^* is a random walk on G with a probability of $\frac{1}{2}$ to stay at the same spot at each step (additional to any such probability already existing for G). The random walk on G^* is sometimes called the *lazy walk* on G. Clearly $\mathbf{Z}^d$ satisfies the volume doubling property and it is easy to see that $\mathbf{Z}^d$ satisfies the weak Poincaré inequality—every group does, see e.g. [PSC01, Lemma 4.1.1] (the other conditions of Delmotte's theorem are also easy to verify, if you prefer). Since the volume doubling property and the weak Poincaré inequality are preserved by rough isometries, G^* satisfies them. Hence, by Delmotte's theorem, it satisfies the parabolic Harnack inequality with respect to the graph metric δ. Hence it satisfies the elliptic Harnack inequality, and since G and G^* have the same harmonic functions, G also satisfies the elliptic Harnack inequality with respect to δ.

Now, to prove Harnack's inequality for the $\mathbf{R}^d$ metric, cover a ball in the $\mathbf{R}^d$ metric $B(v,r)$ by a constant number of balls in the graph metric $B_\delta(w_i,cr)$ such that $B_\delta(w_i,4cr)\subset B(v,2r)$. It is easy to see that for some $c(G)>0$ this can be done with the number of balls uniformly bounded. From the above discussion we have, for every f

harmonic and positive on $B(v,2r)$, that

$$\max\{f : B_\delta(w_i,2cr)\} \leqslant C \min\{f : B_\delta(w_i,2cr)\}$$

for every i. Therefore, if we have a sequence of k balls such that

$$B_\delta(w_j,2cr) \cap B_\delta(w_{j+1},2cr) \neq \varnothing \quad \text{for } j=1,\dots,k,$$

then we get

$$\max\left\{ f : \bigcup_{j=1}^{k} B_\delta(w_j,2cr) \right\} \leqslant C^k \min\left\{ f : \bigcup_{j=1}^{k} B_\delta(w_j,2cr) \right\}.$$

Now, if r is sufficiently large, then the balls $B_\delta(w_i,2cr)$ form a connected graph with respect to intersection, so we get the claimed result. □

LEMMA 2.2. *Let $\mathcal{E}$ and $\mathcal{D}$ be domains in $\mathbf{R}^d$ such that $\bar{\mathcal{E}} \subset \mathcal{D}$. Then Harnack's inequality holds with respect to $\mathcal{E}$ and $\mathcal{D}$, i.e. for any $v \in \mathbf{R}^d$, any $r > C(\mathcal{E},\mathcal{D},G)$ and any function f positive and harmonic on $(r\mathcal{D}+v)\cap G$:*

$$\max\{f : (r\mathcal{E}+v)\cap G\} \leqslant C(\mathcal{E},\mathcal{D},G) \min\{f : (r\mathcal{E}+v)\cap G\}.$$

Further, if $\mathcal{K}$ is a family of $(\mathcal{E},\mathcal{D})$ with $\operatorname{diam}\mathcal{E}$ *bounded above and $d(\mathcal{E},\partial\mathcal{D})$ bounded below, then all $C(\mathcal{E},\mathcal{D},G)$ are bounded by some constant $C(\mathcal{K},G)$.*

Proof. Use the same covering trick as above. □

Remark. Delmotte's theorem is a convenient reference, but for historical accuracy it should be noted that we could have used a number of earlier references. The fact that a graph roughly isometric to $\mathbf{R}^d$ satisfies the elliptic Harnack inequality was proved concurrently by Delmotte [D97] and Holopainen–Soardi [HS97] (who proved it for the p-Laplacian for any p). An analogous result in the setting of manifolds goes back to Kanai [Ka85], who proved that a manifold roughly isometric to $\mathbf{R}^d$ satisfies the (continuous) Harnack inequality by showing that it follows from a d-dimensional isoperimetric inequality.

2.4. The Green function

Let H be any graph (possibly directed). Let $v,w\in H$ and $S\subset H$. The *Green function* with respect to S is defined by

$$G(v,w;S) = \sum_{t=0}^{\infty} \mathbf{P}^v(R(t)=w \text{ and } R[0,t]\subset S).$$

In other words, $G(v,w;S)$ is the expected number of visits to w before leaving S. If $S=H$ we shall omit it in the notation and write $G(v,w)$. In general there is nothing forcing G to be finite.

If G is finite then it is zero outside S and inside S satisfies

$$\Delta G(\cdot\,,w;S)=-\delta_w, \tag{10}$$

i.e. G is harmonic on $S\setminus\{w\}$ and $\Delta G(w,w)=-1$. These conditions uniquely determine $G(v,w;S)$. The reversability of the random walk (3) translates to a symmetry of G in the form

$$\omega(v)G(v,w;S)=\omega(w)G(w,v;S).$$

Lemma 2.3. *Let H be a d-dimensional Euclidean net.*

(i) *If $d=2$ then H is recurrent and the Green function satisfies*

$$G(v,w;S)\leqslant C(H)\log\operatorname{diam} S. \tag{11}$$

(ii) *If $d\geqslant 3$ then H is transient and the Green function satisfies*

$$\begin{aligned} &G(v,w;S)\leqslant C(H)|v-w|^{2-d} \quad \text{for all } v\neq w,\\ &G(v,v;S)\leqslant C(H). \end{aligned} \tag{12}$$

If $B(x,2r)\subset S$ and r is sufficiently large, then inside $B(x,r)$ a lower bound also holds:

$$\begin{aligned} &G(v,w;S)\approx|v-w|^{2-d} \quad \text{for all } v\neq w\in B(x,r),\\ &G(v,v;S)\approx 1. \end{aligned} \tag{13}$$

Remark. For $d=2$ we have (from recurrence) that $G(v,w)=\infty$ for all v and w. The natural analog of $G(v,w)$ in this case is the harmonic potential of H defined by $a(v,w)=\lim_{r\to\infty}G(v,v;B(v,r))-G(v,w;B(v,r))$. It is possible to show that for any two-dimensional Euclidean net $a(v,w)$ is well defined and $a(v,w)\approx\log|v-w|$, but we will have no use for this fact.

Proof. We start with the case $d\geqslant 3$. Let H^* be the lazy version of H as in the proof of Lemma 2.1. By Delmotte's theorem,

$$ct^{-d/2}e^{-C|v-w|^2/t}\leqslant \mathbf{P}^v_{H^*}(R(t)=w)\leqslant Ct^{-d/2}e^{-c|v-w|^2/t}.$$

Summing this over all t, we get

$$G_{H^*}(v,w)\approx|v-w|^{2-d}.$$

Now, $G_H=\frac{1}{2}G_{H^*}$ because one may couple the walks on H and H^* so that at each step of H the walker on H^* walks the same step and then waits for an expected time of 1. Therefore we get $G_H(v,w)\approx|v-w|^{2-d}$. We will not need the graph H^* again, so all the Green functions henceforth are with respect to H.

Now, $G(v,w;S)\leqslant G(v,w)$ gives us (12). To get the lower bound under the assumption $B(x,2r)\subset S$, take $f=f_{S,w}$ to be a harmonic function on S with $f(v)=G(v,w)$ for all $v\in\partial S$. $G(\cdot,w)-f$ will satisfy (10), which defines $G(\cdot,w;S)$, so they are equal. By the maximum principle, we get

$$f(v)\leqslant \max_{y\in\partial S} f(y)\leqslant Cr^{2-d} \quad \text{for all } v\in S,$$

so we get $G(v,w)\approx|v-w|^{2-d}$ inside a ball $B(v,\lambda r)$ for some constant $\lambda=\lambda(G)$ sufficiently small. Using Harnack's inequality (Lemma 2.2) for the domains $B(0,1)\setminus B(0,\lambda)\subset B(0,2)\setminus B\big(0,\frac{1}{2}\lambda\big)$ proves (13).

The two-dimensional case follows from electrical resistance arguments. See [S94] for background on this topic. The maximum principle shows that $G(v,w)\leqslant G(v,v)$ and the latter is equal to the resistance between v and ∂S. The electrical resistance is preserved (up to a constant) by rough isometries, and so we get

$$G(v,v;S)\leqslant G(v,v;B(v,2\operatorname{diam} S))\approx G_{\mathbf{Z}^2}(0,0;B(0,2\operatorname{diam} S))\approx \log\operatorname{diam} S. \qquad \square$$

LEMMA 2.4. *Let G be a Euclidean net and let $v\in G$ and $r>1$. Then*

$$\mathbf{P}^v(R[0,t]\subset B(v,r))\leqslant C(G)e^{-c(G)t/r^2}.$$

Proof. Let G^* be the lazy version of G, as in Lemma 2.1. Again we use Delmotte's theorem and get, for any λ,

$$\mathbf{P}^w_{G^*}(R(\lfloor\lambda r^2\rfloor)=x)\leqslant C\lfloor\lambda r^2\rfloor^{-d/2} \quad \text{for all } w.$$

Now, we have that $\#B(v,r)\leqslant C(G)r^d$, hence summing gives that, for a sufficiently large constant λ (say integer, so we do not have to carry the $\lfloor\cdot\rfloor$ around it),

$$\mathbf{P}^w_{G^*}(R(\lambda r^2)\in B(v,r))\leqslant \tfrac{1}{2} \quad \text{for all } w.$$

Now, a lazy random walk of length t is an average of simple random walks in the following precise sense:

$$\mathbf{P}^w_{G^*}(R(t)\in B(v,r))=\sum_{i=0}^{t} 2^{-t}\binom{t}{i}\mathbf{P}^w_G(R(i)\in B(v,r)),$$

and therefore there exists some $i(w)\in\{0,...,\lambda r^2\}$ such that

$$\mathbf{P}_G^w(R(i(w))\in B(v,r))\leqslant\tfrac{1}{2}.$$

We now define stopping times T_k by $T_0=0$ and $T_{k+1}=T_k+i(R(T_k))$. Clearly we have $\mathbf{P}_G^w(R[0,T_k]\subset B(v,r))\leqslant 2^{-k}$, and since $T_k\leqslant k\lambda r^2$ the lemma is proved. □

Remark. As above, for historical accuracy we should note that several results predating Delmotte could have been used. The fact that for a graph roughly isometric to $\mathbf{R}^d$ one has $\mathbf{P}_{G^*}^v(R(t)=w)\leqslant Ct^{-d/2}$ follows essentially from Varopoulos [V85] for $d\geqslant 3$ and Carlen–Kusuoka–Stroock [CKS87] for all d. See also the very nice approach of using Faber–Krahn inequalities [G94].

The following lemma basically states that a random walk has a positive probability to hit large objects. We will only use the lemma for simple domains with piecewise smooth boundary, so the requirements of clause (iii) will always be satisfied.

Lemma 2.5. *Let H be a d-dimensional Euclidean net.*

(i) *Let $\mathcal{D}$, $\mathcal{S}$ (start) and $\mathcal{H}$ (hit) be bounded domains in $\mathbf{R}^d$ with $\overline{\mathcal{S}},\mathcal{H}\subset\mathcal{D}$, $\mathcal{H}\neq\mathcal{D}$. Then there exists $C_1(\mathcal{D},\mathcal{S},\mathcal{H},H)$ such that for all $r>C_1$, all $v\in H$ and all $w\in(v+r\mathcal{S})\cap H$, if R is a random walk starting from w then*

$$\mathbf{P}(T(\partial(v+r\mathcal{H}))<T(\partial(v+r\mathcal{D})))>c_1(\mathcal{D},\mathcal{S},\mathcal{H},H). \tag{14}$$

(ii) *If $\overline{\mathcal{S}}\cap\overline{\mathcal{H}}=\varnothing$ and $\mathcal{S}$ is a subset of the unbounded component of $\mathbf{R}^d\setminus\overline{\mathcal{H}}$ then in addition*

$$\mathbf{P}(T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D})))>c_2(\mathcal{D},\mathcal{S},\mathcal{H},H). \tag{15}$$

(iii) *Let $\mathcal{K}$ be a family of triplets $(\mathcal{D},\mathcal{S},\mathcal{H})$ such that for every $x\in\mathcal{S}$ there exists a path γ with $\operatorname{len}\gamma$ bounded above leading from x to $\partial_{\mathrm{cont}}\mathcal{H}$ with $d(\gamma,\partial\mathcal{D})$ bounded below. Then C_1 and c_1 are bounded on $\mathcal{K}$. If, in addition, there exists γ from x to $\partial\mathcal{D}$ with $\operatorname{len}\gamma$ bounded above and $d(\gamma,\partial_{\mathrm{cont}}\mathcal{H})$ bounded below, then c_2 is also bounded on $\mathcal{K}$.*

Proof. Let us start with proving (14). Assume first that $\mathcal{D}=B(0,1)$, $\mathcal{H}=B\big(0,\frac{1}{22}\big)$ and $\mathcal{S}=B\big(0,\frac{1}{2}\big)\setminus B\big(0,\frac{2}{22}\big)$, and that $d\geqslant 3$. Examine the Green function

$$G(w)=G(w,v;v+r\mathcal{D}).$$

Let $w\in(v+r\mathcal{S})\cap H$, let R be a random walk starting from w and let T be its stopping time on $\partial(v+r\mathcal{H})\cup\partial(v+r\mathcal{D})$. Then, since G is harmonic on $(v+r\mathcal{D})\setminus(v+r\mathcal{H})$, we get (from (5)) that

$$G(w)=\mathbf{E}G(R(T)).$$

Set $p=\mathbf{P}(R(T)\in\partial(v+r\mathcal{H}))$, which is the probability we want to estimate. Now, $G(x)=0$ for every $x\in\partial(v+r\mathcal{D})$, while for $x\in\partial(v+r\mathcal{H})$ we have from (12) that $G(x)\leqslant C(H)r^{2-d}$. At w itself we have from (13) that $G(w)\geqslant cr^{2-d}$ for some $c(H)$ and r sufficiently large. We get

$$cr^{2-d}\leqslant G(w)=\mathbf{E}G(R(T))\leqslant pCr^{2-d}, \tag{16}$$

so $p\geqslant c$ for r sufficiently large.

To see that the same holds for $d\leqslant 2$, construct an auxiliary graph $H'=H\times\mathbf{Z}^{3-d}$ weighted so that the projection on H of the random walk on H' is (a time change of) the random walk on H. Then, the fact that there is a positive probability to hit $\partial B\big(v,\frac{1}{22}r\big)\subset H'$ before hitting $\partial B(v,r)\subset H'$, immediately implies the same for H.

We now consider general domains. Let $x\in\mathcal{S}\setminus\mathcal{H}$ and choose an $\varepsilon>0$ and a sequence of points $\{x_i\}_{i=0}^n$ with the following properties:

(i) $x_0=x$ and $B(x_n,\varepsilon)\subset\mathcal{H}$;

(ii) $B(x_i,2\varepsilon)\subset\mathcal{D}$;

(iii) $|x_{i+1}-x_i|\in\big]\frac{6}{22}\varepsilon,\frac{7}{22}\varepsilon\big[$.

It is a simple exercise to show that ε and x_i can always be chosen, and furthermore that both ε and n may be bounded on all $\mathcal{S}\setminus\mathcal{H}$, and for part (iii) of the lemma on all $\mathcal{K}$, i.e. for any x in any $\mathcal{S}\setminus\mathcal{H}$ such that $(\mathcal{D},\mathcal{S},\mathcal{H})\in\mathcal{K}$ we have $\varepsilon(x,\mathcal{H},\mathcal{D})>c(\mathcal{K})$ and $n(x,\mathcal{H},\mathcal{D})<C(\mathcal{K})$. Assume now that r is sufficiently large such that the following conditions are satisfied:

(i) every ball of radius $\frac{1}{22}\varepsilon r$ contains at least one point of H;

(ii) $\partial B(w,s)\subset B\big(w,s+\frac{1}{22}\varepsilon r\big)$ for any $s>0$ and $w\in H$;

(iii) εr is sufficiently large so as to satisfy (16).

Clearly this is an assumption of the type $r>C(\mathcal{D},\mathcal{S},\mathcal{H},H)$. Condition (i) allows us to choose a point $w_i\in H\cap B\big(v+rx_i,\frac{1}{22}\varepsilon r\big)$ for every i. Let $T_0=0$ and

$$T_i=\min\big\{t>T_{i-1}:R(t)\in\partial B\big(w_i,\tfrac{1}{22}\varepsilon r\big)\big\}.$$

We wish to use the case already established for the portion of the random walk after T_{i-1}, with the radius being εr instead of r and the center of the balls being w_i instead of v. We may do this because

$$\begin{aligned}R(T_{i-1})\in\partial B\big(w_{i-1},\tfrac{1}{22}\varepsilon r\big)&\subset B\big(w_{i-1},\tfrac{2}{22}\varepsilon r\big)\\&\subset B\big(v+rx_{i-1},\tfrac{3}{22}\varepsilon r\big)\\&\subset B\big(v+rx_i,\tfrac{10}{22}\varepsilon r\big)\setminus B\big(v+rx_i,\tfrac{3}{22}\varepsilon r\big)\\&\subset B\big(w_i,\tfrac{1}{2}\varepsilon r\big)\setminus B\big(w_i,\tfrac{2}{22}\varepsilon r\big).\end{aligned}$$

However, since $B(w_i,\varepsilon r)\subset B(v+rx_i,2\varepsilon r)\subset v+r\mathcal{D}$, then not hitting $\partial B(w_i,\varepsilon r)$ means staying inside $v+r\mathcal{D}$ and so we get

$$\mathbf{P}(T_i<T(\partial(v+r\mathcal{D}))\,|\,T_{i-1}<T(\partial(v+r\mathcal{D})))>c(H).$$

This immediately gives

$$p\overset{(*)}{\geqslant}\mathbf{P}(T_n<T(\partial(v+r\mathcal{D})))>c^n,$$

where $(*)$ comes from the fact that $\partial B\bigl(w_n,\frac{1}{22}\varepsilon r\bigr)\subset B\bigl(v+rx_n,\frac{3}{22}\varepsilon r\bigr)\subset v+r\mathcal{H}$ and hence a random walk starting at an x outside $\mathcal{H}$ and hitting $\partial B\bigl(w_n,\frac{1}{22}\varepsilon r\bigr)$ must cross $\partial\mathcal{H}$. This proves the direction (14) for $x\notin\mathcal{H}$. The case $x\in\mathcal{H}$ is proved identically but taking the x_i's from x to $\mathcal{D}\setminus\mathcal{H}$. For the direction (15) take the x_i's outside, i.e. with $B(x_n,\varepsilon)\cap\mathcal{D}=\varnothing$ and $B(x_i,2\varepsilon)\cap\mathcal{H}=\varnothing$. $\square$

Lemma 2.6. *With the notation of the previous lemma, if $d\geqslant 3$ then* (14) *and* (15) *hold even if $\mathcal{D}$ is allowed to be unbounded. If $d\leqslant 2$ then only* (14) *holds for unbounded $\mathcal{D}$.*

Proof. The only part not following directly from Lemma 2.5 is the proof of (15) when $d\geqslant 3$. We start with the case where $\mathcal{D}=\mathbf{R}^d$ (so $T(\partial(v+r\mathcal{D}))$ is always ∞), $\mathcal{H}=B(0,1)$ and $\mathcal{S}\cap B(0,\lambda)=\varnothing$, where $\lambda(G)>2$ is some constant sufficiently large that will be fixed later. Let $s>\lambda r$ and

$$p(s)=\mathbf{P}^w(T_{v,s}<T_{v,r}),\quad w\in v+r\mathcal{S}.$$

Let $G(x)=G(x,v;B(v,s))$ be the Green function. From (13) we see that for $x\in\partial B(v,r)$ we have $G(x)\geqslant c(H)r^{2-d}$ if r is sufficiently large. At w itself we have $G(w)\leqslant C(H)(\lambda r)^{2-d}$ and by definition $G|_{\partial B(v,s)}\equiv 0$, so

$$C(\lambda r)^{2-d}\geqslant G(w)=\mathbf{E}G(R(T))\geqslant(1-p)c(H)r^{2-d},$$

so $p\geqslant 1-C\lambda^{2-d}$ and for λ sufficiently large this would be $\geqslant\frac{1}{2}$. Fix λ to be some such constant. We get that $\lim_{s\to\infty}p\geqslant\frac{1}{2}$. Hence

$$\mathbf{P}(T(\partial(v+r\mathcal{H}))=\infty)=\lim_{s\to\infty}\mathbf{P}(T(\partial(v+r\mathcal{H}))>T_{v,s})\geqslant\tfrac{1}{2}\tag{17}$$

and this case is finished. For general $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{H}$, let $B(0,\varrho)$ $(\varrho>1)$ be a sufficiently large ball such that $\mathcal{D}\cap B(0,\varrho)$ contains $\mathcal{S}$, $\mathcal{H}$ and a path between them. Let $\mathcal{D}_2$ be the component of $\mathcal{D}\cap B(0,\lambda\varrho)$ containing $\mathcal{S}$ and $\mathcal{H}$. Then, Lemma 2.5 shows that for r sufficiently large we have

$$\mathbf{P}(T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D}_2)))>c(\mathcal{D}_2,\mathcal{S},\mathcal{H},G).\tag{18}$$

If $R(T(\partial(v+r\mathcal{D}_2)))\notin\partial(v+r\mathcal{D})$, then $R(T(\partial(v+r\mathcal{D}_2)))\in\partial B(v,\lambda\varrho r)$ and so the previous case (rescaled by ϱ) with the strong Markov property shows that

$$\begin{aligned}\mathbf{P}(T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D}))\,|\,T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D}_2)))\\ &=\mathbf{P}(T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D})),T(\partial(v+r\mathcal{D}_2))=T_{v,\lambda\varrho r}\,|\\ &\qquad T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D}_2)))\\ &=\mathbf{E}\mathbf{P}^{R(T_{v,\lambda\varrho r})}(T(\partial(v+r\mathcal{H}))\geqslant T(\partial(v+r\mathcal{D})))\\ &\geqslant \mathbf{E}\mathbf{P}^{R(T_{v,\lambda\varrho r})}(T_{v,\varrho r}=\infty)\\ &\overset{(17)}{\geqslant}\mathbf{E}c(G)=c(G)\end{aligned}$$

and together with (18), we get (15). □

LEMMA 2.7. *Let $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{H}$ be domains in $\mathbf{R}^d$ with $\overline{\mathcal{S}},\overline{\mathcal{H}}\subset\mathcal{D}$ and $\overline{\mathcal{S}}\cap\overline{\mathcal{H}}=\varnothing$. Then there exists $C_2(\mathcal{D},\mathcal{S},\mathcal{H},G)$ such that for all $r>C_2$, all $v\in G$, all $w\in(v+r\mathcal{S})\cap G$ and all $x\in(v+r\mathcal{H})\cap G$, if R is a random walk starting from w then*

$$\mathbf{P}(T(\{x\})<T(\partial(v+r\mathcal{D})))\approx\begin{cases}r^{2-d}, & \text{if } d\geqslant 3,\\ 1/\log r, & \text{if } d=2,\end{cases}$$

where the constants implicit in the $\approx$ may depend on $\mathcal{D}$, $\mathcal{S}$, $\mathcal{H}$ and G. Further, if $\mathcal{K}$ is a family of $(\mathcal{D},\mathcal{H},\mathcal{S})$ triplets satisfying the conditions of Lemma 2.5 (iii), *then C_2 and the implicit constants are bounded on $\mathcal{K}$.*

Proof. Let $\varepsilon=\varepsilon(\mathcal{K},G)$ be sufficiently small. Use Lemma 2.5 to show that the probability to hit a ball of radius $r\varepsilon$ around x is ≈1, and then the same Green function calculations as in that lemma, to show that the probability to hit a point before exiting from a ball containing $\mathcal{D}$ are $\approx r^{2-d}$ for $d\geqslant3$ and $\approx1/\log r$ for $d=2$. □

LEMMA 2.8. *Let G be a d-dimensional Euclidean net, $d\geqslant3$. Let $v\in G$, $s>4r>C(G)$, $A\subset B(v,r)$ and $w\in\partial B(v,2r)$. Then*

$$\mathbf{P}^w(T(A)<T_{v,4r})\approx\mathbf{P}^w(T(A)<T_{v,s}).$$

Further, if $B\subset A$ then

$$\mathbf{P}^w(R(T(A\cup\partial B(v,4r)))\in B)\approx\mathbf{P}^w(R(T(A\cup\partial B(v,s)))\in B).$$

Proof. We shall only show the first estimate, the second one is proved identically. Clearly $\mathbf{P}^w(T(A)<T_{v,4r})\leqslant\mathbf{P}^w(T(A)<T_{v,s})$, so we need to show the other direction. Define stopping times $T_0=0$ and

$$\begin{aligned}T_{2i+1}&:=\min\{t>T_{2i}:R(t)\in\partial B(v,4r)\cup A)\},\\ T_{2i}&:=\min\{t>T_{2i-1}:R(t)\in\partial B(v,2r)\cup\partial B(v,s)\}.\end{aligned}$$

Let I be the first time when $R(T_I)\in A$ (for I odd) or $R(T_I)\in\partial B(v,s)$ (for I even). We consider the process stopped at I. From Lemma 2.6 we see that

$$\mathbf{P}(R(T_{2i})\in\partial B(v,s)\mid I>2i-1)\geqslant \mathbf{E}\mathbf{P}^{R(T_{2i-1})}(T_{v,2r}=\infty)\geqslant c(G). \tag{19}$$

From Harnack's inequality we get that

$$\mathbf{P}^w(R(T_{2i+1})\in A\mid I>2i)=\mathbf{E}\mathbf{P}^{R(T_{2i})}(T(A)<T_{v,4r})\approx \min_{x\in\partial B(v,2r)}\mathbf{P}^x(T(A)<T_{v,4r})$$

and hence this is (up to a constant) independent of i. Hence we get

$$\begin{aligned}\mathbf{P}^w(T(A)<T_{v,s}) &= \sum_{i=0}^{\infty}\mathbf{P}^w(I>2i \text{ and } R(T_{2i+1})\in A)\\ &\leqslant C(G)\sum_{i=0}^{\infty}\mathbf{P}^w(R(T_1)\in A)\mathbf{P}(I>2i)\\ &\overset{(19)}{\leqslant} C(G)\sum_{i=0}^{\infty}\mathbf{P}^w(R(T_1)\in A)(1-c(G))^i\\ &= C(G)\mathbf{P}^w(T(A)<T_{v,4r}).\end{aligned}$$

□

2.5. The discrete Beurling projection in three dimensions

The Beurling projection theorem says that for a two-dimensional Brownian motion starting at 0 the probability to hit a given set $K\subset B(0,1)$ before hitting $\partial B(0,1)$ is larger than the probability to hit its *angular projection*, namely the set $\{|z|:z\in K\}$. In particular, if K is connected and intersects both $\partial B(0,\varepsilon)$ and $\partial B(0,1)$ then the probability to avoid it is maximal when $K=[\varepsilon,1]$, and in this case it may be calculated explicitly from the conformal invariance of Brownian motion and is $\approx\sqrt{\varepsilon}$. A discrete version of this result (up to constants) was achieved by Kesten [K87]. In this section we shall prove a three-dimensional variation of this result, namely the following lemma.

LEMMA 2.9. *Let H be a three-dimensional Euclidean net. Then there exists a constant $C(H)$ such that for all $v\in H$, for all $r>C(H)$ and for all connected sets $A\subset H$ that intersect both $B(v,r)$ and $H\setminus B(v,2r)$ one has*

$$\mathbf{P}^v(R[0,T_{v,4r}]\cap A\neq\varnothing)\geqslant\frac{c}{\log r}.$$

While Kesten's version of Beurling's arguments may be applied to three dimensions without much change, in the setting of Lemma 2.9 the notion of capacity, particularly of Martin capacity, can be used to shorten the argument significantly. We shall first give the relevant definitions, and the proof will follow thereafter.

2.5.1. Martin capacity

Definition. Let H be a countable set and let $K(v,w)$ be some function (the "kernel"). The *capacity* of a set $S\subset H$ with respect to K is defined by

$$\mathrm{Cap}_K(S) := \left(\inf_{\mu(S)=1}\int_S\int_S K(v,w)\,d\mu(v)\,d\mu(w)\right)^{-1}.$$

The infimum here is over all probability measures μ supported on S.

Definition. Let H be a directed graph and let $v\in H$. Then the *Martin capacity* of the graph with respect to v is the capacity with respect to the Martin kernel, defined by

$$K(w,x) := \frac{G(w,x)}{G(v,x)},$$

where G is the Green function.

THEOREM. (Benjamini, Pemantle and Peres) *Let H be a directed graph, let $v\in H$ and let $S\subset H$ be such that the Green function $G(w,x)$ is finite for all $w,x\in S$. Let* Cap *be the Martin capacity with respect to v. Then for any S we have*

$$\tfrac{1}{2}\,\mathrm{Cap}(S)\leqslant \mathbf{P}^v(R[0,\infty[\cap S\neq\varnothing)\leqslant \mathrm{Cap}(S).$$

A nice and simple proof may be found in [BPP95, Theorem 2.2].

Proof of Lemma 2.9. Let $\lambda=\lambda(H)$ be some constant such that every edge of H has length $\leqslant\lambda$. Then A intersects every spherical shell $B(v,s+\lambda)\setminus B(v+s)$, $r\leqslant s\leqslant 2r$. Let $a_i\in A\cap(B(v,r+(i+1)\lambda)\setminus B(v,r+i\lambda))$ for $i=0,...,\lfloor r/\lambda\rfloor$. Let $A^*=\{a_i\}_i$. The lemma will be proved if we show that

$$\mathbf{P}^v(T(A^*)<T_{v,4r})\geqslant \frac{c}{\log r}.$$

Let H' be the directed graph given by taking $\overline{H\cap B(v,4r)}$ and making each point of $\partial B(v,4r)$ a "sink", i.e. a point with the only exit being a self-loop. By definition, $G_{H'}(w,x)=G_H(w,x;B(v,4r))$. We get the equivalent formulation

$$\mathbf{P}^v_{H'}(T(A^*)<\infty)\geqslant\frac{c}{\log r}.$$

We now use the Benjamini–Pemantle–Peres theorem on H'. We get that it is enough to estimate $\mathrm{Cap}(A^*)\geqslant c/\log r$. By the definition of capacity, we need to show that there exists a μ on A^* such that

$$\int_{A^*}\int_{A^*}\frac{G(w,x)}{G(v,x)}\,d\mu(w)\,d\mu(x)\leqslant C\log r. \tag{20}$$

Let μ be the uniform measure on A^*. Then by (13) we have that

$$K(a_i,a_j)=\frac{G(a_i,a_j)}{G(v,a_j)}\leqslant C\frac{|a_i-a_j|^{-1}}{r^{-1}}\leqslant Cr|i-j|^{-1}.$$

Summing gives (20) and the lemma. □

2.6. Intersection probabilities

LEMMA 2.10. *Let H be a Euclidean net of dimension $\leqslant 3$ and let $\varepsilon \in]0,1[$. Let $v\in H$ and $r>0$, and let R^1 and R^2 be random walks starting from vertices v^1 and v^2, $v^i \in B(v,(1-\varepsilon)r)$. Then*

$$\mathbf{P}(R^1[0,T^1_{v^1,\varepsilon r}]\cap R^2[0,T^2_{v,r}]\neq\varnothing) > c_3(\varepsilon,H)$$

if $r>C_3(\varepsilon,H)$.

Proof. We shall only show the case $d=3$—the case $d=2$ is identical and will be left to the reader. Let $\lambda=\lambda(\varepsilon,H)$ be some constant whose value will be fixed later. Let $s\geqslant\varepsilon\lambda$ be some number, $w\in H$ and $\delta>0$. For any $x\in B(w,(1-\delta)s)$, let $\Gamma(y)$ be the probability that a random walk starting from y will hit x before hitting $\partial B(w,s)$, and define $\Gamma(x):=1$. Γ is harmonic on $B(w,s)\setminus\{x\}$ and is 0 on $\partial B(w,s)$. Hence $\Gamma(y)= G(y,x;B(w,s))/G(x,x;B(w,s))$, where G is the Green function. Using (12), (13) and Harnack's inequality (Lemma 2.2), we get

$$\Gamma(y)\geqslant \frac{c(\delta,H)}{|y-x|}, \quad y\in B(w,(1-\delta)s)\setminus\{x\},$$
$$\Gamma(y)\leqslant \frac{C(\delta,H)}{|y-x|}, \quad y\in B(w,s)\setminus\{x\},$$

for s sufficiently large.

Define now $A:=B\big(v^1,\tfrac12\varepsilon r\big)\setminus B\big(v^1,\tfrac14\varepsilon r\big)$ and

$$X := A\cap R^1[0,T^1_{v^1,\varepsilon r}]\cap R^2[0,T^2_{v,r}].$$

For any $x\in A$, the preceding calculation (used once for $w=v^1$, $s=\varepsilon r$ and $\delta=\frac12$, and a second time for $w=v^2$, $s=r$ and $\delta=\frac12\varepsilon$) shows that $\mathbf{P}(x\in X)\approx r^{-2}$. The $\approx$ sign here and below may depend on ε and on the Euclidean net structure constants of H. Rough isometry preserves (up to a constant) volumes of balls and shells, hence if εr is sufficiently large we get $\#A\approx r^3$ and hence $\mathbf{E}\#X\approx r$. Next, we want to calculate $\mathbf{E}(\#X)^2$. For any $x\neq y\in A$ we have

$$\mathbf{P}(x,y\in R^i[0,T^i])\approx r^{-1}|x-y|^{-1}. \tag{21}$$

Indeed, this probability is $\geqslant$ the probability to hit x first (which is $\approx 1/r$) and then to hit y (which is $\approx 1/|x-y|$). On the other hand, it is $\leqslant$ the sum of this probability and its symmetric image. So (21) is explained. This shows that $\mathbf{P}(x,y\in X)\approx r^{-2}|x-y|^{-2}$ and summing over y, we get

$$\begin{aligned}\sum_{y:y\neq x}\mathbf{P}(x,y\in X) &\overset{(*)}{=} \sum_{n=-C(H)}^{\lfloor\log_2 r\rfloor}\sum_{y\in B(x,2^{n+1})\setminus B(x,2^n)}\mathbf{P}(x,y\in X)\\ &\leqslant C(\varepsilon,H)r^{-2}\sum_{n=-C(H)}^{\lfloor\log_2 r\rfloor}4^{-n}\#(B(x,2^{n+1})\setminus B(x,2^n))\\ &\overset{(**)}{\leqslant} C(\varepsilon,H)r^{-2}\sum_{n=-C(H)}^{\lfloor\log_2 r\rfloor}2^n\\ &\leqslant C(\varepsilon,H)r^{-1},\end{aligned}$$

where $(*)$ comes from the fact that H is separated in $\mathbf{R}^3$ hence $|x-y|\geqslant c(H)$, and $(**)$ uses again the fact that rough isometry preserves volumes. Summing over x, we get

$$\mathbf{E}(\#X)^2\leqslant \#A\cdot C(\varepsilon,H)r^{-1}+\sum_{x\in A}\mathbf{P}(x\in X)\leqslant C(\varepsilon,H)r^2.$$

Hence the well-known inequality $\mathbf{P}(\#X>0)\geqslant(\mathbf{E}\#X)^2/\mathbf{E}(\#X)^2$ finishes the lemma. □

LEMMA 2.11. *Let G be a Euclidean net of dimension $d\leqslant 3$ and let $v^1,v^2\in G$. Let R^i, $i=1,2$, be random walks starting from v^i and stopped on $\partial B(v^1,r)$. Then*

$$\mathbf{P}(R^1\cap R^2=\varnothing)\leqslant C(G)\left(\frac{|v^1-v^2|}{r}\right)^{c(G)}.$$

Proof. Let $a_j:=2^j|v^1-v^2|$ and assume without loss of generality that $r=a_n$ for some integer n. Let T^i_j be the stopping time of R^i on the sphere $\partial B(v^1,a_j)$. Examine the events

$$\mathcal{E}_j:=\{R^1[T^1_j,T^1_{j+1}]\cap R^2[T^2_j,T^2_{j+1}]\neq\varnothing\}.$$

For $j>C(G)$ we have that $\partial B(v^1,a_j)\subset B\big(v^1,\frac{2}{3}a_{j+1}\big)$, and we may use Lemma 2.10 with $\varepsilon=\frac{1}{3}$ and the strong Markov property to get

$$\mathbf{P}(\mathcal{E}_j\mid R^1(T^1_j)\text{ and }R^2(T^2_j))\geqslant c_3\big(\tfrac{1}{3},G\big)\quad\text{for all }\ j>C(G).$$

However, $\mathcal{E}_j$ may depend on $\mathcal{E}_0,...,\mathcal{E}_{j-1}$ only through $R^i(T^i_j)$, so we get

$$\mathbf{P}(\mathcal{E}_j\mid\mathcal{E}_0,...,\mathcal{E}_{j-1})\geqslant c(G).$$

And hence

$$\mathbf{P}(R^1\cap R^2=\varnothing)\leqslant\mathbf{P}\left(\bigcap_{j=C}^{n-1}\neg\mathcal{E}_j\right)\leqslant(1-c(G))^{n-C}\leqslant C\left(\frac{|v^1-v^2|}{r}\right)^{c(G)}$$

and the lemma is proved. □

This basic proof method is known as the "Wiener shell test".

Remark. Given Theorem 3 below (p. 77), it might be tempting to conjecture that $c(G)$ is in effect ξ, the non-intersection exponent of d-dimensional Brownian motion. However, this is not true. Indeed, the intersection exponent is a "conformally invariant" property rather than a metric property. Unfortunately, I do not know any example sufficiently simple to explain here.

LEMMA 2.12. *Let G be a Euclidean net, let $H\subset\mathbf{R}^d$ be a half-space and let $v\in G\setminus\partial H$. Let R be a random walk starting from v. Then*

$$\mathbf{P}(T_{v,r}<T(\partial H))\leqslant C(G)\left(\frac{d(v,\partial H)}{r}\right)^{c(G)}\quad \textit{for all } r>2d(v,\partial H).$$

Proof. Let $s>2d(v,\partial H)$ and examine a random walk R starting from any point in $\partial B(v,s)$ and stopped on $\partial B(v,2s)$. We use Lemma 2.5 with $\mathcal{D}=B(0,2)$, $\mathcal{S}=B\big(0,\frac{3}{2}\big)$ and $\mathcal{H}=((H-v)/s)\cap\mathcal{D}$. If $s>C(G)$ then $\partial B(v,s)\subset v+s\mathcal{S}$ and $\mathcal{H}$ is non-empty, so the lemma applies to our R. We get that the probability of R to hit $\partial(v+s\mathcal{H})\subset\partial H\cap B(v,2s)$ before hitting $\partial B(v,2s)$ is $\geqslant c_1(\mathcal{D},\mathcal{S},\mathcal{H},G)$ if $s>C_1(\mathcal{D},\mathcal{S},\mathcal{H},G)$. Further, if s is sufficiently large, then the family of possible $\mathcal{H}$'s satisfies the requirements of clause (iii) of Lemma 2.5, and these c's and C's are bounded. The Wiener shell test now gives the lemma. □

Again, it is not necessarily true that $c=1$ as in $\mathbf{Z}^d$. This is only true with additional assumptions, such as isotropicity, see Theorem 2 below (p. 73). A counterexample may be constructed as follows. Let $\phi:\overline{\mathbf{H}}\to\mathbf{C}$ be defined by $\phi(re^{i\theta})=re^{2i\theta}$, where $\mathbf{H}$ is as usual the upper half-space $\{z:\operatorname{Im} z>0\}$. Let $G:=\phi(\mathbf{Z}^2\cap\overline{\mathbf{H}})$ and identify $\phi(n)$ and $\phi(-n)$ so that G contains edges from n to both $\sqrt{n^2+1}\,e^{\pm 2i\arctan 1/n}$. Then it is easy to see that G is a Euclidean net, while the escape probability from, say, $v=i\sqrt{2}$ to $\partial B(v,r)$ without hitting $\partial\mathbf{H}$ is the same as the escape probability of a random walk on $\mathbf{Z}^2$ from a corner, which are well known to be $\approx r^{-2}$. (In this paragraph i was, of course, the imaginary unit. From now on, the letter i will only be used to denote indices.)

An argument identical to that of Lemma 2.12 works for any polyhedron.

LEMMA 2.13. *Let G be a Euclidean net and let $Q\subset\mathbf{R}^d$ be a polyhedron. Let $1<r_1<r_2<s$ be some numbers and let $v\in G$ satisfy $d(v,sQ)\leqslant r_1$. Let R be a random walk starting from v. Then*

$$\mathbf{P}(T_{v,r_2}<T(sQ))\leqslant C(Q,G)\left(\frac{r_1}{r_2}\right)^{c(Q,G)}.$$

We omit the proof.

The next lemma is technical and is only here for completeness. In fact all the graphs we will consider in this paper have no "dangling ends", and it is straightforward to see that for every $v^1 \neq v^2$ one can construct disjoint paths going in opposite directions (so clause (ii) of the lemma is satisfied).

LEMMA 2.14. *Let G be a Euclidean net of dimension $\geqslant 2$, let $0<\varepsilon<\frac{1}{2}$ and $s>0$. Then there exists a $\varkappa=\varkappa(\varepsilon,s,G)$ such that for any $v, v^1 \neq v^2 \in G$ with $|v-v^i| \leqslant s$, one of the following holds*:

(i) *there are no two disjoint paths starting from v^i and ending outside $B(v,\varkappa)$*;

(ii) *for any $r \geqslant \varkappa$ there exist two disjoint simple paths $\gamma^i \subset B(v,r)$ starting from v^i and ending in some w^i such that*

$$B(w^i,(1-\varepsilon)r) \cap \gamma^{3-i} = \varnothing, \quad w^i \in B(v,r) \setminus \overline{B(v,(1-\varepsilon)r)}. \tag{22}$$

Proof. Let $\lambda=\lambda(G)$ be such that any ball in $\mathbf{R}^d$ with radius $\geqslant\lambda$ contains at least one point of G, and such that no edge of G has length $>\lambda$. Let μ be such that for any $x,y\in G$ with $|x-y|\leqslant 4\lambda$ there is a path γ from x to y, $\gamma\subset B(x,\mu)$. Now take any point $x\in G$ and any direction θ and construct an infinitely long "ray" $x\in R\subset G$ by taking a sequence of tangent balls on the half-line in direction θ, taking a point of G in every ball and connecting them by short paths as above. The result is that R is contained in the open infinite cylinder whose basis is a $(d-1)$-dimensional ball of radius $\mu+\lambda$ orthogonal to θ, centered at x. Actually, R is contained only in the half-cylinder starting μ before x.

It is now clear that if $|v^1-v^2|>2\mu$ then we may simply extend such rays in the directions $\pm(v^1-v^2)$, and they will not intersect. This allows us to prove the case $s>2\mu$ given the case $s=2\mu$—this is a simple geometric exercise (it is enough to define $\varkappa(s,\varepsilon)=\big(s+C(G)+\varkappa\big(\frac{1}{10},2\mu\big)\big)/\varepsilon$, but the precise value is of no importance). Hence we will assume $s=2\mu$.

Define now $\nu:=2\mu+2\lambda$ and impose the condition $\varkappa\geqslant 16\nu$. Let now δ^1 and δ^2 be two disjoint paths starting from v^1 and v^2, respectively, and going to a distance of $\varkappa$. The lemma will be proved once we construct γ^i satisfying (ii). Let $x^i=\delta^i(j^i)$ be the first point of δ^i outside $B(v,16\nu)$. A simple exercise in plane geometry shows that, if $|v-x^i|\geqslant\alpha$, then one can find an η satisfying that $|\langle\eta,v\rangle-\langle\eta,x^i\rangle|\geqslant\frac{1}{2}\alpha$ for $i=1,2$. Applying this in our case gives $|\langle\eta,v\rangle-\langle\eta,x^i\rangle|\geqslant 8\nu$. Define now

$$I^i=\Big[\min_{j\leqslant j^i}\langle\eta,\delta^i(j)\rangle,\max_{j\leqslant j^i}\langle\eta,\delta^i(j)\rangle\Big].$$

In the sequel, we will say about a (half-)cylinder $\mathcal{C}$ in a direction orthogonal to η that it is *in elevation* e if

$$\min_{v\in\mathcal{C}}\langle\eta,v\rangle=\langle\eta,x\rangle+e.$$

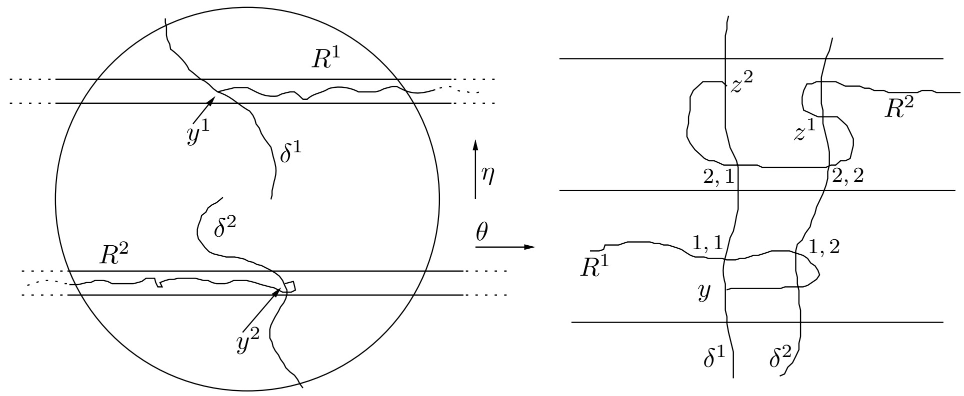

Figure 2. On the left, Case 1, the case of $|I^1 \cap I^2|$ small. On the right, Case 2. The points denoted by i, j are $\delta^j(k^{i,j}) = R^i(m^{i,j})$.

Case 1. Consider the case $|I^1 \cap I^2| < 6\nu$. In this case one of the I^i's—without loss of generality, we may assume I^1—contains $\langle \eta, v\rangle + [\nu, 8\nu]$ and I^2 contains $\langle \eta, v\rangle + [-8\nu, -\nu]$ (otherwise, replace η by $-\eta$). Therefore $\langle \eta, v\rangle \pm [7\nu, 8\nu]$ is contained in $I^1 \setminus I^2$ and $I^2 \setminus I^1$, respectively. Let $y^i = \delta^i(k^i)$ be such that

$$\big| |\langle \eta, y^i\rangle - \langle \eta, v\rangle| - \tfrac{15}{2}\nu \big| \leqslant \tfrac{1}{2}\lambda.$$

Such y^i always exist since every edge in G has length $\leqslant \lambda$. Let θ be some vector orthogonal to η. Let R^1 (resp. R^2) be an infinite path starting from w^1 (resp. w^2) and contained in the half-cylinder of radius $\frac{1}{2}\nu$ in the direction θ (resp. $-\theta$) and in elevation 7ν (resp. -8ν). Since the cylinders are disjoint so are the R^i's. See Figure 2, left. Since $I^1 \cap \langle \eta, v\rangle + [-8\nu, -7\nu] = \varnothing$, we get that $\delta^1 \cap R^2 = \varnothing$, and symmetrically we have $\delta^2 \cap R^1 = \varnothing$. Let l^i be the first time such that $R^i(l^i) \notin B(x, r)$. Then, we define

$$\gamma^i = \mathrm{LE}(\delta^i[0, k^i] \cup R^i[0, l^i - 1]).$$

Clearly $\delta^i \cup R^i$ are disjoint and the operation of taking LE conserves this, hence the γ^i's are simple and disjoint. Condition (22) now follows from simple plane geometry (recall that LE conserves the end points) if only $\varkappa$ is sufficiently large, so this case is finished.

Case 2. We now assume that $|I^1 \cap I^2| \geqslant 6\nu$. Without loss of generality, we may assume that for some $a \geqslant \langle \eta, v\rangle + \nu$ we have $[a, a + 2\nu] \subset I^1 \cap I^2$ (if not, replace η by $-\eta$). Let $y \in \delta^1$ be the first point that satisfies $\big|\langle \eta, y\rangle - \big(a + \frac{1}{2}\nu\big)\big| \leqslant \frac{1}{2}\lambda$ and let $z^i \in \delta^i$ be such that $\big|\langle \eta, z^i\rangle - \big(a + \frac{3}{2}\nu\big)\big| \leqslant \frac{1}{2}\lambda$.

This time define θ to be the projection of $z^1 - z^2$ into the (hyper)plane orthogonal to η (if $z^1 - z^2$ is collinear with η, just pick an arbitrary vector θ orthogonal to η). Let

$\mathcal{C}^2$ be a cylinder of radius $\frac{1}{2}\nu$ in the direction θ with elevation $a+\nu$ such that both z^1 and z^2 are in the middle cylinder of side length λ. Let R^2 be an infinite path in a half of $\mathcal{C}^2$ in the direction θ starting from z^2 and containing z^1. Let $\mathcal{C}^1$ be a similar cylinder in elevation a containing y in its middle, and let R^1 be an infinite path in the half of $\mathcal{C}^1$ in the direction $-\theta$ starting from y. Also let R^1 be simple, which can be done, say by taking LE. See Figure 2, right. Let $k^{2,i}$ be the first time when $\delta^i(k^{2,i})\in R^2$ and let $m^{2,i}$ be such that $R^2(m^{2,i})=\delta^i(k^{2,i})$. Let

$$m^{1,i} := \max\{m : R^1(m) \in \delta^i[0, k^{2,i}[\,\} \quad \text{and} \quad k^{1,i} := \min\{k : \delta^i(k) = R^1(m^{1,i})\}.$$

By definition, R^1 always intersects $\delta^1[0, k^{2,1}[$, hence $m^{1,1}$ and $k^{1,1}$ are well defined. If R^1 does not intersect $\delta^2[0, k^{2,2}[$, we consider $m^{1,2}$ to be $-\infty$ and $k^{1,2}$ to be undefined. As before, define l^i to be the first time when $R^i(l^i)\notin B(v,r)$.

We can now define γ^i by connecting a δ^i to an R^i according to the relation between $m^{1,1}$ and $m^{1,2}$. In formulas, if $m^{1,1}<m^{1,2}$ define

$$\gamma^1 := \mathrm{LE}(\delta^1[0, k^{2,1}[\cup R^2[m^{2,1}, l^2-1]) \quad \text{and} \quad \gamma^2 := \mathrm{LE}(\delta^2[0, k^{1,2}[\cup R^1[m^{1,2}, l^1-1]),$$

while if $m^{1,1}>m^{1,2}$ (which includes $m^{1,2}=-\infty$) define

$$\gamma^1 := \mathrm{LE}(\delta^1[0, k^{1,1}[\cup R^1[m^{1,1}, l^1-1]) \quad \text{and} \quad \gamma^2 := \mathrm{LE}(\delta^2[0, k^{2,2}[\cup R^2[m^{2,2}, l^2-1]).$$

In both cases it is easy to verify that $\gamma^1\cap\gamma^2=\varnothing$ and that (ii) holds, if $\varkappa$ is sufficiently large, just like in Case 1. Hence the lemma is concluded. □

3. Isotropic graphs

3.1. Preliminaries

In this chapter we will need to compare random walk and Brownian motion. For the definition and basic properties of Brownian motion see any standard text book, e.g. [RW94] or [B95].

To avoid confusion with the use of the letter B for a ball in a metric space, we will denote Brownian motion by W, giving homage to Wiener, even though he seems to have only been interested in the one-dimensional case.

The equivalent of the stopping times $T(X)$ and $T_{v,r}$ will be denoted by S, i.e. $S(X)$ is the time when the Brownian motion hits X for the first time, and $S_{v,r}=S(\partial_{\mathrm{cont}}B(v,r))$. Similarly, we shall use $S^i(X)$ and $S^i_{v,r}$ when we have more than one Brownian motion involved.

We will also need, in a few places, the following *Hausdorff distance from a subset*, defined by

$$\subset_{\mathrm{Haus}}(A,B) := \sup_{a\in A} d(a,B) = \inf_{C\subset B} d_{\mathrm{Haus}}(A,C).$$

$\subset_{\mathrm{Haus}}$ is monotone in the sense that if $A_1\subset A_2$ and $B_1\subset B_2$ then

$$\subset_{\mathrm{Haus}}(A_1,B_2) \leqslant \subset_{\mathrm{Haus}}(A_2,B_1).$$

3.2. Background on the non-intersection exponent

We shall need some known results about the non-intersection exponent ξ_d, so let us start with a quick survey of this topic (mostly developed by Lawler and coauthors).

1. Let x and y be two points on $\partial B(0,1)$ and let W^x and W^y be independent Brownian motions in $\mathbf{R}^d$ starting from x and y, respectively, and stopped when hitting $\partial B(0,r)$. Define the non-intersection probability by

$$P(r) := \max_{x,y} p(x,y,r), \quad \text{where } p(x,y,r) := \mathbf{P}(W^x \cap W^y = \varnothing).$$

The scaling invariance of Brownian motion and the strong Markov property easily give that P is submultiplicative in r (i.e. $P(rs)\leqslant P(r)P(s)$, or $\log P$ is subadditive in $\log r$) and we get

$$P(r) = r^{-\xi_d+o(1)}, \quad \text{as } r\to\infty. \tag{23}$$

ξ_d is the well-known *non-intersection exponent*. The invariance of Brownian motion to rotations, scaling and translations allows us to map $\vec{1}:=(1,0\,...,0)$ and $-\vec{1}$ to the x and the y where the maximum occurs and conclude that $p(\vec{1},-\vec{1},r)=r^{-\xi_d+o(1)}$.

We will also need a generalization of this quantity: let W^{x_i} $(i=1,...,k)$ and W^{y_i} $(i=1,...,l)$ be independent Brownian motions in $\mathbf{R}^d$ starting from $x_1,...,x_k$ and $y_1,...,y_l$ in $\partial B(0,1)$, respectively, and stopped on $\partial B(0,r)$. Define equivalently

$$P(k,l,r) := \max_{\substack{x_1,...,x_k\\ y_1,...,y_l}} p(x_1,...,x_k,y_1,...,y_l,r), \tag{24}$$

where

$$p(x_1,...,x_k,y_1,...,y_l,r) := \mathbf{P}\biggl(\biggl(\bigcup_{i=1}^k W^{x_i}\biggr)\cap\biggl(\bigcup_{i=1}^l W^{y_i}\biggr)=\varnothing\biggr).$$

Again, submultiplicativity shows that $P=r^{-\xi_d(k,l)+o(1)}$, and $\xi_d(k,l)$ is called the *k,l-non-intersection exponent*. With this notation, $\xi_d=\xi_d(1,1)$. A simple "choosing the best point" argument shows that the maximum in (24) is achieved, up to a factor $\leqslant kl$, when all the x_i's are the same and all the y_i's are the same. And invariance again shows us that

$$p(\underbrace{\vec{1},...,\vec{1}}_{k},\underbrace{-\vec{1},...,-\vec{1}}_{l},r)=r^{-\xi_d(k,l)+o(1)}.$$

These ξ's are non-trivial only in dimensions 2 and 3. In dimension 1 it follows from the "gambler ruin problem" that $\xi_1(k,l)=k+l$, while in dimensions $\geqslant 4$ Brownian motions almost surely do not intersect [DEK50], so $\xi\equiv 0$. See [L91] for a more detailed explanation of these facts. Hence from now on we will only relate to dimensions 2 and 3.

2. In [BL90a] it was shows that the same $\xi(k,l)$ hold for the equivalent problem for random walks (see also [CM91]). Other relevant variations use Brownian motions (or random walks) W_i^* with fixed length t, or with the length being an exponential variable with expectation t ("a random walk with killing rate $1/t$"). In either case,

$$\mathbf{P}\biggl(\biggl(\bigcup_{i=1}^{k} W_i^x\biggr)\cap\biggl(\bigcup_{i=1}^{l} W_i^y\biggr)=\varnothing\biggr)=t^{-\xi_d(k,l)/2+o(1)}. \tag{25}$$

For example, notice that, if τ_R is the stopping time when W exits $\partial B(0,r)$, then the probability that either $\tau_R>Cr^2\log r$ or $\tau_R<cr^2/\log r$ are negligible, which explains (25).

3. In [L89] it was shown that $\xi_d(2,1)=4-d$. Very roughly, the proof uses the fact that two random walks starting from the same point can be thought of as one bidirectional walk, which allows one to "reduce one parameter" and get an estimate for the probability. We remark that a similar technique was used in [L99, §12.5] to calculate some intersection exponents for combinations of random walks and loop-erased random walks.

4. In [BL90b] it was shown that the $\xi(k,l)$ are strictly increasing, and in particular that $\xi_3(1,1)<1$. The proof uses the Wiener shell test, somewhat like the techniques we will use in §4.

5. In [L96a] the estimate (23) was improved to

$$P(x,y,r)\approx r^{-\xi(1,1)}, \tag{26}$$

i.e. the error was shown to be in a constant only (for better comparison, write $P(x,y,r)=r^{-\xi(1,1)+O(1/\log r)}$). Roughly, this follows by proving "supermultiplicativity" in the sense that $P(rs)\geqslant cP(r)P(s)$. This, in turn, follows after proving that two Brownian motions conditioned not to intersect will also be quite far along the path and in their end points. In [L96b] this result was extended to simple random walk via the so-called Skorokhod embedding, a coupling of Brownian motion and random walk on the same probability space so as to be quite close.

The analog of [L96a] for general $\xi(k,l)$ was proved in [L98], while the analog of [L96b] is [LP00]. See also [LSW02b].

6. Both [L96a] and [L96b] used the estimate (26) to prove the existence of many cut times or cut points for random walk, using relatively straightforward second-moment methods. Since $\xi_3(1,1)<1$, we get that the Hausdorff dimension of the cut points is strictly greater than 1, which implies that the set of cut points of Brownian motion is hittable by a second Brownian motion meaning that the hitting probability is positive. See [L99, §12.4] for the corresponding calculation for random walk.

7. While we will not use it, it is impossible not to mention that in dimension 2 there is a precise formula for $\xi_2(n,k)$, conjectured by Duplantier and Kwon [DK88] and proved by Lawler, Schramm and Werner [LSW01a], [LSW01b], [LSW02a]. Both the heuristic arguments and the final proof depend crucially on the Riemann conformal mapping theorem and are therefore specifically two-dimensional.

3.3. Definition

Let G be a d-dimensional Euclidean net. Let $v\in G$ and $r>0$. Let A be a $(d-1)$-dimensional spherical simplex (since we are only interested in $d=2,3$ we have in fact an arc or a spherical triangle) in $\partial_{\mathrm{cont}}B(v,r)$. Let $|A|$ be the $(d-1)$-volume of A normalized so that $|\partial_{\mathrm{cont}}B(v,r)|=1$. We wish to define discrete versions of A. For this purpose, identify each edge (w,x) of G with the linear segment in $\mathbf{R}^d$ between the two vertices, and say that $w\in A^-$ if $w\in\partial B(v,r)$ and all edges (w,x), $x\in B(v,r)$, intersect A. Say that $w\in A^+$ if $w\in\partial B(v,r)$ and some edge (w,x) intersects A. Any set A^* between A^- and A^+ will be called a *discrete version* of A. Denote by $p_{A^*}=\mathbf{P}^v(R(T_{v,r})\in A^*)$. We call G *isotropic* if

$$|p_{A^*}-|A||\leqslant Kr^{-\alpha} \quad \text{for all } v,\ r,\ A \text{ and } A^*. \tag{27}$$

Here $K>0$ and $\alpha>0$ are parameters of G, so it would be more precise to call G (d,α,K)-isotropic. We will rarely need to do so, though. As in the previous chapter, when we write $C(G)$ we mean a constant that depends only on the isotropicity parameters d, α and K, and the Euclidean net structure constants (see p. 44), but not on other properties of G. Together we call these numbers the *isotropicity structure constants*.

We have not defined whether we are talking about an open, closed or other simplex because by expanding or contracting slightly it is obvious that if (27) holds for one than it holds for any and all. We also remark that by examining triangles intersecting no edge of G, it is obvious that $\alpha\leqslant d-1$, and if G is a grid then $\alpha\leqslant 1$. This last inequality is tight: it is possible to show that the grid $\mathbf{Z}^d$ is isotropic with $\alpha=1$, though we will have no use for this fact. It would be interesting to construct an example in $d \geqslant 3$ of an isotropic graph with $\alpha > 1$, even if one weakens the definition to require that (27) holds only for a specific choice of discrete version of A.

3.4. Coupling with Brownian motion

In this section we shall show how to couple a random walk on G with a Brownian motion on $\mathbf{R}^d$. This will be the main tool for using isotropic graphs and indeed, it is probably possible to define isotropic graphs via the coupling. However, we will need some specific properties of the coupling (see below) that are cumbersome to formulate.

We will construct the coupled walk and motion by considering a random walk R on G and constructing an appropriate Brownian motion W. Let therefore R be given and define inductively a sequence of stopping times: $\tau_0 = 0$ and

$$\tau_i := \min\{t > \tau_{i-1} : |R(t) - R(\tau_{i-1})| > r_i\}, \quad r_i := i^{4/\alpha}. \tag{28}$$

The reason behind the choice of r_i will become evident later on, during the proof of Lemma 3.2—we remark only that the connection between 4 and the dimension d is $4 \geqslant 2(d-1)$. Construct now fixed divisions Δ_i of the sphere $\mathbf{S}^{d-1}$ into $D_i := \lfloor r_i^{\alpha/2} \rfloor + 4$ disjoint spherical simplices of $(d-1)$-normalized volume $\approx 1/D_i$ and diameter $\approx D_i^{-1/(d-1)}$ (associate the boundaries of the simplices with them as you please, this is not important). In two dimensions one may just take Δ_i to be a collection of (half-closed half-open) arcs of length $1/D_i$. In three dimensions it is an easy geometric exercise to show that such a "triangulation" exists, knowing only that D_i is $\geqslant 4$. For every $\delta \in \Delta_i$ define([2]) δ^* which, unlike δ, may depend on v and on the walk up to $R(\tau_{i-1})$, to be a discrete version of δ such that the δ^*'s cover $\partial B(R(\tau_{i-1}), r_i)$ and are disjoint, i.e. if $\delta_1 \neq \delta_2 \in \Delta_i$ then $\delta_1^* \cap \delta_2^* = \varnothing$. Define,

$$p_{i,\delta} := \mathbf{P}(R(\tau_i) \in \delta^* \mid R(\tau_{i-1})). \tag{29}$$

We get from (27) that

$$|p_{i,\delta} - |\delta|| \leqslant K r_i^{-\alpha}. \tag{30}$$

Define therefore $\eta_i := \min_\delta p_{i,\delta}/|\delta|$ and get

$$1 \geqslant \eta_i \geqslant 1 - CKr_i^{-\alpha/2}.$$

We can now construct W, and we shall do so in parts, in parallel with times σ_i which would be the analogs of the τ_i's. Define $W(0) := v$ and $\sigma_0 := 0$. Assume that $R(\tau_i) \in \delta^*$.

([2]) This definition is not unique, but everything we will do will not depend on the choice of which "boundary vertex" to associate with which δ^*. If one prefers a uniquely defined coupling, just order Δ_i and then associate each boundary vertex with the δ^* first in this order.

Throw a random independent coin X_i with probability $\eta_i|\delta|/p_{i,\delta}$ for 1. The definition of η_i ensures that this number is $\in[0,1]$. If $X_i=1$, define W_i' to be a Brownian motion starting from 0 and conditioned to exit $B(0,r_i)$ at $r_i\delta$. If $X_i=0$, let W_i' be an unconditioned Brownian motion. In both cases define σ_i' to be the time when W_i' exits $B(0,r_i)$. Finally define $\sigma_i=\sigma_{i-1}+\sigma_i'$, and W on the interval $]\sigma_{i-1},\sigma_i]$ by

$$W(t):=W(\sigma_{i-1})+W_i'(t-\sigma_{i-1}).$$

LEMMA 3.1. *The W constructed above is a regular Brownian motion.*

Proof. Since $\mathbf{E}\sigma_i'>c>0$ and they are independent, we get that almost surely

$$\sum_i \sigma_i=\infty,$$

and hence W is an almost surely well-defined function $[0,\infty[\to\mathbf{R}^d$. Now compare W to a regular Brownian motion W^*. Let σ_i^* be stopping times defined by

$$\sigma_i^*=\inf\{t>\sigma_{i-1}^*:W^*(t)\notin B(W^*(\sigma_{i-1}^*,r_i))\}.$$

Using the strong Markov property [RW94, p. 21] inductively gives that

$$W^*(t-\sigma_{i-1}^*)-W^*(\sigma_{i-1}^*)$$

is distributed like a Brownian motion starting from 0 and stopped when exiting $B(0,r_i)$. On the other hand, it follows from the definition that each W_i' has probability $\eta_i|\delta|$ to be a Brownian motion conditioned to hit $r_i\delta$ (for every $\delta\in\Delta_i$) and probability $1-\eta_i\left(\sum_i p_i\right)$ to be unconditioned, hence W_i' is also a regular Brownian motion starting from 0 and stopped on $\partial B(0,r_i)$. Hence $W[0,\sigma_i]\sim W^*[0,\sigma_i^*]$ for all i. Taking limit as $i\to\infty$ shows that $W\sim W^*$. □

LEMMA 3.2. *Let G be an isotropic graph and $v\in G$. Let R and W be the coupled walk and motion starting from v. Let r_i, τ_i and σ_i be as in the definition of the coupling* (28). *Then*

$$\mathbf{P}(\textit{there exists } j\leqslant i:|R(\tau_j)-W(\sigma_j)|\geqslant\lambda r_i)\leqslant C(G)\exp(-c(G)\lambda)$$

for any $\lambda>0$.

Proof. We use the notation X_i from the definition of the coupling. If $X_j=1$ for some j, then we get

$$R(\tau_{j-1})-R(\tau_j)\in r_j\delta+B(0,C(G))\quad\text{and}\quad W(\sigma_{j-1})-W(\sigma_j)\in r_j\delta.$$

Hence

$$|R(\tau_{j-1})-R(\tau_j)-W(\sigma_{j-1})+W(\sigma_j)| \leqslant Cr_j^{1-\alpha/2(d-1)}+C(G),$$

and, since $d\leqslant 3$, we may simply write $\leqslant Cr_j^{1-\alpha/4}+C$. Summing, we get that if $X_j=1$ for all $j\leqslant i$ then

$$|R(\tau_i)-W(\sigma_i)| \leqslant C\sum_{j=1}^{i} j^{4/\alpha-1}+C \leqslant C(G)i^{4/\alpha} = C(G)r_i.$$

Hence we need to estimate

$$\Sigma := \sum_{j:X_j=0} |R(\tau_{j-1})-R(\tau_j)-W(\sigma_{j-1})+W(\sigma_j)| \leqslant \sum_{j:X_j=0} 2r_j+C.$$

Divide this sum into blocks

$$\Sigma_k := \sum_{\substack{j:X_j=0 \\ 2^{k-1}<r_j\leqslant 2^k}} 2r_j+C.$$

Now, each Σ_k contains $\leqslant C(G)2^{k\alpha/4}$ summands, and each summand is zero with probability $\geqslant 1-C(G)2^{-k\alpha/2}$ independently, so a very rough estimate gives

$$\mathbf{P}(\Sigma_k > \lambda 2^k) \leqslant C(G)\exp(-c(G)\lambda).$$

Define $l:=\lceil \log_2 r_i \rceil$ and sum over k from 0 to l to get

$$\begin{aligned}\mathbf{P}(\Sigma > \lambda r_i) &\overset{(*)}{\leqslant} \mathbf{P}(\text{there exists } k : \Sigma_k > c\lambda 2^{(l+k)/2}) \\ &\leqslant C(G)\sum_k \exp(-c(G)\lambda 2^{(l-k)/2}) \leqslant C(G)\exp(-c(G)\lambda).\end{aligned}$$

Inequality $(*)$ comes from the fact that if $\Sigma_k \leqslant \lambda 2^{(k+l)/2}$ for all k then

$$\Sigma = \Sigma_0+...+\Sigma_l \leqslant (2+\sqrt{2}\,)\lambda 2^l \leqslant 2(2+\sqrt{2}\,)\lambda r_i,$$

so one may take $c=1/2(2+\sqrt{2}\,)$ on the right-hand side of $(*)$. Since $\Sigma\leqslant\lambda r_i$ implies that $|R(\tau_j)-W(\sigma_j)|\leqslant(\lambda+C(G))r_i$ for all $j\leqslant i$, the lemma is proved. □

Lemma 3.2 is not really convenient to use as it is, because one needs to relate i to more natural events. The following result gives one such useful relation.

LEMMA 3.3. *Let G be an isotropic graph and let $v\in G$. Let R and W be the coupled walk and motion starting from v. Let $r>1$ and let $T=T_{v,r}$ and $S=S_{v,r}$. Let r_i, τ_i and σ_i be as in the definition of the coupling. Let*

$$I := \min\{i : \tau_i \geqslant T \ \textit{and}\ \sigma_i \geqslant S\}.$$

Then, for some constant $c_4(G)$,

$$\mathbf{P}(r_I > \lambda r^{1-c_4(G)}) \leqslant C(G)\exp(-\lambda^{c(G)}).$$

Proof. Since $W(\sigma_i)-W(\sigma_{i-1})$ are independent variables with mean zero and variance cr_i, we get (say by second-moment methods) that for some $c>0$,

$$\mathbf{P}\biggl(|W(\sigma_j)-W(\sigma_i)|>c\biggl(\sum_{k=i}^{j} r_k^2\biggr)^{1/2}\biggr)>c \quad \text{for all } j>i.$$

Lemma 3.2 allows us to replace W by R: we get that for some constants $\mu=\mu(G)$ and $\nu=\nu(G)$,

$$\mathbf{P}\biggl(|R(\tau_j)-R(\tau_i)|>\mu\biggl(\sum_{k=i}^{j} r_k^2\biggr)^{1/2}\biggr)>c \quad \text{for all } j>i+\nu.$$

In particular, if $r_i\sqrt{j-i}>(2/\mu)r$ then $\mathbf{P}(R(\tau_j)\notin B(v,r))>c$ for any $R(\tau_i)\in B(v,r)$. Thus if we define $J:=C(G)r^{1/(4/\alpha+1/2)}$ for some C sufficiently large, we get both $r_J\sqrt{J}>(2/\mu)r$ as well as $J>\nu$. Hence

$$\mathbf{P}(R(\tau_{(n+1)J})\notin B(v,r)\mid R[0,\tau_{nJ}],R(\tau_{nj})\in B(v,r))>c \quad \text{for all } n>1$$

and hence $\mathbf{P}(T>\tau_{nJ})\leqslant Ce^{-cn}$. An identical calculation shows that $\mathbf{P}(S>\sigma_{nJ})\leqslant Ce^{-cn}$. Hence $\mathbf{P}(r_I>r_{nJ})\leqslant Ce^{-cn}$ and since

$$r_{nJ}=(nJ)^{4/\alpha}=n^{C(G)}r^{1-c(G)},$$

the lemma is proved. □

Corollary. *With the notation of Lemma* 3.3,

$$\mathbf{P}(\textit{there exists } j\leqslant I: d_{\text{Haus}}(R[0,\tau_j],W[0,\sigma_j])>\lambda r^{1-c_4})\leqslant C(G)\exp(-\lambda^{c(G)}).$$

Proof. Clearly we may assume that $\lambda>5$. Let i_* be the maximal i such that

$$r_{i_*}\leqslant\sqrt{\lambda}\,r^{1-c_4}.$$

Then Lemma 3.2 shows that

$$\mathbf{P}\bigl(\text{there exists } j\leqslant i_*: |R(\tau_j)-W(\sigma_j)|\geqslant\bigl(\sqrt{\lambda}-2\bigr)r_{i_*}\bigr)\leqslant C(G)\exp\bigl(-c(G)\bigl(\sqrt{\lambda}-2\bigr)\bigr).$$

Now, the points $R(\tau_j)$ are an approximation (in the Hausdorff distance) of the entire path, i.e.

$$d_{\text{Haus}}(R[0,\tau_j],\{R(0),...,R(\tau_j)\})\leqslant r_j,$$

and similarly for W. Thus, we get

$$\mathbf{P}\bigl(\text{there exists } j\leqslant i_*: d_{\text{Haus}}(R[0,\tau_j],W[0,\sigma_j])\geqslant\sqrt{\lambda}\,r_{i_*}\bigr)\leqslant C(G)\exp(-c(G)(\sqrt{\lambda}-2)),$$

and from the definition of i_*,

$$\mathbf{P}(\text{there exists } j\leqslant i_*: d_{\text{Haus}}(R[0,\tau_j],W[0,\sigma_j])\geqslant\lambda r^{1-c_4})\leqslant C(G)\exp(-c(G)(\sqrt{\lambda}-2)).$$

Estimating the probability that $I>i_*$ using Lemma 3.3 proves the corollary. □

LEMMA 3.4. *Let G be an isotropic graph, let $\nu>1$ and let $v\in G$. Let R and W be the coupled walk and motion starting from v. Let $T_r=T_{v,r}$ and $S_r=S_{v,r}$. Then for all $r>\max\{1,s\}$, and for all $\lambda>0$,*

$$\mathbf{P}(\subset_{\mathrm{Haus}}(R[T_s,T_r],W[S_{s/\nu},S_{\nu r}])>\lambda r^{1-c_4(G)})\leqslant C(\nu,G)\exp(-\lambda^{c(G)}), \tag{31}$$

$$\mathbf{P}(\subset_{\mathrm{Haus}}(W[S_s,S_r],R[T_{s/\nu},T_{\nu r}])>\lambda r^{1-c_4(G)})\leqslant C(\nu,G)\exp(-\lambda^{c(G)}). \tag{32}$$

We explicitly include the case $s=0$, where we define $T_0=S_0=0$.

Proof. Let us prove (31). Let r_i, τ_i and σ_i be as in the definition of the coupling (28). Define

$$I_1:=\max\{i:\tau_i<T_s\}\quad\text{and}\quad I_2:=\min\{i:\tau_i\geqslant T_r\}.$$

Then Lemma 3.3 gives that, if $s>1$,

$$\mathbf{P}(r_{I_1}>\lambda s^{1-c_4})\leqslant C(G)\exp(-\lambda^{c(G)}), \tag{33}$$

$$\mathbf{P}(r_{I_2}>\lambda r^{1-c_4})\leqslant C(G)\exp(-\lambda^{c(G)}). \tag{34}$$

Denote by i_1 the last i such that $r_i\leqslant\lambda s^{1-c_4}$ and by i_2 the last i such that $r_i\leqslant\lambda r^{1-c_4}$. Lemma 3.2 shows that

$$\mathbf{P}(\text{there exists } j\leqslant i_k:|R(\tau_j)-W(\sigma_j)|\geqslant\lambda r_{i_k})\leqslant C(G)\exp(-c(G)\lambda),\quad k=1,2. \tag{35}$$

Together with the estimates of r_{I_k}, this gives the following consequence:

$$\mathbf{P}(d_{\mathrm{Haus}}(R[\tau_{I_1},\tau_{I_2}],W[\sigma_{I_1},\sigma_{I_2}])\geqslant\lambda^2 r^{1-c_4})\leqslant C(G)\exp(-\lambda^{c(G)}). \tag{36}$$

We can replace λ^2 by λ in the left-hand side paying only in the constant inside the exponent in the right-hand side.

Case 1. If $s<r^{1-c_4}$ then the fact that $|W(S_{s/\nu})-R(T_s)|\leqslant C(G)+2s$ shows that

$$\subset_{\mathrm{Haus}}(R[T_s,T_r],W[0,S_{\nu r}])\geqslant\subset_{\mathrm{Haus}}(R[T_s,T_r],W[S_{s/\nu},S_{\nu r}])-(C(G)+2s),$$

which allows us to estimate

$$\begin{aligned}\mathbf{P}(\subset_{\mathrm{Haus}}&(R[T_s,T_r],W[S_{s/\nu},S_{\nu r}])>\lambda r^{1-c_4})\\&\leqslant\mathbf{P}(\subset_{\mathrm{Haus}}(R[T_s,T_r],W[0,S_{\nu r}])>(\lambda-C(G))r^{1-c_4})\\&\leqslant\mathbf{P}(\subset_{\mathrm{Haus}}(R[\tau_{I_1},\tau_{I_2}],W[0,\sigma_{I_2}])>(\lambda-C(G))r^{1-c_4})+\mathbf{P}(\sigma_{I_2}>S_{\nu r})\\&\overset{(36)}{\leqslant}C(G)\exp(-(\lambda-C(G))^{c(G)})+\mathbf{P}(\sigma_{I_2}>S_{\nu r}).\end{aligned}$$

Now, to estimate $\mathbf{P}(\sigma_{I_2}>S_{\nu r})$, we use the fact that $|R(\tau_i)-v|\leqslant r$ for any $i<I_2$, and get

$$\mathbf{P}(\sigma_{I_2}>S_{\nu r})\leqslant \mathbf{P}(\text{there exists } j\leqslant i_2 : |W(\sigma_j)-R(\tau_j)|\geqslant (\nu-1)r-\lambda r^{1-c_4})+\mathbf{P}(I_2>i_2)$$
$$\overset{(34,35)}{\leqslant} C(G)\exp(-c(G)((\nu-1)\lambda^{-1}r^{c_4}-1))+C(G)\exp(-\lambda^{c(G)}).$$

Hence, if $\lambda<r^{c(G)}$, (31) is proved.

Case 2. If $s\geqslant r^{1-c_4}$, then we estimate

$$\begin{aligned}&\mathbf{P}(\subset_{\text{Haus}}(R[T_s,T_r],W[S_{s/\nu},S_{\nu r}])>\lambda r^{1-c_4})\\ &\quad\leqslant \mathbf{P}(\subset_{\text{Haus}}(R[\tau_{I_1},\tau_{I_2}],W[\sigma_{I_1},\sigma_{I_2}])>\lambda r^{1-c_4})+\mathbf{P}(\sigma_{I_1}<S_{s/\nu})+\mathbf{P}(\sigma_{I_2}>S_{\nu r})\\ &\quad\overset{(36)}{\leqslant} C(G)\exp(-\lambda^{c(G)})+\mathbf{P}(\sigma_{I_1}<S_{s/\nu})+\mathbf{P}(\sigma_{I_2}>S_{\nu r}).\end{aligned}$$

Now, the estimate of $\mathbf{P}(\sigma_{I_2}>S_{\nu r})$ is as in Case 1. The estimate of $\mathbf{P}(\sigma_{I_1}<S_{s/\nu})$ is similar: using that $|R(\tau_{I_1})-v|\geqslant s-r_{I_1}-C(G)$, we get

$$\begin{aligned}\mathbf{P}(\sigma_{I_1}<S_{s/\nu})&\leqslant \mathbf{P}(|W(\sigma_{I_1})-v|<s/\nu)\\ &\leqslant \mathbf{P}(\text{there exists } j\leqslant i_1 : |W(\sigma_j)-R(\tau_j)|\geqslant s(1-1/\nu)-\lambda s^{1-c_4}-C(G))\\ &\qquad+\mathbf{P}(I_1>i_1)\\ &\overset{(33,35)}{\leqslant} C(G)\exp(-c(G)(1-1/\nu)\lambda^{-1}s^{c_4}-C(G))+C(G)\exp(-\lambda^{c(G)}),\end{aligned}$$

and again, if $\lambda\leqslant s^{c(G)}$, (31) is proved. Since in this case $s>r^{c(G)}$, it is enough to assume that $\lambda\leqslant r^{c(G)}$ in order to get $\lambda\leqslant s^{c(G)}$, and consequently (31).

Case 3. The previous calculations proved the lemma in the case $\lambda\leqslant r^{c_5}$ for some $c_5(G)$. However, this implies that for any $\lambda\leqslant 2r^{c_4}$,

$$\mathbf{P}(\subset_{\text{Haus}}(R[T_s,T_r],W[S_{s/\nu},S_{\nu r}])>\lambda r^{1-c_4(G)})\leqslant C(\nu,G)\exp\big(-\big(\tfrac{1}{2}\lambda\big)^{c(G)c_5/c_4}\big),$$

or, in other words, (31) holds with different constants in the right-hand side. However, for $\lambda>2r^{c_4}$, (31) holds trivially because

$$\subset_{\text{Haus}}(R[T_s,T_r],W[S_{s/\nu},S_{\nu r}])\leqslant d_{\text{Haus}}(R[T_s,T_r],W[S_{s/\nu},S_r])\leqslant 2r,$$

so the probability in (31) is zero. This finishes the proof of (31). The proof of (32) is identical. □

Corollary. *With the notation of Lemma* 3.4, *if* R *and* W *start from* w, *with*

$$|v-w|\leqslant \tfrac{1}{4}s(1-1/\nu),$$

then (31) *and* (32) *still hold, possibly with different constants. Further, this holds if*

$$s=0 \quad \textit{and} \quad |v-w|\leqslant \tfrac{1}{4}r(\nu-1).$$

This follows from Lemma 3.4 and the monotonicity of $\subset_{\text{Haus}}$.

3.5. Hitting of small balls

From now on we will prove "natural" facts about walks on isotropic graphs, natural in the sense that they do not need the coupling (or other special notation) to be stated. In this section we shall prove two lemmas about the hitting probability of "intermediate scale" objects, i.e. of the size r^{1-c} (both will be used in §5). In the next sections we shall focus on more delicate facts.

LEMMA 3.5. *Let G be an isotropic graph, and let $\varepsilon>0$. Let $r>C(\varepsilon,G)$ and let $v,w\in B(x,r(1-\varepsilon))$, $|v-w|>\frac{1}{2}\varepsilon r$. Let*

$$p:=\mathbf{P}^v(T_{w,s}<T_{x,r}) \quad \textit{and} \quad q:=\mathbf{P}^v(S_{w,s}<S_{x,r}),$$

where $r^{1-c_4/2}\leqslant s\leqslant|v-w|-\frac{1}{2}\varepsilon r$ and $c_4(G)$ comes from Lemma 3.3. Then

$$|p-q|\leqslant C(\varepsilon,G)r^{-c(G)}\max\{p,q\}.$$

Proof. The first step is to get a simple lower bound for q. In the case $|v-w|\leqslant\frac{3}{4}d(w,\partial_{\mathrm{cont}}B(x,r))$ then a calculation using the continuous analog of (5) with the Newtonian potential [B95, Chapter II.3] around w shows that

$$q\geqslant c(\varepsilon,d)\begin{cases} s/r, & \text{if } d=3,\\ 1/\log(r/s), & \text{if } d=2\end{cases} \tag{37}$$

(d being the dimension). Removing the condition $|v-w|\leqslant\frac{3}{4}d(w,\partial B(x,r))$, we still have (37), perhaps with a different constant. Indeed, using the continuous Harnack inequality [B95, Chapter II.1] for the domain $B(x,r)\setminus B\big(w,s+\frac{1}{4}\varepsilon r\big)$ shows (37) for all w and v.

Next define

$$q^{\pm}=\mathbf{P}^v(S_{w,s^{\pm}}<S_{x,r}), \quad \text{where } s^{\pm}=s\pm r^{1-(3/4)c_4},$$

so $q^-\leqslant q\leqslant q^+$. Now, the strong Markov property gives us that

$$q-q^-\leqslant\mathbf{P}^v(S_{w,s}<S_{x,r})\max_{y\in\partial_{\mathrm{cont}}B(w,s)}\mathbf{P}^y(W[0,S_{x,r}]\cap B(w,s^-)=\varnothing),$$

and again, similar calculations with the continuous Newtonian potential give, for any $y\in\partial_{\mathrm{cont}}B(w,s)$,

$$\mathbf{P}^y(W[0,S_{x,r}]\cap B(w,s^-)=\varnothing)\leqslant C(\varepsilon,d)\frac{r^{1-(3/4)c_4}}{s}\leqslant Cr^{-c_4/4}.$$

So we get $q-q^-\leqslant Cqr^{-c}$. A similar calculation shows that $q^+-q\leqslant Cq^+r^{-c}$ and for r sufficiently large we may write $q^+-q\leqslant Cqr^{-c}$.

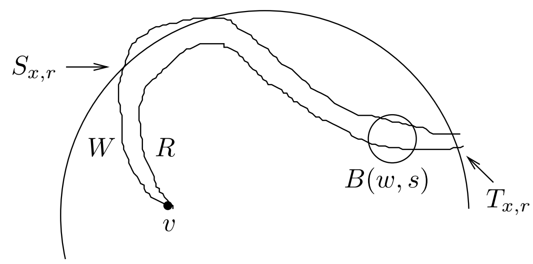

Figure 3. $R[0,T_{x,r}]$ intersects the ball $B(w,s)$ but $W[0,S_{x,r}]$ is quite far from it.

To extract from these inequalities conclusions about $|q-p|$, couple R and W as above. Let r_i, τ_i and σ_i be as in the definition of the coupling (28). Let

$$I := \min\{i : \tau_i \geqslant T_{v,r} \text{ and } \sigma_i \geqslant S_{v,r}\}.$$

When comparing p with q^+ we need to consider two cases: the first is when $R[0,\tau_I]$ and $W[0,\sigma_I]$ are not very close; and the second is as in Figure 3. In a formula: if λ is the length of the longest edge in G then

$$\begin{aligned} p-q^+ &\leqslant \mathbf{P}(\{R[0,T_{x,r}]\cap\partial B(w,s)\neq\varnothing\}\cap\{W[0,S_{x,r}]\cap\partial B(w,s^+)=\varnothing\}) \\ &\leqslant \mathbf{P}(d_{\mathrm{Haus}}(R[0,\tau_I],W[0,\sigma_I])\geqslant r^{1-(3/4)c_4}-\lambda) \\ &\qquad +\mathbf{P}(W[S_{x,r},S_{x,r+r^{1-(3/4)c_4}}]\cap B(w,s^+)\neq\varnothing). \end{aligned} \tag{38}$$

Now, the corollary of Lemma 3.3 for the ball $B(v,2r)$ gives, since $B(x,r)\subset B(v,2r)$, that

$$\mathbf{P}(d_{\mathrm{Haus}}(R[0,\tau_I],W[0,\sigma_I])\geqslant r^{1-(3/4)c_4}-\lambda)\leqslant C(G)\exp(-r^{c(G)}).$$

For the second summand we have from the strong Markov property,

$$\begin{aligned} &\mathbf{P}(W[S_{x,r},S_{x,r+r^{1-(3/4)c_4}}]\cap B(w,s^+)\neq\varnothing) \\ &\qquad \leqslant \max_{y\in\partial_{\mathrm{cont}}B(x,r)} \mathbf{P}^y\big(W[0,S_{x,r+r^{1-(3/4)c_4}}]\cap B\big(x,r\big(1-\tfrac{1}{4}\varepsilon\big)\big)\neq\varnothing\big) \\ &\qquad\qquad \times \max_{y\in\partial_{\mathrm{cont}}B(x,r(1-\varepsilon/4))} \mathbf{P}^y(W[0,S_{x,r+r^{1-(3/4)c_4}}]\cap B(w,s^+)\neq\varnothing) \\ &\qquad \overset{(*)}{\leqslant} Cr^{-(3/4)c_4}\begin{cases} s/r, & \text{if } d=3, \\ 1/\log(r/s), & \text{if } d=2 \end{cases} \\ &\qquad \leqslant Cr^{-c}q. \end{aligned}$$

Both estimates of $(*)$ follow from the continuous Newtonian potential. Note that we assumed here that $\frac{1}{8}\varepsilon r>r^{1-(3/4)c_4}$ which we may, if r is sufficiently large. This finishes the proof that $p\leqslant q+Cr^{-c}q$.

The proof of the other direction is similar. We have

$$\begin{aligned} q^- - p &\leqslant \mathbf{P}(\{W[0,S_{x,r}]\cap\partial B(w,s^-)\neq\varnothing\}\cap\{R[0,T_{x,r}]\cap\partial B(w,s)=\varnothing\}) \\ &\leqslant \mathbf{P}(d_{\mathrm{Haus}}(R[0,\tau_I],W[0,\sigma_I])\geqslant r^{1-(3/4)c_4}) \\ &\quad+\mathbf{P}(W[S_{x,r-r^{1-(3/4)c_4}},S_{x,r}]\cap B(w,s^-)\neq\varnothing), \end{aligned}$$

and an identical calculation finishes this case, and the lemma. □

LEMMA 3.6. *Let G be an isotropic graph and let $\varepsilon>0$. Let $r>C(\varepsilon,G)$ and let $v\in B(x,(1-\varepsilon)r)$. Let Δ be a spherical triangle on $\partial_{\mathrm{cont}}B(x,r)$ and let Δ^* be a discrete version of it. Let $q:=\mathbf{P}^v(W(S_{x,r})\in\Delta)$ and $p:=\mathbf{P}^v(R(T_{x,r})\in\Delta^*)$. Then*

$$|p-q|\leqslant C(G,\varepsilon)r^{-c_6(G)}.$$

(The only difference between Lemma 3.6 and the definition of an isotropic graph is that here the starting point of the walk v might be different from the center of the stopping ball x.)

Proof. The proof is very similar to the proof of the previous lemma, so we indicate only the differences. We define $q^\pm$ as the probabilities of W to hit $\partial B(x,r)$ at $\Delta^\pm$, where

$$\begin{aligned} \Delta^+ &:= (\Delta+B(0,r^{1-c_4/2}))\cap\partial_{\mathrm{cont}}B(x,r), \\ \Delta^- &:= \{x\in\Delta : B(x,r^{1-c_4/2})\cap\partial_{\mathrm{cont}}B(x,r)\subset\Delta\}. \end{aligned}$$

The proof that q^+-q, $q-q^-\leqslant Cr^{-c}$ is direct calculation for $v=x$, and for general v follows from the continuous Harnack inequality. The proof that $p-q^+\leqslant Cr^{-c}$ is similar, except that the last term in the right-hand side of (38) should be replaced, for example, by

$$\begin{aligned} &\mathbf{P}(\{W[0,S_{x,r}]\cap(\Delta+B(0,r^{1-(3/4)c_4}))\neq\varnothing\}\cap\{W(S_{x,r})\notin\Delta^+\}) \\ &\qquad\leqslant \max_{d(y,\Delta)\leqslant r^{1-(3/4)c_4}} \mathbf{P}^y(W(S_{x,r})\notin\Delta^+)\leqslant Cr^{-c}. \end{aligned}$$

The proof that $q^- - p\leqslant Cr^{-c}$ and the rest of the lemma are similar. □

3.6. Escape probabilities

In this section we move from the "intermediate scale" objects of the previous section to single points. This is more delicate, and we shall employ techniques similar to those of Lawler [L96b]. Our main goal is Theorem 2, but first we need to state and prove two simple claims.

Henceforth R and W will always be a random walk and a Brownian motion coupled as above.

Lemma 3.7. *Let $H\subset\mathbf{R}^d$ be a half-space. Let W be a Brownian motion on $\mathbf{R}^d$ starting from some vertex $v\notin\partial_{\mathrm{cont}}H$. Let $r>2d(v,\partial_{\mathrm{cont}}H)$. Then*

$$p := \mathbf{P}(S_{v,r} < S(\partial_{\mathrm{cont}}H)) \approx \frac{d(v,\partial_{\mathrm{cont}}H)}{r}.$$

Proof. By translation, scaling and rotation invariance we may assume that $r=1$, that $v=\varepsilon e_1$ for some $\frac{1}{2}>\varepsilon>0$ (e_1 being the first basis element) and that $H=\{x:\langle x,e_1\rangle>0\}$. Examine two positive harmonic functions: $f(x)=\mathbf{P}^x(S_{v,r}<S(\partial_{\mathrm{cont}}H))$ and $g(x)=\langle x,e_1\rangle$. Both f and g are zero on $\partial_{\mathrm{cont}}H$ so the boundary Harnack principle for Lipschitz domains [B95, Theorem III.1.2, p. 178] shows that $g(v)/f(v)\approx g(x_0)/f(x_0)$, where x_0 is any reference point. But $g(x_0)/f(x_0)$ is just a number, and the lemma is proved. □

Lemma 3.8. *Let $G\subset\mathbf{R}^d$ be an isotropic graph. Let $H\subset\mathbf{R}^d$ be a half-space. Let $v\in G\setminus\partial_{\mathrm{cont}}H$. Let $\frac{1}{2}\varrho>s>r>2d(v,\partial_{\mathrm{cont}}H)$. Let R and W be coupled walk and motion starting from v. Let $\mathcal{E}$ be an event depending on $R[0,T_{v,r}]$ and $W[0,S_{v,r}]$ only. Then*

$$\mathbf{P}(\mathcal{E}\cap\{W[S_{v,r},S_{v,s}]\cap\partial_{\mathrm{cont}}H=\varnothing\})\leqslant C\frac{r}{s}(\mathbf{P}(\mathcal{E})+C(G)\exp(-r^{c(G)})).$$

Proof. Without loss of generality, we may assume that $s>4r$ and (using Lemma 3.7) $r>1$. Let r_i, τ_i and σ_i be as in the definition of the coupling. Define

$$I := \min\{i : \tau_i \geqslant T_{v,r} \text{ and } \sigma_i \geqslant S_{v,r}\}$$

and examine W after σ_I. This is a regular Brownian motion, starting from $W(\sigma_I)$ and *independent of both* $W[0,\sigma_I]$ *and* $R[0,\tau_I]$. This is because the coupling is Markovian at these times, that is, once you know $W(\sigma_I)$ and $R(\tau_I)$ you can ignore all the past—this is not true at any stopping time, just at the σ_i's and the τ_i's. Therefore, if we denote by $\mathcal{F}$ the event $W[\sigma_I,S_{v,s}]\cap\partial_{\mathrm{cont}}H=\varnothing$, we get from Lemma 3.7,

$$\mathbf{P}(\mathcal{F}\mid W(\sigma_I))\leqslant C\frac{d(W(\sigma_I),\partial_{\mathrm{cont}}H)}{d(W(\sigma_I),\partial_{\mathrm{cont}}B(v,s))}. \tag{39}$$

Actually, this holds only if $W(\sigma_I)\in B(v,s)$—in the other case we will simply estimate $\mathbf{P}\leqslant 1$. This shows that,

$$\mathbf{P}(\mathcal{E}\cap\mathcal{F}\cap\{|W(\sigma_I)-v|\leqslant 2r\})\leqslant C\mathbf{P}(\mathcal{E})\frac{3r}{s-2r}\leqslant C\mathbf{P}(\mathcal{E})\frac{r}{s}. \tag{40}$$

Let us now examine the case when $|W(\sigma_I)-v|>\lambda r$ for some $\lambda\geqslant 2$. By the definition of I we have either $|W(\sigma_I)-v|\leqslant r+r_I$ or $|R(\tau_I)-v|\leqslant r+r_I+C(G)$ and in either case

$$|W(\sigma_I)-R(\tau_I)|\geqslant(\lambda-1)r-r_I-C(G). \tag{41}$$

By Lemma 3.3, we have

$$\mathbf{P}(r_I > \lambda r^{1-c_4/2}) \leqslant C(G)\exp(-(\lambda r^{c_4/2})^{c(G)}). \tag{42}$$

On the other hand, if $r_I \leqslant \lambda r^{1-c_4/2}$, then

$$|W(\sigma_I)-R(\tau_I)| \overset{(41)}{\geqslant} (\lambda-1)r-r_I-C(G) \geqslant ((\lambda-1)r^{c_4/2}-C(G))r^{1-c_4/2}.$$

Using Lemma 3.2 for the first i such that $r_i \geqslant r^{1-c_4/2}$ gives

$$\mathbf{P}(r_I \leqslant \lambda r^{1-c_4/2} \text{ and } |W(\sigma_I)| > \lambda r) \leqslant C(G)\exp(-c(G)((\lambda-1)r^{c_4/2}-C(G)))$$

and, with (42),

$$\mathbf{P}(|W(\sigma_I)-v| > \lambda r) \leqslant C(G)\exp(-(\lambda r)^{c(G)}) \quad \text{for all } \lambda \geqslant 2. \tag{43}$$

Hence we get that

$$\begin{aligned} \mathbf{P}(\{|W(\sigma_I)-v|>2r\}\cap\mathcal{F}) &= \sum_{\lambda=3}^{\infty} \mathbf{P}(\{|W(\sigma_I)-v| \in](\lambda-1)r, \lambda r]\}\cap\mathcal{F}) \\ &\overset{(*)}{\leqslant} C(G)\sum_{\lambda=3}^{\infty}\exp(-(\lambda r)^{c(G)})\cdot\begin{cases} 2(\lambda+1)r/s, & \text{if } \lambda r < \frac{1}{2}s, \\ 1, & \text{otherwise} \end{cases} \\ &\leqslant C(G)\exp(-r^{c(G)})\frac{r}{s}, \end{aligned} \tag{44}$$

where inequality $(*)$ comes from (43) for the left factor and (39) for the right factor. Combining (40) and (44) ends the lemma. □

THEOREM 2. *Let $G \subset \mathbf{R}^d$ be an isotropic graph. Let $H \subset \mathbf{R}^d$ be a half-space. Then there exists a constant $C_4(G)$ such that for any such H, any v with $d(v, \partial_{\mathrm{cont}}H) > C_4$ and any $r > 2d(v, \partial_{\mathrm{cont}}H)$ one has that the probability p that a random walk R on G starting from v will hit $\partial B(v,r)$ before hitting ∂H satisfies*

$$p \approx \frac{d(v, \partial_{\mathrm{cont}}H)}{r}.$$

Remember that the constants implicit in the $\approx$ above, like C_4, may depend on the isotropic structure constants of G. Actually, the proof below shows that the lower bound does not depend on G at all, and the upper bound can also easily be shown not to depend on G. However, we will have no use for these facts.

Proof. Let $\mu = d(v,\partial_{\mathrm{cont}}H)/r$. We may assume, without loss of generality, that $\mu = 2^{-M}$ for some integer M. Before starting with estimates for R, we need to know a fact about Brownian motion, roughly speaking that Brownian motion conditioned to have $\{S_{v,4r} < S(\partial_{\mathrm{cont}}H)\}$ also avoids $\partial_{\mathrm{cont}}H$ along (most of) its path. To formulate this precisely, let $\beta<1$ be some parameter which will be fixed later, and set

$$a_k = d(v,\partial_{\mathrm{cont}}H)2^k \quad \text{and} \quad b_k = a_k^{\beta}.$$

Thus $r = a_M$. Define stopping times $S_k := S_{v,a_k}$ for $k \in]-\infty, M+2]$, and let

$$\mathcal{E} := \bigcup_{k=-\infty}^{M+1} \mathcal{E}_k,$$

where

$$\mathcal{E}_k := \{\text{there exists } t \in]S_{k-1}, S_k] : d(W(t),\partial_{\mathrm{cont}}H) \leqslant b_k\} \cap \{S_{M+2} < S(\partial_{\mathrm{cont}}H)\}.$$

Note that because we end at $\mathcal{E}_{M+1}$, we actually ignore the event of getting close to $\partial_{\mathrm{cont}}H$ on the last stretch of the Brownian motion, namely $]S_{M+1}, S_{M+2}]$ (this makes the proof a little simpler). We assume that $C_4 \geqslant 4$ and then $\mathcal{E}_k$ is empty for $k \leqslant -1$. For $0 \leqslant k \leqslant M+1$ we shall use Lemma 3.7 in the form $\mathbf{P}(S_{k-1} < S(\partial_{\mathrm{cont}}H)) \leqslant C2^{-k}$ and again (together with the strong Markov property) to get

$$\mathbf{P}(S_{M+2} < S(\partial_{\mathrm{cont}}H) \mid W[0,t^*]) \leqslant C\frac{b_k}{r}, \quad \text{where } t^* := \inf_{t > S_{k-1}} d(W(t),\partial_{\mathrm{cont}}H) \leqslant b_k$$

(clearly t^* is a stopping time, so we may use the strong Markov property). Hence we get

$$\mathbf{P}(\mathcal{E}_k) \leqslant C2^{-k}\frac{b_k}{r} = C\mu a_k^{\beta-1},$$

and summing (remember that $\beta<1$),

$$\mathbf{P}(\mathcal{E}) \leqslant C\mu \frac{d(v,\partial_{\mathrm{cont}}H)^{\beta-1}}{1-2^{\beta-1}}. \tag{45}$$

We now move to examine the random walk R. Couple W with R as above. Let $T_k = T_{v,a_k}$, that is the R-equivalents of the S_k. Assume that C_4 is sufficiently large so that always $R[0,T_{-1}] \cap \partial H = \varnothing$. We start with a lower bound for p. A little set calculus gives

$$\begin{aligned}\mathbf{P}(T_M < T(\partial H)) \geqslant \mathbf{P}(S_{M+2} < S(\partial_{\mathrm{cont}}H)) - \mathbf{P}(\mathcal{E}) \\ - \mathbf{P}(\{S_{M+2} < S(\partial_{\mathrm{cont}}H)\} \setminus (\{T_M < T(\partial H)\} \cup \mathcal{E})).\end{aligned} \tag{46}$$

Now, if $T_M \geqslant T(\partial H)$ then we have that $R(n)\in\partial H$ for some $n\in[T_{k-1},T_k]$ for some $k\leqslant M$. If, in addition, $S_{M+2}<S(\partial_{\rm cont}H)$ but $\mathcal{E}$ did not happen, then we must have that

$$d(W[S_{k-2},S_{k+1}],\partial_{\rm cont}H)>b_{k-1},$$

and hence $\subset_{\rm Haus}(R[T_{k-1},T_k],W[S_{k-2},S_{k+1}])>b_{k-1}-C$. Thus, we arrive at

$$\begin{aligned}&\mathbf{P}(\{S_{M+2}<S(\partial_{\rm cont}H)\}\setminus(\{T_M<T(\partial H)\}\cup\mathcal{E}))\\&\quad\leqslant\sum_{k=0}^{M}\mathbf{P}(\{\subset_{\rm Haus}(R[T_{k-1},T_k],W[S_{k-2},S_{k+1}])>b_{k-1}-C\}\cap\{S_{M+2}<S(\partial_{\rm cont}H)\}).\end{aligned}\tag{47}$$

The estimate of the right-hand side of (47) follows from Lemma 3.4, but first we need to chose β and we choose $\beta=1-\frac{1}{2}c_4(G)$, where $c_4(G)$ comes from Lemma 3.4. The lemma then claims that

$$\mathbf{P}(\subset_{\rm Haus}(R[T_{k-1},T_k],W[S_{k-2},S_{k+1}])>b_{k-1}-C)\leqslant C(G)\exp(-a_k^{c(G)}).\tag{48}$$

The condition $S_{M+2}<S(\partial_{\rm cont}H)$ can be added via Lemma 3.8, and we get

$$\begin{aligned}\mathbf{P}(\{\subset_{\rm Haus}(R[T_{k-1},T_k],W[S_{k-2},S_{k+1}])>b_{k-1}-C\}\cap\{S_{M+2}<S(\partial_{\rm cont}H)\})&\\ \leqslant C(G)\exp(-a_k^{c(G)})\mu 2^k.&\end{aligned}$$

Plugging this into (47) and summing, we get

$$\mathbf{P}(\{S_{M+2}<S(\partial_{\rm cont}H)\}\setminus(\{T_M<T(\partial H)\}\cup\mathcal{E}))\leqslant C(G)\exp(-d(v,\partial_{\rm cont}H)^{c(G)})\mu.$$

This we may plug into (46) together with (45) and Lemma 3.7 and get

$$\mathbf{P}(T_M<T(H))\geqslant\mu\biggl(c-C\frac{d(v,\partial_{\rm cont}H)^{-c_4(G)/2}}{1-2^{-c_4(G)/2}}-C(G)\exp(-d(v,\partial_{\rm cont}H)^{c(G)})\biggr),$$

and it is now clear that if C_4 is chosen sufficiently large, then $d(v,\partial_{\rm cont}H)>C_4$ would give that everything inside the parenthesis is $>c$ and the direction $p\geqslant c\mu$ is proved.

The proof that $p\leqslant C\mu$ is, generally speaking, a mirror image exchanging the roles of R and W in the proof of $p\geqslant c\mu$. Since our a priori knowledge about the random walk is smaller (it is, essentially, Lemma 2.12), the proof is somewhat rearranged. Here are the details. Define

$$g_i:=2^i\mathbf{P}(T_i<T(\partial H)).$$

Fix one $i\geqslant 4$. For every $j\in[2,i-2]$ examine the event

$$\mathcal{F}_j:=\{W[S_{j-1},S_j]\cap\partial_{\rm cont}H\neq\varnothing\}\cap\{W]S_j,S_{i-2}]\cap\partial_{\rm cont}H=\varnothing\}.$$

The event $\{T_i<T(\partial H)\}\cap\mathcal{F}_j$ has a number of consequences:

(i) $R[0,T_{j-2}]\cap\partial H=\varnothing$. By definition this event has probability $2^{2-j}g_{j-2}$.

(ii) $R[T_{j-2},T_{j+1}]\cap\partial H=\varnothing$. Lemma 3.4 shows that

$$\mathbf{P}(\subset_{\mathrm{Haus}}(W[S_{j-1},S_j],R[T_{j-2},T_{j+1}])>a_j^{1-c_4/2})\leqslant C(G)\exp(-a_j^{c(G)}),$$

and since $W[S_{j-1},S_j]\cap\partial_{\mathrm{cont}}H\neq\varnothing$, we get that, with probability $1-C\exp(-a_j^c)$,

$$d(R[T_{j-2},T_{j+1}],\partial H)\leqslant a_j^{1-c_4/2}+C(G).$$

Denote this event by $\mathcal{G}$. Lemma 2.12 and the strong Markov property now show that

$$\mathbf{P}(\mathcal{G}\cap\{T_{j+2}>T(\partial H)\}\mid R[0,T_{j-2}])\leqslant C(G)\left(\frac{a_j^{1-c_4/2}}{a_j}\right)^{c(G)}.$$

Together with clause (i) we get

$$\begin{aligned}\mathbf{P}(\{R[T_0,T_{j+2}]\cap\partial H=\varnothing\}\cap\{W[S_{j-1},S_j]\cap\partial_{\mathrm{cont}}H\neq\varnothing\})\\ \leqslant C(G)2^{2-j}g_{j-2}a_j^{-c(G)}+C(G)\exp(-a_j^{c(G)})\overset{(*)}{\leqslant}C(G)2^{2-j}g_{j-2}a_j^{-c(G)},\end{aligned}$$

where in $(*)$ we used the lower bound $g_j\geqslant c$ already established.

(iii) $W[S_{j+2},S_{i-2}]\cap\partial_{\mathrm{cont}}H=\varnothing$. Here we employ Lemma 3.8 and get

$$\begin{aligned}\mathbf{P}(\{T_i<T(\partial H)\}\cap\mathcal{F}_j)&\leqslant C2^{j+2-i}(C(G)2^{2-j}g_{j-2}a_j^{-c(G)}+C(G)\exp(-a_j^{c(G)}))\\ &\leqslant C(G)2^{-i}g_{j-2}a_j^{-c(G)}.\end{aligned}$$

With the estimate of $\mathbf{P}(\{T_i<T(\partial H)\}\cap\mathcal{F}_j)$ complete, we only need to sum over j and use Lemma 3.7 to get

$$\mathbf{P}\left(\{T_i<T(\partial H)\}\setminus\bigcup_{j=2}^{i-2}\mathcal{F}_j\right)\leqslant\mathbf{P}(W[S_1,S_{i-2}]\cap\partial_{\mathrm{cont}}H=\varnothing)\leqslant C2^{-i}\leqslant C2^{-i}g_0,$$

so we get

$$g_i\leqslant C(G)\sum_{j=0}^{i-4}g_j(2^{-c(G)})^j.$$

By [L96b, Lemma 4.5], the g_i's are bounded and the bound depends only on the isotropic structure constants of G (we use here that $g_0\leqslant 1$), and the theorem is proved. □

LEMMA 3.9. *With the notation of Theorem 2 (but $d(v,\partial_{\mathrm{cont}}H)\geqslant C_5$), let p be the probability that a random walk R on G starting from v will hit $\partial B(v,r)\cup\partial H$ in the arc*

$$\alpha:=\partial B(v,r)\cap\left\{x:d(x,\partial_{\mathrm{cont}}H)>\tfrac{1}{2}r\right\}.$$

Then $p\approx d(v,\partial_{\mathrm{cont}}H)/r$.

Proof. The fact that $p\leqslant C(G)d(v,\partial_{\text{cont}}H)/r$ is a direct consequence of Theorem 2. For the other direction, first assume without loss of generality that $r>4d(v,\partial_{\text{cont}}H)$, which can be done by Lemma 2.5. Let $\lambda>0$ be some parameter, and let β be the two arcs $\partial B\big(v,\tfrac{1}{2}r\big)\cap\{x:d(x,\partial_{\text{cont}}H)<\lambda r\}$. Examine the event

$$\mathcal{E}:=\big\{R\big(T\big(\partial B\big(v,\tfrac{1}{2}r\big)\cup\partial H\big)\big)\in\beta\big\}\cap\{T_{v,r}<T(\partial H)\}.$$

Theorem 2 shows that

$$\mathbf{P}\big(R\big(T\big(\partial B\big(v,\tfrac{1}{2}r\big)\cup\partial H\big)\big)\in\beta\big)\leqslant\mathbf{P}^v(T_{v,r/2}<T(\partial H))\leqslant C(G)\frac{d(v,\partial_{\text{cont}}H)}{r}.$$

For any $x\in\beta$ we have, again from Theorem 2,

$$\mathbf{P}^x(T_{v,r}<T(\partial H))\leqslant\mathbf{P}^x(T_{x,r/4}<T(\partial H))\leqslant C(G)\lambda$$

if $\lambda<\frac{1}{8}$ and $\lambda r>C(G)$. Hence we get

$$\mathbf{P}(\mathcal{E})\leqslant C(G)\lambda\frac{d(v,\partial_{\text{cont}}H)}{r}.$$

Combining this with the lower bound of Theorem 2, we get

$$\begin{aligned}c(G)\frac{d(v,\partial_{\text{cont}}H)}{r}&\leqslant\mathbf{P}(T_{v,r/2}<T(\partial H))\\&\leqslant\mathbf{P}\big(R\big(T\big(B\big(v,\tfrac{1}{2}r\big)\cup\partial H\big)\big)\in\partial B\big(v,\tfrac{1}{2}r\big)\setminus\beta\big)+\lambda C(G)\frac{d(v,\partial_{\text{cont}}H)}{r}.\end{aligned}$$

Choose some sufficiently small constant $\lambda=\lambda(G)$ and get that

$$\mathbf{P}\big(R\big(T\big(B\big(v,\tfrac{1}{2}r\big)\cup\partial H\big)\big)\in\partial B\big(v,\tfrac{1}{2}r\big)\setminus\beta\big)>c(G)\frac{d(v,\partial_{\text{cont}}H)}{r}.$$

Lemma 2.5 now shows that there is a probability $>c(G)$ to hit $\partial B(v,r)\cup\partial H$ at α if you start from any point of $\partial B\big(v,\tfrac{1}{2}r\big)\setminus\beta$, and we are done. □

3.7. Lower bound for the non-intersection probability

Our aim in this and the next section is to prove the following theorem.

THEOREM 3. *Let G be an isotropic graph of dimension 2 or 3. Then for any $v^1,v^2\in G$ with $|v^1-v^2|>C(G)$, if R^1 and R^2 are two walks with R^i starting from v^i and stopped on $\partial B(v^1,r)$, $r>2|v^1-v^2|$, then*

$$c(G)\left(\frac{|v^1-v^2|}{r}\right)^{\xi}\leqslant\mathbf{P}(R^1\cap R^2=\varnothing)\leqslant C(G)\left(\frac{|v^1-v^2|}{r}\right)^{\xi},\tag{49}$$

where $\xi=\xi_d(1,1)$ is the intersection exponent from (23).

The proof of Theorem 3 is a completely straightforward generalization of Lawler [L96b]. Hence we shall not repeat the argumentation of [L96b], we shall only note the pieces that require changes. The reader unfamiliar with [L96b] might benefit from a comparison with the proof of Theorem 2 which was also modeled, roughly, on [L96b]. The proof of Theorem 2 used three basic steps: showing that Brownian motion conditioned not to hit ∂H actually stays quite far from it (equation (45)), showing the same for random walk, and then coupling the two processes. We wish to employ a similar strategy for the intersection exponent. The main new difficulty is that the (second) Brownian motion/random walk (which takes the role of ∂H in Theorem 2) is highly non-homogenous. Thus one needs an estimate that the second object is "hittable" at *every point* of its path. These are Lemmas 2.4 (Brownian motion) and 2.6 (random walk) from [L96b]. We only need to reprove the random walk result and we shall do so promptly—see Lemma 3.11.

We start with the following lemma, which is a replacement for (7) in Lemma 2.5 of [L96b].

LEMMA 3.10. *Let $\varepsilon>0$ and let G be an isotropic graph. Then there exists $\delta=\delta(\varepsilon,G)$ such that for any $r>C_6(\varepsilon,G)$, any $v\in G$ and any $w\in G$ with $|v-w|\leqslant r$ one has*

$$\mathbf{P}^{1,w}\Big(\inf_{|z-v|<r}\mathbf{P}^{2,z}(R^2[0,T^2_{v,2r}]\cap R^1[T^1_{v,r},T^1_{v,2r}]\neq\varnothing \mid R^1[T^1_{v,r},T^1_{v,2r}])<\delta\Big)<\varepsilon,$$

where R^1 and R^2 are two independent random walks.

In words, consider a path to be δ*-hittable from* z if the probability of a random walk (R^2) starting from z to hit it is $\geqslant\delta$. Then what we prove here is that the random walk (R^1) is, with probability $1-\varepsilon$, δ-hittable from any $z\in B(v,r)$. To understand the formula formally, remember that the conditional probability $\mathbf{P}(\cdot\,|\,*)$ is a function of $*$ and note that the inf relates to a pointwise infimum of these functions.

Proof. Let $x\in G$ and let $s>\lambda$, where $\lambda=\lambda(G)$ is some sufficiently large constant that will be fixed later. Assume for now that $\lambda>C_3\big(\frac{1}{8},G\big)$, where C_3 comes from Lemma 2.10 of the present paper. Hence we use Lemma 2.10 and get for any $y\in B\big(x,\frac{7}{4}s\big)$,

$$\mathbf{P}^{1,x,2,y}(R^1[0,T^1_{x,s/4}]\cap R^2[0,T^2_{x,2s}]\neq\varnothing)>c(G). \tag{50}$$

For any path γ from x to $\partial B\big(x,\frac{1}{4}s\big)$ define $Y(y,\gamma):=\mathbf{P}^{2,y}(\gamma\cap R^2[0,T^2_{x,2s}]\neq\varnothing)$. Then (50) implies that

$$\mathbf{P}^{1,x}(Y(y,R^1[0,T^1_{x,s/4}])>c(G))>c(G)\quad\text{for all }\ y\in B\big(x,\tfrac{7}{4}s\big).$$

If λ is sufficiently large, then we have $\partial B\big(x,\frac{1}{4}s\big)\subset B\big(x,\frac{1}{2}s\big)$ and then $Y(\cdot\,,\gamma)$ is harmonic on $\big\{y:\frac{1}{2}s\leqslant|x-y|\leqslant\frac{3}{2}s\big\}$, and we may use Harnack's inequality (Lemma 2.2) to show that for some constant $\mu=\mu(G)$,

$$\mathbf{P}^{1,x}\Big(\inf_{3s/4\leqslant|x-y|\leqslant 5s/4} Y(y, R^1[0,T^1_{x,s/2}])>\mu\Big)>c(G). \tag{51}$$

Denote by $Z(\gamma)$ the event

$$\inf_{3s/4\leqslant|x-y|\leqslant 5s/4} Y(y,\gamma)>\mu,$$

where x is the beginning of the path γ.

Next, let $N=N(\varepsilon,G)$ be an integer parameter which will be fixed later. For $i=1,...,N-1$ define $T_i:=T^1_{v,(1+i/N)r}$. Let $x_i=R^1(T_i)$. Let $s=r/4N$. Define U_i to be the stopping times

$$U_i:=\min\big\{t>T_i:R^1(t)\in\partial B\big(x_i,\tfrac14 s\big)\big\}.$$

Finally define $Z_i=Z(R^1[T_i,U_i])$. Then (51) says that $\mathbf{P}(Z_i|x_i)>c(G)$. Since the only effect of $Z_1,...,Z_{i-1}$ on Z_i is through x_i, we get in fact that

$$\mathbf{P}(Z_i\,|\,Z_1,...,Z_{i-1})>c(G),$$

and hence

$$\mathbf{P}\bigg(\bigcap_{i=1}^{N-1}\neg Z_i\bigg)<(1-c(G))^{N-2}.$$

Let $\mathcal{Z}:=\bigcup_{i=1}^{N-1}Z_i$ and choose our parameter N such that $\mathbf{P}(\mathcal{Z})>1-\varepsilon$. Lemma 2.5 shows that for r larger than some constant $\nu(N,G)$ we have that the probability of R^2 to hit $\partial B(x_i,s)$ for any i and for any starting point z of R^2 is $\geqslant c(N,G)$. If λ is sufficiently large then $\partial B(x_i,s)\subset B\big(x_i,\frac54 s\big)$. Hence we get for any $i\in\{1,...,N\}$,

$$\begin{aligned}
&\mathbf{P}(R^2[0,T^2_{v,2r}]\cap R^1[T^1_{v,r},T^1_{v,2r}]\neq\varnothing\,|\,Z_i)\\
&\qquad\geqslant \mathbf{P}(R^2[0,T^2_{v,2s}]\cap R^1[T_i,U_i]\neq\varnothing\,|\,Z_i)\\
&\qquad\overset{(*)}{\geqslant}\mathbf{P}(T^2_{x_i,s}<T^2_{v,2r})\mathbf{E}(Y(R^2(T^2_{x_i,s}),R^1(T_i,U_i))\,|\,Z_i)\\
&\qquad\geqslant \mu\mathbf{P}(T^2_{x_i,s}<T^2_{v,2s})\\
&\qquad\geqslant \mu c(N,G),
\end{aligned}$$

where $(*)$ comes from the strong Markov property at the stopping time $T^2_{x_i,s}$. Hence,

$$\mathbf{P}(R^2[0,T^2_{v,2s}]\cap R^1[T^1_{v,s},T^1_{v,2s}]\,|\,\mathcal{Z})\geqslant \mu c(N,G).$$

This finishes the lemma: we fix λ and define $\delta(\varepsilon,G):=\mu c(N,G)$ and

$$C_6(\varepsilon,G):=\max\{8N\lambda,\nu(N,G)\},$$

and we are done. □

Lemma 3.11. *Let G be an isotropic graph and let M, K and ε be some parameters. Then, there exist $\delta(M,K,\varepsilon,G)$ and $C_7(M,K,\varepsilon,G)$ such that for all $v,w\in G$, and all $r>|v-w|$,*

$$\mathbf{P}^{1,w}\Big(\inf_{(*)}\mathbf{P}^{2,z}(R^2[0,T^2_{v,2r}]\cap R^1[0,T^1_{v,2r}]\neq\varnothing\,|\,R^1[0,T^1_{v,2r}])\geqslant r^{-\delta}\Big)<C_7r^{-M},$$

where the $(*)$ stands for all the $z\in B(v,r)$ such that $d(z,R^1[0,T^1_{v,2r}])\leqslant Kr^{1-\varepsilon}$.

The proof is identical to that of [L96b, Lemma 2.6] and we shall omit it. Very roughly, it uses the previous lemma and the Wiener shell test.

The next element in the proof of Theorem 3 is a coupling of random walk and Brownian motion, and we shall use, naturally, our coupling. For the benefit of those familiar with [L96b] let us make the following remark: Lawler uses the Skorokhod embedding to couple random walk and Brownian motion (this is done in [L96b, Chapter 3]). The Skorokhod embedding has the convenient property that the random walk and the Brownian motion have comparable times, that is $|R(t)-W(t)|\ll t^{1/4+\varepsilon}$ (after linear calibration). This is just not true in our case, or anyway would require non-linear adaptive calibration which is not worth messing with—measuring the Hausdorff distance between R and W is a completely adequate replacement. Hence we shall make no effort to give analogs of the results of Chapter 3 of [L96b] and continue immediately to Chapter 4. Lemma 3.13 is a replacement for Lawler's Lemma 4.1, but we first give an auxiliary result.

Lemma 3.12. *Let $G\subset\mathbf{R}^d$ be an isotropic graph. Let R^1,W^1 and R^2,W^2 be two independent pairs of coupled random walk and Brownian motion on G starting from v^1 and v^2, respectively. Let $s>r>2|v^1-v^2|$. Let $\mathcal{E}$ be an event depending on $R^i[0,T^i_{v,r}]$ and $W^i[0,S^i_{v,r}]$ only. Then*

$$\mathbf{P}(\mathcal{E}\cap\{W^1[S^1_{v,r},S^1_{v,s}]\cap W^2[S^2_{v,r},S^2_{v,s}]=\varnothing\})\leqslant C\Big(\frac{r}{s}\Big)^{\xi}(\mathbf{P}(\mathcal{E})+C(G)\exp(-r^{c(G)})),$$

where ξ is from (23).

The proof is identical to that of Lemma 3.8 of the present paper, with the use of Lemma 3.7 (probability of escape from a half-space) replaced by estimates for the non-intersection probability of two Brownian motions, see [L96a, (2)]. We omit the details.

Lemma 3.13. *Let G be an isotropic graph, let $v\in G$ and let R^1,W^1 and R^2,W^2 be two independent pairs of coupled random walk and Brownian motion on G starting from v^1 and v^2, respectively, $|v-v^i|\leqslant 2^m$, and stopped on $\partial B(v,2^n)$. Define $T^i_j:=T^i_{v,2^j}$,*

$S^i_j{:=}S^i_{v,2^j}$ *and*

$$Q^i_j := \{\subset_{\mathrm{Haus}}(R^i[T^i_{j-1},T^i_j],W^i[S^i_{j-2},S^i_{j+1}]) \geqslant 2^{j(1-c_4/2)}\} \\ \cup\{\subset_{\mathrm{Haus}}(W^i[S^i_{j-1},S^i_j],R^i[T^i_{j-2},T^i_{j+1}]) \geqslant 2^{j(1-c_4/2)}\},$$

$$Q^i_* := \{\subset_{\mathrm{Haus}}(R^i[0,T^i_{m+2}],W^i[0,S^i_{m+3}]) \geqslant 2^{(m+2)(1-c_4/2)}\} \\ \cup\{\subset_{\mathrm{Haus}}(W^i[0,S^i_{m+2}],R^i[0,T^i_{m+3}]) \geqslant 2^{(m+2)(1-c_4/2)}\},$$

$$\mathcal{Q} := Q^1_*\cup Q^2_*\cup \bigcup_{j=m+3}^{n-1}(Q^1_j\cup Q^2_j).$$

Then

$$\mathbf{P}(\mathcal{Q}\cap\{W^1[0,S^1_{n+1}]\cap W^2[0,S^2_{n+1}]=\varnothing\}) \leqslant C(G)\exp(-2^{mc(G)})2^{-(n-m)\xi}.$$

Proof. The corollary of Lemma 3.4 with $\nu=2$ shows that

$$\mathbf{P}(Q^i_j) \leqslant C(G)\exp(-2^{jc(G)}).$$

Next, Lemma 3.12 shows that

$$\mathbf{P}(Q^i_j\cap\{W^1[0,S^1_{n+1}]\cap W^2[0,S^2_{n+1}]=\varnothing\}) \leqslant C(G)\exp(-2^{jc(G)})2^{-(n-j)\xi}.$$

The Q^i_*'s have a similar estimate. Summing over i and j, we get the lemma. □

We now prove a lemma, the equivalent of [L96b, Corollary 4.2], somewhat stronger than the direction $\mathbf{P}(R^1\cap R^2=\varnothing)\geqslant c(|v_1-v_2|/r)^{-\xi}$ of (49). We will need the strengthening in the next chapter.

LEMMA 3.14. *Let G be an isotropic graph of dimension 2 or 3 and let $v\in G$ and $s\geqslant C_8(G)$. Let $v^1,v^2\in G\cap\big(B(v,s)\setminus B\big(v,\tfrac78 s\big)\big)$, $|v^1-v^2|>\tfrac14 s$, let $r>4s$, let η be a unit vector in $\mathbf{R}^d$ and define two subsets of G:*

$$\begin{aligned} U^1 &:= \big(B\big(v,\tfrac12 r\big)\setminus B(v,s)\big)\cup B\big(v^1,\tfrac14 s\big)\cup\big(\overline{B(v,r)}\cap\big\{w:\langle w-v,\eta\rangle\geqslant\tfrac14 r\big\}\big),\\ U^2 &:= \big(B\big(v,\tfrac12 r\big)\setminus B(v,s)\big)\cup B\big(v^2,\tfrac14 s\big)\cup\big(\overline{B(v,r)}\cap\big\{w:\langle w-v,\eta\rangle\leqslant-\tfrac14 r\big\}\big).\end{aligned} \tag{52}$$

Let R^1 and R^2 be two walks with R^i starting from v^i and stopped on $\partial B(v,r)$. Then

$$\mathbf{P}(\{R^1\cap R^2=\varnothing\}\cap\{R^i\subset U^i, i=1,2\}) \geqslant c(G)\Big(\frac{s}{r}\Big)^{\xi}. \tag{53}$$

Proof. This is now immediate. Indeed, consider slightly smaller domains (but extended outward):

$$V^1 := \big(B\big(v,\tfrac{5}{12}r\big)\setminus B\big(v,\tfrac{25}{24}s\big)\big)\cup B\big(v^1,\tfrac{5}{24}s\big)\cup\big\{w\in B(v,2r):\langle w-v,\eta\rangle\geqslant\tfrac13 r\big\},$$
$$V^2 := \big(B\big(v,\tfrac{5}{12}r\big)\setminus B\big(v,\tfrac{25}{24}s\big)\big)\cup B\big(v^2,\tfrac{5}{24}s\big)\cup\big\{w\in B(v,2r):\langle w-v,\eta\rangle\leqslant-\tfrac13 r\big\}.$$

Consider also the event $\mathcal{F}$ that W^1 and W^2 are reasonably far apart along their paths, namely

$$\mathcal{F}^i_j := \{d(W^i[S^i_{v,2^{j-3}},S^i_{v,2^j}],W^{3-i}[0,S^{3-i}_{v,2^j}])>2^{j(1-c_4/4)}\},$$
$$\mathcal{F} := \bigcap_{i=1}^{2}\ \bigcap_{j=\lfloor\log_2 s\rfloor}^{\lceil\log_2 2r\rceil}\mathcal{F}^i_j.$$

Then it follows, using techniques similar to [L96a, Corollaries 3.9, 3.11 and 3.12] and [L96b, Lemma 2.8], that for $s>C(G)$,

$$\mathbf{P}(\mathcal{F}\cap\{W^i[0,2r]\subset V^i, i=1,2\})\geqslant c\Big(\frac{s}{r}\Big)^{\xi}.$$

We couple W^i to R^i such that (R^1,W^1) is independent from (R^2,W^2), and consider the event $\mathcal{Q}$ from Lemma 3.13. If $\mathcal{Q}$ did not occur, then R^i is sufficiently close to W^i so that $W^i[0,S^i_{v,2r}]\subset V^i$ implies $R^i[0,T^i_{v,r}]\subset U^i$, if $s>C(G)$. Further, $\mathcal{F}\setminus\mathcal{Q}$ also implies that $R^1[0,T^1_{v,r}]\cap R^2[0,T^2_{v,r}]=\varnothing$. Finally, Lemma 3.13 shows that if $s>C(G)$ then $\mathbf{P}(\mathcal{Q})\leqslant\frac12 c(s/r)^{\xi}$, which finishes the lemma. □

Corollary. *Let G be an isotropic graph, let $v^1,v^2\in G$ and let $r>4|v^1-v^2|$. Let R^1 and R^2 be two walks starting from v^i and stopped on $\partial B(v^1,r)$. Then*

$$\mathbf{P}(R^1\cap R^2=\varnothing)\begin{cases} >c(G)(|v^1-v^2|/r)^{\xi},\\ =0,\end{cases}$$

where the notation means that $\mathbf{P}(R^1\cap R^2=\varnothing)$ is (depending on v_1, v_2 and r) either 0 or larger than $c(G)(|v^1-v^2|/r)^{\xi}$.

Proof. Using Lemma 3.14, we can fix a constant $\lambda=\lambda(G)$ such that for all $|v^1-v^2|>\lambda$ the first choice happens. Hence assuming that $|v^1-v^2|\leqslant\lambda$ and using Lemma 2.14 with $\varepsilon=\frac18$ and $s=\lambda$, we get that either one of the following two cases occurs:

(i) there are no two disjoint paths going from v^1 and v^2 to the exterior of $B\big(v^1,\varkappa\big(\frac18,\lambda,G\big)\big)$, where $\varkappa$ is from Lemma 2.14 (in this case the probability is 0 for every $r>\varkappa$);

(ii) for $\mu=\max\big\{\varkappa\big(\frac18,\lambda,G\big),C_8\big\}$ there are two disjoint simple paths γ^i starting from v^i and ending at w^i satisfying $w^i\in B(v^1,\mu)\setminus\overline{B\big(v^1,\frac78\mu\big)}$ and $B\big(w^i,\frac78\mu\big)\cap\gamma^{3-i}=\varnothing$.

In the second case, we use Lemma 3.14 and get

$$\mathbf{P}^{1,w^1,2,w^2}((\gamma^1\cup R^1[0,T^1_{v,r}])\cap(\gamma^2\cup R^2[0,T^2_{v,r}])=\varnothing)>c(G)\left(\frac{\mu}{r}\right)^{\xi},$$

where $\gamma^i\cap R^{3-i}=\varnothing$ is satisfied because $B\big(w^i,\frac{1}{4}\mu\big)\cap\gamma^{3-i}=\varnothing$, $\gamma^i\subset B(v^1,\mu)$ and the event of Lemma 3.14 includes that $R^i\cap B\big(v^1,\mu\big)\subset B\big(w^i,\frac{1}{4}\mu\big)$. In the case that R^i start from v^i, the probability that both follow γ^i until its end is $>c(G)$, which proves the corollary for $r>4\mu$. For $r<4\mu$ the lemma will hold automatically for a sufficiently small constant in its definition. □

3.8. The upper bound

Having settled the lower bound in Theorem 3, we need only the following lemma, which is slightly stronger than the upper bound (again, we will need the stronger version in the next chapter).

LEMMA 3.15. *Let G be an isotropic graph of dimension* 2 *or* 3. *Then for any $v^1,v^2\in G$, if R^1 and R^2 are two walks with R^i starting from v^i and stopped on $\partial B(v^1,r)$ then*

$$\mathbf{P}(R^1\cap R^2=\varnothing)\leqslant C(G)\left(\frac{|v^1-v^2|}{r}\right)^{-\xi}.$$

Proof. Assume that $\mathbf{P}(R^1\cap R^2=\varnothing)>0$ (in particular that $v^1\neq v^2$). Also assume, without loss of generality, that $r>4|v^1-v^2|$. Let

$$a_j=2^j|v^1-v^2|,\quad b_j=a_j^{1-c_4/2}\quad\text{and}\quad T^i_j:=T^i_{v^1,a_j}.$$

Define

$$g_j:=2^{j\xi}\mathbf{P}(R^1[0,T^1_j]\cap R^2[0,T^2_j]=\varnothing).$$

The corollary of Lemma 3.14 shows that $g_j>c(G)$. We need to show that $g_j\leqslant C(G)$. Let W^1 and W^2 be Brownian motions coupled to R^1 and R^2, respectively, i.e. the couples (R^1,W^1) and (R^2,W^2) are independent. Let $S^i_j=S^i_{v^1,a_j}$. Examine the event $\mathcal{F}_j$ that j is the last step where the W^i's intersect, namely,

$$\mathcal{F}^1_j=\{W^1]S^1_j,S^1_n]\cap W^2]S^2_j,S^2_n]=\varnothing\}\cap\{W^1[S^1_{j-1},S^1_j]\cap W^2[0,S^2_j]\neq\varnothing\}.$$

Define $\mathcal{F}^2_j$ replacing the roles of W^1 and W^2. The event $\{R^1[0,T^1_n]\cap R^2[0,T^2_n]=\varnothing\}\cap\mathcal{F}^1_j$ has a number of consequences:

(i) $R^1[0,T^1_{j-2}]\cap R^2[0,T^2_{j-2}]=\varnothing$. By definition, this event has probability $2^{(2-j)\xi}g_{j-2}$.

(ii) Next, we use the fact that the W^i's intersect, whereas the R^i's do not. The corollary of Lemma 3.4 shows that

$$\mathbf{P}(\subset_{\mathrm{Haus}}(W^1[S^1_{j-1},S^1_j],R^1[T^1_{j-2},T^1_{j+1}])>b_j)\leqslant C(G)\exp(-a_j^{c(G)}),$$
$$\mathbf{P}(\subset_{\mathrm{Haus}}(W^2[0,S^2_j],R^2[0,T^2_{j+1}])>b_j)\leqslant C(G)\exp(-a_j^{c(G)}),$$

and hence if we define $\mathcal{A}:=\{d(R^1[T^1_{j-2},T^1_{j+1}],R^2[0,T^2_{j+1}])\leqslant 2b_j\}$ ($\mathcal{A}$ standing for "almost intersecting"), we get

$$\mathbf{P}(\{W^1[S^1_{j-1},S^1_j]\cap W^2[0,S^2_j]\neq\varnothing\}\setminus\mathcal{A})\leqslant C(G)\exp(-a_j^{c(G)}). \tag{54}$$

Next, define the event $\mathcal{N}:=\{R^1[T^1_{j-2},T^1_{j+2}]\cap R^2[0,T^2_{j+2}]=\varnothing\}$ ($\mathcal{N}$ standing for "non-intersecting"). Lemma 3.11 allows us to estimate $\mathbf{P}(\mathcal{A}\cap\mathcal{N})$: we use it with the parameters $v=v^1$, $w=R^1(T^1_{j-2})$, $r=a_{j+1}$, $M=2\xi$, $K=2$ and $\varepsilon=\frac{1}{2}c_4$. We get that with probability $\geqslant 1-C_7\bigl(2\xi,2,\frac{1}{2}c_4,G\bigr)a_{j+1}^{-2\xi}$ in R^2,

$$\mathbf{P}^1(\mathcal{A}\cap\mathcal{N}\mid R^2[0,T^1_{j+2}],R^1(T^1_{j-2}))\leqslant a_j^{-\delta(2\xi,2,c_4/2,G)}. \tag{55}$$

Notice that we used the strong Markov property for the stopping time

$$\min\{t>T^1_{j-2}:R^1(t)\in B(v^1,a_{j+1})\text{ and }d(R^1(t),R^2[0,T^1_{j+2}])\leqslant 2b_j\}.$$

Since the events that R^i do not intersect up to T^i_{j-2} and everything that happens after the T^i_{j-2} are dependent only through $R^i(T^i_{j-2})$, and since (55) holds for any values of $R^1(T^1_{j-2})$, we get

$$\begin{aligned}\mathbf{P}(\mathcal{A}\cap\{R^1[0,T^1_{j+2}]\cap R^2[0,T^2_{j+2}]=\varnothing\})&\leqslant C(G)a_{j+1}^{-2\xi}+2^{(2-j)\xi}g_{j-2}a_j^{-c(G)}\\&\overset{(*)}{\leqslant} C(G)2^{-j\xi}g_{j-2}a_j^{-c(G)},\end{aligned}$$

where in $(*)$ we used the lower bound. Adding (54), we get

$$\begin{aligned}&\mathbf{P}(\{W^1[S^1_{j-1},S^1_j]\cap W^2[0,S^2_j]\neq\varnothing\}\cap\{R^1[0,T^1_{j+1}]\cap R^2[0,T^2_{j+1}]=\varnothing\})\\&\qquad\leqslant C(G)2^{-j\xi}g_{j-2}a_j^{-c(G)}+C(G)\exp(-a_j^{c(G)})\leqslant C(G)2^{-j\xi}g_{j-2}a_j^{-c(G)}.\end{aligned} \tag{56}$$

(iii) Finally, we use the condition $W^1]S^1_{j+2},S^1_n]\cap W^2]S^2_{j+2},S^2_n]=\varnothing$. Here, we employ Lemma 3.12 and together with (56) we get

$$\mathbf{P}(\{R^1[0,T^1_n]\cap R^2[0,T^2_n]\neq\varnothing\}\cap\mathcal{F}^1_j)\leqslant C(G)2^{-n\xi}g_{j-2}a_j^{-c(G)}. \tag{57}$$

An identical calculation holds for $\mathcal{F}_j^2$. We are almost done! We only need to remark that

$$\mathbf{P}\big(\{R^1[0,T_n^1]\cap R^2[0,T_n^2]=\varnothing\}\setminus\bigcup_{2\leqslant j\leqslant n-2,i=1,2}\mathcal{F}_j^i\big)$$
$$\leqslant \mathbf{P}(W^1[S_1^1,S_{n-2}^1]\cap W^2[S_1^2,S_{n-2}^2]=\varnothing)\leqslant C2^{-n\xi}\leqslant C2^{-n\xi}g_0,$$

and we get

$$g_n\leqslant C(G)\sum_{j=0}^{n-4}g_j(2^{-c(G)})^j.$$

By [L96b, Lemma 4.5], the g_i's are bounded and the bound depends only on the isotropic structure constants of G (we use here that $g_0\leqslant 1$), so the lemma and Theorem 3 are proved. □

4. Quasi-loops

Let γ be a path in a d-Euclidean net and let $v\in\mathbf{R}^d$. We say that γ has an (s,r)-*quasi-loop* near v if there exist a couple of points $\gamma(i),\gamma(j)\in B(v,s)$ such that $\operatorname{diam}\gamma[i,j]\geqslant r$. In this case we write $v\in\mathcal{QL}(s,r,\gamma)$. We take the v's in a grid such that the balls $B(v,s)$ cover $\mathbf{R}^d$, and define

$$\mathrm{QL}(s,r,\gamma):=\#\bigg(\mathcal{QL}(s,r,\gamma)\cap\frac{1}{d}s\mathbf{Z}^d\bigg).$$

Our purpose in this chapter is to prove that loop-erased random walk has no quasi-loops in the following sense.

THEOREM 4. *Let G be an isotropic graph of dimension* 2 *or* 3, *and let* $0<\varepsilon<1$. *Then there exists $\delta=\delta(\varepsilon,G)>0$ such that for all $v\in G$, all $r>C(\varepsilon,G)$ and any subset $v\in\mathcal{D}\subset B(v,r)$,*

$$\mathbf{E}^v\,\mathrm{QL}(r^{1-\varepsilon},r^{1-\delta},\mathrm{LE}(R[0,T(\partial\mathcal{D})]))\leqslant C(\varepsilon,G)r^{-\delta}.$$

Dimensions 2 and 3 are very different. The proof for dimension 2 was done in the case of $\mathbf{Z}^2$ by Schramm [S00, Lemma 3.4] and is practically the same in our more general settings ([K, Lemma 18] is another variation on Schramm's argument). Therefore we shall only sketch the required elements in the end of the chapter. We shall concentrate on dimension 3. It turns out that the techniques we use will rely heavily on the non-intersection exponent and therefore work only for isotropic graphs. Hence an interesting conjecture appears.

CONJECTURE. *Theorem* 4 *holds for any Euclidean net.*

Again, this is true in dimension 2, hence the interesting case is dimension 3.

It will be convenient in many places to consider discontinuous paths. Therefore, if $\gamma:\{1,...,n\}\to G$ is some function (without the restriction that $\gamma(i)$ and $\gamma(i+1)$ are neighbors), $\mathrm{LE}(\gamma)$ will be defined using formula (6) literally, and is a simple discontinuous path. Likewise we will define $\gamma_1\cup\gamma_2$ even if $\gamma_1(\operatorname{len}\gamma_1)$ is not a neighbor of $\gamma_2(1)$. If γ is a (possibly discontinuous) path and A is some set, then $\gamma\cap A$ would stand for the discontinuous path created in the natural way from the parts of γ inside A, in order.

Here and below when we say "γ is a discontinuous path", we do not exclude the possibility that it is in effect continuous.

4.1. Cut times

For any path γ we define

$$\operatorname{cut}\gamma := \{\gamma(i):\gamma[0,i]\cap\gamma[i+1,\operatorname{len}\gamma]=\varnothing\}.$$

The i's satisfying the condition will be called *cut times* and the $\gamma(i)$'s will be called *cut points*. It is clear that $\operatorname{cut}\gamma\subset\mathrm{LE}(\gamma)$, indeed $\operatorname{cut}\gamma$ is contained in any connected subset of γ containing $\gamma(0)$ and $\gamma(\operatorname{len}\gamma)$. It will also be convenient to define

$$\operatorname{cut}(\gamma;t) := \{\gamma(i): i<t \text{ and } \gamma[0,i]\cap\gamma[i+1,\operatorname{len}\gamma]=\varnothing\}.$$

It has the useful property that $\operatorname{cut}(\gamma;t)$ is increasing in t and decreasing as γ is extended.

For a random walk R, $\operatorname{cut}R$ is intimately related to the non-intersection exponent ξ via time symmetry. Lemma 4.2 below has the details, but first we need some simple preparations.

LEMMA 4.1. *Let G be a three-dimensional Euclidean net, let $v\in\mathbf{R}^3$ and let $r>C(G)$. Let R^i be independent random walks on G starting from points in $B(v,r)$. Let $\mathcal{E}$ be an event which depends only on $R^i[0,T^i_{v,r}]$ and let $\mathcal{F}$ be an event that depends only on $R^i[T^i_{v,2r},\infty[$. Then*

$$\mathbf{P}(\mathcal{E}\cap\mathcal{F})\approx\mathbf{P}(\mathcal{E})\mathbf{P}(\mathcal{F}).$$

The constant implicit in the $\approx$ notation may depend on the number of walks, and on the isotropic structure constants.

Proof. For every $x\in\partial B(v,r)$ and $y\in\partial B(v,2r)$ let $\pi_{x,y}$ be the probability that a random walk starting from x will hit $\partial B(v,2r)$ at y, and let π_y be the probability that $R(T_{v,2r})=y$. By Harnack's inequality (Lemma 2.1) we have that $\pi_{x,y}\approx\pi_{x',y}$ for any $x,x'\in\partial B(v,r)$. Hence

$$\pi_y\approx\pi_{x,y}.$$

This gives

$$\begin{aligned}\mathbf{P}(\mathcal{E}\cap\mathcal{F}) &= \sum_{x^i,y^i} \mathbf{P}(\mathcal{E}\cap\{R^i(T^i_{v,r})=x^i \text{ for all } i\}) \prod_i \pi_{x^i,y^i} \mathbf{P}(\mathcal{F} \mid R^i(T^i_{v,2r})=y^i \text{ for all } i) \\ &\approx \sum_{x^i,y^i} \mathbf{P}(\mathcal{E}\cap\{R^i(T^i_{v,r})=x^i \text{ for all } i\}) \prod_i \pi_{y^i} \mathbf{P}(\mathcal{F} \mid R^i(T^i_{v,2r})=y^i \text{ for all } i) \\ &= \mathbf{P}(\mathcal{E})\mathbf{P}(\mathcal{F}). \end{aligned}$$

□

Lemma 4.2. *Let G be a three-dimensional isotropic graph. Let $v\in G$ and $r>C(G)$. Define the annulus $A:=B(v,2r)\setminus\overline{B(v,r)}$. Let $w\in B\big(v,\frac{1}{2}r\big)$ and let R^1 be a random walk starting from w. Let*

$$\mathcal{C} := \operatorname{cut}(R^1[0,\infty[;T^1_{v,4r}).$$

Let $z\in B\big(v,\frac{1}{2}r\big)$ and let R^2 be a random walk starting from z and stopped on $\partial B(v,4r)$. Then

$$\mathbf{P}(\mathcal{C}\cap R^2[0,T^2_{v,4r}]\cap A\neq\varnothing) > c_7(G).$$

The proof is a relatively straightforward application of second-moment methods, but is quite long. Hence we shall divide it into several shorter claims.

Sublemma 4.2.1. *There exists $C_9(G)$ such that for any $x\in G$ one of the following conditions holds*:

(i) *there are no two disjoint paths leading from x to $\partial B(x,C_9)$;*

(ii) *for any $r\geqslant C_9$ there exist two disjoint simple paths $\gamma^i\subset B(x,r)$ that satisfy that if y^i is the end point of γ^i then*

$$B\big(y^i,\tfrac{1}{4}r\big)\cap\gamma^{3-i}=\varnothing \quad \text{and} \quad y^i\in B(x,r)\setminus\overline{B\big(x,\tfrac{7}{8}r\big)}. \tag{58}$$

"Disjoint paths" here means except for the point x common to both.

Subproof. Let $\lambda=\lambda(G)$ satisfy that any edge in G has length $\leqslant\lambda$. Then Lemma 2.14 for $\varepsilon=\frac{1}{8}$, $s=\lambda$ and all neighbors of x gives the result with $C_9=\varkappa\big(\frac{1}{8},\lambda,G\big)$. □

Points x for which there exist two disjoint paths leading outside $B(x,C_9)$ will be called *$\mathcal{C}$-capable*.

Sublemma 4.2.2. *There exists a constant $C_{10}(G)$ such that any ball B of radius C_{10} contains at least one $\mathcal{C}$-capable point x.*

Subproof. Let γ be a path in $B\cap G$ such that the distance between its two ends y^1 and y^2 is $\geqslant 2C_{10}-C(G)$. We may assume that γ is simple (say by taking its loop-erasure). Let x be the point of γ closest to the plane exactly between y^1 and y^2. Then clearly the portions of γ up to x and from x on are disjoint paths that lead to distance at least $C_{10}-C(G)$, which proves the sublemma, if C_{10} is sufficiently large. □

Sublemma 4.2.3. *Let $x\in G$ and $\varrho>C(G)$. Let R^1 and R^2 be two random walks starting from x. Define subsets similar to* (52) *as follows*:

$$\begin{aligned} V^1 &:= B\big(x,\tfrac12\varrho\big)\cup\big(\overline{B(x,\varrho)}\cap\{y:\langle y-x,(1,0,0)\rangle\geqslant\tfrac14\varrho\}\big),\\ V^2 &:= B\big(x,\tfrac12\varrho\big)\cup\big(\overline{B(x,\varrho)}\cap\{y:\langle y-x,(1,0,0)\rangle\leqslant-\tfrac14\varrho\}\big). \end{aligned} \tag{59}$$

Notice that $\overline{B(x,\varrho)}$ above refers to closure in G. Define further the events

$$\mathcal{V}:=\{R^i[0,T^i_{x,\varrho}]\subset V^i\}_{i=1,2}\quad\textit{and}\quad\mathcal{N}:=\{R^1[0,T^1_{x,\varrho}]\cap R^2[1,T^2_{x,\varrho}]=\varnothing\}.$$

Then

$$\mathbf{P}^x(\mathcal{N})\approx\mathbf{P}^x\big(\mathcal{N}\cap\mathcal{V}\big)\begin{cases}\approx\varrho^{-\xi}, & \textit{if } x \textit{ is } \mathcal{C}\textit{-capable},\\ =0, & \textit{otherwise.}\end{cases}$$

Subproof. The case where x is not $\mathcal{C}$-capable is obvious if $\varrho>C_9$. In the second case, use Sublemma 4.2.1 with its r equal to $\sigma:=\max\{C_9,C_8\}$ (with C_8 from Lemma 3.14) and get two disjoint paths γ^i ending in y^i satisfying (58). This allows us to use Lemma 3.14 with walks starting from the y^i's, the v, s and r of Lemma 3.14 being equal to x, σ and $\frac12\varrho$, respectively, and with $\eta=(1,0,0)$. We get

$$\mathbf{P}^{1,y_1,2,y_2}(\mathcal{N}\cap\{R^i[0,T^i_{x,\varrho}]\subset U^i\}_{i=1,2})\approx\varrho^{-\xi}, \tag{60}$$

where U^i is defined in (52). In particular, $R^i\subset U^i$ shows that $R^i\cap B(x,\sigma)\subset B\big(y^i,\frac14\sigma\big)$ and hence, from (58), $R^i\cap\gamma^{3-i}=\varnothing$. Further, $R^i\subset U^i$ implies that $\gamma^i\cup R^i\subset V^i$. Finally, since the probability that the R^i's starting from x follow γ^i until y^i is ≈1, we get

$$\mathbf{P}(\mathcal{N}\cap\mathcal{V})\approx\varrho^{-\xi}.$$

To finish the sublemma, notice that $\mathbf{P}(\mathcal{N})\leqslant C(G)\varrho^{-\xi}$ follows from Lemma 3.15. □

Sublemma 4.2.4. *Let $x\in A$ and let R^1 and R^2 be two random walks starting from x. Define $\mathcal{H}:=T^1(w)<T^1_{v,4r}$ and $\mathcal{N}':=\{R^1[0,T^1(w)]\cap R^2[1,\infty]=\varnothing\}$. Then*

$$\mathbf{P}(\mathcal{N}'\cap\mathcal{H})\begin{cases}\geqslant c(G)r^{-1-\xi}, & \textit{if } x \textit{ is } \mathcal{C}\textit{-capable},\\ =0, & \textit{otherwise.}\end{cases}$$

Subproof. We use Sublemma 4.2.3 with $\varrho=\frac14r$ and get that (assuming that x is $\mathcal{C}$-capable)

$$\mathbf{P}(\mathcal{N}\cap\mathcal{V})\approx r^{-\xi}. \tag{61}$$

Examining the structure of the V^i's, it is not difficult to see that one may construct six domains $\mathcal{S}^i,\mathcal{H}^i,\mathcal{D}^i\subset B(0,5)$ with the following properties (see Figure 4):

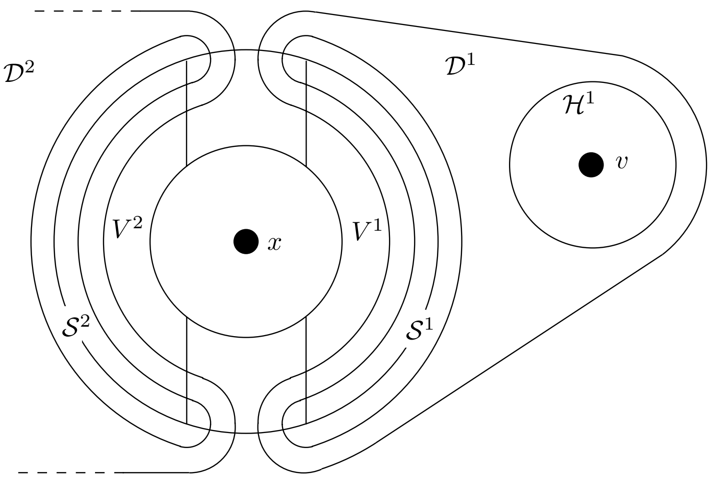

Figure 4. The $\mathcal{S}^i$, $\mathcal{H}^i$ and $\mathcal{D}^i$ in the figure are actually $v+r\mathcal{S}^i$, $v+r\mathcal{H}^i$ and $v+r\mathcal{D}^i$, respectively. $\mathcal{H}^2$ is not shown, imagine it "far away" inside $\mathcal{D}^2$.

(i) $\overline{\mathcal{S}}^i, \overline{\mathcal{H}}^i \subset \mathcal{D}^i$ and $\overline{\mathcal{S}}^i \cap \overline{\mathcal{H}}^i = \varnothing$;

(ii) if $r > C(G)$ then $\partial V^i \cap \partial B\big(x, \frac{1}{4}r\big) \subset v + r\mathcal{S}^i$;

(iii) $\mathcal{D}^1 \cap \mathcal{D}^2 = \varnothing$ and if $r > C(G)$ then $V^i \cap (v + r\mathcal{D}^{3-i}) = \varnothing$;

(iv) $\mathcal{D}^1 \subset B(0,3)$, $B\big(0, \frac{1}{2}\big) \subset \mathcal{H}^1$ and $B(0,4) \cap \mathcal{H}^2 = \varnothing$;

(v) the collection of $\mathcal{S}^i$, $\mathcal{H}^i$ and $\mathcal{D}^i$, for all x, v and r, satisfies the conditions of Lemma 2.5 (iii).

Condition (ii) ensures that under the event $\mathcal{V}$ we have $R^i(T^i_{x,\varrho}) \in v + r\mathcal{S}^i$. Hence we can apply Lemma 2.7 with $\mathcal{D}^1$, $\mathcal{S}^1$ and $\mathcal{H}^1$, and Lemma 2.5 with $\mathcal{D}^2$, $\mathcal{S}^2$ and $\mathcal{H}^2$ for the continuation of R^i after $T^i_{x,\varrho}$. We get that

$$\mathbf{P}(T^1(w) < T^1(\partial(v + r\mathcal{D}^1)) \mid R^1[0, T^1_{x,\varrho}] \subset V^1) > \frac{c(G)}{r},$$
$$\mathbf{P}(T^2_{v,4r} < T^2(\partial(v + r\mathcal{D}^2)) \mid R^2[0, T^2_{x,\varrho}] \subset V^2) > c(G).$$

On the other hand, condition (iii) ensures that if $R^i[T^i_{x,\varrho}, T(v + r\mathcal{H}^i)] \subset v + r\mathcal{D}^i$ then $R^1[0, T(w)] \cap R^2[T^2_{x,\varrho}, T(v + r\mathcal{H}^2)] = \varnothing$ and vice versa. Together with (61), we get

$$\mathbf{P}(\mathcal{N}'' \cap \{R^1[0, T^1(w)] \subset B(v, 3r)\}) > c(G) r^{-1-\xi},$$

where

$$\mathcal{N}'' := \{R^1[0, T^1(\{w\})] \cap R^2[1, T^2_{v,4r}] = \varnothing\}.$$

This ends the sublemma, since Lemma 2.6 with $\mathcal{D} = \mathbf{R}^3$ and $\mathcal{H} = B(0,3)$ shows that the probability of R^1 to never hit $B(v, 3r)$ after hitting $\partial B(v, 4r)$ is $\geqslant c(G)$. $\square$

Let $\mathcal{E}$ be an event on a space of curves. We say that $\mathcal{E}$ is *loop-monotone* if $\mathcal{E}(\gamma)\Rightarrow\mathcal{E}(\gamma')$ whenever γ is γ' with some loops added. In other words, adding loops can only hurt $\mathcal{E}$. A typical example of a loop-monotone event is $\{x\in\mathcal{C}\}$ for some x ($\mathcal{C}$ from the statement of Lemma 4.2). We shall use loop-monotonicity to encapsulate the idea of time reversal in a convenient way in the following sublemma.

SUBLEMMA 4.2.5. *Let $v,w,x\in G$ and let $\mathcal{E}\subset\{T(x)<T_{v,r}\}$ be a loop-monotone event on the space of curves on G starting from w. Then*

$$\mathbf{P}^w(\mathcal{E}(R[0,\infty[))\approx\mathbf{P}^{1,x,2,x}(\mathcal{E}(\overleftarrow{R^1}\cup R^2)\cap\{T^1(w)<T^1_{v,r}\}),$$

where the notation $\overleftarrow{R^1}\cup R^2$ means taking $R^1[0,T^1(w)]$, reversing it (so that it starts from w and ends at x), and concatenating $R^2[1,\infty]$ at its end x.

Subproof. Let $\mathcal{W}=\#\{t\geqslant 1:R(t)=w\}$. The loop-monotonicity of $\mathcal{E}$ gives

$$\mathbf{P}^w(\mathcal{E}\mid\mathcal{W}=k)\leqslant\mathbf{P}(\mathcal{E}\mid\mathcal{W}=0)\quad\text{for all }\ k>0, \tag{62}$$

since conditioning by $\mathcal{W}=k$ is equivalent to adding k closed paths from w to itself and then starting a walk conditioned to have $\mathcal{W}=0$. Hence we get

$$\mathbf{P}(\mathcal{E})=\sum_{k\geqslant 0}\mathbf{P}(\mathcal{E}\mid\mathcal{W}=k)\mathbf{P}(\mathcal{W}=k)\overset{(62)}{\leqslant}\mathbf{P}(\mathcal{E}\mid\mathcal{W}=0)\overset{(*)}{\leqslant}C(G)\mathbf{P}(\mathcal{E}\cap\{\mathcal{W}=0\}),$$

where $(*)$ comes from the transience of G. Since $\mathbf{P}(\mathcal{E})\geqslant\mathbf{P}(\mathcal{E}\cap\{\mathcal{W}=0\})$, we get

$$\mathbf{P}(\mathcal{E})\approx\mathbf{P}(\mathcal{E}\cap\{\mathcal{W}=0\}). \tag{63}$$

Next, we use the time-symmetry of random walk in the form (4) for the portion of the walk between w and x. We get

$$\mathbf{P}^w(\mathcal{E}\cap\{\mathcal{W}=0\})\approx\mathbf{P}^{1,x,2,x}(\mathcal{E}(\overleftarrow{R^1}\cup R^2)\cap\{T^1(w)<T^1(\{x\}\cup\partial B(v,r))\}), \tag{64}$$

where the $\approx$ sign hides the bounded quantity $\omega(w)/\omega(x)$.

The last step is defining

$$\mathcal{G}:=\mathcal{E}(\overleftarrow{R^1}\cup R^2)\cap\{T^1(w)<T^1_{v,r}\}\quad\text{and}\quad\mathcal{X}:=\#\{t\in[1,T^1(w)]:R^1(t)=x\}.$$

The loop-monotonicity of $\mathcal{E}$ gives

$$\mathbf{P}^{1,x,2,x}(\mathcal{G}\mid\mathcal{X}=k\text{ and }R^2=\gamma)\leqslant\mathbf{P}(\mathcal{G}\mid\mathcal{X}=0\text{ and }R^2=\gamma)\quad\text{for all }\ k>0\text{ and }\ \gamma,$$

which gives, like (63),

$$\mathbf{P}(\mathcal{G}\mid R^2=\gamma)\approx\mathbf{P}(\mathcal{G}\cap\{\mathcal{X}=0\}\mid R^2=\gamma)\quad\text{for all }\ \gamma,$$

and summing over all paths γ starting from x, we get

$$\mathbf{P}(\mathcal{G})\approx\mathbf{P}(\mathcal{G}\cap\{\mathcal{X}=0\}). \tag{65}$$

Equations (63), (64) and (65) together finish the proof. □

SUBLEMMA 4.2.6. *For any* $x \in A$,

$$\mathbf{P}(x \in \mathcal{C}) \begin{cases} > c(G) r^{-1-\xi}, & \text{if } x \text{ is } \mathcal{C}\text{-capable}, \\ = 0, & \text{otherwise.} \end{cases}$$

Subproof. This is an immediate consequence of Sublemmas 4.2.5 and 4.2.4. □

This completes what we would need for the estimate of the first moment, and we move to the second moment, which is not really all that more complicated—the complications from the fact that it is a second moment are partially compensated by the fact that we need an upper bound rather than a lower.

SUBLEMMA 4.2.7. *For any* $x \in A$,

$$\mathbf{P}(x \in \mathcal{C}) \leqslant C(G) r^{-1-\xi}.$$

Subproof. Sublemma 4.2.5 shows that

$$\begin{aligned} \mathbf{P}^w(x \in \mathcal{C}) &\approx \mathbf{P}^{1,x,2,x}(\{T^1(w) < T^1_{v,4r}\} \cap \{R^1[0, T^1(w)] \cap R^2[1, \infty[= \varnothing\}) \\ &\leqslant \mathbf{P}^{1,x,2,x}(\{T^1(w) < T^1_{v,4r}\} \cap \{R^1[0, T^1_{x,r/4}] \cap R^2[1, T^2_{x,r/4}] = \varnothing\}) \end{aligned}$$

and Lemma 4.1 shows that

$$\approx \mathbf{P}^{1,x}(T^1(w) < T^1_{v,4r}) \mathbf{P}^{1,x,2,x}(R^1[0, T^1_{x,r/4}] \cap R^2[1, T^2_{x,r/4}] = \varnothing).$$

Sublemma 4.2.3 shows that the term on the right is $\leqslant C(G) r^{-\xi}$, while (13) shows that the term on the left is $\approx r^{-1}$. □

SUBLEMMA 4.2.8. *For any* $x_1, x_2 \in A$,

$$\mathbf{P}(x_1, x_2 \in \mathcal{C}) \leqslant C(G)(r|x_1 - x_2|)^{-1-\xi}.$$

Subproof. First let us note that it is possible to assume that $|x_1 - x_2| > C(G)$, since otherwise $\mathbf{P}(x_1, x_2 \in \mathcal{C}) \leqslant \mathbf{P}(x_1 \in \mathcal{C})$, and then Sublemma 4.2.7 applies. Moreover, it is enough to prove that

$$\mathbf{P}(\{x_1, x_2 \in \mathcal{C}\} \cap \mathcal{O}) \leqslant C(G)(r|x_1 - x_2|)^{-1-\xi},$$

where $\mathcal{O} := \{T_1 < T_2\}$ and T_i is the last time R is in x_i. The other case is just a renaming of x_1 and x_2.

Define now $\varrho := \frac{1}{16}|x_1 - x_2|$ and $x = \frac{1}{2}(x_1 + x_2)$. Let $\mathcal{X} = \{x_1, x_2 \in \mathcal{C}\} \cap \mathcal{O}$. $\mathcal{X}$ is loop-monotone, hence we may use Sublemma 4.2.5 for x_1 and get

$$\mathbf{P}^w(\mathcal{X}) \approx \mathbf{P}^{1,x_1,2,x_1}(\{T^1(w) < T^1_{v,4r}\} \cap \{R^1 \cap R^2 = \varnothing\} \\ \cap \{\text{there exists } t < T^2_{v,4r} : \{R^2(t) = x_2\} \cap \{(R^1 \cup R^2[0,t])) \cap R^2[t+1,\infty[= \varnothing\}\}),$$

where the R^i's in the expression $R^1 \cap R^2 = \varnothing$ stand for the walks until their natural endings, namely $R^1[0, T^1(w)]$ and $R^2[1, \infty[$, respectively. Denote $R^2(T^2_{x_1, 2\varrho})$ by y and "stop" R^2 there, and consider the rest of R^2 as a new random walk R^3 starting from y. We get

$$\begin{aligned} \mathbf{P}^w(\mathcal{X}) \approx \sum_{y \in \partial B(x_1, 2\varrho)} \mathbf{P}^{1,x_1,2,x_1,3,y}(&\{T^1(w) < T^1_{v,4r}\} \cap \{R^2(T^2_{x_1,2\varrho}) = y\} \\ &\cap \{R^1 \cap (R^2 \cup R^3) = \varnothing\} \\ &\cap \{\text{there exists } t < T^3_{v,4r} : \{R^3(t) = x_2\} \\ &\quad \cap \{(R^1 \cup R^2 \cup R^3[0,t]) \cap R^3[t+1, \infty[= \varnothing\}\}), \end{aligned} \tag{66}$$

where R^2 stands for $R^2[1, T^2_{x_1, 2\varrho}]$ and R^3 stands for $R^3[0, \infty[$. We use Sublemma 4.2.5 again, this time for the random walk R^3 and the point x_2 (it is easy to see that the corresponding event is loop-monotone for any value of R^1 and R^2). We get

$$\begin{aligned} \mathbf{P}^w(\mathcal{X}) \approx \sum_y \mathbf{P}^{1,x_1,2,x_1,3,x_2,4,x_2}(&\{T^1(w) < T^1_{v,4r}\} \cap \{R^2(T^2_{x_1,2\varrho}) = y\} \\ &\cap \{T^3(y) < T^3_{v,4r}\} \cap \{R^1 \cap (R^2 \cup R^3 \cup R^4) = \varnothing\} \\ &\cap \{(R^1 \cup R^2 \cup R^3) \cap R^4 = \varnothing\}), \end{aligned} \tag{67}$$

where R^3 stands for $R^3[0, T^3(y)]$ and R^4 stands for $R^4[1, \infty[$. Reducing slightly the non-intersecting sections, we may write

$$\begin{aligned} \mathbf{P}^w(\mathcal{X}) \leqslant C(G) \sum_y \mathbf{P}(&\{T^1(w) < T^1_{v,4r}\} \cap \{R^2(T^2_{x_1,2\varrho}) = y\} \cap \{T^3(y) < T^3_{v,4r}\} \\ &\cap \{R^1[0, T^1_{x_1,\varrho}] \cap R^2[1, T^2_{x_1,\varrho}] = \varnothing\} \cap \{R^3[0, T^3_{x_2,\varrho}] \cap R^4[1, T^4_{x_2,\varrho}] = \varnothing\} \\ &\cap \{R^1[T^1_{x,18\varrho}, T^1_{x,r/4}] \cap R^4[T^4_{x,18\varrho}, T^4_{x,r/4}] = \varnothing\}). \end{aligned}$$

Denote the three non-intersection events above by $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$, respectively. We understand that if $18\varrho > \frac{1}{4}r$ then $\mathcal{N}_3$ is considered to always be satisfied. Now, Sublemma 4.2.3 shows that

$$\mathbf{P}(\mathcal{N}_1) \leqslant C(G)\varrho^{-\xi} \quad \text{and} \quad \mathbf{P}(\mathcal{N}_2) \leqslant C(G)\varrho^{-\xi},$$

and since these events are independent the probability of their intersection is $\leqslant C(G)\varrho^{-2\xi}$. Assume for a moment that $18\varrho \leqslant \frac{1}{4}r$. Then we use Lemma 4.1 for the ball $B(x, 9\varrho)$ and get

$$\mathbf{P}(\mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3) \approx \mathbf{P}(\mathcal{N}_1 \cap \mathcal{N}_2)\mathbf{P}(\mathcal{N}_3),$$

and Theorem 3 shows that $\mathbf{P}(\mathcal{N}_3)\leqslant C(G)(\varrho/r)^{\xi}$, so in total

$$\mathbf{P}(\mathcal{N}_1\cap\mathcal{N}_2\cap\mathcal{N}_3)\leqslant C(G)(r\varrho)^{-\xi}. \tag{68}$$

If $18\varrho>\frac{1}{4}r$, then $(r\varrho)^{-\xi}\approx\varrho^{-2\xi}$ and (68) is again satisfied, so we can continue without the assumption $18\varrho\leqslant\frac{1}{4}r$.

Finally we need to accommodate the various hitting and exit conditions in (67). Let $\mathcal{E}$ be the end points of the portions of the R^i's needed for the $\mathcal{N}_i$'s, namely

$$\mathcal{E}:=(R^1(T^1_{x,r/4}),R^2_{x_1,\varrho},R^3_{x_2,\varrho}).$$

For the condition $T^3(y)<T^3_{v,4r}$ we use the fact that for any point z where R^3 exits $B(x_2,\varrho)$ we have $|z-y|\geqslant\varrho$, and therefore the estimate of the harmonic potential (12) gives

$$\mathbf{P}(T^3(y)<T^3_{v,4r}\,|\,\mathcal{E})\leqslant\mathbf{P}(T^3(y)<\infty\,|\,\mathcal{E})\leqslant C(G)\varrho^{-1}. \tag{69}$$

A similar argument for R^1 gives

$$\mathbf{P}(T^1(w)<T^1_{v,4r}\,|\,\mathcal{E})\leqslant C(G)r^{-1}. \tag{70}$$

Conditioning over $\mathcal{E}$ the events of (69) and (70), $\mathbf{P}(R^2(T^2_{x_1,2\varrho})=y)$ and $\mathcal{N}_1\cap\mathcal{N}_2\cap\mathcal{N}_3$ are all independent. Hence we get

$$\begin{aligned}\mathbf{P}^w(\mathcal{X})&\overset{(*)}{\leqslant}\sum_E C(G)\varrho^{-1}r^{-1}\mathbf{P}(\mathcal{N}_1\cap\mathcal{N}_2\cap\mathcal{N}_3\cap\{\mathcal{E}=E\})\sum_{y\in\partial B(x_1,2\varrho)}\mathbf{P}(R^2(T^2_{x_1,2\varrho})=y\,|\,\mathcal{E}=E)\\&=C(G)\varrho^{-1}r^{-1}\mathbf{P}(\mathcal{N}_1\cap\mathcal{N}_2\cap\mathcal{N}_3)\overset{(68)}{\leqslant}C(G)(\varrho r)^{-1-\xi},\end{aligned}$$

where in $(*)$ we used (69), (70) and independence. □

Proof of Lemma 4.2. Let

$$\mathcal{X}=\#\{\mathcal{C}\cap R^2[0,T^2_{v,4r}]\cap A\}.$$

Sublemma 4.2.6 shows that

$$\mathbf{E}\mathcal{X}=\sum_{x\in A}\mathbf{P}(x\in\mathcal{C})\mathbf{P}(x\in R^2[0,T^2_{v,4r}])\overset{(*)}{\geqslant}c(G)r^{-2-\xi}\#\{x\in A:x\text{ is }\mathcal{C}\text{ capable}\}\overset{(**)}{\geqslant}c(G)r^{1-\xi},$$

where in $(*)$ we used Sublemma 4.2.6 to estimate $\mathbf{P}(x\in\mathcal{C})$ and formula (13) to estimate $\mathbf{P}(x\in R^2[0,T^2_{v,4r}])$; and $(**)$ follows from Sublemma 4.2.2. Correspondingly, we have

$$\begin{aligned}\mathbf{E}\mathcal{X}^2&=\sum_{x_1,x_2\in A}\mathbf{P}(x_1,x_2\in\mathcal{C})\mathbf{P}(x_1,x_2\in R^2[0,T^2_{v,4r}])\\&\overset{(*)}{\leqslant}C(G)\sum_{x_1,x_2\in A}(r|x_1-x_2|)^{-2-\xi}\\&\overset{(**)}{\leqslant}C(G)r^{-2-\xi}\sum_{x_1\in A}\sum_{n=1}^{\log_2 r}2^{n(1-\xi)}\\&\overset{(\dagger)}{\leqslant}C(G)r^{2-2\xi}.\end{aligned}$$

Here, inequality (∗) follows from Sublemma 4.2.8 for $\mathbf{P}(x_1, x_2 \in \mathcal{C})$ and from (12) for $\mathbf{P}(x_1, x_2 \in R^2[0, T^2_{v,4r}])$; inequality (∗∗) comes from the volume estimate

$$\#\{x_2 : |x_1 - x_2| \in [2^n, 2^{n+1}[\} \approx 2^{3n}$$

for $n > C(G)$, since our graph G is roughly isometric to $\mathbf{R}^3$; and inequality (†) comes from the same volume estimate since $r > C(G)$, and (finally!) from $\xi < 1$. The well-known inequality $\mathbf{P}(\mathcal{X} > 0) \geqslant (\mathbf{E}\mathcal{X})^2 / \mathbf{E}\mathcal{X}^2$ now finishes the lemma. □

Corollary. *Under the assumptions of Lemma* 4.2,

$$\mathbf{P}^{1,w}\big(\mathbf{P}^{2,z}(\mathcal{C} \cap R^2[0, T^2_{v,4r}] \cap A \neq \varnothing \mid R^1[0, \infty[) > c(G) \textit{ for all } z \in B\big(v, \tfrac{1}{2}r\big)\big) > c(G).$$

Proof. Denote the event inside the inner $\mathbf{P}$ symbol by $\mathcal{E}$. Then Lemma 4.2 shows that $\mathbf{P}^{1,w,2,v}(\mathcal{E}) \geqslant c(G)$. This shows that

$$\mathbf{P}^{1,w}(\mathbf{P}^{2,v}(\mathcal{E} \mid R^1) > c(G)) > c(G).$$

Now, for any infinite path γ starting from w, the function

$$f(z) = \mathbf{P}^{2,z}(\mathcal{E} \mid R^1[0, \infty[= \gamma)$$

is harmonic outside A and in particular in $B\big(v, \frac{1}{2}r\big)$. Hence, Harnack's inequality (Lemma 2.1) shows that $\min_{z \in B(v, r/2)} f(z) \geqslant c f(v)$, which proves the corollary. □

4.2. Conditioned random walks

Lemma 4.3. *Let G be a three-dimensional isotropic graph. Let $v \in G$ and let $H \subset \mathbf{R}^3$ be a closed half-space with $v \in \partial_{\mathrm{cont}} H$. Let $r > C(G)$ and let $\Gamma \subset B(v, r)$, $d(\Gamma, H) > C_5$. Let R be a random walk starting from v. Then*

$$\mathbf{P}(R(T_{v,r}) \in H \mid R[0, T_{v,r}] \cap \Gamma = \varnothing) \geqslant c_8(G).$$

(The constant C_5 is from Lemma 3.9, p. 76. In particular $d(\Gamma, H) > C_5$ implies that the set of paths from v to $\partial B(v, r)$ not intersecting Γ is non-empty—use the lemma for a translation of H by C_5.)

Proof. The equivalent question for a Brownian motion can be solved by reflecting through $\partial_{\mathrm{cont}} H$ the last section of the motion not intersecting $\partial_{\mathrm{cont}} H$, with the result that the corresponding probability is $\geqslant \frac{1}{2}$. Our proof is a discrete version of this idea.

Formally, denote by $\mathcal{H}$ (resp. $\mathcal{H}^-$) the space of all paths from v to $\partial B(v,r)\cap H$ (resp. $\partial B(v,r)\setminus H$) not intersecting Γ. We shall dissect $\mathcal{H}^-$ into disjoint sets N_γ^- indexed by $\mathcal{G}$:

$$\mathcal{H}^- = \bigcup_{\gamma\in\mathcal{G}} N_\gamma^-,$$

and map each N_γ^- into a set $N_\gamma\subset\mathcal{H}$ such that the following holds:

(i) $\mathbf{P}(N_\gamma)\geqslant c(G)\mathbf{P}(N_\gamma^-)$;

(ii) every path $h\in\mathcal{H}$ is contained in at most $C(G)$ different N_γ's.

Together these properties show that $\mathbf{P}(\mathcal{H})\geqslant c(G)\mathbf{P}(\mathcal{H}^-)$, or equivalently

$$\mathbf{P}(R(T_{v,r})\in H \mid R[0,T_{v,r}]\cap\Gamma=\varnothing)\geqslant c(G)\mathbf{P}(R(T_{v,r})\notin H \mid R[0,T_{v,r}]\cap\Gamma=\varnothing),$$

which would conclude the lemma.

The set $\mathcal{G}$ is the set of all paths γ in $B(v,r)$ avoiding Γ such that $x:=\gamma(\operatorname{len}\gamma)\notin H$ but its neighbor $x':=\gamma(\operatorname{len}\gamma-1)\in H$. For each $\gamma\in\mathcal{G}$ the set N_γ^- is the set of all paths that follow γ until its end and then avoid hitting $H\cup\Gamma$ until hitting $\partial B(v,r)$. It is clear that $\{N_\gamma^-\}_{\gamma\in\mathcal{G}}$ are disjoint sets covering $\mathcal{H}^-$. Take one $\gamma\in\mathcal{G}$, let x be its end and denote $\varrho:=d(x,\partial B(v,r))$. Clearly $\mathbf{P}(N_\gamma^-)$ is the probability that R follows γ (denote it by p_γ) multiplied by the escape probability

$$\mathbf{P}^x(T_{v,r}<T(\Gamma\cup H))\leqslant\mathbf{P}^x(T_{v,r}<T(H))\leqslant\mathbf{P}^x(T_{x,\varrho}<T(H))\overset{(*)}{\leqslant}\frac{C(G)}{\varrho},$$

where $(*)$ comes from Theorem 2.

We shall now construct N_γ under the assumption that ϱ is larger than some constant $\varrho_0(G)$. The value of ϱ_0 will be fixed later on, but for now we need $\varrho_0>4C_5$. We use Lemma 3.9 with the point x', the radius $4C_5$ and the half-space $H'=H+B(0,C_5)$, and we get that there exists a simple path $\delta'\subset\overline{B(x',4C_5)}\cap H'$ from x' to

$$\partial B(x',4C_5)\cap\{y\in H: d(y,\partial_{\mathrm{cont}}H)>C_5\}.$$

Let $\delta=\gamma\cup(x,x')\cup\delta'$. Note that $\Gamma\cap H'=\varnothing$, and therefore also $\Gamma\cap\delta=\varnothing$. Let N_γ be the family of all paths that follow δ until its end and then stay inside H until they exit $B(v,r)$. If $\varrho\leqslant\varrho_0$ simply let δ' be the shortest path from x to $\partial B(v,r)\cap H$ not intersecting Γ and let N_γ contain only the path $\gamma\cup\delta'$.

The lemma will be concluded once we show (i) and (ii). To see (i), first note that the case where $\varrho\leqslant\varrho_0$ is obvious since then $\mathbf{P}(N_\gamma^-)\approx p_\gamma\approx\mathbf{P}(N_\gamma)$. In the case $\varrho>\varrho_0$, the length of δ' is $\leqslant C(G)$, so the probability to follow δ is $\geqslant c(G)p_\gamma$. We use Lemma 3.9 again to get that for a random walk starting from $y:=\delta(\operatorname{len}\delta)$, the probability to hit

$$\alpha:=\partial B(y,\varrho)\cap\bigl\{z: d(z,\partial_{\mathrm{cont}}H)>\tfrac{1}{2}\varrho\bigr\}$$

before hitting ∂H is $\geqslant C(G)/\varrho$. Finally, Lemma 2.5 shows that for any $z\in\alpha$ a random walk starting from z has a probability $>c(G)$ to exit $B(v,r)$ before hitting ∂H. To use Lemma 2.5 we need to assume that ϱ is large enough, and this is the condition for ϱ_0 which can now be fixed. All three together give (i).

As for (ii), it is easy to see that every $h\in\mathcal{H}$ can belong to only boundedly many N_γ for which $\varrho\leqslant\varrho_0$. Hence, examine the case $\varrho>\varrho_0$ and let $h\in\mathcal{H}$. If $h\in N_\gamma$ then $x\notin H$, but after y all points of h are in H and the path between x and y is in $B(x,C(G))$. Therefore if we define $e(h)$ as the last vertex in $h\setminus H$, we know that $x\in B(e(h),C(G))$, and in particular has just $C(G)$ possibilities. Since γ is simply the part of h up to x, we see that it too has only $C(G)$ possibilities, which shows (ii) and the lemma. □

LEMMA 4.4. *Let G be a three-dimensional isotropic graph and let $\varepsilon>0$. Then there exist $q=q(G)>0$ and $\delta=\delta(\varepsilon,G)>0$ such that the following holds. Let $v\in G$, $r>C(\varepsilon,G)$ and $s\in[r,2r-\varepsilon r]$. Let $\Gamma\subset B(v,s)$ be some set such that*

$$\mathbf{P}^v(R[0,T_{v,4r}]\cap\Gamma\neq\varnothing)\leqslant\delta. \tag{71}$$

Let $w\in\partial B(v,s)$ be admissible (see below). Then

$$\mathbf{P}^{1,w}(\mathbf{P}^{2,y}(\mathcal{I})>q \text{ for all } y\in B(w,\varepsilon r)\mid R^1[0,T^1_{v,4r}]\cap\Gamma=\varnothing)>q, \tag{72}$$

where

$$\mathcal{I}:=\{\operatorname{cut}(R^1[0,T^1_{v,4r}];T^1_{w,\varepsilon r})\cap R^2[0,T^2_{v,4r}]\neq\varnothing\}.$$

We call w *admissible* if there exists a path $\gamma\subset\overline{B(w,16C_5)}$ starting from w and ending outside $B(v,|v-w|+2C_5)$ which does not intersect Γ (the constant $16C_5$ will be used in Lemma 4.5 below to show that many admissible points exist).

In words, the lemma says that the fact that $\operatorname{cut}(R^1)$ is hittable does not change if one conditions on not hitting Γ, even if one starts very close to Γ—the only condition is that Γ is not very hittable ($<\delta$) from far away (v). The fact that ε affects only δ but not q will play a significant role later on.

Proof. Let $\lambda=\lambda(G)$ be some parameter that will be fixed later. Let also $\mu:=\varepsilon/4\lambda$ and $\varrho=\mu r$. Let γ be the path from the definition of admissibility of w and assume, without loss of generality, that it is simple (say by taking LE). We use Lemma 4.3 with the starting point being $w':=\gamma(\operatorname{len}\gamma)$; with H' being the half-space orthogonal to the segment $[v,w]$ such that $w'\in\partial_{\mathrm{cont}}H'$ and $v\notin H'$; and with radius ϱ' to be fixed later. (Note that the condition of Lemma 4.3, $d(\Gamma,H')>C_5$, will be fulfilled if r is sufficiently large). We get that if $\varrho'>C(G)$ then

$$\mathbf{P}^{1,w'}(R^1(T^1_{w',\varrho'})\in H'\mid R^1[0,T^1_{w',\varrho'}]\cap\Gamma=\varnothing)\geqslant c(G).$$

Let H be the translation of H' such that $w\in\partial_{\rm cont}H$. On one hand, the probability that a random walk R^1 starting from w will follow γ until w' is $\geqslant c(G)$. On the other hand, if $\varrho'=\varrho-C(G)$ for some $C(G)$ sufficiently large, then for any point $x\in\partial B(w',\varrho')\cap H'$ there is a probability $\geqslant c(G)$ that a random walk starting from x will hit $\partial B(w,\varrho)\cup\Gamma$ in $\partial B(w,\varrho)\cap H$. All this allow us to drop the $'$ notation and we get

$$\mathbf{P}^{1,w}(R^1(T^1_{w,\varrho})\in H \mid R^1[0,T^1_{w,\varrho}]\cap\Gamma=\varnothing)\geqslant c(G). \tag{73}$$

Denote the event $R^1(T^1_{w,\varrho})\in H$ by $\mathcal{H}$.

Next define $\mathcal{C}=\mathrm{cut}(R^1[T^1_{w,\varrho},T^1_{v,4r}];T^1_{w,\varepsilon r})$ and $A=B\big(w,\tfrac12\varepsilon r\big)\setminus\overline{B\big(w,\tfrac14\varepsilon r\big)}$. The corollary of Lemma 4.2 shows that, for any $x\in\partial B(w,\varrho)$ and some $c(G)$,

$$\begin{aligned}\mathbf{P}^{1,x}\big(&\mathbf{P}^{2,y}(\mathcal{C}\cap R^2\cap A\neq\varnothing \mid R^1)>c(G)\text{ for all } y\in B\big(w,\tfrac18\varepsilon r\big)\big)\\ &\geqslant \mathbf{P}^{1,x}\big(\mathbf{P}^{2,y}(\mathrm{cut}(R^1[0,\infty[;T^1_{w,\varepsilon r})\cap R^2[0,T^2_{w,\varepsilon r}]\cap A\neq\varnothing \mid R^1)>c(G)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{for all } y\in B\big(w,\tfrac18\varepsilon r\big)\big)\\ &\geqslant c(G),\end{aligned}\tag{74}$$

where the notation R^i (e.g. R^2 above) stands for $R^i[0,T^i_{v,4r}]$, and where the $T^1_{w,\varrho}$ in the definition of $\mathcal{C}$ is considered to be 0 when starting from x. Since I promised to prove the lemma for any $y\in B(w,\varepsilon r)$, just note that for any such y we have that with probability $>c(G)$ the walk R^2 hits $B\big(w,\frac18\varepsilon r\big)$, and therefore (74) holds for the larger ball too, i.e.

$$\mathbf{P}^{1,x}(\mathbf{P}^{2,y}(\mathcal{C}\cap R^2\cap A\neq\varnothing \mid R^1)>c(G)\text{ for all } y\in B(w,\varepsilon r))\geqslant c(G). \tag{75}$$

Denote this event by $\mathcal{J}$.

Next, we take into consideration λ. Using (5) with Green function $G(\,\cdot\,,w;B(v,4r))$ shows that

$$\mathbf{P}^{1,x}(R^1[T^1_{w,\varepsilon r/4},T^1_{v,4r}]\cap B(w,\varrho)\neq\varnothing)\leqslant\frac{C(G)}{\lambda}. \tag{76}$$

Denote this event by $\mathcal{K}$.

The next step is saying, roughly, "if Γ is not hittable, then conditioning on not hitting Γ has no effect". Formally, we assume that, for some $\nu=\nu(G)$ to be fixed later

$$\mathbf{P}^{1,x}(R^1\cap\Gamma\neq\varnothing)\leqslant\nu\quad\text{for all } x\in\partial B(w,\varrho)\cap H. \tag{77}$$

As we shall see later, this assumption will be satisfied with a proper choice of δ. For now, this allows us to preform the following calculation, which will return us to a walk starting from w:

$$\begin{aligned}
&\mathbf{P}^{1,w}(\mathcal{J}\setminus\mathcal{K}\,|\,R^1\cap\Gamma=\varnothing)\\
&\qquad\geqslant \mathbf{P}^{1,w}((\mathcal{J}\setminus\mathcal{K})\cap\mathcal{H}\,|\,R^1\cap\Gamma=\varnothing)\\
&\qquad= \sum_{x\in\partial B(w,\varrho)\cap H} \mathbf{P}^{1,w}((\mathcal{J}\setminus\mathcal{K})\cap\{R^1(T^1_{w,\varrho})=x\}\,|\,R^1\cap\Gamma=\varnothing)\\
&\qquad\overset{(*)}{=} \sum_{x\in\partial B(w,\varrho)\cap H} \mathbf{P}^{1,w}(R^1(T^1_{w,\varrho})=x\,|\,R^1[0,T^1_{w,\varrho}]\cap\Gamma=\varnothing)\\
&\qquad\qquad\qquad\times\frac{\mathbf{P}^{1,x}((\mathcal{J}\setminus\mathcal{K})\cap\{R^1\cap\Gamma=\varnothing\})}{\mathbf{P}^{1,w}(R^1\cap\Gamma=\varnothing\,|\,R^1[0,T^1_{w,\varrho}]\cap\Gamma=\varnothing)} \qquad (78)\\
&\qquad\overset{(**)}{\geqslant}\left(c(G)-\frac{C(G)}{\lambda}-\nu\right)\sum_x \mathbf{P}^{1,w}(R^1(T^1_{w,\varrho})=x\,|\,R^1[0,T^1_{w,\varrho}]\cap\Gamma=\varnothing)\\
&\qquad=\left(c(G)-\frac{C(G)}{\lambda}-\nu\right)\mathbf{P}^{1,w}(\mathcal{H}\,|\,R^1[0,T^1_{w,\varrho}]\cap\Gamma=\varnothing)\\
&\qquad\overset{(73)}{\geqslant}\left(c(G)-\frac{C(G)}{\lambda}-\nu\right)c(G),
\end{aligned}$$

where $(*)$ comes from the definition of conditioned probability, and $(**)$ comes from the estimates (75) for $\mathcal{J}$ and (76) for $\mathcal{K}$, from the assumption (77) and from bounding the denominator by 1. Picking λ sufficiently large and ν sufficiently small, we get that the result of the computation is positive and dependent on the isotropic structure constants of G only.

Finally, notice that if $\mathcal{K}$ does not occur, i.e. if $R^1[T^1_{w,\varepsilon r/4},T^1_{v,4r}]$ does not return to the ball $B(w,\varrho)$, then

$$\operatorname{cut}(R^1[T^1_{w,\varrho},T^1_{v,4r}];T^1_{w,\varepsilon r})\cap A\subset \operatorname{cut}(R^1[0,T^1_{v,4r}];T^1_{w,\varepsilon r}),$$

so (78) gives us (72) with an appropriate choice of q. Hence we need only justify (77).

To see (77) we use Harnack's inequality (Lemma 2.2) for the family of domains (see Figure 5)

$$\begin{aligned}
\mathcal{D}_z&=B(0,2)\setminus\overline{B(0,|z|)},\\
\mathcal{E}_z&=\big(B(z,2\mu)\setminus\overline{B\big(z,\tfrac12\mu\big)}\big)\cap\{y:\langle y,z\rangle>|z|^2\},
\end{aligned}$$

defined for $z\in B\big(0,2-\frac12\varepsilon\big)\setminus B(0,1)$. The function $f(x)=\mathbf{P}^x(R[0,T_{v,4r}]\cap\Gamma\neq\varnothing)$ is harmonic in x in the domain $v+r\mathcal{D}_z$, $z=(w-v)/r$, so we get for $r>C(\mu,G)$ that it is (up to constants depending on G and μ) independent of $x\in v+r\mathcal{E}_z$. Note that for $r>C(\varepsilon,G)$,

$$|z|\leqslant 2-\varepsilon-\frac{C(G)}{r}\leqslant 2-\frac{\varepsilon}{2}\quad\text{and}\quad \partial B(w,\varrho)\cap H\subset v+r\mathcal{E}_z.$$

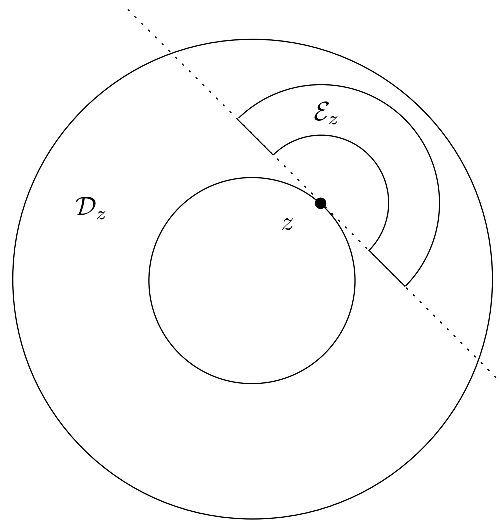

Figure 5. The domains $\mathcal{D}_z$ and $\mathcal{E}_z$.

We want to compare $f(x)$ with $f(v)$. We use (clause (iii) of) Lemma 2.5 with $\mathcal{D}=B(0,2)$, $\mathcal{S}=B\left(0,\frac{1}{2}\right)$ and $\mathcal{H}=\mathcal{E}_z$. We get

$$\begin{aligned} f(v) &\overset{(*)}{\geqslant} \mathbf{P}^v(T(v+r\mathcal{E}_z)<T_{v,4r})\mathbf{E}f(R(T(v+r\mathcal{E}_z))) \\ &\overset{(**)}{\geqslant} c(\mu,G)\mathbf{E}f(R(T(v+r\mathcal{E}_z))) \\ &\overset{(\dagger)}{\geqslant} c(\mu,G)f(x) \qquad \text{for all } x\in\partial B(w,\varrho)\cap H, \end{aligned} \tag{79}$$

where $(*)$ comes from the strong Markov property, $(**)$ comes from Lemma 2.5 and $(\dagger)$ comes from Harnack's inequality. This finishes the lemma—choose $\delta(\varepsilon,G)=\nu c(\mu,G)$ and the assumption (71) will imply (77). $\square$

4.3. Wiener's shell test

For the next lemma we need to introduce some notation. Let $v\in G$ and $r>1$ be some number and let

$$A_1:=B(v,2r)\setminus\overline{B(v,r)} \quad\text{and}\quad A_2:=B(v,4r)\setminus\overline{B\left(v,\tfrac{1}{2}r\right)}. \tag{80}$$

The notation $\overline{B(v,r)}$ relates here to closure in G, not in $\mathbf{R}^3$. We shall let

$$\partial A_1=\partial B(v,2r)\cup\partial B(v,r)$$

(the general definition of boundary in G might be a little smaller). Both conventions apply to any annulus in this section.

If $\gamma_1, ..., \gamma_n$ are discontinuous paths (usually we will consider two or three) then we will consider $\mathrm{LE}(\gamma_1 \cup ... \cup \gamma_n)$ as composed of n pieces, each one "coming from some γ_i", and will denote them by $\mathrm{LE}_i(\gamma_1 \cup ... \cup \gamma_n)$. Formally, let $t_0 = 1$, $t_n = \operatorname{len} \mathrm{LE}(\gamma_1 \cup ... \cup \gamma_n)$, and let t_i, $i = 1, ..., n-1$, be the first t such that $j_t > \operatorname{len}(\gamma_1) + ... + \operatorname{len}(\gamma_{i-1})$ (where j_t comes from the definition of LE, equation (6)). Then

$$\mathrm{LE}_i(\gamma_1 \cup ... \cup \gamma_n) := \begin{cases} \mathrm{LE}(\gamma_1 \cup ... \cup \gamma_n)[t_{i-1}, t_i - 1], & \text{if } t_i > t_{i-1}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Note that LE_i are simple and disjoint, and that $\mathrm{LE}(\gamma_1 \cup ... \cup \gamma_n) = \mathrm{LE}_1 \cup ... \cup \mathrm{LE}_n$.

Lemma 4.5. *Let G be a three-dimensional isotropic graph and let $\varepsilon > 0$ and $\eta > 0$. Then there exists a $\delta = \delta(\varepsilon, \eta, G)$ such that the following holds. Let $v \in G$, $r > C(\varepsilon, \eta, G)$ and $s \in [r, 2r - \eta r]$. Let A_1 and A_2 be as in* (80). *Let $\gamma \subset A_2$ be a discontinuous path starting from $\partial B\big(v, \frac{1}{2}r\big)$ and let R^1 be a random walk starting from some point in ∂A_1 and stopped at ∂A_2. Let*

$$L = \mathrm{LE}(\gamma \cup R^1), \quad L_1 = \mathrm{LE}_1(\gamma \cup R^1) \quad \text{and} \quad L_2 = \mathrm{LE}_2(\gamma \cup R^1).$$

Let $\mathcal{X}$ be the event that the following three events hold:

(i) $L_1 \subset B(v, s)$;

(ii) $L_2 \not\subset B(v, s + \eta r)$;

(iii) $\mathbf{P}^{2,v}(R^2[0, T^2_{v,4r}] \cap L' \neq \varnothing \mid L') < \delta$, *where $L' = L[1, t]$ and t is the first time when L hits $\partial B(v, s + \eta r)$, i.e. $t := \min\{i : L(i) \notin B(v, s + \eta r)\}$.*

Then

$$\mathbf{P}(\mathcal{X}) < \varepsilon.$$

In words, the probability that R^1 extends γ even by a little (η) without being (δ-)hittable, is small. To get a clearer geometric picture, think about γ as the restriction of a continuous path to A_2, i.e. as a sequence of paths coming in and (except for the last one) ending in ∂A_2; and think about R^1 as starting from the end of γ.

We remark that γ may be empty, in which case (i) always holds, and in (ii) one may replace L_2 by L.

Proof. Let $q = q(G)$ be from Lemma 4.4. Let $K = K(\varepsilon, G) > 2$ be an integer such that

$$(1-q)^K < \varepsilon.$$

Now fix the parameter ε of Lemma 4.4 to be $\eta/2K$ and denote by $\lambda = \lambda(\varepsilon, G)$ the result.

Lemma 4.4	ε	q	δ
Here	$\eta/2K$	q	λ

Table 1. Use of Lemma 4.4 in Lemma 4.5

Define

$$s_i = s + \frac{r\eta i}{K+2} \quad \text{for } i = 1, ..., K+1.$$

Let j_k be as in the definition of loop-erasure (6) so that $R^1[j_k+1, T^1(\partial A_2)]$ is a random walk conditioned not to hit $\mathrm{LE}(\gamma \cup R^1[0, j_k])$. Define

$$\tau_i' := \max\{j_k \leqslant T^1(\partial A_2) : \mathrm{LE}(\gamma \cup R^1[0, j_k]) \subset B(v, s_i)\}.$$

As in Lemma 4.4, let j_k be admissible if there exists a path $\delta \subset \overline{B(R^1(j_k), 16C_5)}$ from $R^1(j_k)$ to $\partial B(v, |v-R^1(j_k)|+2C_5)$ which does not intersect $\mathrm{LE}(\gamma \cup R^1[0, j_k])$, and define

$$\tau_i := \min\{j_k \geqslant \tau_i' : j_k \text{ admissible}\}.$$

We note that if $L_1 \subset B(v, s)$ and $L_2 \not\subset B(v, s_i + C_5)$, then τ_i is well defined. Indeed, let $t > \tau_i'$ (τ_i' is obviously well defined) be the first j_k such that $R^1(t) \notin B(v, s_i + C_5)$, and let $s^* = |R^1(t) - v| - C_5$ so that $s^* > s_i$. We use Lemma 3.9 with the radius being $8C_5$ and with H being the half-space tangent to $B(v, s^*)$ and orthogonal to the segment $[v, R^1(t)]$. We get that there exists a path $\delta' \subset B(R^1(t), 8C_5) \setminus \partial H$ from $R^1(t)$ to some $x \in \partial B(R^1(t), 8C_5) \cap \{x : d(x, \partial_{\mathrm{cont}} H) > 4C_5\}$. If r is sufficiently large then this implies that $d(x, v) > s^* + 3C_5$. Let

$$q_0 := \max\{q : \delta'(q) \in \mathrm{LE}(\gamma \cup R^1[0, t])\}$$

and let t_0 be the j_k such that $R^1(t_0) = \delta'(q_0)$. Clearly $\delta'(q_0) \notin B(v, s_i)$ so $t_0 > \tau_i'$. Further, $\delta := \delta'[q_0, \operatorname{len} \delta']$ is a path from $R^1(t_0)$ to x not intersecting $\mathrm{LE}(\gamma \cup R^1[0, t_0])$. Since

$$R^1(t_0) \in B(v, s^* + C_5) \quad \text{and} \quad x \notin B(v, s^* + 3C_5)$$

and since $B(R^1(t), 8C_5) \subset B(R^1(t_0), 16C_5)$, we see that all the admissibility requirements are satisfied. Therefore t_0 is admissible and hence τ_0 is well defined and $\leqslant t_0$.

Fix some i and assume that τ_i is well defined and that $\tau_i < T^1(\partial A_2)$. Examine R^1 after τ_i. The definition of τ_i considers only $\mathrm{LE}(R[0, \tau_i])$, therefore it only affects R^1 after τ_i by conditioning it to not intersect $\Gamma_i := \mathrm{LE}(\gamma \cup R^1[0, \tau_i])$. Hence Lemma 4.4 applies. We get

$$\mathbf{P}(\mathcal{B}_i \mid \tau_i, R^1[0, \tau_i]) < 1 - q, \tag{81}$$

where $\mathcal{B}_i$ is the event that

(a) τ_i is well defined;

(b) $\mathbf{P}^{2,v}(R^2[0,T^2_{v,4r}]\cap\Gamma_i\neq\varnothing|\Gamma_i)\leqslant\lambda$;

(c) for some $y\in B(w_i,\varrho)$, $\varrho:=\eta r/2K$,

$$\mathbf{P}^{2,y}(\operatorname{cut}(R^1[\tau_i,T^1_{v,4r}];T^*_i)\cap R^2[0,T^2_{v,4r}]\neq\varnothing)\leqslant q, \tag{82}$$

where $T^*_i:=\min\{t>\tau_i:R^1(t)\in\partial B(w_i,\varrho)\}$.

The notation in (81) might deserve some explanation: we are conditioning here on the fact that τ_i is well defined (so (a) is satisfied automatically), on its value which gives some information on R beyond τ_i and on the entire path from 0 to τ_i (which gives Γ_i and w_i).

It will be convenient to replace (a) by the stronger condition

(a′) τ_{i+1} is well defined,

and then replace (82) by a slightly stronger condition:

$$\mathbf{P}^{2,y}(\operatorname{cut}(R^1[\tau_i,\tau_{i+1}];T^*_i)\cap R^2[0,T^2_{v,4r}]\neq\varnothing)\leqslant q. \tag{83}$$

This is possible since $T^*_i<\tau_{i+1}$ by our choice of ϱ. Denote $\mathcal{B}_i$, with (a) and (82) replaced by (a′) and (83) respectively, by $\mathcal{B}'_i$ and get $\mathbf{P}(\mathcal{B}'_i|\tau_i,R^1[0,\tau_i])<1-q$. $\mathcal{B}'_i$ depends only on τ_{i+1} and $R[0,\tau_{i+1}]$, and hence we can write

$$\mathbf{P}(\mathcal{B}'_i\,|\,\mathcal{B}'_0,...,\mathcal{B}'_{i-1})=\mathbf{E}\mathbf{P}(\mathcal{B}'_i\,|\,\tau_1,...,\tau_i,R[0,\tau_i])=\mathbf{E}\mathbf{P}(\mathcal{B}'_i\,|\,\tau_i,R[0,\tau_i])<1-q$$

(the $\mathbf{E}$ signs above stand for conditional expectation with respect to $\mathcal{B}'_{i-1},...,\mathcal{B}'_0$). Hence

$$\mathbf{P}\biggl(\bigcap_{i=0}^{K-1}\mathcal{B}'_i\biggr)<(1-q)^K<\varepsilon.$$

The lemma will be finished when we show that for an appropriate choice of δ we have $\mathcal{X}\subset\bigcap_{i=0}^{K-1}\mathcal{B}'_i$. As explained above, conditions (i) and (ii) in the definition of $\mathcal{X}$ show that all the τ_i's are well defined. Hence, condition (a′) in the definition of $\mathcal{B}'_i$ is satisfied for all i. Setting $\delta<\lambda$ will ensure condition (b) for all i, since $\Gamma_i\subset L'$ for all i. Finally, Lemma 2.5 shows that for some $\nu(\varepsilon,\eta,G)$,

$$\mathbf{P}^v(T(B(w_i,\varrho))<T_{v,4r})>\nu \quad \text{for all } i \text{ and for all } w_i\in\partial B(v,s_i),$$

which shows that setting $\delta<q\nu$ ensures condition (c) (we used here (83) and the fact that $\operatorname{cut}(R^1[\tau_i+1,\tau_{i+1}];T^*_i)\subset L'$). Hence $\mathcal{X}\subset\bigcap_{i=0}^{K-1}\mathcal{B}'_i$ and the lemma is proved. □

LEMMA 4.6. *Let G be a three-dimensional isotropic graph, and let K be some number. Then there exists $\delta=\delta(K,G)>0$ such that the following holds. Let $v\in G$ and let R^1 be a random walk starting from v. Let $r>s>1$. Let $\mathcal{B}$ be the event that there exists some $T\geqslant 0$ such that*

(i) $\mathrm{LE}(R^1)\not\subset B(v,r)$, *where* $R^1:=R^1[0,T]$, *and*

(ii) $\mathbf{P}^{2,v}(R^2[0,T^2_{v,r}]\cap\mathrm{LE}(R^1)\cap(B(v,r)\setminus\overline{B(v,s)})=\varnothing\,|\,R^1)>(s/r)^\delta$.

Then $\mathbf{P}(\mathcal{B})<C(K,G)(s/r)^K$.

It is not difficult to see that the probability that $\mathrm{LE}(R^1)\subset B(v,s)$ is $\approx s/r$ (for $T\approx r^2$, which is the range that interests us). Therefore in fact $\mathrm{LE}(R^1)$ has a rather high probability to be "unhittable" because it is small. The point about the lemma is that if it is not small, this probability is negligible.

Proof. We may assume, without loss of generality, that both r/s and s are sufficiently large (and the bound may depend on K). A Brownian motion starting from a point in $\partial B(0,1)$ has probability $\frac{2}{3}$ to reach $\partial B(0,2)$ before $\partial B\big(0,\frac{1}{2}\big)$. Hence, by Lemma 3.5, if $\varrho>C(G)$, then a random walk starting from $x\in\partial B(v,\varrho)$ has probability $>\frac{7}{12}$ to reach $\partial B(v,2\varrho)$ before $\partial B\big(v,\frac{1}{2}\varrho\big)$. Hence, if we define $\varrho_i=s2^i$ and stopping times $\tau'_0=0$,

$$\tau'_j=\min\{t>\tau'_{j-1}:R^1(t)\in\partial B(v,\varrho_{i'(j)})\text{ for }i'(j)\text{ such that }R^1(\tau'_{j-1})\notin\partial B(v,\varrho_{i'(j)})\},$$

then the process $i'(j)$ dominates a random walk on $\mathbf{N}$ with a drift to infinity. In particular, it follows that if we let

$$n'_i:=\#\{j:i'(j)=i\}\quad\text{and}\quad N':=\sum_{i=0}^{I}n'_i,\tag{84}$$

then there exists a $\lambda=\lambda(K)$ such that

$$\mathbf{P}(N'>\lambda I)\leqslant C2^{-2IK}.\tag{85}$$

The estimate (85) holds for any value of I, but we will define $I:=\lfloor\log_2 r/s\rfloor$, since we are interested in what happens until r, and get $\mathbf{P}(N'>\lambda I)\leqslant C(s/r)^K$.

Next, we take the I annuli between s and r and denote the even ones as A_1's, i.e.

$$A_{1,i}:=B(v,2\varrho_{2i})\setminus\overline{B(v,\varrho_{2i})}\quad\text{and}\quad A_{2,i}:=B(v,4\varrho_{2i})\setminus\overline{B\big(v,\tfrac{1}{2}\varrho_{2i}\big)},$$

and define stopping times τ_j "from one A_1 to the next", i.e. $\tau_1:=\tau'_1$ and

$$\tau_j=\min\{t>\tau_{j-1}:R^1(t)\in\partial A_{1,i(j)}\text{ for some }i(j)\text{ such that }R^1(\tau_{j-1})\notin\partial A_{1,i(j)}\}.$$

It is clear that the τ_j's are a subsequence of the τ'_j's, hence if we define n_i and

$$N = \sum_{i=0}^{\lfloor (I-1)/2 \rfloor} n_i,$$

analogously to (84), then the analog of (85) will also hold. Define now $M := \lceil 6\lambda \rceil$ and get

$$\mathbf{P}\big(\#\big\{i < \tfrac{1}{2}I : n_i > M\big\} > \tfrac{1}{6}I\big) \leqslant C\Big(\frac{s}{r}\Big)^K.$$

Next, use Lemma 4.5 with

$$\eta = \frac{1}{M} \quad \text{and} \quad \varepsilon = \frac{\mu}{M^2},$$

where $\mu=\mu(K)$ is some parameter which will be fixed later. Call the δ of Lemma 4.5 ν. Fix one j and some $m \in \{0, ..., M-1\}$. Let $r_j = \varrho_{i(j)}$ and $s_{j,m} = r_j(1+m/M)$. Define $\gamma_{i,j} := \mathrm{LE}(R^1[0, \tau_j]) \cap A_{2,i}$ and $\gamma_j := \gamma_{i(j),j}$. Denote by $\mathcal{X}_{j,m}$ the event that the following conditions are all fulfilled:

(i) $L_{1,j} \subset B(v, s_{j,m})$, where

$$L_{k,j} := \mathrm{LE}_k(\gamma_j \cup R^1[\tau_j, \tau_{j+1}]), \quad k = 1, 2;$$

(ii) $L_{2,j} \not\subset B(v, s_{j,m+1})$;

(iii) $\mathbf{P}^{2,v}(R^2[0, T^2_{v,4r_j}] \cap L'_{j,m} \neq \varnothing \,|\, L'_{j,m}) < \nu$, where $L'_{j,m} = L_{1,j} \cup L_{2,j}[1, t]$, and t is the first time when $L_{2,j}$ hits $\partial B(v, s_{j,m+1})$, i.e. $t := \min\{i : L_{2,j}(i) \notin B(v, s_{j,m+1})\}$.

This completes all the parameters of Lemma 4.5 (see Table 2) and we may use it to get that $\mathbf{P}(\mathcal{X}_{j,m} \,|\, R[0, \tau_j]) < \mu/M^2$, and hence

$$\mathbf{P}\bigg(\bigcup_m \mathcal{X}_{j,m} \,\bigg|\, R[0, \tau_j]\bigg) < \frac{\mu}{M}.$$

Denote this event by $\mathcal{X}_j$. The $\mathcal{X}_j$'s are therefore dominated by a sequence of independent random variables with probability μ/M. A simple (and standard) calculation now shows that if μ is taken sufficiently small, one has

$$\mathbf{P}\bigg(\#\{j \leqslant \lambda I : \mathcal{X}_j\} > \frac{\lambda I}{M}\bigg) \leqslant C(G)\Big(\frac{s}{r}\Big)^K.$$

Fix μ satisfying this condition. Thus, we may define $\mathcal{G}$ as the event that $N \leqslant \lambda I$ and $\mathcal{X}_j$ occurred less than $(\lambda/M)I \leqslant \frac{1}{6}I$ times before λI. Our calculations show that

$$\mathbf{P}(\neg\mathcal{G}) \leqslant C\Big(\frac{s}{r}\Big)^K.$$

Lemma 4.5	ε	η	δ	r	s	A_1
Here	μ/M^2	$1/M$	ν	r_j	$s_{j,m}$	$A_{1,i(j)}$

Lemma 4.5	A_2	γ	R^1	L_k	L'	$\mathcal{X}$
Here	$A_{2,i(j)}$	γ_j	$R^1[\tau_j,\tau_{j+1}]$	$L_{k,j}$	$L'_{j,m}$	$\mathcal{X}_{j,m}$

Table 2. Use of Lemma 4.5 in Lemma 4.6.

The lemma will be finished once we show that with an appropriate choice of δ one has $\mathcal{B}\cap\mathcal{G}=\varnothing$.

Assume therefore from now on that both $\mathcal{B}$ and $\mathcal{G}$ occurred. There are $\lceil\frac{1}{2}I\rceil$ different $A_{1,i}$'s, and under the assumption $\mathcal{G}$, in $\leqslant\frac{1}{6}I$ of them we have $n_i>M$ and in $\leqslant\frac{1}{6}I$ of them we have that some $\mathcal{X}_j$ occurred when $R(\tau_j)\in\partial A_{1,i}$. Hence, we get that at least $\frac{1}{6}I$ are "good", in the sense that they are visited $\leqslant M$ times and are "$\mathcal{X}_j$-free". Fix i to be one such good index. We will now show that $A_{1,i}\cap\mathrm{LE}(R^1)$ is "hittable", and we shall show that by induction.

SUBLEMMA 4.6.1. *For any j define*

$$n_{i,j}:=\#\{k<j:R^1(\tau_k)\in A_{1,i}\},$$

$$u_{i,j}:=\varrho_{2i}\Big(1+\frac{n_{i,j}}{M}\Big)\quad\textit{and}\quad U_{i,j}:=B(v,u_{i,j})\setminus\overline{B(v,\varrho_{2i})}.$$

Let also $\gamma^*_{i,j}$ be $\gamma_{i,j}[0,t_{i,j}]$, where $t_{i,j}$ is the first time when $\gamma_{i,j}(t)\in\partial B(v,u_{i,j})$, or empty if $\gamma_{i,j}\cap\partial B(v,u_{i,j})=\varnothing$. Then for all j, either $\gamma^*_{i,j}=\varnothing$ or

$$\mathbf{P}^{2,w}(\gamma^*_{i,j}\cap U_{i,j}\cap R^2[0,T^2_{4\varrho_{2i}}]=\varnothing)\geqslant\nu.$$

Note that $n_i\leqslant M$ (because i is a good index), and hence $u_{i,j}\leqslant 2\varrho_{2i}$ throughout the induction.

Subproof. We use induction over j. If $R^1(\tau_j)\notin\partial A_{1,i}$, then $R^1[\tau_j,\tau_{j+1}]$ does not enter $A_{1,i}$, and hence can only affect $\gamma^*_{i,j}\cap A_{1,i}$ by removing some components from its end. In this case it must remove the last component of $\gamma^*_{i,j}\cap A_{1,i}$, which is the component intersecting $\partial B(w,u_{i,j})$, and all the components of $\gamma_{i,j}$ after $\gamma^*_{i,j}$. Hence we get that $\gamma_{i,j+1}\cap\partial B(w,u_{i,j})=\varnothing$ (here $u_{i,j}=u_{i,j+1}$) and the induction holds in this case. Therefore, we need only be interested in the case $R^1(\tau_j)\in\partial A_{1,i}$. We know that $\mathcal{X}_j$ did not occur, and in particular $\mathcal{X}_{j,n_{i,j}}$ did not occur. Hence, one of the three constituents of $\mathcal{X}_{j,n_{i,j}}$ must have failed. Let us review them in order.

(i) If $L_{1,j}\not\subset B(v,u_{i,j})$ (note that $s_{j,n_{i,j}}=u_{i,j}$) then

- $\gamma_{i,j}\cap\partial B(v,u_{i,j})\neq\varnothing$, and by the induction hypothesis, $\gamma^*_{i,j}$ is hittable;
- R^1 must have hit $\gamma_{i,j}$, if at all, after exiting $B(v,u_{i,j})$; therefore, if $\gamma^*_{i,j+1}\neq\varnothing$, then it contains $\gamma^*_{i,j}$, so $\gamma^*_{i,j+1}\cap U_{i,j+1}$ is hittable and the induction holds.

(ii) If $L_{1,j}\cup L_{2,j}\subset B(v,u_{i,j+1})$, then $\gamma_{i,j+1}\cap\partial B(v,u_{i,j+1})=\varnothing$ and the induction holds.

(iii) Finally, if (i) and (ii) of the definition of $\mathcal{X}_j$ happened, then we must have

$$\mathbf{P}^{2,w}(R^2[0,T^2_{w,4\varrho_{2i}}]\cap L'_{j,n_{i,j}}\neq\varnothing\mid L'_{v,n_{i,j}})\geqslant\nu,$$

and then the induction holds because $L'_{j,n_{i,j}}=\gamma^*_{i,j+1}$.

Hence the sublemma is proved. □

Assume now that $\mathcal{B}$ happened and let T be the "bad" time. Let J be such that $\tau_J\leqslant T<\tau_{J+1}$. As in the sublemma, the part of the walk on $[\tau_j,T]$ can affect in an adverse way only the part of the walk inside $A_{1,i(J)}$. Hence the sublemma shows that for any good $i\neq i(J)$, either $\mathrm{LE}(R^1)$ does not contain a crossing of $A_{1,i}$, or this crossing is hittable. However, if $\mathcal{B}$ happened then $\mathrm{LE}(R^1)$ crosses $B(v,r)\setminus\overline{B(v,s)}$, and hence all the $A_{1,i}$'s, so we get for all good $i\neq i(J)$ that

$$\mathbf{P}^{2,v}(\mathrm{LE}(R^1)\cap A_{1,i}\cap R^2[0,T^2_{v,4\varrho_{2i}}]\neq\varnothing)\geqslant\nu.$$

Harnack's inequality (Lemma 2.2) shows that

$$\mathbf{P}^{2,w}(\mathrm{LE}(R^1)\cap A_{1,i}\cap R^2[0,T^2_{v,4\varrho_{2i}}]\neq\varnothing)>c\nu\quad\text{for all }w\in B\big(v,\tfrac12\varrho_{2i}\big).\tag{86}$$

Hence, let $i_1<i_2<...<i_n$ be good indices with $i_{k+1}-i_k\geqslant2$, $i_k\neq I,i(J)$. We can take $n:=\big\lfloor\frac{1}{12}I\big\rfloor-2$. Then

$$\begin{aligned}
&\mathbf{P}^{2,v}(R^2[0,T^2_{v,r}]\cap\mathrm{LE}(R^1)\cap(B(v,r)\setminus\overline{B(v,s)})=\varnothing)\\
&\quad\leqslant\prod_{k=1}^{n-1}\mathbf{P}^{2,v}(R^2[T^2_{v,\varrho_{2i_k}/2},T^2_{v,4\varrho_{2i_k}}]\cap\mathrm{LE}(R^1)=\varnothing\mid R^2[0,T^2_{v,4\varrho_{2i_{k-1}}}]\cap\mathrm{LE}(R^1)=\varnothing)\\
&\quad\leqslant\prod_{k=1}^{n-1}\max_{w\in B(v,\varrho_{2i_k}/2)}\mathbf{P}^{2,w}(R^2[0,T^2_{v,4\varrho_{2i_k}}]\cap\mathrm{LE}(R^1)=\varnothing)\\
&\quad\overset{(86)}{<}\prod_{k=1}^{n-1}(1-c\nu)\\
&\quad\leqslant C(1-c\nu)^{I/12}\\
&\quad\leqslant C\Big(\frac{s}{r}\Big)^{(\log(1-c\nu))/12\log2}.
\end{aligned}$$

Therefore, taking $\delta=(\log(1-c\nu))/24\log2$ and r/s sufficiently large, we get a contradiction, so $\mathcal{G}\cap\mathcal{B}=\varnothing$, and the lemma is proved. □

4.4. Hittable sets

The next step (Lemmas 4.7 and 4.8) is to show that $\mathrm{LE}(R)$ is "hittable" in a rather strong sense.

Definition. Let $\lambda>0$ and $1\leqslant\varrho<\sigma$. We define $\mathcal{H}^{\mathrm{out}}(\lambda,\varrho,\sigma)$ to be the family of paths β satisfying that for every w and every subpath $\gamma\subset\beta$ which is an outgoing crossing of $B(w,\sigma)\setminus\overline{B(w,\varrho)}$, i.e. $\gamma(0)\in\partial B(w,\varrho)$ and $\gamma(\operatorname{len}\gamma)\in\partial B(w,\sigma)$, one has

$$\mathbf{P}^w(R[0,T_{w,\sigma}]\cap\gamma)=\varnothing)\leqslant\lambda.$$

Define $\mathcal{H}^{\mathrm{out}}(\lambda,\mu)=\bigcap_{\varrho\geqslant 1}\mathcal{H}^{\mathrm{out}}(\lambda,\varrho,\mu\varrho)$.

Similarly, we define $\mathcal{H}^{\mathrm{in}}$ to be the set where this holds for γ incoming, i.e. $\gamma(0)\in\partial B(w,\sigma)$ and $\gamma(\operatorname{len}\gamma)\in\partial B(w,\varrho)$. We define $\mathcal{H}:=\mathcal{H}^{\mathrm{out}}\cap\mathcal{H}^{\mathrm{in}}$. Note that these properties are hereditary: if $\beta\subset\gamma\in\mathcal{H}^{\mathrm{out/in}}$ then $\beta\in\mathcal{H}^{\mathrm{out/in}}$.

LEMMA 4.7. *Let G, ε, v, r and $\mathcal{D}$ be as in Theorem 4 and let $K>0$. Then there exists $\delta=\delta(\varepsilon,K,G)$ such that*

$$\mathbf{P}^{1,v}(\textit{there exists } T\leqslant T^1(\partial\mathcal{D}):\mathrm{LE}(R^1[0,T])\notin\mathcal{H}^{\mathrm{out}}(r^{-\delta},r^{\varepsilon}))<C(\varepsilon,K,G)r^{-K}.$$

Proof. Clearly we may assume that r is sufficiently large (and the bound may depend on K and ε). Since $\mathcal{H}^{\mathrm{out}}(r^{-\delta},\varrho,\sigma)$ is increasing in σ and decreasing in ϱ, it is enough to show that for any fixed $\sigma=\frac{1}{2}\varrho r^{\varepsilon}$ we have

$$\mathbf{P}^{1,v}(\text{there exists } T\leqslant T^1(\partial\mathcal{D}):\mathrm{LE}(R^1[0,T])\notin\mathcal{H}^{\mathrm{out}}(r^{-\delta},\varrho,\sigma))\leqslant Cr^{-K-1}. \tag{87}$$

Once (87) is established, we can apply it for $\varrho=1,2,4,...,2^{\lfloor\log_2 r\rfloor}$ and bound the probability that (87) holds for one such ϱ by the sum, which is $\leqslant Cr^{-K-1}\log r\leqslant Cr^{-K}$, and the lemma would be proved.

Fix therefore one ϱ and $\sigma=\frac{1}{2}\varrho r^{\varepsilon}$. Let $w\in\mathcal{D}$ be some point and let

$$A=B(w,\sigma)\setminus\overline{B(w,\varrho)}.$$

Let $x\in\overline{B(w,\varrho)}$ be some point, and let $\varrho':=\max\{2\varrho,\varrho_0\}$ for some constant $\varrho_0(\varepsilon,K,G)$ to be fixed later. Let $A':=B\big(x,\frac{1}{2}\sigma\big)\setminus\overline{B(x,\varrho')}$ so that $A'\subset A$ for all sufficiently large r. Let t be some number. Use Lemma 4.6 with the parameters and notation in Table 3. We get, for the event $\mathcal{B}_x$ that there exists some t such that

(i) $\mathrm{LE}(R^1[0,t])\not\subset B\big(x,\frac{1}{2}\sigma\big)$ and

(ii) $\mathbf{P}^{2,x}(R^2[0,T^2_{x,\sigma/2}]\cap\mathrm{LE}(R^1[0,t])\cap A'=\varnothing\,|\,R^1)>C(\varepsilon,K,G)r^{-\lambda\varepsilon}$,

Lemma 4.6	K	δ	v	T	r	s	$\mathcal{B}$
Here	$(K+10)/\varepsilon$	λ	x	t	$\frac{1}{2}\sigma$	ϱ'	$\mathcal{B}_x$

Table 3. Use of Lemma 4.6 in Lemma 4.7.

that $\mathbf{P}^{1,x}(\mathcal{B}_x)<C(\varepsilon,K,G)r^{-K-10}$. If ϱ_0 is sufficiently large, we can use Harnack's inequality to change in (ii) the starting point of R^2 from x to w, and pay only by increasing the constant.

Returning to a random walk starting from v, we define an event $\mathcal{E}=\mathcal{E}(t_1,t_2,x)$ by

(i) $R^1(t_1)=x$,

(ii) $\mathrm{LE}(R^1[0,t_1])\cap R^1[t_1+1,t_2]=\varnothing$,

(iii) $\mathrm{LE}(R^1[t_1+1,t_2])\not\subset B(x,\sigma)$,

(iv) $\mathbf{P}^{2,w}(R^2[0,T^2_{w,\sigma}]\cap\mathrm{LE}(R^1[t_1+1,t_2])\cap A=\varnothing\,|\,R^1)>c(\varepsilon,K,G)r^{-\lambda\varepsilon}$,

and get $\mathbf{P}^{1,v}\bigl(\bigcup_{t_2\geqslant t_1}\mathcal{E}\bigr)\leqslant C(\varepsilon,K,G)r^{-K-10}$. Summing, we get for any U,

$$\mathbf{P}\Bigl(\bigcup\mathcal{E}(t_1,t_2,x)\Bigr)\leqslant Cr^{-K-4}U, \tag{88}$$

where the union is over all $w\in\mathcal{D}$, $x\in\overline{B(w,\varrho)}$, $t_1\leqslant U$ and $t_2\geqslant t_1$. Now, by Lemma 2.4,

$$\mathbf{P}(T(\partial\mathcal{D})>\mu r^2\log r)\leqslant\mathbf{P}(T_{v,r}>\mu r^2\log r)\leqslant C(G)e^{-c(G)\mu\log r},$$

so, for $\mu=\mu(K,G)$ sufficiently large,

$$\mathbf{P}(T(\partial\mathcal{D})>\mu r^2\log r)\leqslant Cr^{-K-1}.$$

Therefore, using $U=\mu r^2\log r$ in (88) gives

$$\mathbf{P}\Bigl(\bigcup\mathcal{E}(t_1,t_2,x)\Bigr)\leqslant Cr^{-K-1},$$

where the union is now over all w,x and $t_1\leqslant t_2\leqslant T^1(\partial\mathcal{D})$. This finishes the lemma by taking $\delta=\frac{1}{2}\varepsilon\lambda$ and r sufficiently large, since if $\mathrm{LE}(R[0,T])\notin\mathcal{H}^{\mathrm{out}}$ occurred then one of the $\mathcal{E}(t_1,t_2,x)$ must have occurred. Namely, let γ be a subpath of $\mathrm{LE}(R[0,T])$ satisfying the requirements from the definition of $\mathcal{H}^{\mathrm{out}}$ and being minimal (basically this means $\gamma\subset\bar{A}$) and assume that $\gamma=\mathrm{LE}(R[0,T])[l,m]$. Then, we may take $t_1:=j_l$ (where j comes from the definition of LE, equation (6)), $t_2:=j_m$ and $x=R^1(t_1)$, and directly from the definitions, $\mathcal{E}(t_1,t_2,x)$ will be satisfied. Therefore (87) holds for any appropriate ϱ and σ, and the lemma is proved. □

The following lemma will be used only formally—in effect the previous lemma is enough. We include it here mainly for completeness.

Lemma 4.7	v	$\mathcal{D}$	K	δ
Here	x	$\mathcal{D}$	$2K+7$	δ

Table 4. Use of Lemma 4.7 in Lemma 4.8.

LEMMA 4.8. *Lemma* 4.7 *holds with* $\mathcal{H}^{\mathrm{out}}$ *replaced by* $\mathcal{H}^{\mathrm{in}}$.

Proof. Let $x\in\overline{\mathcal{D}}$ satisfy that

$$\mathbf{P}^v(R(T(\partial\mathcal{D}\cup\{x\}))=x)\geqslant r^{-K-3}. \tag{89}$$

The transience of G shows that (compare with (63))

$$\mathbf{P}^v(R(T(\partial\mathcal{D}\cup\{v,x\}))=x)\geqslant c(G)r^{-K-3}.$$

Using the symmetry of random walk in the form (4), we get that

$$\mathbf{P}^x(R(T(\partial\mathcal{D}\cup\{v,x\}))=v)\geqslant c(G)r^{-K-3}. \tag{90}$$

Next, use Lemma 4.7 with the parameters in Table 4 (if $x\in\partial\mathcal{D}$ then use the lemma with each neighbor x' of x in $\mathcal{D}$ serving as the starting point instead of x). For any time or stopping time t, define an event

$$\mathcal{O}(t)=\{\mathrm{LE}(R[0,t])\in\mathcal{H}^{\mathrm{out}}(r^{-\delta},r^{\varepsilon})\}. \tag{91}$$

With this notation, the conclusion of Lemma 4.7 is that

$$\mathbf{P}^x(\text{there exists } t\leqslant T(\partial\mathcal{D}):\mathcal{O}(t))\leqslant C(K,G)r^{-2K-7}.$$

In particular, $\mathbf{P}^x(\mathcal{O}(T(\partial\mathcal{D}\cup\{v,x\})))\leqslant Cr^{-2K-7}$. Hence

$$\mathbf{P}^x(\mathcal{O}(T(v))\mid R(T(\partial\mathcal{D}\cup\{v,x\}))=v)\leqslant\frac{\mathbf{P}^x(\mathcal{O}(T(\partial\mathcal{D}\cup\{v,x\})))}{\mathbf{P}^x(R(T(\partial\mathcal{D}\cup\{v,x\}))=v)}\overset{(90)}{\leqslant}Cr^{-K-4}.$$

Now, loop-erased random walk conditioned on the end vertex is symmetric (see e.g. [K, Lemma 2]) so,

$$\mathbf{P}^v(\mathcal{I}(T(x))\mid R(T(\partial\mathcal{D}\cup\{v,x\}))=x)=\mathbf{P}^x(\mathcal{O}(T(v))\mid R(T(\partial\mathcal{D}\cup\{v,x\}))=v),$$

where $\mathcal{I}$ is $\mathcal{O}$ for the reversed path, or, in other words, with $\mathcal{H}^{\mathrm{out}}$ replaced by $\mathcal{H}^{\mathrm{in}}$ in (91). Now, since $\mathrm{LE}(R[0,T(\partial\mathcal{D}\cup\{x\})])$ does not depend on how many times we returned to v, we get

$$\mathbf{P}^v(\mathcal{I}(T(x))\mid R(T(\partial\mathcal{D}\cup\{x\}))=x)=\mathbf{P}^v(\mathcal{I}(T(x))\mid R(T(\partial\mathcal{D}\cup\{v,x\}))=x).$$

Lemma 1.3 now shows that, if $T_n = T_n(x)$ is the time of the nth return to x before hitting $\partial\mathcal{D}$, then $\mathrm{LE}(R[0,T_1]) \sim \mathrm{LE}(R[0,T_n])$ and therefore

$$\mathbf{P}^v(\mathcal{I}(T_n) \mid T_n < T(\partial\mathcal{D})) \leqslant Cr^{-K-4}, \quad n = 1, 2, \dots . \tag{92}$$

To finish the lemma we need only sum over n and x. The transience of G shows that

$$\mathbf{P}^v(T_n > \lambda) \leqslant Ce^{-c\lambda},$$

and therefore for $\lambda = C(G)\log r$ with C sufficiently large we have that this probability is $\leqslant Cr^{-K-3}$. Therefore we get

$$\begin{aligned}\mathbf{P}\biggl(\bigcup_{n=1}^{\infty}\{T_n < T(\partial\mathcal{D})\} \cap \mathcal{I}(T_n)\biggr) &\leqslant Cr^{-K-3} + \sum_{n=1}^{\lambda} \mathbf{P}(\mathcal{I}(T_n) \mid T_n < T(\partial\mathcal{D})) \\ &\leqslant Cr^{-K-3} + (C\log r) r^{-K-4} \\ &\leqslant Cr^{-K-3}.\end{aligned}$$

Denoting by $\mathcal{X}$ the set of the x's satisfying (89), we can sum over x and get

$$\begin{aligned}&\mathbf{P}\biggl(\bigcup_{n,x}\{T_n(x) < T(\partial\mathcal{D})\} \cap \mathcal{I}(T_n)\biggr) \\ &\qquad \leqslant \sum_{x\in\mathcal{X}} \mathbf{P}\biggl(\bigcup_{n=1}^{\infty}\{T_n(x) < T(\partial\mathcal{D})\} \cap \mathcal{I}(T_n)\biggr) + \sum_{x\notin\mathcal{X}} \mathbf{P}(T_1(x) < T(\partial\mathcal{D})) \\ &\qquad \leqslant \#\mathcal{X} \cdot Cr^{-K-3} + \#(\overline{\mathcal{D}} \setminus \mathcal{X}) \cdot Cr^{-K-3} \\ &\qquad \leqslant Cr^{-K}.\end{aligned}$$

This finishes the lemma, since the event $\bigcup_{n,x}\{T_n(x) < T(\partial\mathcal{D})\} \cap \mathcal{I}(T_n)$ is exactly the desired event. □

The combination of Lemmas 4.7 and 4.8 is that for an appropriate $\delta = \delta(\varepsilon, K, G)$ we have

$$\mathbf{P}(\text{there exists } t \leqslant T(\partial\mathcal{D}) : \mathrm{LE}(R[0,t]) \in \mathcal{H}(r^{-\delta}, r^{\varepsilon})) \leqslant Cr^{-K}. \tag{93}$$

4.5. Proof of Theorem 4 in three dimensions

The proof works in three different scales, which we will denote by $\varrho \ll \sigma \ll \tau \ll r$. The scale $\varrho = r^{1-\varepsilon}$ is the scale of the "closeness" of the two ends of the quasi-loop. The scale τ is the scale of the diameter of the quasi-loop, so after all parameters are fixed we shall fix some $\delta > 0$ and then define $\tau := r^{1-\delta}$. The scale σ is an auxiliary scale—rather than estimate for some point $w \in \frac{1}{3}\varrho\mathbf{Z}^3$ that the probability that $w \in \mathcal{QL}(\varrho, \tau, \mathrm{LE}(R[0, T(\partial\mathcal{D})]))$ is small, we shall show it simultaneously for all points in $B(w, \sigma) \cap \frac{1}{3}\varrho\mathbf{Z}^3$.

LEMMA 4.9. *Let G be a three-dimensional isotropic graph, and let $\varepsilon>0$ and $\delta>0$ be given. Let $1\leqslant\varrho\leqslant\sigma\leqslant\frac{1}{2}\tau\leqslant\frac{1}{2}r$ and $\sigma\geqslant 2\varrho(r^{\varepsilon}+2)$. Let w be some vertex and let $b\subset B(w,\tau)$ be some set. Let $\gamma\in\mathcal{H}=\mathcal{H}(r^{-\delta},r^{\varepsilon})$ be a path starting from some point in $\overline{B(w,\tau)}$ and ending on $\partial B(w,\tau)$. Let R be a random walk starting from some $x\in\partial B(w,\sigma)$ and stopped at $\partial B(w,\tau)\cup b$. Define*

$$\Delta(\gamma):=\#\bigl\{y\in\tfrac{1}{3}\varrho\mathbf{Z}^3\cap B(w,\sigma): d(y,\gamma)\leqslant\varrho \text{ and } d(y,R)\leqslant\varrho\bigr\}\cdot\mathbf{1}_{\{R\cap\gamma=\varnothing\}}.$$

Then

$$\mathbf{E}^x\Delta(\gamma)\cdot\mathbf{1}_{\{R\cap b=\varnothing\}}\leqslant C(G)r^{-\delta}\log^2 r, \tag{94}$$

$$\mathbf{E}^x\Delta(\gamma)\cdot\mathbf{1}_{\{R\cap b\neq\varnothing\}}\leqslant C(G)\mathbf{P}^x(R\cap b\neq\varnothing)\log^2 r. \tag{95}$$

Proof. Denote the left-hand sides in (94) and (95) by $\Delta_1(\gamma)$ and $\Delta_2(\gamma)$, respectively. Define stopping times $s_1\leqslant s_1^*\leqslant s_2\leqslant\ldots\leqslant T(\partial B(w,\tau)\cup b)$ by

$$s_1:=\min\{t: d(R(t),\gamma\cap B(w,\sigma+\varrho))\leqslant 2\varrho\},$$

and for $i\geqslant 1$,

$$s_i^*:=\min\{t\geqslant s_i: R(t)\in\partial B(R(s_i),8\varrho)\},$$
$$s_{i+1}:=\min\{t\geqslant s_i^*: d(R(t),\gamma\cap B(w,\sigma+\varrho))\leqslant 2\varrho\}$$

(for clarity we removed the conditions $s_i, s_i^*\leqslant T(\partial B(w,\tau)\cup b)$—if R hits $\partial B(w,\tau)\cup b$ after any of them, consider the sequence stabilized at this point). Let I be the number of s_i's defined before the process is stopped, i.e. $R(s_{I+1})\in\partial B(w,\tau)\cup b$. Lemma 2.9 gives for $i<I$ that the probability to intersect γ between s_i and s_i^* is $\geqslant c(G)/\log\varrho\geqslant c(G)/\log r$. This gives

$$\mathbf{P}(\{I\geqslant i\}\cap\{R[0,s_i]\cap\gamma=\varnothing\})\leqslant\left(1-\frac{c(G)}{\log r}\right)^{i-1}\quad\text{for all } i. \tag{96}$$

Further, for the time period between s_i and s_{I+1} the definition of $\mathcal{H}$ gives that

$$\mathbf{P}^x(\{R[s_i,s_{I+1}]\cap\gamma=\varnothing\}\cap\{R(s_{I+1})\notin b\}\mid I\geqslant i \text{ and } R[0,s_i])\leqslant r^{-\delta}.$$

Here is where we use the condition $\sigma\geqslant 2\varrho(r^{\varepsilon}+2)$, since

$$d(R(s_i),\partial B(w,\tau))\geqslant\tau-\sigma-3\varrho\geqslant\sigma-3\varrho.$$

Finally, it is clear that $\Delta(\gamma)<CI$. Therefore

$$\begin{aligned}\mathbf{E}^x\Delta_1(\gamma) &= \sum_{i=0}^{\infty}\mathbf{E}^x(\Delta_1(\gamma)\cdot\mathbf{1}_{\{I=i\}})\\ &\leqslant \sum_{i=0}^{\infty} Ci\mathbf{P}^x(\{\Delta_1(\gamma)>0\}\cap\{I\geqslant i\})\\ &\leqslant C\sum_{i=1}^{\infty} i\mathbf{P}^x(\{R(s_{I+1})\notin b\}\cap\{R\cap\gamma=\varnothing\}\cap\{I\geqslant i\})\\ &\leqslant C\sum_{i=1}^{\infty} i\left(1-\frac{c(G)}{\log r}\right)^{i-1}Cr^{-\delta}\\ &\leqslant C(G)r^{-\delta}\log^2 r.\end{aligned} \tag{97}$$

As for $\mathbf{E}^x\Delta_2(\gamma)$, we first notice that if b is empty then there is nothing to prove. If b is not empty, then the estimate of the Green function (Lemma 2.3) shows that

$$\mathbf{P}(R\cap b\neq\varnothing) > \frac{C}{r}.$$

Therefore, we may use (96) to show that if $\lambda=\lambda(G)$ is a sufficiently large constant, then

$$\mathbf{P}(\{I>\lambda\log^2 r\}\cap\{R\cap\gamma=\varnothing\})\leqslant r^{-4}.$$

Hence, we get

$$\begin{aligned}\mathbf{E}^x\Delta_2(\gamma) &\leqslant \mathbf{E}^x\Delta_2(\gamma)\cdot\mathbf{1}_{\{I>\lambda\log^2 r\}}+\mathbf{E}^x\Delta_2(\gamma)\cdot\mathbf{1}_{\{I\leqslant\lambda\log^2 r\}}\\ &\leqslant C\left(\frac{\sigma}{\varrho}\right)^3 r^{-4}+C\lambda(\log r)^2\mathbf{P}^x(R\cap b\neq\varnothing)\\ &\leqslant C\mathbf{P}^x(R\cap b\neq\varnothing)\log^2 r.\end{aligned}$$

□

Returning to the proof of Theorem 4, we shall from this point on assume that $2r^{1-\varepsilon/2}\leqslant\sigma\leqslant r^{1-2\delta}$. We fix some $w\in\mathcal{D}$ and some stopping time T, and define

$$\mathcal{X}(T):=\mathcal{X}_w(T):=\#\big(\mathcal{QL}(\varrho,\tau,\mathrm{LE}(R[0,T]))\cap B(w,\sigma)\cap\tfrac{1}{3}\varrho\mathbf{Z}^3\big),$$

so $\mathcal{X}:=\mathcal{X}(T(\partial\mathcal{D}))$ is what we need to estimate. Define exit and entry stopping times by $T_0=0$ and

$$\begin{aligned}T_{2i+1} &= \min\{t\geqslant T_{2i} : R(t)\in\partial B(w,2\sigma)\cup\partial\mathcal{D}\},\\ T_{2i} &= \min\{t\geqslant T_{2i-1} : R(t)\in\partial B(w,4\sigma)\cup\partial\mathcal{D}\}.\end{aligned}$$

Let I be the first i such that $R(T_i)\in\partial\mathcal{D}$. We note that Lemma 2.6 shows that

$$\mathbf{P}(R(T_{2i+1})\in\partial\mathcal{D}\mid R[0,T_{2i}])\geqslant\mathbf{P}^{R(T_{2i})}(T_{w,2\sigma}=\infty)\geqslant c(G),$$

and hence we get that the probability that I is large drops exponentially, and hence if $\lambda=\lambda(G)$ is sufficiently large we get

$$\mathbf{P}(I>\lambda\log r)\leqslant\frac{1}{r^4}. \tag{98}$$

Denote $M:=\lfloor\lambda\log r\rfloor$. With this notation we can write

$$\mathbf{E}\mathcal{X}\leqslant C\left(\frac{\sigma}{\varrho}\right)^3\mathbf{P}(I>M)+\sum_{i=1}^{M}\mathbf{E}(\mathcal{X}\cdot\mathbf{1}_{\{I=i\}})\overset{(98)}{\leqslant}Cr^{-1}+\sum_{i=1}^{M}\mathbf{E}(\mathcal{X}\cdot\mathbf{1}_{\{I=i\}}). \tag{99}$$

In other words, we can ignore the first summand in (99). Let us therefore define the number of quasi-loops up to the ith time $\mathcal{X}_i:=\mathcal{X}(T_i)$. The process of loop-erasing between T_{2i} and T_{2i+1} can only destroy quasi-loops in $B(w,\sigma)$, so we get

$$\mathcal{X}\cdot\mathbf{1}_{\{I=2i+1\}}\leqslant\mathcal{X}_{2i}\cdot\mathbf{1}_{\{I=2i+1\}}\leqslant\mathcal{X}_{2i}\cdot\mathbf{1}_{\{I>2i\}}.$$

With this in mind, we define

$$\Delta_i:=(\mathcal{X}_{2i+2}-\mathcal{X}_{2i})\cdot\mathbf{1}_{\{I>2i+2\}}.$$

LEMMA 4.10. *With the notation above and*

$$\eta:=\min\bigl\{\delta\bigl(\tfrac{1}{2}\varepsilon,4,G\bigr),1\bigr\}, \tag{100}$$

where $\delta(\cdot)$ is given by (93),

$$\mathbf{E}\Delta_i\leqslant Cir^{-\eta}\log^2 r.$$

Proof. Examine some y,

$$y\in\mathcal{QL}(\varrho,\tau,\mathrm{LE}(R[0,T_{2i+2}]))\setminus\mathcal{QL}(\varrho,\tau,\mathrm{LE}(R[0,T_{2i}])).$$

Directly from the definitions, we must have that

(i) y is ϱ-near at least one component γ of $\mathrm{LE}(R[0,T_{2i}])\cap B(w,4\sigma)$;

(ii) $R[T_{2i},T_{2i+1}]$ gets ϱ-near y and then fails to intersect at least one of the segments γ from (i), as well as $\partial\mathcal{D}\cap B(w,4\sigma)$.

In other words, the number of such y's in $\frac{1}{3}\varrho\mathbf{Z}^3 \cap B(w,\sigma)$ with respect to a specific γ can be bounded by $\Delta(\gamma)$, Δ from Lemma 4.9, with the parameters in the following table:

Lemma 4.9	ϱ	σ	τ	ε	δ	b
Here	ϱ	2σ	4σ	$\frac{1}{2}\varepsilon$	η	$\partial\mathcal{D}\cap B(w,4\sigma)$

Note that (93) shows that with probability $\geqslant 1-Cr^{-4}$ we have $\gamma\in\mathcal{H}=\mathcal{H}(r^{-\eta},r^{\varepsilon/2})$ for all $\gamma\subset \mathrm{LE}(R[0,T_{2i+1}])$.

With this in mind, we denote by Γ_i the collection of connected components γ of $\mathrm{LE}(R[0,T_{2i+1}])\cap B(w,4\sigma)$ satisfying $\gamma\cap B(w,\sigma+\varrho)\neq\varnothing$ and get

$$\mathbf{E}\Delta_i\cdot\mathbf{1}_{\{\Gamma_i\subset\mathcal{H}\}} \leqslant \max_{x\in\partial B(w,2\sigma)} \sum_{\gamma\in\Gamma_i} \mathbf{E}^x\Delta(\gamma)\cdot\mathbf{1}_{\{R\cap\partial\mathcal{D}=\varnothing\}} \overset{(94)}{\leqslant} C(\#\Gamma_i)r^{-\eta}\log^2 r,$$
$$\mathbf{E}\Delta_i\cdot\mathbf{1}_{\{\Gamma_i\not\subset\mathcal{H}\}} \leqslant C\left(\frac{\sigma}{\varrho}\right)^3 \mathbf{P}(\Gamma_i\not\subset\mathcal{H}) \leqslant Cr^3r^{-4}.$$

It easy to see that $\#\Gamma_i\leqslant i$, and the lemma is finished. □

Summing Lemma 4.10 up to i, we get

$$\mathbf{E}\mathcal{X}_{2i}\cdot\mathbf{1}_{\{I>2i\}} \leqslant Cr^{-\eta}(\log r)^2 i^2$$

(η being defined by (100)). Another summation, up to M, will give us

$$\mathbf{E}\mathcal{X} \leqslant Cr^{-\eta}\log^5 r + \mathbf{E}\Delta', \quad \Delta' := (\mathcal{X}_I - \mathcal{X}_{I-2})\cdot\mathbf{1}_{\{I \text{ even}\}}. \tag{101}$$

Thus we are left with the estimate of $\mathbf{E}\Delta'$, which is the behavior near the boundary—if $\partial\mathcal{D}\cap B(w,4\sigma)=\varnothing$ then of course I is always odd and we get $\Delta'\equiv 0$. It is at this point that we utilize the difference between τ and σ.

LEMMA 4.11. *Let $\omega=\omega(v,\mathcal{D})$ be the discrete harmonic measure from v, i.e.*

$$\omega(A) := \mathbf{P}^v(R(T(\partial\mathcal{D}))\in A) \quad \textit{for all } A\subset\partial\mathcal{D}.$$

Then

$$\mathbf{E}^v(\Delta') \leqslant C\frac{\sigma}{\tau}\omega(B(w,16\sigma))\log^2 r.$$

Proof. Define stopping times $U_0\leqslant U_1\leqslant\ldots$ using $U_0=0$ and

$$U_{2j+1} := \min\{t\geqslant U_{2j} : R(t)\in\partial B(w,8\sigma)\cup\partial\mathcal{D}\}, \qquad j=0,1,\ldots,$$
$$U_{2j} := \min\{t\geqslant U_{2j-1} : R(t)\in\partial B(w,\tfrac{1}{2}\tau)\cup\partial\mathcal{D}\}, \quad j=1,2,\ldots.$$

We assume here that $\sigma<\frac{1}{32}\tau$, which will hold if r is sufficiently large. Define J to be the first j such that $R(U_j)\in\partial\mathcal{D}$. Our first target is to connect J and $\omega(B(w,16\sigma))$. Let

$$p(x):=\mathbf{P}^x(T(\partial\mathcal{D}\cap B(w,4\sigma))<T_{w,\tau/2}),\quad x\in\partial B(w,8\sigma).$$

We have

$$\begin{aligned}\omega(B(w,16\sigma))&=\sum_{j=1}^{\infty}\mathbf{P}(\{J=j\}\cap\{R(U_j)\in\partial\mathcal{D}\cap B(w,16\sigma)\})\\&\geqslant\mathbf{P}(\{J=2\}\cap\{R(U_2)\in\partial\mathcal{D}\cap B(w,16\sigma)\})\\&\geqslant\mathbf{P}(J>1)\mathbf{E}\mathbf{P}^{R(U_1)}(T(\partial\mathcal{D})<T_{w,16\sigma})\\&\geqslant\mathbf{P}(J>1)\min_{x\in\partial B(w,8\sigma)}\mathbf{P}^x(T(\partial\mathcal{D}\cap B(w,4\sigma))<T_{w,16\sigma})\\&\overset{(*)}{\geqslant}c(G)\mathbf{P}(J>1)\min p(x),\end{aligned}\tag{102}$$

where $(*)$ comes from using Lemma 2.8 to change from $T_{w,16\sigma}$ to $T_{w,\tau/2}$. The "min" here (and later on also "max") always refers to the minimum (maximum) over $\partial B(w,8\sigma)$.

The estimate of Δ' follows from the clause (95) of Lemma 4.9, which we now use with the parameters as follows:

Lemma 4.9	ϱ	σ	τ	ε	δ	b
Here	ϱ	8σ	$\frac{1}{2}\tau$	$\frac{1}{2}\varepsilon$	η	$\partial\mathcal{D}\cap B(w,4\sigma)$

An argument identical to that of the previous lemma gives that

$$\mathbf{E}(\Delta'\mid J>2j-1)\leqslant C(G)(\#\Gamma'_j)\max p(x)\log^2 r,$$

where Γ'_j is the collection of the connected components γ of $\mathrm{LE}(R[0,U_{2j-1}])\cap B\big(w,\frac{1}{2}\tau\big)$ satisfying $\gamma\cap B(w,\sigma+\varrho)\neq\emptyset$. Again, it is easy to see that $\#\Gamma'_j\leqslant j-1$ and in particular $\#\Gamma'_1=0$. Hence, we get

$$\mathbf{E}(\Delta')=\sum_{j=1}^{\infty}\mathbf{E}\Delta'\cdot\mathbf{1}_{\{J=2j\}}\leqslant C(G)\max p(x)\log^2 r\sum_{j=2}^{\infty}(j-1)\mathbf{P}(J>2j-1).\tag{103}$$

The Green function estimate (12) also shows that

$$\begin{aligned}\mathbf{P}(J>2j+1\mid J>2j)&=\mathbf{E}\mathbf{P}^{R(U_{2i})}(R(T(\partial B(w,8\sigma)\cup\partial\mathcal{D}))\in B(w,8\sigma))\\&\leqslant\mathbf{E}\mathbf{P}^{R(U_{2i})}(T_{w,8\sigma}<\infty)\leqslant C\frac{\sigma}{\tau}.\end{aligned}$$

Hence, we have (for r sufficiently large)

$$\sum_{j=1}^{\infty} j\mathbf{P}(J>2j+1) \leqslant \mathbf{P}(J>1)\sum_{j=1}^{\infty} j\Big(C\frac{\sigma}{\tau}\Big)^j \leqslant C\mathbf{P}(J>1)\frac{\sigma}{\tau},$$

and together with (102), (103) and Harnack's inequality (Lemma 2.2), it shows that $\max p(x)\leqslant C(G)\min p(x)$, we get

$$\mathbf{E}(\Delta') \leqslant C\frac{\sigma}{\tau}\omega(B(w,16\sigma))\log^2 r. \qquad \square$$

Lemma 4.11 and (101) give

$$\mathbf{E}(\mathcal{X}_w) \leqslant Cr^{-\eta}\log^5 r + C\Big(\frac{\sigma}{\tau}\Big)\omega(B(w,16\sigma))\log^2 r. \tag{104}$$

For any point in $z\in B(v,r)\cap\frac{1}{3}\sigma\mathbf{Z}^3$, let w_z be the point closest to z in G. Clearly, if r is sufficiently large, the balls $B(w_z,\sigma)$ form a cover of $\mathcal{D}$ and furthermore

$$\sum_z \omega(B(w_z,16\sigma)) \leqslant C\omega(\partial\mathcal{D}) = C.$$

Hence, we can sum (104) over z and get

$$\mathbf{E}\,\mathrm{QL}(\varrho,\tau,\mathrm{LE}(R[0,T(\partial\mathcal{D})])) \leqslant C\Big(\frac{r}{\sigma}\Big)^3 r^{-\eta}\log^5 r + C\Big(\frac{\sigma}{\tau}\Big)\log^2 r.$$

Now is the time to pick δ. We take $\delta=\min\{\frac{1}{11}\eta,\frac{1}{7}\varepsilon\}$ and define $\tau=r^{1-\delta}$ and $\sigma=r^{1-3\delta}$. For r sufficiently large the condition $\sigma>2r^{1-\varepsilon/2}$ would be fulfilled. We get

$$\mathbf{E}\,\mathrm{QL}(r^{1-\varepsilon},r^{1-\delta},\mathrm{LE}(R[0,T(\partial\mathcal{D})])) \leqslant Cr^{-2\delta}\log^5 r + Cr^{-2\delta}\log^2 r \leqslant Cr^{-\delta},$$

and the theorem is proved.

4.6. Proof of Theorem 4 in two dimensions

Lemma 4.12. *Let G be a two-dimensional Euclidean net. Let $v\in G$, let $r>C(G)$ and let γ be a path from $\partial B(v,r)$ to $\partial B(v,2r)$. Let $w\in B(v,r)$ and let R be a random walk starting from w and stopped on $\partial B(v,2r)$. Then*

$$\mathbf{P}(R\cap\gamma\neq\varnothing) > c(G).$$

Proof. It is easy to see, applying Lemma 2.5, say three times, that there is a probability $>c(G)$ that R does a loop around the annulus $B\big(v, \frac{3}{2}r\big) \setminus \overline{B(v,r)}$. Two-dimensional geometry shows that in this case the linear extensions of R and γ intersect (the *linear extension* of a path in G is a path in $\mathbf{R}^2$ which is composed of all points of γ connected by linear segments). Since the length of edges in G is bounded, we get

$$\mathbf{P}\big(\text{there exists } t \leqslant T_{v,2r} : d\big(R(t), \gamma \cap B\big(v, \tfrac{3}{2}r\big)\big) < C(G)\big) > c(G).$$

However, the first such t is a stopping time, so we can consider the walk after it as a regular random walk, and of course it has a positive probability to hit γ. □

LEMMA 4.13. *Let G be a two-dimensional Euclidean net. Let $v \in G$ and let $s>r>1$. Let γ be a path from $\partial B(v,r)$ to $\partial B(v,s)$. Let R be a random walk starting from v. Then*

$$\mathbf{P}(R \cap \gamma = \varnothing) \leqslant C(G) \left(\frac{r}{s}\right)^{c(G)}.$$

This follows immediately from the previous lemma and the Wiener shell test.

The proof of Theorem 4 now proceeds exactly as in the three-dimensional case, with Lemma 4.12 serving as a replacement for Lemma 2.9, and Lemma 4.13 serving as a replacement for Lemmas 4.7 and 4.8. The only complication is Lemma 4.11, which no longer holds as stated. It is necessary at this point to estimate $\mathbf{P}(J>2i+1 \mid J>2i)$ by Lemma 4.13 for the incoming walk, with the result that one gets only $(\sigma/\tau)^c$ in the formulation of the lemma. See [K, Sublemmas 18.2 and 18.3] for details.

4.7. Continuity in the starting point

We will need one additional corollary of the techniques of this chapter.

LEMMA 4.14. *Let G be an isotropic graph and let $v, w \in \mathcal{E} \subset \mathcal{D} \subset G$, $\mathcal{D}$ finite. Let*

$$p^x := \mathbf{P}^x(\mathrm{LE}(R[0, T(\partial\mathcal{D})]) \subset \mathcal{E}), \quad x = v, w.$$

Then

$$|p^v - p^w| \leqslant C(G) \left(\frac{|v-w|}{d(v, \partial\mathcal{E})}\right)^{c(G)}.$$

Proof. Let $\mu = |v-w|/d(v, \partial\mathcal{E})$. We may assume, without loss of generality, that μ is sufficiently small. Let H be the graph generated by taking $\overline{\mathcal{D}}$ and identifying all the points of $\partial\mathcal{D}$ (this process is often called "wiring the boundary"). Let T be the uniform spanning tree on H. Then by Pemantle [P91], the distribution of the path in T from x to $\partial\mathcal{D}$ is identical to the distribution of a loop-erased random walk on H starting from x

and stopped on $\partial\mathcal{D}$, which is identical to the distribution on the graph G. Hence the two branches β^v and β^w of T from v and w to $\partial\mathcal{D}$ is a coupling of the two loop-erased walks. Denote by γ^x the portion of β^x from x until the unique intersection of β^v and β^w; and denote by p the probability that $\gamma^v\cup\gamma^w\subset\mathcal{E}$. Then, clearly,

$$|p^v-p^w|\leqslant 1-p.$$

Now, by Wilson's algorithm [W96], γ^w may be constructed by first constructing β^v and then taking a random walk starting from w, stopping it when it first hits β^v (possibly at time 0) and performing loop-erasure on the result. Hence, we need to show that

$$\mathbf{P}^w(T(\beta^v)>T(\partial\mathcal{E}))\leqslant C(G)\mu^{c(G)}.$$

In three dimensions we use Lemma 4.6. Let $K=\frac{1}{2}$ and let δ be the $\delta\left(\frac{1}{2},G\right)$ of Lemma 4.6. Let the r and s of Lemma 4.6 be $d(w,\partial\mathcal{E})$ and $\max\{|v-w|,1\}$, respectively—if μ is sufficiently small we would get $r>s$. We get that (except for probability $C\mu^{1/2}$ in the walk starting from v)

$$\mathbf{P}^{2,w}(R^{2,w}[0,T^2_{w,r}]\cap \mathrm{LE}(R^1[0,T^1(\partial\mathcal{D})])\cap B(w,r)\setminus\overline{B(w,s)}=\varnothing)\leqslant C\mu^\delta.$$

Therefore $\mathbf{P}(\gamma^w\not\subset\mathcal{E})\leqslant C\mu^\delta+C\mu^{1/2}$. The same holds for $\mathbf{P}(\gamma^v\not\subset\mathcal{E})$ and the three-dimensional case is finished. The two-dimensional case follows similarly from Lemma 4.13. □

Remark. The use of Lemma 4.6 to estimate the probability that a loop-erased random walk and a random walk starting from close points will hit is somewhat an overkill. For example, in $\mathbf{Z}^3$, if they start from the same point then the nice symmetry argument of [L99] can show that this probability is $>1-Cr^{-1/3}$. Presumably, an equivalent argument would work in our case as well. The arguments of [AB99] should also give a usable estimate.

5. Isotropic interpolation

The purpose of this chapter is to compare random walks on two or more graphs all of which are isotropic, with uniformly bounded structure constants. We shall call such a collection $\mathcal{G}$ an *isotropic family* and denote by $C(\mathcal{G})$, $c(\mathcal{G})$ etc. constants which depend only on the maximum of the isotropic structure constants of all $G\in\mathcal{G}$.

5.1. Hitting probabilities

In this section we will compare probabilities by proving inequalities of the sort

$$|p-q| \leqslant \varepsilon \max\{p, q\}.$$

It will be convenient to denote this by

$$p \overset{\varepsilon}{\simeq} q.$$

When $\varepsilon = Cr^{-c}$ for some constants $C(\mathcal{G})$ and $c(\mathcal{G})$, we will usually omit it, and just write $p \simeq q$ (r will be clear from the context). Occasionally we will prove instead that $|p-q| \leqslant \varepsilon p$ or $|p-q| \leqslant \varepsilon \min\{p, q\}$. We will always assume that $\varepsilon \leqslant \frac{1}{2}$, and then they are all equivalent up to constants.

We will often use the following version of differentiation of product: assume that

$$p^1 \overset{\alpha}{\simeq} p^2 \text{ and } q^1 \overset{\beta}{\simeq} q^2, \quad \alpha, \beta \leqslant \tfrac{1}{2}.$$

Then

$$\begin{aligned} |p^1 q^1 - p^2 q^2| &\leqslant |p^1 - p^2| q^1 + p^2 |q^1 - q^2| \\ &\leqslant \alpha q^1 \max\{p^1, p^2\} + \beta p^2 \max\{q^1, q^2\} \\ &\leqslant \alpha q^1 (1+2\alpha) p^1 + \beta p^2 (1+2\beta) q^2 \\ &\leqslant (\alpha+\beta+2(\alpha^2+\beta^2)) \max\{p^1 q^1, p^2 q^2\} && (105) \\ &\leqslant 2(\alpha+\beta) \max\{p^1 q^1, p^2 q^2\}. && (106) \end{aligned}$$

Another useful fact is that if $p^i = \sum_n q_n^i$ and $q_n^1 \overset{\beta}{\simeq} q_n^2$, then

$$\begin{aligned} |p^1 - p^2| &\leqslant \sum_n |q_n^1 - q_n^2| \leqslant \sum_n \beta \max\{q_n^1, q_n^2\} \leqslant \sum_n \beta (q_n^1 + q_n^2) = \beta(p^1 + p^2) \\ &\leqslant 2\beta \max\{p^1, p^2\}. \end{aligned} \qquad (107)$$

LEMMA 5.1. *Let G be an isotropic graph and let $r_1 > s > r_2 > 1$. Let $v \in G$ and let $v^1, v^2 \in \partial B(v, s)$ and $w \in A := \partial B(v, r_1) \cup \partial B(v, r_2)$. Let*

$$p^i = \mathbf{P}^{v^i}(R(T(A)) = w).$$

Then

$$|p^1 - p^2| \leqslant C(G) \Big(\max\Big\{ \frac{s}{r_1}, \frac{r_2}{s} \Big\} \Big)^{c(G)} \max\{p_1, p_2\}. \qquad (108)$$

Similarly, if $A := \partial B(v, r_1) \cup \{v\}$ then

$$|p_1 - p_2| \leqslant C \left(\max\left\{ \frac{s}{r_1}, \frac{1}{s} \right\} \right)^{c(G)} \max\{p_1, p_2\}.$$

Proof. Assume first that $r_2 \geqslant r_1^{1-c_4/2}$, where $c_4(G)$ is from Lemma 3.3, and assume also that $r_1 = 2^N s = 2^{2N} r_2$ for some integer N (removing both assumptions is easy, and we do it in the end of the lemma). Assume also that $w \in \partial B(v, r_2)$. Define

$$a_j := s2^{-j} \text{ and } X_j := \partial B(v, a_j) \cup \partial B(v, r_1), \quad j = 0, 1, ..., N.$$

Define stopping times $T_0 = 0$ and

$$T_j := \min\{t \geqslant T_{j-1} : R(t) \in X_j\}.$$

Define probability measures μ_j^i on $\partial B(v, a_j)$ by

$$\mu_j^i(x) = \mathbf{P}^{v^i}(R(T_j) = x \mid R(T_j) \in \partial B(v, a_j)).$$

We proceed by examining how μ_j^i evolves with j. It will be useful to use L^1 estimates during intermediate stages, so define

$$\varepsilon_j := \sum_{x \in \partial B(v, a_j)} |\mu_j^1(x) - \mu_j^2(x)|.$$

Clearly $\varepsilon_0 = 2$. For $x \in \partial B(v, a_j)$ and $y \in \partial B(v, a_{j+1})$ we let

$$\pi(x, y) = \mathbf{P}^x(R(T(X_{j+1})) = y \mid R(T(X_{j+1})) \in \partial B(0, a_{j+1})),$$

and then get

$$\mu_{j+1}^i(y) = \sum_{x \in \partial B(0, a_j)} \mu_j^i(x) \pi(x, y).$$

Let now

$$A^+ := \{x \in \partial B(0, a_j) : \mu_j^1(x) \geqslant \mu_j^2(x)\}, \quad A^- := \partial B(0, a_j) \setminus A^+,$$

and

$$D^\pm(y) := \sum_{x \in A^\pm} |\mu_j^1(x) - \mu_j^2(x)| \pi(x, y).$$

Clearly,

$$\sum_y D^\pm(y) = \sum_{x \in A^\pm} |\mu_j^1(x) - \mu_j^2(x)| \sum_y \pi(x, y) = \sum_{x \in A^\pm} |\mu_j^1(x) - \mu_j^2(x)| = \tfrac{1}{2}\varepsilon_j. \tag{109}$$

Next, Harnack's inequality (Lemma 2.2) shows that $\pi(x, y) \approx \pi(x', y)$ for any $x, x' \in \partial B(0, a_j)$, so $D^+(y) \approx D^-(y)$ ($\pi(x, y)$ is a quotient of two harmonic functions and we use Harnack's inequality for both). This gives that

$$|D^+(y) - D^-(y)| \leqslant (1 - c(G))(D^+(y) + D^-(y)),$$

and hence

$$\varepsilon_{j+1}=\sum_{y\in\partial B(v,a_{j+1})}|D^+(y)-D^-(y)|\leqslant(1-c)\sum_y(D^+(y)+D^-(y))\overset{(109)}{=}(1-c)\varepsilon_j.$$

Therefore we get $\varepsilon_{N-1}\leqslant 2(1-c)^{N-1}=C(s/r_1)^c$. This establishes the L^1 estimate. We now use the last step (from $N-1$ to N) to move to a uniform estimate. Return to our $w\in\partial B(v,r_2)$. We have

$$\mu_N^i(w)=\sum_{x\in\partial B(v,a_{N-1})}\mu_{N-1}^i(x)\pi(x,w)\geqslant\min_x\pi(x,w).$$

On the other hand

$$|\mu_N^1(w)-\mu_N^2(w)|\leqslant\sum_{x\in\partial B(v,a_{N-1})}|\mu_{N-1}^1(x)-\mu_{N-1}^2(x)|\pi(x,w)\leqslant C\Big(\frac{s}{r_1}\Big)^c\max_x\pi(x,w)$$

and, by Harnack's inequality (Lemma 2.2),

$$|\mu_N^1(w)-\mu_N^2(w)|\leqslant C\Big(\frac{s}{r_1}\Big)^c\max\{\mu_N^1(w),\mu_N^2(w)\}. \tag{110}$$

The difference between $\mu_N^i(w)$ and p^i is just that $\mu_N^i(w)$ is conditioned, i.e.

$$p^i=\mu_N^i(w)\mathbf{P}^{v^i}(R(T_N)\in\partial B(v,r_2)).$$

To estimate the second term we use Lemma 3.5. The probability q^i that a Brownian motion starting from v^i hits $\partial B(v,r_2)$ before $\partial B(v,r_1)$ is

$$q^i=\frac{a(|v^i|)-a(r_1)}{a(r_2)-a(r_1)},\quad\text{where } a(t)=\begin{cases}t^{2-d}, & \text{if } d\geqslant 3,\\ \log t, & \text{if } d=2,\end{cases} \tag{111}$$

and in either case we get that $|q^1-q^2|\leqslant C(G)s^{-1}\max\{q^1,q^2\}$. Therefore Lemma 3.5 (take e.g. the ε of Lemma 3.5 to be $\frac{1}{2}$) gives that

$$\mathbf{P}^{v^1}(R(T_N)\in\partial B(0,r_2))\simeq\mathbf{P}^{v^2}(R(T_N)\in\partial B(0,r_2)). \tag{112}$$

Together with (110) the lemma is proved in this case.

The case where $w\in\partial B(v,r_1)$ is identical, with this time defining $a_j=s2^j$ and

$$X_j=\partial B(v,a_j)\cup\partial B(v,r_2).$$

The argument about the exponential decrease of the ε_j works identically. Finally, (111) shows that $\mathbf{P}^{v^i}(R(T_N)\in\partial B(v,r_1))>c$ and therefore (112) implies an identical estimate for $\mathbf{P}^{v^i}(R(T_N)\in\partial B(v,r_1))$. Hence this case is finished too.

Finally, assume that one of the assumptions on r_1, s and r_2 fails. If $r_1/s<2$ or $s/r_2<2$ the lemma holds trivially for sufficiently large constants. Hence assume that both are $\geqslant 2$. It now follows easily that one can find u_i such that $r_1>u_1>s>u_2>r_2$, $u_1/s=s/u_2=2^N$ and $u_2\geqslant u_1^{1-c_4/2}$, and further that

$$\frac{s}{u_1}\leqslant\left(\max\left\{\frac{s}{r_1},\frac{r_2}{s}\right\}\right)^{c(G)}.$$

We use the case already established and find that for any $x\in\partial B(v,u_1)\cup\partial B(v,u_2)$ one has that

$$|p^1(x)-p^2(x)|\leqslant C(G)\left(\frac{s}{u_1}\right)^c\max\{p^1(x),p^2(x)\},$$

where we define $p^i(x):=\mathbf{P}^{v^i}(R(T(\partial B(v,u_1)\cup\partial B(v,u_2)))=x)$. Let

$$\pi(x)=\mathbf{P}^x(R(T(\partial B(v,r_1)\cup\partial B(v,r_2)))=w)$$

and get

$$p^i=\sum_x p^i(x)\pi(x).$$

Therefore

$$\begin{aligned}|p^1-p^2|&\leqslant\sum_x|p^1(x)-p^2(x)|\pi(x)\leqslant C(G)\left(\frac{s}{u_1}\right)^c\sum_x(p^1(x)+p^2(x))\pi(x)\\&=C(G)\left(\frac{s}{u_1}\right)^c(p^1+p^2)\leqslant C\left(\max\left\{\frac{s}{r_1},\frac{r_2}{s}\right\}\right)^c\max\{p^1,p^2\}.\end{aligned}$$

The case of a single point is proved identically. □

LEMMA 5.2. *Let G, r_1, s, v, v^1 and v^2 be as in Lemma* 5.1, *and let $w\in\partial B(v,r_1)$. Let $p^i=\mathbf{P}^{v^i}(R(T_{v,r_1})=w)$. Then*

$$|p^1-p^2|\leqslant C(G)\left(\frac{s}{r_1}\right)^{c(G)}\max\{p^1,p^2\}.$$

The proof is a simplified version of the proof of Lemma 5.1 and we shall omit it.

COROLLARY. *The conclusion of Lemma* 5.2 *holds if $v^1,v^2\in B(0,s)$ (and not on its boundary).*

Proof. Apply Lemma 5.2 after the stopping time on $\partial B(0,s)$. □

LEMMA 5.3. *Let $\mathcal{G}=(G^1,G^2)$ be an isotropic family, let $v\in\mathbf{R}^d$ and assume that for some $r>C(\mathcal{G})$ one has*

$$\overline{B_{G^1}(v,4r)}\setminus B_{G^1}(v,r)=\overline{B_{G^2}(v,4r)}\setminus B_{G^2}(v,r).\tag{113}$$

Let $w\in\overline{B\big(v,\frac{5}{2}r\big)}\setminus B\big(v,\frac{3}{2}r\big)$, $x\in\overline{B(v,3r)}\setminus B(v,r)$ *and let* R^i *be a random walk on* G^i *starting from* x *(which is contained in both* G^1 *and* G^2*) and stopped on* $\partial B(v,4r)$*. Let*

$$p^i=\mathbf{P}(w\in R^i[1,T^i_{v,4r}]).$$

Then $|p^1-p^2|\leqslant C(\mathcal{G})r^{-c(\mathcal{G})}\min\{p^1,p^2\}$.

In (113), $\overline{B_{G^i}(v,4r)}$ is an open ball in the Euclidean metric, with the closure operation of G^i. We consider $\overline{B_{G^i}(v,4r)}\setminus B_{G^i}(v,r)$ as an induced subgraph of G^i as well as a subset of $\mathbf{R}^d$. Equality in (113) is both as graphs and as subsets of $\mathbf{R}^d$.

Notice that in the definition of p^i we consider that w was hit only starting from the first step. Hence the lemma gives a non-trivial estimate even if $x=w$ (a case that is actually important).

Proof. Let $X=\partial B(v,4r)\cup\{w\}$. Let $s=r^{1-c_4(\mathcal{G})/2}$, where $c_4(\mathcal{G})=\min\{c_4(G_1),c_4(G_2)\}$ and $c_4(G)$ is from Lemma 3.3. Define stopping times as follows: $T^i_0=0$ and

$$\begin{aligned} T^i_{2n+1}&:=\min\{t\geqslant T^i_{2n}:R^i(t)\in\partial B(w,s)\cup X\}, && n=0,1,\dots,\\ T^i_{2n}&:=\min\big\{t\geqslant T^i_{2n-1}:R^i(t)\in\partial B\big(w,\tfrac14 r\big)\cup X\big\}, && n=1,2,\dots. \end{aligned}$$

Let $p^i_n=\mathbf{P}(R^i(T^i_n)\notin X)$. The core of the lemma is showing that

$$|p^1_n-p^2_n|\leqslant Knr^{-k}\max\{p^1_n,p^2_n\} \tag{114}$$

for any n such that $Kn^2r^{-k}\leqslant\frac14$. Here $K=K(\mathcal{G})$ and $k=k(\mathcal{G})$ are some *sufficient* constants (by which we mean that K is sufficiently large and $k>0$ sufficiently small) which will be fixed later.

The proof of (114) will be done by induction over n. For $n=1$, (114) follows from Lemma 3.5 with $\varepsilon=\frac18$, if r, K and k are sufficient. Next, if $n\geqslant1$, define

$$\mu^i_{2n}(y):=\mathbf{P}(R^i(T^i_{2n})=y\,|\,T^i_{2n-1}\notin X),\quad y\in\partial B\big(w,\tfrac14 r\big)\cup\{w\},$$

and then Lemma 5.1 gives that the portion of the walk between T_{2n-1} and T_{2n}, which is the same since $B_{G^1}\big(v,\frac14 r\big)=B_{G^2}\big(v,\frac14 r\big)$, erases most of the difference between the μ^i's, and we have

$$\mu^1_{2n}(y)\simeq\mu^2_{2n}(y). \tag{115}$$

Let $\alpha^i=\sum_{y\neq w}\mu^i_{2n}(y)$, so $\alpha^1\simeq\alpha^2$. Now, $p^i_{2n}=p^i_{2n-1}\alpha^i$ so we can use (105), note that $Cr^{-c}<\frac12$ for r sufficiently large and

$$K(2n-1)r^{-k}\leqslant\frac{2n-1}{4(2n)^2}<\frac{1}{8n}<\frac12,$$

and we get

$$\begin{aligned}|p^1_{2n}-p^2_{2n}| &\overset{(105)}{\leqslant}(K(2n-1)r^{-k}+Cr^{-c}+2((K(2n-1)r^{-k})^2+(Cr^{-c})^2))\max\{p^1_{2n},p^2_{2n}\}\\ &\leqslant \big(K\big(2n-\tfrac{1}{2}\big)r^{-k}+Cr^{-c}\big)\max\{p^1_{2n},p^2_{2n}\}\\ &\overset{(*)}{\leqslant} 2Knr^{-k}\max\{p^1_{2n},p^2_{2n}\},\end{aligned}\tag{116}$$

where $(*)$ holds if only K and k are sufficient. Hence, the induction holds when going from $2n-1$ to $2n$.

For the case of going from $2n$ to $2n+1$, let, for any $y\in\partial B\big(w,\frac{1}{4}r\big)$,

$$\pi^i(y)=\mathbf{P}^y(R^i(T^i(\partial B(w,s)\cup X))\in\partial B(w,s)).$$

Then

$$p^i_{2n+1}=p^i_{2n}\sum_y \mu^i_{2n}(y)\pi^i(y).$$

As in the previous part, we define $\alpha^i:=\sum_y \mu^i_{2n}(y)\pi^i(y)$. Lemma 3.5 shows that

$$\pi^1(y)\simeq\pi^2(y).$$

Together with (115), we get that

$$|\alpha^1-\alpha^2|\overset{(106)}{\leqslant}\sum_y Cr^{-c}\max\{\mu^1_{2n}(y)\pi^1(y),\mu^2_{2n}(y)\pi(y)\}\overset{(107)}{\leqslant}Cr^{-c}\max\{\alpha^1,\alpha^2\}$$

for r sufficiently large. Thus, we can repeat the calculations of (116) and get again that (114) is preserved if only K and k are sufficient. Hence the proof of (114) is complete and we may fix the values of K and k.

Next, we need to ask how many n's are actually relevant. Lemma 2.5 shows that

$$p^i_{2n+1}\leqslant(1-c(\mathcal{G}))p^i_{2n}\quad\text{for all } n\geqslant 1.$$

Hence, the p^i decrease exponentially. On the other hand, the Green function estimates (11) and (12) show that

$$p^i>c(\mathcal{G})\begin{cases} r^{2-d}, & \text{if } d\geqslant 3,\\ 1/\log r, & \text{if } d=2.\end{cases}$$

Define therefore $N=C(\mathcal{G})\log r$ for some sufficiently large C and get

$$\sum_{n=N+1}^{\infty} p^i_n<r^{-1}p^i.$$

We can now calculate

$$p^i=\mathbf{P}(R^i(T_1^i)=w)+\sum_{n=1}^{\infty}p_{2n-1}^i\mu_{2n}^i(w).$$

The first summand is non-zero only if $x\in B(w,s)$ and in this case it is independent of i. Hence, we may write

$$\begin{aligned}|p^1-p^2| &\overset{(106)}{\leqslant}\sum_{n=1}^{N}(2Knr^{-k}+Cr^{-c})\max\{p_{2n-1}^1\mu_{2n}^1(w),p_{2n-1}^2\mu_{2n}^2(w)\}+2r^{-1}\max\{p^1,p^2\}\\ &\overset{(107)}{\leqslant}(4KNr^{-k}+Cr^{-c}+2r^{-1})\max\{p^1,p^2\}\end{aligned}$$

if only r is sufficiently large so that we have $KN^2r^{-k}<\frac{1}{4}$. This finishes the lemma. □

Remark. In three dimensions this lemma may be simplified significantly, since the probability to hit w after T_2 is significantly smaller than between 0 and T_2, so there is no need for the induction.

LEMMA 5.4. *Let $\mathcal{G}$, r, v, w and R^i be as in Lemma* 5.3 (*perhaps with a different constant bounding r from below*). *Let $x\in\partial B(v,4r)$ and let*

$$p^i:=\mathbf{P}^w(R^i(T_{v,4r}^i)=x).$$

Then $p^1\simeq p^2$.

Proof. The symmetry of random walks in the form (4) shows that

$$p^i=\frac{\mathbf{P}^x(w\in R^i[1,T_{v,4r}^i])}{1-\mathbf{P}^w(w\in R^i[1,T_{v,4r}^i])}\nu, \tag{117}$$

where the constant ν is the ratio of the degrees of w and x and is independent of i. Denote the denominator by $1-a^i$. Lemma 5.3 shows that $a^1\simeq a^2$. The Green function estimates (11) and (12) show that $1-a^i\geqslant c/\log r$, and therefore

$$\left|\frac{1}{1-a^1}-\frac{1}{1-a^2}\right|\leqslant Cr^{-c}\log r\max\left\{\frac{1}{1-a^1},\frac{1}{1-a^2}\right\}, \tag{118}$$

and we can drop the log factor from (118) and pay in the constants only.

Next denote the numerator of (117) by b^i. For any $y\in\partial B(v,3r)$ let

$$\pi(y):=\mathbf{P}^x(R^i(T^i(\partial B(v,4r)\cup\partial B(v,3r)))=y)\quad\text{and}\quad \varrho^i(y):=\mathbf{P}^y(w\in R^i[1,T_{v,4r}^i])$$

(π does not depend on i because G^1 and G^2 are identical on the relevant annulus). Hence we get $b^i=\sum_y\pi(y)\varrho^i(y)$. Lemma 5.3 shows that $\varrho^1(y)\simeq\varrho^2(y)$, and therefore by (107) we get $b^1\simeq b^2$. Together with (118) the lemma is proved (again we use here (106)). □

Corollary. *Let $\mathcal{G}$, r, v and x be as in Lemma* 5.4, *and let $v^1, v^2 \in \partial B(v, 2r)$. Let*

$$p^i := \mathbf{P}^{v^i}(R^i(T^i_{v,4r}) = x).$$

Then

$$|p^1 - p^2| \leqslant C(\mathcal{G}) \left(\frac{|v^1 - v^2|}{r} \right)^{c(\mathcal{G})} \max\{p^1, p^2\}.$$

Proof. If $|v^1 - v^2| \leqslant \frac{1}{8} r$, then we can use Lemma 5.2 to show that the walk up to $B\big(v^1, \frac{1}{4} r\big)$ erases the difference between v^1 and v^2, and then use Lemma 5.4 to show that the fact that the graphs are different has a small effect. If $|v^1 - v^2| > \frac{1}{8} r$, then for a sufficiently large C, there is nothing to prove. □

Lemma 5.5. *Lemma* 5.4 *holds also when $w \in B\big(v, \frac{3}{2} r\big)$.*

Here w is some point in $\mathbf{R}^d$ and the notation $\mathbf{P}^w$ refers to a random walk starting from the point of the relevant graph closest to w.

Proof. Let $\{\Delta_j\}$ be a triangulation of $B(v, 2r)$ by spherical triangles such that

$$|\Delta_j| \geqslant c r^{-c_6/2} \quad \text{and} \quad \operatorname{diam} \Delta_j \leqslant C r^{1 - c_6/2d},$$

where c_6 is from Lemma 3.6 and $|\cdot|$ is the normalized volume. Let Δ^*_j be disjoint discrete versions of the Δ_j's covering $\partial B(v, 2r)$. Let w^i be the point of G^i closest to w, let $p^i_j := \mathbf{P}^{w^i}(R^i(T_{v,2r}) \in \Delta^*_j)$ and $q^i_j := \mathbf{P}^{w^i}(W(S_{v,2r}) \in \Delta_j)$ be the Brownian motion analogues. Now, q^i_j has a formula given from the surface integral over the Poisson kernel [B95, II, Theorem 1.17]:

$$q^i_j = r^{d-2} \int_{\Delta_j} \frac{r^2 - \|w^i\|^2}{\|x - w^i\|^d} \, dx.$$

This immediately shows, since $\|w^1 - w^2\| \leqslant C(G)$, that $q^1_j \simeq q^2_j$. Lemma 3.6 shows that

$$|q^i_j - p^i_j| \leqslant C r^{-c_6} \leqslant C r^{-c_6/2} |\Delta_j| \leqslant C r^{-c_6/2} q^i_j,$$

so also $p^1_j \simeq p^2_j$.

Next, let $y^1, y^2 \in \Delta^*_j$. Let $\pi^i(y) = \mathbf{P}^y(R^i(T^i_{v,4r}) = x)$. Because $\operatorname{diam} \Delta^*_j \leqslant C r^{1-c}$, we can use the corollary after Lemma 5.4 to get $\pi^1(x^1) \simeq \pi^2(x^2)$. This finishes the lemma, since the probabilities to hit a given Δ^*_j are $C r^{-c}$-similar, and the point in which you hit is unimportant up to a $C r^{-c}$-error. □

5.2. Definition

Let G^1 and G^2 be two d-dimensional isotropic graphs, and let $\alpha>0$. We say that G^1 and G^2 have an *α-isotropic interpolation* if the following holds. Let L and M, $M\geqslant 1$ integer, satisfy

$$L > C(G^i) \quad \text{and} \quad M \leqslant L^{\alpha}. \tag{119}$$

We always assume that $C(G^i)\geqslant 2$ always. Let $\xi\in\{1,2\}^{M^d}$ be any configuration. Then there exist graphs $G(L,M,\xi)$ such that

(i) if all the coordinates of ξ are i then $G(L,M,\xi)\equiv G^i$;

(ii) if

$$I := [a_1,b_1]\times...\times[a_d,b_d]\subset\{0,...,M-1\}^d$$

is some box in the configuration space, and if $\xi_1|_I\equiv\xi_2|_I$, then $G(L,M,\xi_1)$ is equal to $G(L,M,\xi_2)$ on a corresponding box in $\mathbf{R}^d$, i.e.

$$G(L,M,\xi_1)\cap J = G(L,M,\xi_2)\cap J,$$

where

$$J := \big[L\big(a_1+\tfrac{1}{3}\big), L\big(b_1+\tfrac{2}{3}\big)\big]\times...\times\big[L\big(a_d+\tfrac{1}{3}\big), L\big(b_d+\tfrac{2}{3}\big)\big]; \tag{120}$$

as above, equality is both as graphs, with the graph structure induced from $G(L,M,\xi_i)$, and as subsets of $\mathbf{R}^d$;

(iii) $G(L,M,\xi)$ is isotropic with the isotropic structure constants bounded independently of L, M and ξ.

Notice that this definition gives special importance to the point zero. This is just for convenience—in practice, this fact will have no significance.

The core of this paper is the proof of the following theorem.

THEOREM 5. *Let G^1 and G^2 be two d-dimensional graphs with an α-isotropic interpolation, $d=2,3$. Let $\mathcal{D}\subset\big[\frac{1}{4},\frac{3}{4}\big]^d$ be an open polyhedron and let $\mathcal{E}$ be some open set. Let $a\in\mathcal{E}\cap\mathcal{D}$ be some point. Let $s>0$ be some number, and let R^i be random walks on G^i starting from sa and stopped when hitting $\partial(s\mathcal{D})$. Then*

$$\mathbf{P}(\mathrm{LE}(R^i)\subset s(\mathcal{E}+B(0,Cs^{-c}))) > \mathbf{P}(\mathrm{LE}(R^{3-i})\subset s\mathcal{E}) - Cs^{-c}.$$

Remember that "a random walk starting from sa", means that it starts from the point of G^i closest to sa; that $\mathcal{E}+B(0,Cs^{-c})$ refers to the set of points within distance Cs^{-c} from $\mathcal{E}$; and that an "open polyhedron" is any open set whose boundary is made of non-degenerate linear polyhedra of dimension $d-1$, and that we do not require that the boundary of the polyhedron be connected, but we do not allow slits.

A comment is due on the use of constants here. They all depend on a, $\mathcal{D}$ and on the graphs G^1 and G^2 (in fact they do not really depend on a but we will have no use for this fact). Like in previous chapters, they depend only on α and the global bound for the isotropic structure constants over all of $G(L,M,\xi)$ and not on other properties of G^i. However, there is no need to continue to point this fact out—we only did so in the previous chapters in order to be able to analyze walks on $G(L,M,\xi)$ simultaneously, and we will not have families of isotropic interpolations in the future.

5.3. Proof of Theorem 5

The arguments in this section are very similar to those of [K, §4], so we will be brief.

LEMMA 5.6. *Let G^i, a and s be as in Theorem 5. Assume that $s\leqslant LM$ with L and M satisfying (119) and L sufficiently large (in addition to the restriction of (119)). Let ξ^1 and ξ^2 be two configurations which differ only in one point z, so in particular*

$$H := G(L,M,\xi^i)\cap\left(\left[\tfrac{1}{3}L, s-\tfrac{1}{3}L\right]^d \setminus B(Lz,16L)\right)$$

does not depend on i. Let $\mathcal{B}\subset H$ be any subset such that a is in a finite component of $G(L,M,\xi^i)\setminus\mathcal{B}$ (think about $\mathcal{B}$ as the boundary of some $\mathcal{D}$, $a\in\mathcal{D}$). Let R^i be a random walk on $G(L,M,\xi^i)$ starting from sa. Let $b\in\mathcal{B}$ and define

$$p^i = \mathbf{P}(R^i(T^i(\mathcal{B})) = b).$$

Then

$$|p^1-p^2| \leqslant C(\mathcal{G})L^{-c(\mathcal{G})}\max\{p^1,p^2\}.$$

Proof sketch. Define stopping times T^i_j on $\partial B(Lz,3L)$ and $\partial B(Lz,12L)$ alternately. The graphs $G(L,M,\xi^i)$ are identical outside $B(Lz,3L)$, hence Lemma 5.5 shows that the transition probabilities, up to CL^{-c}, do not depend on i. Further, the probability to reach T^i_j without hitting $\mathcal{B}$ drops like e^{-cj} in three dimensions and like $e^{-cj/\log M}$ in two dimensions. Harnack's inequality on $B(Lz,16L)$ shows that the probabilities to hit b after T_j, conditioned on not hitting it before, are, up to a constant, independent of j. Hence these CL^{-c}-errors do not accumulate to more than $CL^{-c}\log M\leqslant CL^{-c}$. See [K, Lemma 16] for a detailed argument. □

LEMMA 5.7. *Let G^i, a, s, L, M, ξ^i, z, $\mathcal{B}$ and b be as in Lemma 5.6. Let R^i be a random walk on $G(L,M,\xi^i)$ starting from sa and conditioned to hit $\mathcal{B}$ in b. Let ζ^i be the segment of $\mathrm{LE}(R^i)$ until hitting $\partial B(Lz,16L)$ for the first time, or all of $\mathrm{LE}(R^i)$ if $\mathrm{LE}(R^i)\cap\partial B(Lz,16L)=\varnothing$.*

Then

$$\sum_{\gamma} |\mathbf{P}(\zeta^1=\gamma)-\mathbf{P}(\zeta^2=\gamma)| \leqslant C(\mathcal{G})L^{-c(\mathcal{G})},$$

where the sum is over all simple paths γ from sa to $\partial B(Lz,16L)\cup\partial(s\mathcal{D})$.

Proof sketch. The crucial point here is that ζ^i depends only on what happens outside $B(Lz,16L)$ (quite unlike the other portion of $\mathrm{LE}(R^i)$). Therefore the same argument as in the previous lemma works here. Conditioning on all the entry and exit points from $\partial B(Lz,3L)$ and $\partial B(Lz,12L)$, the probabilities are identical, and in average in γ it is enough to consider $\log^2 L$ such points. See [K, Lemma 17] for a detailed argument. □

The proof of Theorem 5 is also detailed in [K], where it is called the "main lemma". Since this is a crucial part of the argument, I prefer to bring it here in full.

Since the theorem is symmetric in i, fix it to be 1. Let $\varepsilon>0$ be some constant (depending on $\mathcal{G}$) which will be fixed later. We define L by

$$34L=s^{1-\varepsilon} \tag{121}$$

and $M=\lceil s/L\rceil$. We assume that L and M satisfy the requirements (119) of the isotropic interpolations, which will hold if $\varepsilon<\min\left\{\frac{1}{2}\alpha,\frac{1}{2}\right\}$ and L is large enough. We will also assume that L is sufficiently large for Lemmas 5.6 and 5.7 to hold, and also that all edges of all graphs $G(L,M,\xi)$ are shorter than L, and every ball of radius L in $\mathbf{R}^d$ contains at least one point from every $G(L,M,\xi)$. All these requirements translate to $s>C(\varepsilon,\mathcal{G})$.

Let $\delta=\delta(\varepsilon,\mathcal{G})$ be the quantity given by Theorem 4 (p. 85) for the ε from (121), for all the graphs $G(L,M,\xi)$ simultaneously. In other words, we have, if $s>C(\varepsilon,\mathcal{G})$,

$$\mathbf{E}^{sa}\,\mathrm{QL}(34L,s^{1-\delta},\mathrm{LE}(R[0,T(\partial\mathcal{D})]))\leqslant C(\mathcal{G})s^{-\delta},$$

which holds for a random walk R on any $G(L,M,\xi)$. With this δ, define "bad" subsets of $\{x\in\{0,...,M-1\}^d : x/M\in\mathcal{D}\}$ as follows:

$$\Phi:=\{x: d(Lx,\partial_{\mathrm{cont}}s\mathcal{E})\leqslant s^{1-\delta}+18L\}, \tag{122}$$

$$\Psi:=\{x: d(Lx,\partial_{\mathrm{cont}}s\mathcal{D})\leqslant 17L\}, \tag{123}$$

$$\Theta:=\{x: d(Lx,sa)\leqslant 17L\}.$$

It should be noticed that any $x\notin\Psi$ satisfies $d(Lx,\partial_{G(L,M,\xi)}(s\mathcal{D}))>17L$ and any $x\notin\Phi$ satisfies $d(Lx,\partial_{G(L,M,\xi)}(s\mathcal{E}))>s^{1-\delta}+17L$, both for any configuration ξ.

LEMMA 5.8. *With the definitions above, let $Y\subset\{0,...,M-1\}^d\setminus(\Phi\cup\Psi\cup\Theta)$. Let ξ^1 and ξ^2 be two configurations such that $\xi^i|_Y\equiv i$ but $\xi^1|_{Y^c}\equiv\xi^2|_{Y^c}$. Let R^i be a random walk on $G(L,M,\xi^i)$ starting from sa and stopped on $\partial\mathcal{D}$. Let*

$$p^i:=\mathbf{P}(\mathrm{LE}(R^i)\subset s\mathcal{E}).$$

Then

$$|p^1-p^2|\leqslant C(\mathcal{G})s^{-\delta}\log\#Y+C(\mathcal{G})L^{-c(\mathcal{G})}\#Y.$$

Proof. For every $0\leqslant k\leqslant\#Y$, let D_k be a random subset of Y of size k, let ξ_k be the configuration which is 2 on D_k, 1 on $Y\setminus D_k$ and identical to ξ^1 outside Y. Define $H_k:=G(L,M,\xi_k)$. Let a_k be the point of H_k closest to sa. Let S_k be a random walk on H_k starting from a_k and stopped on $B_k:=\partial_{H_k}s\mathcal{D}$. Let

$$p_k:=\mathbf{P}(\mathrm{LE}(S_k)\subset s\mathcal{E}),$$

where $\mathbf{P}$ here is over both the walk and the randomness of the graph (notice that $p^1=p_0$ and $p^2=p_{\#Y}$). The lemma will be proved once we show that

$$|p_k-p_{k+1}|\leqslant Cs^{-\delta}\left(\frac{1}{k+1}+\frac{1}{\#Y-k}\right)+CL^{-c}. \tag{124}$$

For this purpose, couple D_k and D_{k+1} such that $D_k\subset D_{k+1}$. Let $\Delta_k\subset\Delta_{k+1}$ be subsets of Y of sizes k and $k+1$, respectively, and let $z=\Delta_{k+1}\setminus\Delta_k$. For most of the rest of the lemma, we condition on the event (denote it by $\mathcal{X}$) that $D_k=\Delta_k$ and $D_{k+1}=\Delta_{k+1}$. Let $Z=B(Lz,16L)$. We construct $\mathrm{LE}(S_k)$ as follows:

- let b_k be a random point on B_k chosen with the hitting probabilities of S_k;
- let $\breve{S}_k$ be a random walk from a_k to B_k conditioned to hit b_k;
- let $\breve{\gamma}_k$ be a random simple path from a_k to $\partial Z\cup\{b_k\}$, which has the same distribution as the segment of $\mathrm{LE}(\breve{S}_k)$ until ∂Z (including the first point in ∂Z), or all of $\mathrm{LE}(\breve{S}_k)$ if $\mathrm{LE}(\breve{S}_k)\cap\partial Z=\varnothing$ (notice that $a_k\notin Z$ from the requirement $Y\cap\Theta=\varnothing$);
- let c_k be the point where $\breve{\gamma}_k$ hits ∂Z, if it does;
- let $\breve{T}_k$ be a random walk on H_k starting from b_k and conditioned to hit $B_k\cup\breve{\gamma}_k$ in c_k; set $\breve{T}_k=\varnothing$ if $\breve{\gamma}_k$ never hits ∂Z;
- let $\gamma_k=\breve{\gamma}_k\cup\mathrm{LE}(\breve{T}_k)$.

An easy application of Lemma 1.1 (symmetry of conditioned loop-erased random walks) shows that $\gamma_k\sim\mathrm{LE}(S_k)$. Lemma 5.6 shows that

$$\sum_b|q_k^1-q_{k+1}^1|\leqslant CL^{-c}, \tag{125}$$

where

$$q_k^1(b):=\mathbf{P}(b_k=b\,|\,\mathcal{X}).$$

Next, we use Lemma 5.7 for the random walk on H_k starting from a_k, stopped on B_k, and conditioned to hit b_k. The definition of Ψ (123) ensures the condition $B_k\cap Z=\varnothing$ required by Lemma 5.7. This shows that

$$\sum_\gamma|q_k^2-q_{k+1}^2|\leqslant CL^{-c}\quad\text{for all }\ b\in B_k, \tag{126}$$

where

$$q_k^2(b,\gamma) := \mathbf{P}(\breve{\gamma}_k = \gamma \mid \mathcal{X}, b_k = b).$$

Thirdly, we again use Lemma 5.7, this time for a random walk starting from b_k, stopped on $B_k \cup \breve{\gamma}_k$ and conditioned to hit $\check{c}_k$, to show that, when $\breve{\gamma}'_k$ is the portion of $\mathrm{LE}(\breve{T}_k)$ up to Z, one has

$$|q_k^3 - q_{k+1}^3| \leqslant CL^{-c} \quad \text{for all } b \text{ and } \gamma, \tag{127}$$

where

$$q_k^3(b,\gamma) := \mathbf{P}(\breve{\gamma}'_k \subset s\mathcal{E} \mid \mathcal{X}, b_k = b, \breve{\gamma}_k = \gamma).$$

Summing (125), (126) and (127) gives

$$\begin{aligned} |\mathbf{P}((\breve{\gamma}_k \cup \breve{\gamma}'_k) \subset s\mathcal{E} \mid \mathcal{X}) - \mathbf{P}((\breve{\gamma}_{k+1} \cup \breve{\gamma}'_{k+1}) \subset s\mathcal{E} \mid \mathcal{X})| & \\ = \Bigg| \sum_{b,\gamma \subset s\mathcal{E}} q_k^1 q_k^2 q_k^3 - q_{k+1}^1 q_{k+1}^2 q_{k+1}^3 \Bigg| & \\ \leqslant CL^{-c} + \Bigg| \sum_{b,\gamma \subset s\mathcal{E}} (q_k^1 q_k^2 - q_{k+1}^1 q_{k+1}^2) q_k^3 \Bigg| & \\ \leqslant \ldots \leqslant CL^{-c}. & \end{aligned} \tag{128}$$

In other words, we have proved that the probabilities that both segments of $\mathrm{LE}(S_k)$, leading up to Z and from Z to B_k, are in $s\mathcal{E}$, are close for k and $k+1$. Thus the only case we have not covered is that $\breve{\gamma}_k \cup \breve{\gamma}'_k \subset s\mathcal{E}$ but $\mathrm{LE}(S_k) \not\subset s\mathcal{E}$. But Lz is far from $\partial s\mathcal{E}$ (because $Y \cap \Phi = \varnothing$ and by the definition of Φ, (122)) so we get a quasi-loop near Lz, namely

$$Lz \in \mathcal{QL}(17L, s^{1-\delta}, \mathrm{LE}(S_k)).$$

Let therefore

$$q_k^4(x) := \mathbf{P}(Lx \in \mathcal{QL}(17L, s^{1-\delta}, \mathrm{LE}(S_k)) \mid \mathcal{X})$$

and then write (128) as

$$|\mathbf{P}(\mathrm{LE}(S_k) \subset s\mathcal{E} \mid \mathcal{X}) - \mathbf{P}(\mathrm{LE}(S_{k+1}) \subset s\mathcal{E} \mid \mathcal{X})| \leqslant CL^{-c} + q_k^4(z) + q_{k+1}^4(z).$$

We now integrate over $\mathcal{X}$ and get

$$|p_k - p_{k+1}| \leqslant CL^{-c} + \mathbf{E} q_k^4(z) + \mathbf{E} q_{k+1}^4(z). \tag{129}$$

The estimate of (129) is where the random choice of the sets D_k plays its part. Theorem 4 gives us that

$$\sum_{x \in Y \setminus D_k} q_k^4(x) \leqslant \sum_{x \in Y} q_k^4(x) \leqslant C \mathbf{E}\, \mathrm{QL}(34L, s^{1-\delta}, \mathrm{LE}(R[0, T(\partial\mathcal{D})])) \leqslant Cs^{-\delta}.$$

Now, if we think about the coupling of D_k and D_{k+1} as "D_{k+1} is the addition of a random z to D_k", then z is is obviously independent of the walk on $G(r,M,\xi_k)$, so we have

$$\mathbf{E}q_k(z) \leqslant \frac{Cs^{-\delta}}{\#(Y \setminus D_k)} = \frac{Cs^{-\delta}}{\#Y-k}.$$

For $q_{k+1}(z)$ we similarly think about D_k as the removal of a random z from D_{k+1} and get

$$\mathbf{E}q_{k+1}(z) \leqslant \frac{Cs^{-\delta}}{k+1},$$

and the lemma is proved. □

Continuing the proof of the theorem, we first apply Lemma 5.8 to the configurations $\xi^1 \equiv 1$ and

$$\xi^2 = \begin{cases} 1, & \text{if } \Phi \cup \Psi \cup \Theta, \\ 2, & \text{otherwise.} \end{cases}$$

Let

$$p^i := \mathbf{P}^{sa}_{G(L,M,\xi^i)}(\mathrm{LE}(R) \subset s\mathcal{E})$$

(R here is $R[0,T(\partial\mathcal{D})]$ and will stay so for a while). We get

$$|p^1 - p^2| \leqslant Cs^{-\delta} \log s + CL^{-c}M^d. \tag{130}$$

Next, we wish to remove Φ. For this purpose we define

$$p^3 := \mathbf{P}^{sa}_{G(L,M,\xi^2)}(\mathrm{LE}(R) \subset s\mathcal{E}_2),$$

where

$$\mathcal{E}_2 := \mathcal{E} + B\left(0, 2s^{-\delta} + \frac{37L}{s}\right),$$

and get $p^3 > p^2$. The definition of $\mathcal{E}_2$ gives us that

$$d(Lx, \partial_{\mathrm{cont}} s\mathcal{E}_2) > s^{1-\delta} + 18L \quad \text{for all } x \in \Phi,$$

so we can use Lemma 5.8 with ξ^2,

$$\xi^3 = \begin{cases} 1, & \text{if } \Psi \cup \Theta, \\ 2, & \text{otherwise,} \end{cases}$$

and the domain $\mathcal{E}_2$ to get

$$|p^3 - p^4| \leqslant Cs^{-\delta} \log s + CL^{-c}M^d, \tag{131}$$

where $p^4=\mathbf{P}^{sa}_{G(L,M,\xi^3)}(\mathrm{LE}(R)\subset s\mathcal{E}_2)$.

The third step is to get rid of Θ. For this purpose, we define

$$\mathcal{E}_3:=\mathcal{E}_2+B(0,s^{-\varepsilon/2})$$

and $p^5=\mathbf{P}^{sa}_{G(L,M,\xi^3)}(\mathrm{LE}(R)\subset s\mathcal{E}_3)$, so that $p^5>p^4$. Find some point $b\in\mathcal{E}_3$ with

$$|a-b|=\frac{35L}{s}$$

(which can always be done if s is sufficiently large) and use Lemma 4.14 to find that

$$\begin{aligned}|p^5-p^6|&\leqslant C\biggl(\frac{35L}{sd(a,\partial_{\mathrm{cont}}(\mathcal{E}_3\cap\mathcal{D}))}\biggr)^c\\&\leqslant C\biggl(\frac{35L}{\min\{s^{1-\varepsilon/2},sd(a,\partial_{\mathrm{cont}}\mathcal{D})\}}\biggr)^c\\&=C(\varepsilon,a,\mathcal{D},\mathcal{G})s^{-c(\varepsilon,a,\mathcal{D},G)},\end{aligned}\tag{132}$$

where $p^6=\mathbf{P}^{sb}_{G(L,M,\xi^3)}(\mathrm{LE}(R)\subset s\mathcal{E}_3)$. We can now apply Lemma 5.8 with ξ^3,

$$\xi^4=\begin{cases}1, & \text{if } \Psi,\\ 2, & \text{otherwise,}\end{cases}$$

the domain $\mathcal{E}_3$ and the point b to get

$$|p^6-p^7|\leqslant Cs^{-\delta}+CL^{-c},\tag{133}$$

where $p^7=\mathbf{P}^{sb}_{G(L,M,\xi^4)}(\mathrm{LE}(R)\subset s\mathcal{E}_3)$. We apply Lemma 4.14 again to return to a: we get $|p^7-p^8|\leqslant Cs^{-c}$, where $p^8=\mathbf{P}^{sa}_{G(L,M,\xi^4)}(\mathrm{LE}(R)\subset s\mathcal{E}_3)$.

Finally, we need to get rid of Ψ. Let $\mathcal{D}_2$ be a shrinking of $\mathcal{D}$ by $21L/s$, i.e.

$$\mathcal{D}_2:=\biggl\{x:B\biggl(x,\frac{21L}{s}\biggr)\subset\mathcal{D}\biggr\},$$

so we have $L\Psi\cap s\mathcal{D}_2=\varnothing$. Let δ_2 be the value given by Theorem 4 for $\frac{1}{2}\varepsilon$, where ε comes from (121), for all the graphs $G(L,M,\xi)$ simultaneously, and define

$$\mathcal{E}_4:=\mathcal{E}_3+B(0,s^{-\delta_2}+s^{-\varepsilon/2}).$$

Let

$$p^9=\mathbf{P}^{sa}_{G(L,M,\xi^4)}(\mathrm{LE}(R[0,T(\partial\mathcal{D}_2)])\subset s\mathcal{E}_4).$$

Lemma 5.9. *With the definitions above*,

$$p^9 > p^8 - Cs^{-c}, \tag{134}$$

where C and c may depend on $\mathcal{D}$ and ε in addition to $\mathcal{G}$.

Proof. $\mathbf{R}^d \setminus \mathcal{D}$ has a finite number of connected components, $\{Q_i\}_i$, which are all polyhedra. For each Q_i we may use Lemma 2.13 to get that, for every $1<r_1<r_2<s$, every ξ and every $v \in G(L,M,\xi)$, with $d(v,sQ_i) \leqslant r_1$, we have that

$$\mathbf{P}(T_{v,r_2} < T(sQ_i)) \leqslant C(\mathcal{G},Q_i)\left(\frac{r_1}{r_2}\right)^{c(\mathcal{G},Q_i)}. \tag{135}$$

Let $K = \max_i C(\mathcal{G},Q_i)$ and $k = \min_i c(\mathcal{G},Q_i)$.

Now, p^8 and p^9 measure walks on the same graph stopped at $\partial s\mathcal{D}$ and $\partial s\mathcal{D}_2$, respectively. Therefore, we may couple these walks so that the first is a continuation of the second. In other words, define $t_1 > t_2$ to be the stopping times of R on $\partial s\mathcal{D}$ and $\partial s\mathcal{D}_2$ (define $t_2 = 0$ if $a \notin s\mathcal{D}_2$) then the question reduces to an estimate of

$$\mathbf{P}(\{\mathrm{LE}(R[0,t_2]) \not\subset s\mathcal{E}_4\} \cap \{\mathrm{LE}(R[0,t_1]) \subset s\mathcal{E}_3\}). \tag{136}$$

Now, the definition of $\mathcal{D}_2$ gives that the distance of $R(t_2)$ from the closest connected component of $\mathbf{R}^d \setminus s\mathcal{D}$ is $\leqslant 21L$. From the definition of K and k, we get

$$\mathbf{P}\big(R[t_2,t_1] \text{ exits } B\big(R(t_2), \tfrac{1}{2}s^{1-\varepsilon/2}\big)\big) \leqslant K\left(\frac{42L}{s^{1-\varepsilon/2}}\right)^k = C(\mathcal{D},\mathcal{G})s^{-\varepsilon c(\mathcal{D},\mathcal{G})}. \tag{137}$$

On the other hand, if $R[t_2,t_1] \subset B\big(R(t_2), \frac{1}{2}s^{1-\varepsilon/2}\big)$, and in addition the event of (136) holds, then we can conclude that $R(t_2) \in \mathcal{QL}\big(\frac{1}{2}s^{1-\varepsilon/2}, s^{1-\delta_2}, \mathrm{LE}(S_3[0,t_2])\big)$, which means that

$$\mathrm{QL}(s^{1-\varepsilon/2}, s^{1-\delta_2}, \mathrm{LE}(S_3[0,t_2])) \geqslant 1.$$

By Theorem 4 and Markov's inequality, the probability for this is $\leqslant Cs^{-\delta_2}$. This ends the lemma. □

The definition of isotropic interpolation shows that

$$\overline{G(L,M,\xi^4) \cap \mathcal{D}_2} = \overline{G^2 \cap \mathcal{D}_2}.$$

Therefore, we may define R to be a random walk on G^2 starting from sa and get that $p^9 = \mathbf{P}(\mathrm{LE}(R[0,T(\partial s\mathcal{D}_2)]) \subset s\mathcal{E}_4)$. We only need to return from $\mathcal{D}_2$ to $\mathcal{D}$, so write

$$p^{10} := \mathbf{P}(\mathrm{LE}(R[0,T(\partial s\mathcal{D})]) \subset s\mathcal{E}_5),$$

where

$$\mathcal{E}_5 := \mathcal{E}_4 + B(0, s^{-\varepsilon/2}).$$

LEMMA 5.10. *With the definitions above,*

$$p^{10} > p^9 - Cs^{-c}, \tag{138}$$

where again C and c may depend on $\mathcal{D}$ and ε in addition to $\mathcal{G}$.

Proof. As in Lemma 5.9, it is enough to show that

$$\mathbf{P}(\mathrm{LE}(R[0,t_1]) \not\subset s\mathcal{E}_5, \mathrm{LE}(R[0,t_2]) \subset s\mathcal{E}_4) \leqslant Cs^{-c},$$

with the same t_1 and t_2. Unlike in Sublemma 5.9, this requires no recourse to Theorem 4, but rather follows directly from Lemma 2.13 since this event implies that

$$R[t_2,t_1] \not\subset R(t_2) + B(0, s^{1-\varepsilon/2}),$$

whose probability can be bounded by

$$K\left(\frac{21L}{s^{1-\varepsilon/2}}\right)^k = Cs^{-c},$$

with the same K and k as in Lemma 5.9. □

Summing up (130)–(134) and (138), we get, for some $c_{11}(\mathcal{G})$,

$$p^{10} > p^1 - C(\varepsilon, a, \mathcal{D}, \mathcal{G}) s^{-c(\varepsilon,a,\mathcal{D},\mathcal{G})} - C(\mathcal{G}) M^d L^{-c_{11}}. \tag{139}$$

The only thing left now is to choose ε. For $\varepsilon < c_{11}/2d$, we will have $M^d L^{-c_{11}} \leqslant Cs^{-c_{11}/2}$. This finishes the proof of Theorem 5.

5.4. The limit process

In this section we derive consequences of Theorem 5: we will prove the following theorems.

THEOREM 6. *Let G be a d-dimensional isotropic graph with an isotropic interpolation to $2G$. Let $\mathcal{D} \subset \mathbf{R}^d$ be a polyhedron and let $a \in \mathcal{D}$. Let $\mathbf{P}_n$ be the distribution of the loop-erasure of a random walk on $2^n\mathcal{D} \cap G$ starting from $2^n a$ and stopped when hitting $\partial 2^n \mathcal{D}$, multiplied by 2^{-n}. Then the $\mathbf{P}_n$'s converge in the space $\mathcal{M}(\mathcal{H}(\overline{\mathcal{D}}))$.*

Recall that $\mathcal{H}(\mathcal{X})$ is the space of compact subsets of $\mathcal{X}$ with the Hausdorff metric, and $\mathcal{M}(\mathcal{X})$ is the space of measures on $\mathcal{X}$ with the topology of weak convergence. $2G$ is the graph obtained by stretching G by 2 uniformly and possibly multiplying all weights by a constant α (this last action does not change the process of course). In other words, the theorem holds if for any α there is an isotropic interpolation between G and $2G$. Strangely enough, in three dimensions α will usually not be 1.

The limit of the $\mathbf{P}_n$'s is called the *scaling limit* (of the loop-erased random walk) and G is said to *have a scaling limit*.

Theorem 7. *Let G and H be two d-dimensional isotropic graphs with an isotropic interpolation, and assume that G has a scaling limit. Then H has a scaling limit and the scaling limits are identical.*

We start with a proof of Theorem 6. In the following we shall abuse notation by letting, for any subset A in a metric space X and any $\varepsilon>0$,

$$\begin{aligned}A+B(\varepsilon)&:=\{x\in X: d(x,A)<\varepsilon\},\\ A+\overline{B(\varepsilon)}&:=\{x\in X: \text{there exists } a\in A \text{ with } d(x,a)\leqslant\varepsilon\}.\end{aligned}$$

We shall also need the following notation: for a relatively open set $E\subset\overline{\mathcal{D}}$, define the following subsets of $\mathcal{H}(\overline{\mathcal{D}})$:

$$\mathcal{S}(E):=\{K\subset E\} \quad\text{and}\quad \mathcal{I}(E):=\{K\cap E\neq\varnothing\}. \tag{140}$$

Lemma 5.11. *Let G, a and $\mathcal{D}$ be as in Theorem 6 and let $\mathcal{E}_1,...,\mathcal{E}_k\subset\mathcal{D}$ be open or closed. Then, for almost every $\varepsilon>0$, the limit*

$$\lim_{n\to\infty}\mathbf{P}_n\biggl(\mathcal{S}\biggl(\bigcap_{i=1}^{k}(\mathcal{E}_i+\overline{B(\varepsilon)})\biggr)\biggr)$$

exists.

Here and below "almost every" can be replaced with "except for a countable set".

Proof. We may replace $\overline{B(\varepsilon)}$ by $B(\varepsilon)$: for $\mathcal{E}_i$ open, one has $\mathcal{E}_i+\overline{B(\varepsilon)}=\mathcal{E}_i+B(\varepsilon)$, and for $\mathcal{E}_i$ closed, one has $\mathcal{E}_i+\overline{B(\varepsilon)}=\overline{\mathcal{E}_i+B(\varepsilon)}$, and there is only a countable number of ε's for which the closing operation affects the problem at all. Further, we may also assume that all $\mathcal{E}_i$ are open, since replacing each by $\mathcal{E}_i+B(\delta)$ and then letting δ tend to zero will prove the general case.

Denote by LE the expression $\mathrm{LE}(R[0,T(\partial 2^n\mathcal{D})])$ when n is assumed to be clear from the context. Let

$$\overline{p}(\mathcal{E}):=\varlimsup_{n\to\infty}\mathbf{P}_n(\mathcal{S}(\mathcal{E})),$$

and similarly $\underline{p}(\mathcal{E})$. By Theorem 5, we have

$$\begin{aligned}\mathbf{P}_G^{2^n a}\biggl(\mathrm{LE}\subset 2^n\biggl(\bigcap_{i=1}^{k}\mathcal{E}_i\biggr)\biggr)&=\mathbf{P}_{2G}^{2^{n+1}a}\biggl(\mathrm{LE}\subset 2^{n+1}\biggl(\bigcap_{i=1}^{k}\mathcal{E}_i\biggr)\biggr)\\ &<\mathbf{P}_G^{2^{n+1}a}\biggl(\mathrm{LE}\subset 2^{n+1}\biggl(\biggl(\bigcap_{i=1}^{k}\mathcal{E}_i\biggr)+B(C2^{-nc})\biggr)\biggr)+C2^{-nc},\end{aligned}$$

and inductively for any m,

$$\begin{aligned}
&< \mathbf{P}_G^{2^{n+m}a}\Bigg(\mathrm{LE}\subset 2^{n+m}\Bigg(\Bigg(\bigcap_{i=1}^{k}\mathcal{E}_i\Bigg)+B(C(2^{-nc}+...+2^{-(n+m)c}))\Bigg)\Bigg) \\
&\qquad +C(2^{-nc}+...+2^{-(n+m)c}) \\
&< \mathbf{P}_G^{2^{n+m}a}\Bigg(\mathrm{LE}\subset 2^{n+m}\Bigg(\Bigg(\bigcap_{i=1}^{k}\mathcal{E}_i\Bigg)+B\Bigg(\frac{C2^{-nc}}{1-2^{-c}}\Bigg)\Bigg)\Bigg)+\frac{C2^{-nc}}{1-2^{-c}} \\
&\leqslant \mathbf{P}_G^{2^{n+m}a}\Bigg(\mathrm{LE}\subset 2^{n+m}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(C2^{-nc}))\Bigg)\Bigg)+C2^{-nc}.
\end{aligned}$$

Taking m to ∞, we get

$$\mathbf{P}_G^{2^n a}\Bigg(\mathrm{LE}\subset 2^n\Bigg(\bigcap_{i=1}^{k}\mathcal{E}_i\Bigg)\Bigg)\leqslant \underline{p}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(C2^{-nc}))\Bigg)+C2^{-nc},$$

and letting n tend to ∞ gives

$$\overline{p}\Bigg(\bigcap_{i=1}^{k}\mathcal{E}_i\Bigg)\leqslant \lim_{\varepsilon\to 0^+}\underline{p}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(\varepsilon))\Bigg).$$

Now, $\underline{p}\big(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(\varepsilon))\big)$ is a monotone function, hence it is continuous except at a countable number of points. At each point x of continuity we have

$$\begin{aligned}
\overline{p}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(x))\Bigg) &\leqslant \lim_{\varepsilon\to 0^+}\underline{p}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(x+\varepsilon))\Bigg) \\
&= \underline{p}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(x))\Bigg)\leqslant \overline{p}\Bigg(\bigcap_{i=1}^{k}(\mathcal{E}_i+B(x))\Bigg),
\end{aligned}$$

and therefore all the inequalities are equalities. □

Remark. It is not very difficult to construct an example of an open set $\mathcal{E}$ (say with $G=\mathbf{Z}^d$, $\mathcal{D}=\left[\frac{1}{4},\frac{3}{4}\right]^d$ and $a=\vec{\frac{1}{2}}$) such that $\mathbf{P}_n(\mathcal{S}(\mathcal{E}))$ does not converge. For example, note that one can construct a collection of small holes in $\mathcal{E}$ which will be invisible for small n, will pierce $\mathcal{E}$ completely for intemediate n's (so that $\mathbf{P}_n(\mathcal{S}(\mathcal{E}))$ is very close to zero) and will again become almost irrelevant for very large n. Repeating this in different areas (converging to the boundary, so that $\mathcal{E}$ stays open) with a sequence of n's will provide the example.

Lemma 5.12. *Let G, a and $\mathcal{D}$ be as in Theorem* 6. *Let $\mathcal{O}\subset\mathcal{H}(\overline{\mathcal{D}})$ be an open set and let $\varepsilon>0$. Then there exists an open set $\mathcal{V}\subset\mathcal{H}(\overline{\mathcal{D}})$ with $\mathcal{O}\subset\mathcal{V}\subset\mathcal{O}+B(\varepsilon)$ such that*

$$\lim_{n\to\infty}\mathbf{P}_n(\mathcal{V}) \tag{141}$$

exists.

Proof. Let $E\subset\overline{\mathcal{D}}$ be relatively open, let $v_1,...,v_k\in\overline{\mathcal{D}}$ and let $a>b>0$. We define

$$\mathcal{Q}(E,v_1,...,v_k,a,b):=\mathcal{S}(E+B(a+b))\cap\bigcap_{i=1}^{k}\mathcal{I}(B(v_i,a-b)).$$

Our main goal is to show that for $\mathcal{Q}_1,...,\mathcal{Q}_l$ with a common a and b,

$$\mathcal{Q}_i=\mathcal{Q}(E_i,v_{1,i},...,v_{k_i,i},a,b),$$

one has, for every a and almost every b, that

$$\lim_{n\to\infty}\mathbf{P}_n\biggl(\bigcap_{i=1}^{l}\mathcal{Q}_i\biggr) \tag{142}$$

exists. Collect all the $v_{i,j}$'s into a single list, $v_1,...,v_m$, and let $I\subset\{1,...,m\}$ be arbitrary. We use Lemma 5.11 for $E_1+B(a),...,E_l+B(a)$ and for $\mathbf{R}^d\setminus B(v_i,a)$ for all $i\in I$. We get that for almost every $0<b<a$,

$$\lim_{n\to\infty}\mathbf{P}_n\biggl(\mathcal{S}\biggl(\biggl(\bigcap_{i=1}^{l}(E_i+B(a+b))\biggr)\setminus\bigcup_{i\in I}B(v_i,a-b)\biggr)\biggr) \tag{143}$$

exists. Now take some b such that the limit (143) exists for all I. Subtracting (143) for a given I from (143) for $I=\varnothing$ gives that the limit

$$\lim_{n\to\infty}\mathbf{P}_n\biggl(\mathcal{S}\biggl(\bigcap_{i=1}^{l}(E_i+B(a+b))\biggr)\cap\bigcup_{i\in I}\mathcal{I}(B(v_i,a-b))\biggr)$$

exists. Since this holds for all I, we can use the inclusion-exclusion principle to show that the limit

$$\lim_{n\to\infty}\mathbf{P}_n\biggl(\mathcal{S}\biggl(\bigcap_{i=1}^{l}(E_i+B(a+b))\biggr)\cap\bigcap_{i=1}^{m}\mathcal{I}(B(v_i,a-b))\biggr)$$

exists, which is equivalent to (142).

Proving the lemma is now easy. Take a finite set of $\{K_i\}_{i=1}^{L}\in\mathcal{O}$ such that the balls $B\big(K_i,\frac{1}{4}\varepsilon\big)$ cover $\mathcal{O}$ (this is possible from the compactness of $\mathcal{H}$). For each K_i one can take $E_i=K_i+B\big(\frac{1}{4}\varepsilon\big)$ and $v_{1,i},...,v_{k_i,i}$ to be a $\frac{1}{4}\varepsilon$-net in K_i, and get that

$$B\big(K_i,\tfrac{1}{4}\varepsilon\big)\subset\mathcal{Q}\big(E_i,v_{1,i},...,v_{k_i,i},\tfrac{1}{2}\varepsilon,b\big)\subset B(K_i,\varepsilon)\quad\text{for all } 0<b<\tfrac{1}{4}\varepsilon.$$

Denote this set by $\mathcal{Q}_i$. For every $I\subset\{1,...,L\}$ we use (142) to see that for almost every $0<b<\frac{1}{4}\varepsilon$,

$$\lim_{n\to\infty}\mathbf{P}_n\bigg(\bigcap_{i\in I}\mathcal{Q}_i\bigg)$$

exists. Take a b such that the limit exists for all I. By the inclusion-exclusion principle, we get that

$$\lim_{n\to\infty}\mathbf{P}_n\bigg(\bigcup_{i=1}^{L}\mathcal{Q}_i\bigg)$$

exists. Define $\mathcal{V}=\bigcup_{i=1}^{L}\mathcal{Q}_i$ and the lemma is finished. □

Proof of Theorem 6. Let $f:\mathcal{H}(\overline{\mathcal{D}})\to[0,\infty)$ be a continuous function. Let $\varepsilon>0$ and define

$$\mathcal{O}_i=f^{-1}[\varepsilon i,\infty).$$

Then

$$\varepsilon\sum_{i=1}^{\infty}\mathbf{1}_{\mathcal{O}_i}\leqslant f<\varepsilon\bigg(1+\sum_{i=1}^{\infty}\mathbf{1}_{\mathcal{O}_i}\bigg).$$

By the compactness of $\mathcal{H}$, there exists some $\delta>0$ such that $\mathcal{O}_i+B(\delta)\subset\mathcal{O}_{i-1}$ for all $i\geqslant1$ (note that $\mathcal{O}_i=\varnothing$ for i sufficiently large). For every $i\geqslant1$ such that $\mathcal{O}_i\neq\varnothing$ use Lemma 5.12 to find $\mathcal{O}_i\subset\mathcal{V}_i\subset\mathcal{O}_{i-1}$ such that the limit (141) exists (for larger i's define $\mathcal{V}_i=\varnothing$). We get

$$\begin{aligned}\varlimsup_{n\to\infty}\int f\,d\mathbf{P}_n&\leqslant\varlimsup_{n\to\infty}\int\varepsilon\bigg(1+\sum_{i=1}^{\infty}\mathbf{1}_{\mathcal{V}_i}\bigg)\,d\mathbf{P}_n=\varliminf_{n\to\infty}\int\varepsilon\bigg(1+\sum_{i=1}^{\infty}\mathbf{1}_{\mathcal{V}_i}\bigg)\,d\mathbf{P}_n\\&\leqslant\varliminf_{n\to\infty}\int\varepsilon\bigg(1+\sum_{i=1}^{\infty}\mathbf{1}_{\mathcal{O}_{i-1}}\bigg)\,d\mathbf{P}_n\leqslant2\varepsilon+\varliminf_{n\to\infty}\int f\,d\mathbf{P}_n.\end{aligned}$$

Since ε was arbitrary, we see that the limit $\int f\,d\mathbf{P}_n$ exists for any positive f. Any function f can be written as f^+-f^-, hence we see that the limit exists for any continuous f. This finishes the theorem since, by compactness, if $\mathbf{P}_n$ does not converge it must have two subsequences converging to different values, which is a contradiction. □

We now move to the proof of Theorem 7.

Lemma 5.13. *Let G be a graph with a scaling limit and let a, $\mathcal{D}$ and $\mathcal{E}_1, ..., \mathcal{E}_k$ be as in Lemma* 5.11. *Then for almost every $\varepsilon>0$,*

$$\lim_{n\to\infty} \mathbf{P}_n\biggl(\mathcal{S}\biggl(\bigcap_{i=1}^k (\mathcal{E}_i + \overline{B(\varepsilon)})\biggr)\biggr)$$

exists.

Proof. As in Lemma 5.11, we may assume that all the $\mathcal{E}_i$'s are open and replace $\bar{B}$ by B. Define $\overline{p}(\mathcal{E})$ and $\underline{p}(\mathcal{E})$ as in Lemma 5.11. If they are different on $\bigcap_{i=1}^k (\mathcal{E}_i + B(x))$ for an uncountable number of x's, then one of them would be a continuity point for both $\overline{p}$ and $\underline{p}$. Hence we get an interval $[\alpha, \beta]$ such that

$$\underline{p}\biggl(\bigcap_{i=1}^k (\mathcal{E}_i + B(x))\biggr) \leqslant A < B \leqslant \overline{p}\biggl(\bigcap_{i=1}^k (\mathcal{E}_i + B(x))\biggr) \quad \text{for all } x \in [\alpha, \beta].$$

Define now

$$\mu(K) := \min\biggl\{x : K \subset \bigcap_{i=1}^k (\mathcal{E}_i + B(x))\biggr\}, \quad K \in \mathcal{H}(\overline{\mathcal{D}}),$$

and

$$f(K) := \begin{cases} 1, & \text{if } \mu(K) \leqslant \alpha, \\ 0, & \text{if } \mu(K) \geqslant \beta, \\ \text{linear in } \mu(K), & \text{otherwise.} \end{cases}$$

It is easy to see that f is continuous on $\mathcal{H}$. Therefore, by the definition of scaling limit, one must have that

$$\lim_{n\to\infty} \int f\, d\mathbf{P}_n$$

exists. However,

$$\varliminf_{n\to\infty} \int f\, d\mathbf{P}_n \leqslant \varliminf_{n\to\infty} \mathbf{P}_n\biggl(\mathcal{S}\biggl(\bigcap_{i=1}^k (\mathcal{E}_i + B(\beta))\biggr)\biggr) \leqslant A,$$

$$\varlimsup_{n\to\infty} \int f\, d\mathbf{P}_n \geqslant \varlimsup_{n\to\infty} \mathbf{P}_n\biggl(\mathcal{S}\biggl(\bigcap_{i=1}^k (\mathcal{E}_i + B(\alpha))\biggr)\biggr) \geqslant B,$$

which is a contradiction. □

Lemma 5.14. *Let G and H be as in Theorem* 7 *and let a, $\mathcal{D}$, $\mathcal{E}_1, ..., \mathcal{E}_k$ and ε be as in Lemma* 5.11. *Then for almost all $\varepsilon>0$,*

$$\lim_{n\to\infty} \mathbf{P}_G^{2^n a}\biggl(2^{-n}\,\mathrm{LE} \subset \bigcap_{i=1}^k (\mathcal{E}_i + \overline{B(\varepsilon)})\biggr) = \lim_{n\to\infty} \mathbf{P}_H^{2^n a}\biggl(2^{-n}\,\mathrm{LE} \subset \bigcap_{i=1}^k (\mathcal{E}_i + \overline{B(\varepsilon)})\biggr).$$

In particular both limits exist.

Proof sketch. Letting

$$\overline{p}_G(x) := \overline{\lim_{n\to\infty}} \mathbf{P}_G^{2^n a}\left(2^{-n}\,\mathrm{LE} \subset \bigcap_{i=1}^{k} (\mathcal{E}_i + B(x))\right),$$

a calculation analogous to that of Lemma 5.11 gives that

$$\underline{p}_G(x) \leqslant \lim_{\varepsilon\to 0^+} \underline{p}_H(x+\varepsilon) \quad \text{and} \quad \overline{p}_H(x) \leqslant \lim_{\varepsilon\to 0^+} \overline{p}_G(x+\varepsilon),$$

hence the lemma holds for every x which is a continuity point for all $\overline{p}$ and $\underline{p}$, and for which $\underline{p}_G(x) = \overline{p}_G(x)$. Lemma 5.13 is used here to show that this last condition holds almost everywhere. □

LEMMA 5.15. *Let G and H be as in Theorem* 7 *and let $\mathcal{O}$ and ε be as in Lemma* 5.12. *Then, there exists $\mathcal{O} \subset \mathcal{V} \subset \mathcal{O} + B(\varepsilon)$ such that*

$$\lim_{n\to\infty} \mathbf{P}_G^{2^n a}(2^{-n}\,\mathrm{LE} \in \mathcal{V}) = \lim_{n\to\infty} \mathbf{P}_H^{2^n a}(2^{-n}\,\mathrm{LE} \in \mathcal{V}).$$

The proof of this lemma is a complete analogue of the proof of Lemma 5.12. Concluding Theorem 7 from it is similar to the conclusion of Theorem 6 from Lemma 5.12. We shall omit both.

6. Examples

In this chapter we finally descend from the generalities of "Euclidean nets" and "isotropic graphs" to concrete examples. Specifically, we shall demonstrate an isotropic interpolation between $\mathbf{Z}^d$ and $2\mathbf{Z}^d$, which will prove Theorem 1 from Theorem 6. Further we shall show an isotropic interpolation between $\mathbf{Z}^3$ and $3\mathbf{Z}^3$ which will show, using Theorem 7, that the scaling limit is invariant under arbitrary dilations; and an isotropic interpolation between $\mathbf{Z}^3$ and a rotated $\mathbf{Z}^3$ which will show that the scaling limit is invariant under rotations.

We start with a lemma on continuous functions.

LEMMA 6.1. *Let f be harmonic on $B(0,1)$ and continuous on $\overline{B(0,1)}$, and assume that on $\partial B(0,1)$, f is C^2 and satisfies*

$$\|f\|_\infty \leqslant 1, \quad \|f\|_{1,\infty} \leqslant N \quad \textit{and} \quad \|f\|_{2,\infty} \leqslant N^2$$

for some $N \geqslant 1$. Then $|\partial f/\partial x_i| \leqslant CN$ inside $B(0,1)$.

The notation $\|f\|_{k,p}$ stands for the Sobolev norm, which in this case simply means $\|f\|_{1,\infty} = \max_{x,i} |(\partial f/\partial\theta_i)(x)|$, and similarly for second derivatives.

Proof. The value of f inside $B(0,1)$ is related to the value on the boundary by the Poisson kernel:

$$f(x)=\int_{\partial B(0,1)} f(y)\frac{1-\|x\|^2}{\|x-y\|^d}\,dy,$$

where dy is the surface area measure on $\partial B(0,1)$ normalized to be a probability measure. This immediately gives an estimate in the ball $B\bigl(0,\frac{1}{2}\bigr)$, since

$$\frac{\partial f}{\partial x_i}=\int_{\partial B(0,1)} f(y)\frac{\partial}{\partial x_i}\frac{1-\|x\|^2}{\|x-y\|^d}\,dy\leqslant C\int_{\partial B(0,1)} f(y)\,dy\leqslant C.$$

Hence, we need to estimate the derivatives only near the boundary.

Let now $x\in B(0,1)\setminus B\bigl(0,\frac{1}{2}\bigr)$, let θ be some direction on the sphere and let T_ε be a rotation by ε around the pole orthogonal to x and $x+\varepsilon\theta$ in the direction θ. Then the rotational invariance of the Poisson kernel allows us to write

$$\frac{\partial f}{\partial \theta}(x)=\lim_{\varepsilon\to 0}\int_{\partial B(0,1)}\frac{f(y)-f(T_\varepsilon y)}{\varepsilon}\frac{1-\|x\|^2}{\|x-y\|^2}\,dy\leqslant\int_{\partial B(0,1)} N\frac{1-\|x\|^2}{\|x-y\|^2}\,dy=N. \qquad (144)$$

Since $\|x\|\geqslant\frac{1}{2}$, we get that the tangent derivatives are $\leqslant 2N$. Therefore, the lemma will be proved once we get a similar estimate for the radial derivative.

A calculation similar to (144) shows that $|\partial^2 f/\partial x_i^2|\leqslant 4N^2$ for a tangent direction x_i. Hence, by the harmonicity of f, we get that $|\partial^2 f/\partial r^2|\leqslant 4(d-1)N^2$. Examine now a point $x\in\partial B(0,1)$. Since $|f(x)|\leqslant 1$ and $|f(x(1-1/N))|\leqslant 1$, there must be a point y on the interval $[x,x(1-1/N)]$ such that $|(\partial f/\partial r)(y)|\leqslant 2N$. With the bound on $\partial^2 f/\partial r^2$, this gives $|(\partial f/\partial r)(x)|\leqslant CN$. This allows us to bound $\partial f/\partial r$ everywhere, since

$$\begin{aligned}\frac{\partial f}{\partial r}(x)&=\lim_{\varepsilon\to 0^+}\int_{\partial B(0,1)}\frac{f(y)-f(y-\varepsilon/\|x\|)}{\varepsilon}\frac{1-\|x\|^2}{\|x-y\|^2}\,dy\\&=\frac{1}{\|x\|}\int_{\partial B(0,1)}\frac{\partial f}{\partial r}(y)\frac{1-\|x\|^2}{\|x-y\|^2}\,dy\leqslant\frac{CN}{\|x\|}\leqslant CN.\end{aligned}$$

□

Our purpose at this point is to prove that $\mathbf{Z}^d$ and $2\mathbf{Z}^d$ have an isotropic interpolation, which will allow us to invoke Theorem 6. We start with the case $d=3$, for which we weight $2\mathbf{Z}^3$ by 2, i.e. assume that all the edges have weight 2 (the statement of Theorem 6, p. 135, allows us to do so). We shall show this for $\alpha=\frac{1}{9}$ (the α from the definition of isotropic interpolation, (119) on p. 127). Therefore, assume that $L>6$ and $M\leqslant L^{1/9}$ (6 is our choice for the C from (119)). Let $\xi\in\{1,2\}^{M^3}$. We need to construct a graph $G=G(L,M,\xi)$. We do it as follows.

Vertices. For every $(x,y,z)\in\{0,...,M-1\}^3$ such that $\xi(x,y,z)=1$ we take every point of $\mathbf{Z}^3\cap([Lx,Lx+L)\times[Ly,Ly+L)\times[Lz,Lz+L))$ to be a vertex of G. We call such

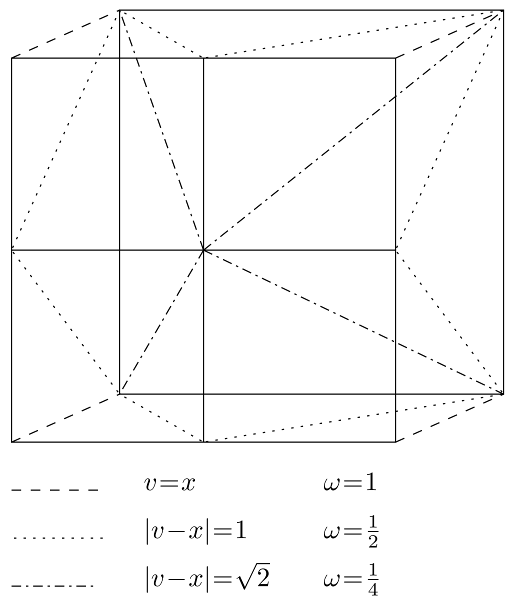

Figure 6. The lower-left square has vertices of type 1, whereas the upper-right square has vertices of type 2. The weights ω are under the assumption that $|w-x|=1$; otherwise divide all weights by 2.

vertices *vertices of type* 1. If $\xi(x,y,z)=2$ we take every point of $2\mathbf{Z}^3\cap([Lx,Lx+L)\times[Ly,Ly+L)\times[Lz,Lz+L))$ to be a vertex of G and call it a *vertex of type* 2. Outside $[0,LM)^3$ we choose between $\mathbf{Z}^3$ and $2\mathbf{Z}^3$ by a majority vote on ξ (or in any other way that gives 1 for $\xi\equiv1$ and 2 for $\xi\equiv2$).

Edges. Two vertices of type 1 will have an edge if and only if their distance is 1, and in this case the edge will have weight 1. Two vertices of type 2 will have an edge if and only if their distance is 2 and in this case the edge will have weight 2. If v is of type 1 and w of type 2, then we need to find the vertex x of type 1 closest to w. If $|x-w|$ is not 1 or 2, set $\omega(v,w)$ to 0. Otherwise, define the weight by

$$\omega(v,w)=\frac{1}{|w-x|}\begin{cases}1, & \text{if } v=x,\\ \frac{1}{2}, & \text{if } |v-x|=1,\\ \frac{1}{4}, & \text{if } |v-x|=\sqrt{2},\\ 0, & \text{otherwise}\end{cases}\tag{145}$$

(0 here means no edge). See Figure 6.

Requirements (i) and (ii) from the definition of an isotropic interpolation (p. 127) are obvious. Hence, the only thing we need to show is that these graphs are isotropic. It is clear that G is always a Euclidean net, so we need to verify the more delicate condition on the exit probabilities in the definition of an isotropic graph (p. 62). This will follow from a comparison of the continuous and discrete Laplacian. Let therefore f be a continuously harmonic function in a ball of radius 3 around some vertex v. Let Δ be the discrete Laplacian. Then, the following facts hold:

(i) If $d(v,X)>2$ for any square X of the form $\{Ln\}\times[0,LM]\times[0,LM]$, $[0,LM]\times\{Ln\}\times[0,LM]$ or $[0,LM]\times[0,LM]\times\{Ln\}$, $n\in\{0,...,M\}$, then $(\Delta f)(v)\leqslant C\|f\|_{4,\infty}$, the Sobolev norm being on the ball $B(v,3)$. In a ball of radius r there are no more than $C(r^3/L+r^2)$ points not satisfying this condition.

(ii) If $d(v,X)>2$ for any segment X of the form $\{Ln\}\times\{Lm\}\times[0,LM]$, $\{Ln\}\times[0,LM]\times\{Lm\}$ or $[0,LM]\times\{Ln\}\times\{Lm\}$, $n,m\in\{0,...,M\}$, then $(\Delta f)(v)\leqslant C\|f\|_{2,\infty}$. In a ball of radius r there are no more than $C(r^3/L^2+r)$ points not satisfying this condition.

(iii) For any v, $(\Delta f)(v)\leqslant C\|f\|_{1,\infty}$.

Seeing (i)–(iii) is not difficult. Write a Taylor expansion for f around v of order 4, 2 or 1, respectively, write

$$(\Delta f)(v)=\sum_{w\sim v}\omega(v,w)(f(w)-f(v))$$

and calculate and see that all terms except the error term vanish. Note that case (ii) is the one where all the fancy weights in the "stitching" between $\mathbf{Z}^3$ and $2\mathbf{Z}^3$ are needed, and also the reason we had to take $2\mathbf{Z}^3$ to have all the weights 2. In fact (iii) holds in any graph with bounded degree.

Lemma 6.2. *A three-dimensional Euclidean net satisfying* (i)–(iii) *is isotropic.*

Proof. By the definition of an isotropic graph, we need to take some $v\in G$, some $r>0$, and a spherical triangle $A\subset\partial_{\mathrm{cont}}B(v,r)$. We may assume that $r>1$, since otherwise (27) holds automatically for $K\geqslant 2$. Let

$$A^-=\{x\in A:d(x,\partial A)\geqslant r^{4/5}\}\quad\text{and}\quad A^+:=\{x\in\partial_{\mathrm{cont}}B(v,r):d(x,A)\leqslant r^{4/5}\}.$$

A^- could be empty and A^+could be all of $\partial_{\mathrm{cont}}B(v,r)$, but anyway we always have

$$\int \mathbf{1}_{A^-}+Cr^{-1/5}\geqslant\int\mathbf{1}_A\geqslant\int\mathbf{1}_{A^+}-Cr^{-1/5},$$

where $\int$ here is with respect to the surface area measure on $\partial_{\mathrm{cont}}B(v,r)$, normalized to have total area 1. The first step is to find two C^5 functions f^- and f^+ on $\partial_{\mathrm{cont}}B(v,r)$ such that $\|f^-\|_{k,\infty},\|f^+\|_{k,\infty}\leqslant Cr^{-4k/5}$ for $k=1,...,5$, and such that

$$f^-\leqslant\mathbf{1}_{A^-}\leqslant\mathbf{1}_{A^+}\leqslant f^+\quad\text{and}\quad\int f^-+Cr^{-1/5}\geqslant\int\mathbf{1}_A\geqslant\int f^+-Cr^{-1/5}.\tag{146}$$

It is easy to construct such $f^\pm$. For example, start with a spherically symmetric function η which is 1 in a spherical cap of radius $r^{4/5}$ and supported in a spherical cap of radius $2r^{4/5}$, $\|\eta\|_{k,\infty}\leqslant Cr^{-4k/5}$. Cover the sphere with a locally finite family of translations $\{T_1, ..., T_m\}$ of the $r^{4/5}$-cap (so that $\sum_j T_j(\eta)\geqslant 1$), and define $\nu_i:=T_i(\eta)/\sum_j T_j(\eta)$, so that the ν_i's form a partition of unity. Define f^- to be the sum of all the ν_i's supported inside A^- and f^+ to be the sum of all the ν_i's such that $\operatorname{supp}\nu_i\cap A^+\neq\varnothing$. Verifying all the properties of f^- and f^+ is easy.

We wish to discretize (146). Let $A^*\subset\partial B(v,r)$ be a discrete version of A as in the definition of isotropic graphs. We want to find functions F^- and F^+ satisfying (146) (with the integral being with respect to the discrete harmonic measure of $\partial B(v,r)$ starting from v). We shall only show the construction of F^-—the construction of F^+ is identical.

Stretch f^- to $\partial B(v,r+\lambda)$, where λ is some constant so that $\partial B(v,r)\subset B(v,r+\lambda)$—for $\mathbf{Z}^3$ and $2\mathbf{Z}^3$ we can take $\lambda=3$. Extend f^- to a harmonic function on $B(v,r+\lambda)$ continuous on $\overline{B(v,r+\lambda)}$. Call this extension g^1 and notice that $\|g^1\|_{k,\infty}\leqslant Cr^{-4k/5}$ for $k=1,...,4$ (this follows from using Lemma 6.1 for rescaled versions of g^1 and its derivatives). Let $a(w,x):=-G(w,x;\overline{B(v,r)})$ be the discrete Green function. Define the following "correction" for g^1:

$$D^1(w):=\sum_{x\in B(v,r)}(\Delta g^1)(x)a(w,x).$$

Because $\Delta a(\cdot\,,x)$ is a delta function at x, we get that $g^2:=g^1-D^1$ is discretely harmonic on $B(v,r)$. We also note that $g^2\equiv g^1$ on $\partial B(v,r)$. What we need is to estimate D^1 at v. Recall the estimate $a(v,w)\leqslant C|v-w|^{-1}$ from Lemma 2.3, equation (12). Summing over points of type (i), we get that

$$\begin{aligned}\sum_{x\text{ of type (i)}}|(\Delta g^1)(x)a(v,x)|&\leqslant Cr^{-16/5}\sum_{s=0}^{\lfloor\log_2 r\rfloor}\sum_{2^s\leqslant|v-x|<2^{s+1}}a(v,x)\\&\leqslant Cr^{-16/5}\sum_{s=0}^{\lfloor\log_2 r\rfloor}4^s\leqslant Cr^{-6/5}.\end{aligned}$$

Points of type (ii) have a worse estimate, but are fewer. In particular, a ball of radius 2^s around v will contain no more than $C\min\{8^s/L+4^s, L^2M^3\}$ such points. Assume first that $r\leqslant LM$, which implies that $L\geqslant r^{9/10}$. We get that

$$\begin{aligned}\sum_{x\text{ of type (ii)}}|(\Delta g^1)(x)a(v,x)|&\leqslant Cr^{-8/5}\sum_{s=0}^{\lfloor\log_2 r\rfloor}\sum_{2^s\leqslant|v-x|<2^{s+1}}a(v,x)\\&\leqslant Cr^{-8/5}\sum_{s=0}^{\lfloor\log_2 r\rfloor}2^s+\frac{4^s}{L}\leqslant C\left(r^{-3/5}+\frac{r^{2/5}}{L}\right)\leqslant Cr^{-1/2}.\end{aligned}$$

If $r>LM$ (which implies that $M\leqslant r^{1/10}$) a similar calculation shows that

$$\sum_{x \text{ of type (ii)}} |(\Delta g^1)(x)a(v,x)| \leqslant Cr^{-8/5}\Bigg(LM^2+\sum_{s=\lfloor \log_2 LM\rfloor}^{\lfloor \log_2 r\rfloor} L^2M^3 2^{-s}\Bigg) \leqslant Cr^{-8/5}LM^2 \leqslant Cr^{-1/2}.$$

Finally, for points of type (iii) we get that a ball of radius 2^s will contain no more than $C\min\{8^s/L^2+2^s, LM^3\}$ such points, and an identical calculation shows that

$$\sum_{w \text{ of type (iii)}} |(\Delta g^1)(v)a(v,w)| \leqslant Cr^{-4/5}\log r + Cr^{-3/5}.$$

Summing these three terms, we get that $|D^1(v)|\leqslant Cr^{-1/2}$.

The only reason not to use g^2 directly is that $g^2 \not\leqslant \mathbf{1}_{A^*}$ on $\partial B(v,r)$. We do have, on $\partial B(v,r)$, that $g^2\equiv g^1\leqslant 1$, but we need to correct g^2 to be 0 outside A^*. Let

$$w\in \partial B(v,r)\setminus A^* \quad \text{and} \quad w' = w\frac{r}{r+\lambda}.$$

Then

$$g^2(w)=g^1(w)=f^-(w')=r^{-1}\int_{\partial_{\mathrm{cont}}B(v,r)} f^-(x)\frac{r^2-\|wr'\|^2}{\|w'-x\|^3}\,dx \leqslant r^{-1}\int_{A^-}\frac{r^2-\|w'\|^2}{\|w'-x\|^3}\,dx \leqslant \frac{C}{d(w',A^-)} \leqslant Cr^{-4/5}.$$

Hence, defining a second correction D^2 on $\partial B(v,r)$ by

$$D^2(w)=\begin{cases} f^2(w), & \text{if } w\notin A^*,\\ 0, & \text{otherwise,}\end{cases}$$

and extending it to a discretely harmonic function on $B(v,r)$, we get, from the discrete maximum principle, that $D^2(v)\leqslant Cr^{-4/5}$.

Defining $F^-=g^2-D^2$, the lemma is now easy: by the definition of the harmonic measure, $\int F^-=F^-(v)$ (in the discrete case) and $\int f^-=f^-(v)$ (in the continuous case). We get that

$$|F^-(v)-f^-(v)|\leqslant |D^1(v)|+|D^2(v)|\leqslant Cr^{-1/2}.$$

Therefore

$$p_{A^*}=\int \mathbf{1}_{A^*} \geqslant \int F^- = F^-(v) \geqslant f^-(v)-Cr^{-1/2} = \int f^- - Cr^{-1/2} \overset{(146)}{\geqslant} \int \mathbf{1}_A - Cr^{-1/5} = |A|-Cr^{-1/5}.$$

A similar calculation with the similarly defined F^+ will show that $p_A\leqslant |A|+Cr^{-1/5}$, so (27) is proved, G is isotropic and the lemma is proved. □

Conclusion. Lemma 6.2, the discussion before it and Theorem 6 together show that $\mathbf{Z}^3$ has a scaling limit. Hence Theorem 1 is proved in three dimensions.

For the case of $\mathbf{Z}^2$, we define $2\mathbf{Z}^2$ to have all the weights 1. We construct the graphs $G(L,M,\xi)$ equivalently, but with the weights defined as follows: if v and w are of the same type then the edges between them are as in the three-dimensional cases and all the weights are 1; if v is of type 1 and w is of type 2 we again find the vertex x of type 1 closest to w, and then define, if $|w-x|$ is 1 or 2,

$$\omega(v,w)=\frac{1}{|w-x|}\begin{cases}1, & \text{if } v=x,\\ \frac{1}{2}, & \text{if } |v-x|=1,\\ 0, & \text{otherwise}\end{cases}$$

(this is the equivalent of (145)). A figure can be found in [K, Figure 1, p. 10]. The equivalents of (i)–(iii) from p. 144 are as follows:

(i) If $d(v,X)>2$ for any segment X of the form $\{Ln\}\times[0,LM]$ or $[0,LM]\times\{Ln\}$, $n\in\{0,...,M\}$, then $(\Delta f)(v)\leqslant C\|f\|_{4,\infty}$. In a ball of radius r there are no more than $C(r^2/L+r)$ points not satisfying this condition.

(ii) If $d(v,X)>2$ for any point X of the form $\{Ln\}\times\{Lm\}$, $n,m\in\{0,...,M\}$, then $(\Delta f)(v)\leqslant C\|f\|_{2,\infty}$. In a ball of radius r there are no more than $C(r^2/L^2+1)$ points not satisfying this condition.

(iii) For any v, $(\Delta f)(v)\leqslant C\|f\|_{1,\infty}$.

These conditions are easy to verify, and as in the three-dimensional case, the definition of ω at the stitches is used only for (ii). The equivalent of Lemma 6.2 is proved in the same way (indeed, it is simpler as the construction of f and g is easier; and since the estimate $a(v,w)\approx\log|v-w|$ means there is no reason to divide into shells of size 2^s as in the three-dimensional case). This shows that $\mathbf{Z}^2$ has a scaling limit, and concludes Theorem 1.

CONJECTURE. *Any non-trivial stitching of* $\mathbf{Z}^d$ *and* $2\mathbf{Z}^d$ *is an isotropic graph.*

In other words, while condition (ii) obviously does not hold unless we insert weights in a manner similar to (145), the conjecture states that the graph $G(L,M,\xi)$ would be isotropic even if we, for example, just connect every vertex of type 2 to the nearest vertex of type 1 and give the edge weight $18\frac{7}{10}$. Be forewarned that the weighting of $2\mathbf{Z}^d$ by 2 (in the three-dimensional case) is very much needed. A simple resistance calculation would show that, for example for $\xi\equiv1$ on $[0,M/2]\times[0,M-1]^2$ and 2 otherwise, the graph $G(L,M,\xi)$ can never be isotropic (no matter what you put in the connecting layer), unless the edges of length 2 are weighted by 2.

6.1. Invariance

The definition of the scaling limit immediately implies that it is invariant under multiplication by 2, meaning that if $\mu(\mathcal{D},a)$ is the scaling limit of a loop-erased random walk on $\mathbf{Z}^3\cap 2^n\mathcal{D}$ starting from $2^n a$, then $\mu(2\mathcal{D},2a)=2\mu(\mathcal{D},a)$, where the notation "$2\mu$" stands for a stretching of μ by 2 in the natural way.

In this section we give a few examples of additional invariances that μ satisfies. The first example is multiplication by 3. In other words, we want to show that

$$\mu(3\mathcal{D},3a)=3\mu(\mathcal{D},a).$$

A moments reflection shows that this will follow if we show that $\mathbf{Z}^3$ and $3\mathbf{Z}^3$ have the same scaling limit, which would follow by Theorem 7 if we show that they have an isotropic interpolation. We follow the same guidelines as in the previous section. Define $G(L,M,\xi)$ as $\mathbf{Z}^3$ and $3\mathbf{Z}^3$ with weight 3 on the internals of the cubes. In the stitches we let x be the vertex of type 1 closest to w and then define, if $|x-w|\in\{1,2,3\}$,

$$\omega(v,w)=\frac{1}{|x-w|}\begin{cases}1, & \text{if } v=x,\\ \frac{2}{3}, & \text{if } |v-x|\in\{1,\sqrt{2}\},\\ \frac{1}{3}, & \text{if } |v-x|\in\{2,\sqrt{8}\}.\end{cases}$$

As in the previous section, a calculation verifies (ii) which implies Lemma 6.2, isotropic interpolation and, with Theorem 7, the invariance of μ.

Since (as is well known) the numbers 2^k3^{-n} are dense in $\mathbf{R}$, this shows that μ is in fact invariant under a dense set of multiplications. We shall now sketch a simple continuity argument which shows that μ is in fact invariant under all multiplications. Let $\alpha>1$ and examine the situation of Theorem 5, i.e. we have the graphs $\mathbf{Z}^3$ and $\alpha\mathbf{Z}^3$, some $a\in\mathcal{E}\cap\mathcal{D}$ and some $s>0$. We use Theorem 5 for $\mathbf{Z}^3$ and $2\mathbf{Z}^3$ repeatedly and rescale (as in the proof of Lemma 5.11) to get that for any k,

$$\mathbf{P}^{2^k sa}_{\mathbf{Z}^3}(\mathrm{LE}(R)\subset 2^k s(\mathcal{E}+B(0,Cs^{-c})))>\mathbf{P}^{sa}_{\mathbf{Z}^3}(\mathrm{LE}(R)\subset s\mathcal{E})-Cs^{-c}.$$

Next, we use Theorem 5 for $\mathbf{Z}^3$ and $3\mathbf{Z}^3$ repeatedly and rescale in the opposite direction to get that, as long as $2^k3^{-n}>1$,

$$\mathbf{P}^{2^k3^{-n}sa}_{\mathbf{Z}^3}(\mathrm{LE}(R)\subset 2^k3^{-n}s(\mathcal{E}+Cs^{-c}B(0,1)))>\mathbf{P}^{sa}_{\mathbf{Z}^3}(\mathrm{LE}(R)\subset s\mathcal{E})-Cs^{-c}.$$

Notice that the various constants do not depend on k and n—in fact we can get as close as we want to α, until $2^k3^{-n}\mathcal{D}\cap\mathbf{Z}^3=\alpha\mathcal{D}\cap\mathbf{Z}^3$ (and the same for $\mathcal{E}$), with no price to pay. There might be a problem that the point of $2^k3^{-n}\mathbf{Z}^3$ closest to a might be different by C

from the point of $\alpha\mathcal{D}\cap\mathbf{Z}^3$ closest to a, but we have Lemma 4.14 to show us that this affects the relevant probabilities by no more than Cs^{-c} as well. Thus the conclusion of Theorem 5 holds for $\mathbf{Z}^3$ and $\alpha\mathbf{Z}^3$, and therefore so does Theorem 7.

We next move to rotations. Examine the lattice G in $\mathbf{R}^3$ spanned by the vectors

$$(4,3,0),\quad (3,-4,0)\quad \text{and}\quad (0,0,5).$$

Since the vectors are orthogonal and of equal length, condition (i) will continue to hold. By now the reader should have only technical difficulties in producing a stitching of G and, for example, $25\mathbf{Z}^3$ with the weights being 5, which will satisfy condition (ii): in fact our lattice is invariant under translations by $25\mathbf{Z}^3$, which reduces the verification of (ii) to a small number of cases. Therefore G and $25\mathbf{Z}^3$ have an isotropic interpolation and the same scaling limit. This shows that the scaling limit is invariant under rotations by $\arctan\frac{3}{4}$ around the z-axis. As is well known and not difficult to see, $\left(\arctan\frac{3}{4}\right)/\pi$ is irrational, so the semigroup generated by this rotation is dense, and a similar continuity argument can be used to show that μ is invariant under any rotation around the z-axis. Since μ is clearly invariant under a change of coordinates, and since any rotation is a combination of three rotations around the axes, we see that μ is invariant under all rotations.

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