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Generalized hyperbolicity

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Introduction

Let $x=(x_1, x_2, \dots x_n)$ be coordinates in R^n with the scalar product $(x,x')=\sum_{j=1}^n x_j x_j'$ and the norm $|x|$. We define

$$D=\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \dots \frac{1}{i}\frac{\partial}{\partial x_n}\right), \quad D^\alpha=\prod_{\alpha_k\neq 0}\left(\frac{1}{i}\frac{\partial}{\partial x_k}\right)^{\alpha_k} \quad \text{and} \quad |\alpha|=\sum_{k=1}^n \alpha_k,$$

where $\alpha=(\alpha_1, \alpha_2, \dots \alpha_n)$ is a multiindex with non-negative integer components. As in Schwartz [1], let $\mathcal{E}(O)$ be the Fréchet space of all infinitely differentiable functions on the open non-empty set $O\subset R^n$ topologized by the semi-norms $\sup_{x\in K}|D^\alpha\varphi(x)|$, where K is compact in O. A complex polynomial P is called hyperbolic with respect to $N\in R^n$ if $P(D)$ has a fundamental solution, locally in the dual space $\mathcal{E}'(R^n)$, with support in a cone $(x, N)\geqslant\varepsilon|x|$, $\varepsilon>0$. Let P_m be the principal part of P. Then, according to Gårding [1], P is hyperbolic with respect to N if and only if there is a constant C such that $P_m(N)\neq 0$ and $P(\xi+i\tau N)\neq 0$ when $\xi\in R^n$ and $\tau\leqslant -C$. We shall here investigate hyperbolicity in other suitable distribution spaces.

For fixed $d>1$ we consider in $\mathcal{E}(O)$ the quasi-norms

$$|\varphi, K|_{d,l}=\sup_{\substack{\alpha\\ x\in K}} l^{-|\alpha|}|\alpha|^{-|\alpha|d}|D^\alpha\varphi(x)|,$$

where $l>0$ and K is compact in O. Set

$$G(d, O)=\{\varphi;\, |\varphi, K|_{d,l}<\infty \text{ for every } l>0 \text{ and every compact } K\subset O\}$$

topologized by the semi-norms $|\varphi, K|_{d,l}$ (cf. Hörmander [1], p. 146). We observe some simple properties of $G(d, O)$ and related spaces. For instance, $G(d, O)$ is a Fréchet space and it contains non-vanishing functions with compact support exactly when $d>1$. Let H be the half space $(x, N)\geqslant 0$ and denote by $\overline{G_0(d, H)}$ the subspace of all functions in $G(d, R^n)$ supported by H. We prove that the mapping

$$P(D)\colon \overline{G_0(d, H)}\to\overline{G_0(d, H)}$$

is injective and has a continuous inverse if and only if there is a constant C such that $P_m(N)\neq 0$ and $P(\xi+i\tau N)\neq 0$ when $\xi\in R^n$ and $\tau\leqslant -C(1+|\xi|^{1/d})$. This is also the precise condition for the existence of a fundamental solution of $P(D)$, locally in the dual space $G'(d, R^n)$, with support in a cone $(x, N)\geqslant\varepsilon|x|$, $\varepsilon>0$. We call such polynomials d-hyperbolic with respect to N. When $d=\infty$, we get formally the hyperbolic

case and generally, the theory of d-hyperbolic polynomials parallels that of hyperbolic polynomials. For instance, if P is d-hyperbolic with respect to N, then P is also d-hyperbolic with respect to every N' in the open cone Γ which is the largest connected N-component of $\{\xi; P_m(\xi) \neq 0\}$. The above fundamental solution of $P(D)$ is supported by the dual cone of Γ. Further, if $\xi \in R^n$, then $P_m(\xi+\tau N)$ has only real zeros τ when P is d-hyperbolic with respect to N. A special feature of d-hyperbolicity is that $P_m(\xi+\tau N)$ has at most a s-fold zero τ for ξ non-proportional to N if and only if P_m+Q is d-hyperbolic with respect to N for all Q of order $\leqslant l$ where $1/d+(m-l)/s=1$.
The presentation mainly follows Hörmander [1] which we often refer to.

The generalized distribution spaces[1]

We use the notations $\mathcal{E}(O)$, D^α and $|\alpha|$ as in the introduction. For fixed $d \geqslant 0$ we consider in $\mathcal{E}(O)$ the quasi-norms

$$|\varphi, K|_{d,l} = \sup_{\substack{\alpha \\ x\in K}} l^{-|\alpha|} |\alpha|^{-|\alpha| d} |D^\alpha \varphi(x)|,$$

where $l>0$ and K is compact in O. They are continuous from below, i.e.

$$\varphi_j \to \varphi \text{ in } \mathcal{E}(O) \Rightarrow \underline{\lim} |\varphi_j, K|_{d,l} \geqslant |\varphi, K|_{d,l},$$

and they have a countable basis obtained by taking sequences $l_k \searrow 0$ and $K_k \nearrow O$.

Definition 1. Let $G(d, O)$ be the space

$$\{\varphi; |\varphi, K|_{d,l} < \infty \text{ for every } l>0 \text{ and every compact } K \subset O\}$$

with the topology given by the quasi-norms $|\varphi, K|_{d,l}$. Let further

$$G_0(d, O) = \bigcup_{K\subset O} G_0(d, K)$$

be the inductive limit of all

$$G_0(d, K) = \{\varphi; \varphi \in G(d, O), \operatorname{supp} \varphi \subset K\},$$

where K is compact in O and $G_0(d, K)$ is topologized by our quasi-norms $|\varphi, K|_{d,l}$. If $O=R^n$ we omit R^n and write $G(d)$ and $G_0(d)$ respectively.

Clearly, $G(1, O)$ is the set of all entire analytic functions on C^n and $G(d_1, O) \subset G(d_2, O)$ if and only if $d_1 \leqslant d_2$. Thus $G_0(d, O)$ only contains the null function for $d \leqslant 1$. When $d>1$, we have the following theorem.

Theorem 1. *If $d>1$, there exist functions $\varphi \in G_0(d, O)$ with the support in an arbitrarily given open set of O such that $\varphi \geqslant 0$ and $\int \varphi(x)dx=1$. $G(d, O)$ and $G_0(d, O)$ are algebras under pointwise multiplication.*

Proof. The existence part of the theorem is a consequence of the Denjoy–Carleman theorem. For a direct proof see Lemma 5.7.1, p. 146 in Hörmander [1].
In the following we only consider $d>1$.

[1] Cf. for instance the spaces in Beurling [1], Gelfand–Shilov [1] and Roumieu [1]. Se also Gevrey [1].

We observe that $G(d, O)$ is a Fréchet space. In fact, the quasi-norms $|\varphi, K|_{d,l}$ have a countable basis and every Cauchy sequence $\{\varphi_j\}_{j=1}^{\infty}$ in $G(d, O)$ has a limit φ in $\mathcal{E}(O)$ which belongs to $G(d, O)$ since

$$|\varphi_j - \varphi, K|_{d,l} \leqslant \varliminf_{k\to\infty} |\varphi_j - \varphi_k, K|_{d,l}.$$

The quasi-norms
$$\sum_{j=1}^{\infty} c_j |\varphi, K_{j+1} \cap \complement K_j|_{d,l},$$

where $\{c_j\}_{j=1}^{\infty}$ is an arbitrary sequence of positive numbers and $K_j \nearrow O$, define the topology of $G_0(d, O)$.

$G(d, O)$ and $G_0(d, O)$ have properties analogous to the spaces $\mathcal{E}(O)$ and $\mathcal{D}(O)$ in Schwartz [1]. In this connection it is even natural to write $\mathcal{E}(O) = G(\infty, O)$ and $\mathcal{D}(O) = G_0(\infty, O)$. The dual spaces $G'(d, O)$ and $G_0'(d, O)$ are considered under the weak and strong topology. They are analogous to the Schwartz spaces $\mathcal{E}'(O)$ and $\mathcal{D}'(O)$ respectively. For instance, $G'(d, O)$ is the set of all elements in $G_0'(d, O)$ which have compact support in O. Further, a sequence $(\varphi_\nu)_{\nu=1}^{\infty}$ converges to 0 in $G_0(d, O)$ if and only if $\bigcup_\nu$ supp φ_ν is contained in a fixed compact set $K \subset O$ and $\varphi_\nu \to 0$ in $G_0(d, K)$. From the general theory of topological spaces we know that a linear form T on $G_0(d, O)$ is continuous precisely when T is continuous on $G_0(d, K)$ for every compact K in O. This implies that a linear form T on $G_0(d, O)$ is contained in $G_0'(d, O)$ if and only if $T(\varphi_\nu) \to 0$ for every sequence $(\varphi_\nu)_{\nu=1}^{\infty}$ which tends to 0 in $G_0(d, O)$. Another consequence is

Theorem 2. *A linear form T on $G_0(d, O)$ belongs to $G_0'(d, O)$ if and only if to every compact set $K \subset O$ there are constants l and $C > 0$ that such*

$$|T(\varphi)| \leqslant C|\varphi, K|_{d,l} \quad \textit{when } \varphi \in G_0(d, K).$$

Mainly according to this theorem and Hahn–Banach, $T \in G_0'(d, O)$ exactly when $T = \sum_\alpha D^\alpha \mu_\alpha$ where μ_α are measures on O satisfying $(\int_K |d\mu_\alpha|)^{1/|\alpha|} = O(|\alpha|^{-d})$ for every compact $K \subset O$.

Convolutions. To be able to work with convolutions we give some definitions and theorems, well-known in the Schwartz case. We write

$$A \overset{+}{\underset{(-)}{}} B = \{x \overset{+}{\underset{(-)}{}} y;\ x \in A,\ y \in B\}, \quad \text{where } A \text{ and } B \text{ are sets in } R^n.$$

Definition 2. Let $T \in G_0'(d)$ and $\varphi \in G(d)$ with supp $T \cap (K\text{-supp}\,\varphi)$ compact for every compact set K. We then define

$$(T * \varphi)\ (x) = T_y(\varphi(x-y)) = T_y(\chi(y)\varphi(x-y)),$$

where $\chi \in G_0(d)$ and $\chi \equiv 1$ on a neighborhood of supp $T \cap (x\text{-supp}\,\varphi)$.

It is immediate that the definition is independent of χ. If we write $\varphi(x-y) = \check{\varphi}_x(y)$, we have

$$(T * \varphi)\ (x) = T(\chi\check{\varphi}_x) = T(\check{\varphi}_x).$$

The requirements of the definition are fulfilled, for instance, when $T \in G_0'(d)$, $\varphi \in G(d)$ and supp T, supp $\varphi \subset \{x;\ (x, N) \geqslant 0\}$ with one of the supports in a cone $(x, N) \geqslant \varepsilon|x|$ where $\varepsilon > 0$.

Theorem 3. *Let T and φ have the properties stated in Definition 2. Then $D^\alpha(T*\varphi)=(D^\alpha T)*\varphi=T*D^\alpha\varphi$ and $\operatorname{supp} T*\varphi\subset\operatorname{supp} T+\operatorname{supp}\varphi$. Further, $T*\varphi$ belongs to $G(d)$ and $T*\varphi_\nu\to T*\varphi$ in $G(d)$ when $\varphi_\nu\to\varphi$ in $G(d)$ and $\bigcup_\nu(\operatorname{supp} T\cap[K-\operatorname{supp}\varphi_\nu])$ is bounded for every bounded K.*

Proof. We consider first $D^\alpha(T*\varphi)=(D^\alpha T)*\varphi=T*D^\alpha\varphi$ where $D^\alpha T$, defined by $D^\alpha T(\varphi)=(-1)^{|\alpha|}T(D^\alpha\varphi)$, belongs to $G'(d)$. Set $D_k=\dfrac{1}{i}\dfrac{\partial}{\partial x_k}$. It is enough to prove that $D_k(T*\varphi)=T*D_k\varphi$.

Let e be the unit vector along the x_k-axis.

$$D_k(T*\varphi)(x)=\lim_{h\to 0}T\left(\frac{1}{ih}[\check{\varphi}_{x+he}-\check{\varphi}_x]\right).$$

Now $1/ih\,[\check{\varphi}_{x+he}-\check{\varphi}_x]$ tends to $(D_k\varphi)_x^\vee$ in $G(d)$ for the mean value theorem implies

$$\left|\frac{1}{ih}[\check{\varphi}_{x+he}-\check{\varphi}_x]-(D_k\varphi)_x^\vee,K\right|_{d,l}\leqslant|h|\,|(D_k^2\varphi)_x^\vee,K'|_{d,l}$$

when $0\neq|h|\leqslant 1$ and $K'=K-\{t\,e;\ |t|\leqslant 1\}$. Since $\operatorname{supp} T\cap\operatorname{supp}[\check{\varphi}_{x+he}-\check{\varphi}_x]$ is compact when $|h|\leqslant 1$, this gives

$$D_k(T*\varphi)=T*D_k\varphi.$$

In order to prove that $T*\varphi\in G(d)$, take an arbitrary compact set K and choose χ in $G_0(d)$ so that $\chi\equiv 1$ in a neighborhood of $\operatorname{supp} T\cap[K-\operatorname{supp}\varphi]$. We write $\operatorname{supp}\chi=K_0$. From Theorem 2 we then obtain constants l_0 and C_0 such that

$$|(T*\varphi)(x)|=|T(\chi\check{\varphi}_x)|\leqslant C_0|\chi\check{\varphi}_x,K_0|_{d,l_0}$$

when $x\in K$. This implies

$$\begin{aligned}|D^\alpha(T*\varphi)(x)|&=|T(\chi(D^\alpha\varphi)_x^\vee)|\leqslant C_0|\chi(D^\alpha\varphi)_x^\vee,K_0|_{d,l_0}\\&\leqslant C_0 l^{|\alpha|}|\alpha|^{|\alpha|d}|l^{-|\alpha|}|\alpha|^{-|\alpha|d}\chi(D^\alpha\varphi)_x^\vee,K_0|_{d,l_0}\\&\leqslant C_0' l^{|\alpha|}|\alpha|^{|\alpha|d}|\check{\varphi}_x,K_0|_{d,l'}\end{aligned}$$

for all $x\in K$ where $l'=2^{-1}e^{-d}\min(l,l_0)$. Hence $T*\varphi\in G(d)$. The same estimate gives also that $T*\varphi_\nu\to T*\varphi$ in $G(d)$ when $\varphi_\nu\to\varphi$ in $G(d)$ and $\bigcup_\nu[\operatorname{supp} T\cap(K-\operatorname{supp}\varphi_\nu)]$ is bounded for every compact K. Finally it remains to localize the support of $T*\varphi$. $(T*\varphi)(x)\neq 0$ only if $\operatorname{supp} T$ meets $\operatorname{supp}\check{\varphi}_x$, i.e. only if there is $y\in\operatorname{supp} T$ such that $x-y\in\operatorname{supp}\varphi$, which means that $x\in\operatorname{supp} T+\operatorname{supp}\varphi$. The proof is complete.

The following three theorems are easy generalizations of theorems for $\mathcal{D}'$ (cf. Hörmander [1], pp. 14–17). We omit the proofs.

Theorem 4. *Let T and φ have the properties in Definition 2 above and let $\psi\in G_0(d)$. Then*

$$(T*\varphi)*\psi=T*(\varphi*\psi)=(T*\psi)*\varphi.$$

Theorem 5. *Let V be a linear mapping from $G_0(d)$ to $G(d)$ which commutes with translations and is continuous in the sense that $V\varphi_j\to 0$ in $G(d)$ if $(\varphi_j)_{j=1}^\infty$ tends to 0 in $G_0(d)$. Then there is one and only one $T\in G_0'(d)$ such that $V\varphi=T*\varphi$ when $\varphi\in G_0(d)$.*

Let now T_1 and T_2 belong to $G_0'(d)$ with supp $T_1 \cap (K - \text{supp } T_2)$ compact for every compact K. Then, according to Theorem 3,

$$G_0(d) \ni \varphi \to T_1 * (T_2 * \varphi) \in G(d)$$

satisfies the requirements of Theorem 5. Hence, there is a unique distribution T in $G_0'(d)$ such that

$$T_1 * (T_2 * \varphi) = T * \varphi.$$

We use this for the definition of the convolution $T_1 * T_2$.

Definition 3. The convolution T of two distributions T_1 and T_2 in $G_0'(d)$ with supp $T_1 \cap (K - \text{supp } T_2)$ compact for every compact K is defined by

$$T_1 * (T_2 * \varphi) = T * \varphi$$

and denoted by $T_1 * T_2$.

If $T_3 \in G'(d)$, we can define $(T_1 * T_2) * T_3$ and $T_1 * (T_2 * T_3)$. We obtain

$$(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3).$$

Finally we note that our results give

Theorem 6. *Let T_1 and T_2 have the properties in Definition 3. Then* $T_1 * T_2 = T_2 * T_1$ *and* supp $T_1 * T_2 \subset$ supp $T_1 +$ supp T_2.

Clearly, $D^\alpha T = (D^\alpha \delta) * T$ where δ is the Dirac measure. Together with the associativity and the commutativity of the convolution this implies

$$D^\alpha (T_1 * T_2) = (D^\alpha T_1) * T_2 = T_1 * D^\alpha T_2.$$

Fourier–Laplace transforms. We are also interested in the Fourier–Laplace transform of the elements in $G_0(d)$ and $G'(d)$. For $\zeta \in C^n$ we write $\zeta = \xi + i\eta$, where ξ and $\eta \in R^n$, and

$$\hat{\varphi}(\zeta) = \int e^{-ix\zeta} \varphi(x)\, dx,$$

where $x\zeta = \sum_{k=1}^n x_k \zeta_k$. Further, we use the notation

$$|\varphi|_\lambda = \int |\hat{\varphi}(\xi)|\, e^{\lambda|\xi|^{1/d}}\, d\xi.$$

We have the following characterization (cf. Hörmander [1], p. 21 and p. 147).

Theorem 7. *Let Φ be an entire analytic function and K a closed convex set in R^n. Define $S(\eta) = \sup_{x \in K} (x, \eta)$. Then, Φ is the Fourier–Laplace transform of a function in $G_0(d)$ with support in K if and only if to every real number λ there is a constant C_λ such that*

$$|\Phi(\zeta)| \leqslant C_\lambda \exp\,(S(\eta) - \lambda |\xi|^{1/d}). \tag{7.1}$$

Further, Φ is the Fourier–Laplace transform of an element in $G'(d)$ with support in K if and only if for some constant λ_0 there is to every $\varepsilon > 0$ a constant C_ε such that

$$|\Phi(\zeta)| \leqslant C_\varepsilon \exp\,(S(\eta) + \varepsilon|\eta| + \lambda_0|\xi|^{1/d}). \tag{7.2}$$

Proof. Let $\varphi \in G_0(d, K)$. It is clear that $\hat{\varphi}$ is entire analytic. Obviously,

$$\zeta^\alpha \hat{\varphi}(\zeta) = \int e^{-ix\zeta}\, D^\alpha \varphi(x)\, dx$$

implies
$$|\zeta^\alpha|\,|\hat{\varphi}(\zeta)| \leqslant C\, e^{S(\eta)}\, l^{|\alpha|}\,|\alpha|^{|\alpha| d} \sup_{\substack{x\in K\\ \alpha}} l^{-|\alpha|}\,|\alpha|^{-|\alpha|d}\,|D^\alpha\varphi(x)|$$
$$= C\, e^{S(\eta)}\, l^{|\alpha|}\,|\alpha|^{|\alpha|d}\,|\varphi, K|_{d,l}$$

so that
$$|\zeta|^k\,|\hat{\varphi}(\zeta)| \leqslant C\,|\varphi, K|_{d,l}\,(nl)^k\, k^{kd}\, e^{S(\eta)},$$

where C is the measure of K. Hence

$$|\hat{\varphi}(\zeta)| \leqslant C\,|\varphi, K|_{d,l}\,(nl\, k^d\,|\zeta|^{-1})^k\, e^{S(\eta)}.$$

Let k be the largest integer $\leqslant |\zeta|^{1/d}(nel)^{-1/d}$. Then,

$$|\hat{\varphi}(\zeta)| \leqslant C\,|\varphi, K|_{d,l}\, e^{-k}\, e^{S(\eta)}.$$

Because $k > |\zeta|^{1/d}(nel)^{-1/d} - 1$, we obtain

$$|\hat{\varphi}(\zeta)| \leqslant Ce\,|\varphi, K|_{d,l} \exp\,(S(\eta) - \lambda|\zeta|^{1/d}), \tag{7.3}$$

where $\lambda = (nel)^{-1/d}$. This proves the necessity of (7.1). In particular we observe that

$$|\varphi|_\lambda \leqslant C'\,|\varphi, K|_{d,l}, \tag{7.4}$$

where $\lambda = (nel)^{-1/d} - 1$ and C' only depends on the measure of K.

We turn to the sufficiency of (7.2). Suppose that the entire function Φ satisfies this inequality. Consider the linear form

$$T(\varphi) = (2\pi)^{-n} \int \Phi(\xi)\, \hat{\varphi}(-\xi)\, d\xi \tag{7.5}$$

on $G_0(d)$. Because of (7.2), (7.4) and Theorem 2, T belongs to $G_0'(d)$. Set $K_\varepsilon = K + \{x;\ |x| \leqslant \varepsilon\}$ and consider $x_0 \notin K_\varepsilon$. We can choose $a > 0$ and $v \in R^n$ such that $|v| = 1$ and K_ε is contained in $(x - x_0, v) \leqslant -2a$. Let $\varphi \in G_0(d, O)$ where $O = \{x;\ |x - x_0| < a\}$. According to (7.3), (7.2) and the analyticity, we can shift the integration of (7.5) into the complex domain which gives

$$T(\varphi) = (2\pi)^{-n} \int \Phi(\xi + i\eta)\, \hat{\varphi}(-\xi - i\eta)\, d\xi,$$

where η is arbitrarily fixed in R^n. Thus,

$$|T(\varphi)| \leqslant C_{\lambda,\varepsilon} \exp\,(S(\eta) + (a+\varepsilon)\,|\eta| - (x_0, \eta)) \int e^{(\lambda_0 - \lambda)\,|\xi|^{1/d}}\, d\xi.$$

In particular, for $\lambda > \lambda_0$ and $\eta = vt$ we obtain

$$|T(\varphi)| \leq C\, e^{-at} \to 0 \quad \text{when} \quad t \to +\infty.$$

Hence supp $T \subset K_\varepsilon$ for every $\varepsilon > 0$ which implies supp $T \subset K$. It is also easily seen that $T_x(e^{-ix\zeta}) = \Phi(\zeta)$ so (7.2) is sufficient.

For the proof of the necessity of (7.2), assume that $T \in G_0'(d)$ with supp $T \subset K$. Take ψ in $G_0(d, 0)$ so that $\psi \equiv 1$ on $K_{\varepsilon/2}$ and supp $\psi \subset K_\varepsilon$. According to Theorem 2 we have

$$|T_x(e^{-ix\zeta})| = |T_x(\psi(x)\, e^{-ix\zeta})| \leq C\, |e^{-ix\zeta}\psi(x),\, K_\varepsilon|_{d,\, l}$$

for some l and C. This gives (7.2). Since $\sum_{k=0}^{N} (-ix\zeta)^k/k!$ tends to $e^{-ix\zeta}$ in $G(d, R^n)$, it is also clear that $T_x(e^{-ix\zeta})$ is entire analytic.

Finally we have to prove that (7.1) is sufficient. The sufficiency of (7.2) implies that every entire function Φ, which satisfies (7.1), is the Fourier–Laplace transform of a T in $G'(d, R^n)$ with support in K. From (7.5) it follows that T is the infinitely differentiable function

$$(2\pi)^{-n} \int \Phi(\xi)\, e^{ix\xi}\, d\xi.$$

According to the assumption, $|T|_\lambda < \infty$ for every λ. Further,

$$|D^\alpha T(x)| \leq (2\pi)^{-n} \int |\xi^\alpha|\, |\hat{T}(\xi)|\, d\xi \leq (2\pi)^{-n} |T|_\lambda \sup\, (|\xi|^{|\alpha|} \exp(-\lambda |\xi|^{1/d})$$

$$\leq (2\pi)^{-n} \left(\frac{d}{\lambda e}\right)^{d|\alpha|} |\alpha|^{|\alpha| d} |T|_\lambda = (2\pi)^{-n}\, l^{|\alpha|}\, |\alpha|^{|\alpha| d} |T|_\lambda$$

when $l = d^d (\lambda e)^{-d}$. This implies

$$|T, K|_{d,\, l} \leq (2\pi)^{-n} |T|_\lambda \tag{7.6}$$

for an arbitary compact set K. The proof is complete.

Remark. If we define the singular support of $T \in G_0'(d, O)$ as the set of points in O having no neighborhood where T is in $G(d)$, it is possible to prove a result analogous to the last theorem for the singular support.

We observe that (7.6) and (7.4) give

$$|\varphi, K|_{d,\, l} \leq (2\pi)^{-n} |\varphi|_\lambda \quad \text{and} \quad |\varphi|_\lambda \leq C\, |\varphi, K|_{d,\, l'}$$

when $\varphi \in G_0(d, K)$. Thus, the semi-norms $|\varphi, K|_{d,\, l}$ and $|\varphi|_\lambda$ define the same topology on $G_0(d, K)$ and by that the same inductive limit on $G_0(d, O)$ (cf. Beurling [1]). Write finally $|\varphi|_{\lambda,\psi} = |\psi\varphi|_\lambda$ for fixed ψ in $G_0(d, O)$ when $\varphi \in G(d, O)$. It is immediate that the semi-norms

$$\{|\varphi|_{\lambda,\psi};\ \psi \in G_0(d, O),\ \lambda > 0\}$$

are equivalent to the semi-norms

$$\{|\varphi, K|_{d,\, l};\ l > 0 \text{ and } K \text{ compact in } O\}.$$

Hence we can define the topology of the Fréchet space $G(d, O)$ by the semi-norms $|\varphi|_{\lambda,\psi}$.

The necessity of d-hyperbolicity

As in the introduction, let H be the half space $(x, N) \geq 0$ and $\overline{G_0(d, H)}$ the set of those functions in $G(d)$ which have the support in H. Set $\inf_t |\eta - t N| = |\eta|_N$ when $\eta \in R^n$.

Theorem 8. *Assume that the mapping $\varphi \to P(D)\varphi$ in $\overline{G_0(d, H)}$ is injective and that its inverse is continuous. Then there is a constant $C > 0$ such that*

$$P(\zeta) = P(\xi + i\eta) \neq 0 \text{ if } (\eta, N) \leq -C(1 + |\eta|_N + |\xi|^{1/d}).$$

Proof. We use the semi-norms $|\varphi|_{\lambda, \psi}$ of $G(d)$. The continuity of $P(D)\varphi \to \varphi$ in $\overline{G_0(d, H)}$ means that to every $\lambda > 0$ and $\psi \in G_0(d)$ there are constants $C, \lambda_0 > 0$ and $\psi_0 \in G_0(d)$ such that

$$|\varphi|_{\lambda, \psi} \leq C |P(D)\varphi|_{\lambda_0, \psi_0} \text{ when } \varphi \in \overline{G_0(d, H)}.$$

Let $\psi \in G_0(d)$ with $\psi(N) = 1$. Then

$$|\varphi(N)| = |\varphi(N)\psi(N)| \leq |\varphi|_{0, \psi}$$

which together with the continuity implies

$$|\varphi(N)| \leq C |P(D)\varphi|_{\lambda_0, \psi_0}$$

for some constants C and $\lambda_0 > 0$ and a fixed $\psi_0 \in G_0(d)$. Take $\chi \in \overline{G(d, R)}$ so that $\chi(t) = 0$ for $t \leq 2^{-2}(N, N)$ and $\chi(t) = 1$ for $t \geq 2^{-1}(N, N)$. We can then apply the inequality to $\varphi(x) = e^{i(x-N, \zeta)} \chi((x, N))$ and get

$$\begin{aligned} 1 &\leq C |P(D)\, e^{i(x-N, \zeta)} \chi((x, N))|_{\lambda_0, \psi_0} \\ &= C |\psi_0(x)\, P(D)\, e^{i(x-N, \zeta)} \chi((x, N))|_{\lambda_0}. \end{aligned} \tag{8.1}$$

When $P(\zeta) = 0$, we have

$$\psi_0(x)\, P(D)\, e^{i(x-N, \zeta)} \chi((x, N)) = \sum_{\gamma \neq 0} \frac{1}{\gamma!} P^{(\gamma)}(\zeta)\, e^{i(x-N, \zeta)} \psi_0(x)\, D^\gamma \chi((x, N)).$$

Here the support of $g_\gamma(x) = \psi_0(x) D^\gamma \chi((x, N))$ is contained in a bounded set B of $\{x;\ 2^{-2}(N, N) \leq (x, N) \leq 2^{-1}(N, N)\}$ when $\gamma \neq 0$. According to (7.1), there is thus to every $\lambda > 0$ a constant $C > 0$ so that

$$|\hat{g}_\gamma(\zeta)| \leq C \exp(S(\eta) - \lambda |\xi|^{1/d})$$

for $\gamma \neq 0$ where $S(\eta) = \sup_{x \in B} (x, \eta)$. This gives for $\alpha \in R^n$

$$\begin{aligned} \left| \int e^{-i\alpha x} g_\gamma(x)\, e^{i(x-N, \zeta)} dx \right| &= e^{(\eta, N)} |\hat{g}_\gamma(\alpha - \zeta)| \leq C \exp((\eta, N) + S(-\eta) - \lambda |\alpha - \xi|^{1/d}) \\ &\leq C \exp((\eta, N) + S(-\eta) + \lambda |\xi|^{1/\alpha} - \lambda |\alpha|^{1/d}) \end{aligned}$$

Hence (8.1) implies that there is a polynomial Q such that

$$1 \leq Q(|\zeta|) \exp((\eta, N) + S(-\eta) + 2\lambda_0 |\xi|^{1/d}). \tag{8.2}$$

In order to estimate $S(-\eta)$ we write $x = sN + y$ where $(y, N) = 0$. Then $2^{-2} \leqslant s \leqslant 2^{-1}$ and $|y| \leqslant D$ for some fixed D if $x \in B$. When $(\eta, N) < 0$, we obtain

$$S(-\eta) = \sup_{x \in B} (x, -\eta) \leqslant \sup_{2^{-2} \leqslant s \leqslant 2^{-1}} s(N, -\eta) + \sup_{|y| \leqslant D} (y, -\eta) \leqslant -2^{-1}(\eta, N) + D \inf_t |\eta - tN|.$$

From (8.2) it hence follows that

$$0 \leqslant (\eta, N) + C(1 + |\eta|_N + |\xi|^{1/d})$$

for some constant $C > 0$ when $P(\zeta) = 0$ and $(\eta, N) < 0$. Consequently, $P(\zeta) \neq 0$ when $(\eta, N) \leqslant -C(1 + |\eta|_N + |\xi|^{1/d})$ and the proof is complete.

We let m be the order of P and denote the principal part by P_m.

Theorem 9. $P_m(N) \neq 0$ *if there exists a constant* C *such that* $P(\xi + i\eta) \neq 0$ *when* $(\eta, N) \leqslant -C(1 + |\eta|_N + |\xi|^{1/d})$.

Proof. Assume that $N = (1, 0, \ldots 0)$ and $P_m(N) = 0$. Since $P_m \not\equiv 0$, there are constants $(\alpha_j)_{j=2}^n$ so that $P_m(1, \alpha_2, \ldots \alpha_n) \neq 0$. We consider the polynomial

$$Q(\lambda, \mu) = P(\lambda, \lambda\mu\alpha_2 \ldots \lambda\mu\alpha_n) = \sum_{\nu=0}^{m} \lambda^\nu R_\nu(\mu),$$

where $R_m(\mu) = P_m(1, \mu\alpha_2, \ldots \mu\alpha_n) \not\equiv 0$ according to the choice of $(\alpha_j)_{j=2}^n$. Because of the assumption, the zeros $\lambda(\mu)$ of $Q(\lambda, \mu)$ satisfy

$$\operatorname{Im} \lambda(\mu) \geqslant -C(1 + |\mu\, \lambda(\mu)| + |\operatorname{Re} \lambda(\mu))|^{1/d}) \tag{9.1}$$

for a suitable constant $C > 0$ when $|\mu| \leqslant 1$. As $R_m(\mu) \not\equiv 0$, we further know that the zeros can be developed into a Puiseux series around $\mu = 0$. We obtain

$$Q(\lambda, \mu) = R_m(\mu) \prod_{j=1}^{m} (\lambda - \lambda_j(\mu)),$$

where every $\lambda_j(\mu)$ for some positive integer p is an analytic function of $\mu^{1/p}$ when $0 < |\mu| < \delta$, without any essential singularity at $\mu^{1/p} = 0$, i.e.

$$\lambda_j(\mu) = \sum_{k=N_j}^{\infty} a_k \mu^{(1/p)\cdot k},$$

where N_j is a whole number.

We have assumed $R_m(0) = 0$. Because of (9.1) at least one $R_\nu(0) \neq 0$. Hence, if $\mu \to 0$ so that $R_m(\mu) \neq 0$, at least one quotient $R_\nu(\mu)/R_m(\mu)$ tends to infinity. Consequently, $|\lambda_{j_0}(\mu)| \to \infty$ for some j_0 when $\mu \to 0$, i.e. $N_{j_0} = N$ is a negative integer. Thus $\lambda_{j_0}(\mu)$ behaves asymptotically as $a_N(\mu^{1/p})^N$ when $\mu \to 0$, which is a contradiction to (9.1) since $d > 1$. The theorem is proved.

Remark. If $P_m(N) = 0$, we can construct functions $0 \neq \varphi \in \overline{G_0(d, H)}$ such that $P(D)\varphi = 0$ (cf. Hörmander [1], p. 121). Hence $P_m(N) \neq 0$ is properly a direct consequence of the injectiveness of the considered mapping.

If $P(\xi+i\eta)\neq 0$ when $(\eta, N)\leqslant -C(1+|\eta|_N+|\xi|^{1/d})$, we obtain, in the special case $\eta=\tau N$, $\tau\in R$, that $P(\xi+i\tau N)\neq 0$ when $\xi\in R^n$ and $\tau(N,N)\leqslant -C(1+|\xi|^{1/d})$. According to the last theorem, such polynomials also satisfy $P_m(N)\neq 0$. We make the following definition.

Definition 4. A polynomial P is called d-hyperbolic with respect to N if there is a constant C such that $P_m(N)\neq 0$ and $P(\xi+i\tau N)\neq 0$ when $\xi\in R^n$ and $\tau\leqslant -C(1+|\xi|^{1/d})$. We consider $1<d\leqslant\infty$ with the convention that $|\xi|^{1/\infty}=1$ so that $d=\infty$ is formally the Gårding case. According to Lemmas 1 below, $d=1$ is the Cauchy–Kovalevsky case. The following theorem is now immediate.

Theorem 10. *P is d-hyperbolic with respect to N if $P(D)\varphi\to\varphi$ is a continuous mapping in $\overline{G_0(d, H)}$.*

We also have

Theorem 11. *P is d-hyperbolic with respect to N if the mapping $\varphi\to P(D)\varphi$ is bijective in $\overline{G_0(d, H)}$, i.e. if the equation $P(D)\varphi=\psi$ has a unique solution $\varphi\in\overline{G_0(d, H)}$ for every $\psi\in\overline{G_0(d, H)}$.*

Proof. Since $\overline{G_0(d, H)}$ is a closed subspace of the Fréchet space $G(d)$, $\overline{G_0(d, H)}$ is itself a Fréchet space. The mapping $\varphi\to P(D)\varphi$ is continuous in $\overline{G_0(d, H)}$. According to Banach's theorem the inverse is then continuous too. The application of Theorem 10 completes the proof.

Algebraic properties of d-hyperbolic polynomials

The following theorems, which give some algebraic properties of our polynomials, are easy generalizations of the corresponding theorems for ∞-hyperbolic polynomials (cf. Hörmander [1], p. 132). We need the following lemma.

Lemma 1. *If $P_m(N)\neq 0$, there is a constant C such that $|\tau|\leqslant C(1+|\zeta|)$ when $\tau\in C$, $\zeta\in C^n$ and $P(\zeta+\tau N)=0$.*

Proof. It is no restriction to assume $P_m(N)=1$. Then $P(\zeta+\tau N)=\tau^m+\sum_{\nu=0}^{m-1}P_\nu(\zeta)\tau^\nu$ where the order of $P_\nu\leqslant m-\nu$. Hence, there is a constant C such that $|P_\nu(\zeta)|\leqslant(C2^{-1}(1+|\zeta|))^{m-\nu}$, which gives

$$\left|\sum_{\nu=0}^{m-1}P_\nu(\zeta)\,\tau^\nu\right|\leqslant|\tau|^m\sum_{\nu=0}^{m-1}2^{\nu-m}<|\tau|^m \text{ if } |\tau|>C(1+|\zeta|).$$

This proves the lemma.

For the sake of completeness we also prove the converse of Lemma 1.

Lemma 2. *$P_m(N)\neq 0$ if P is of order m and $|\tau|\leqslant C(1+|\zeta|)$ for some constant C when $\tau\in C$, $\zeta\in C^n$ and $P(\zeta+\tau N)=0$.*

Proof. Assume that $P_m(N)=0$. Then

$$P(\zeta+\tau N)=\sum_{\nu=0}^{\mu}P_\nu(\zeta)\,\tau^\nu,$$

where $\mu < m$ and the order of $P_\nu = m - \nu$ for at least one $\nu = \nu_0$ since the order of P is m. First we prove that $P_\mu(\zeta)$ is a constant. The polynomials P_ν cannot have a common zero since this violates our assumption. If P_μ depends on ζ, it has a zero ζ_0. Let ζ tend to ζ_0 so that $P_\mu(\zeta) \neq 0$. Then at least one quotient

$$\frac{P_\nu(\zeta)}{P_\mu(\zeta)}, \quad \nu < \mu,$$

tends to infinity and by that also at least one zero $\tau(\zeta)$ of $P(\zeta + \tau N)$. This is again a contradiction to the assumption so that $P_\mu(\zeta)$ is a constant. Now we know that P_{ν_0} is the sum of all possible $(\mu - \nu_0)$-products of the roots of $P(\zeta + \tau N) = 0$. We have assumed that the roots satisfy $|\tau| \leq C(1 + |\zeta|)$ for a suitable constant C. With another constant C we thus get

$$|P_{\nu_0}(\zeta)| \leq C(1 + |\zeta|)^{\mu - \nu_0}$$

which contradicts that the order of P_{ν_0} is $m - \nu_0$. The proof is complete.

Let P be d-hyperbolic with respect to N. Then $P_m(N) \neq 0$, and $P(\xi + i\tau N) = 0$ implies $\operatorname{Re}\tau \geq -C(1 + |\xi|^{1/d} + |\operatorname{Im}\tau|^{1/d})$ for a suitable fixed $C > 0$ when $\xi \in R^n$. According to Lemma 1, there is another C such that $|\tau| \leq C(1 + |\xi|)$ when $P(\xi + i\tau N) = 0$. Hence, if P is d-hyperbolic with respect to N, we have a constant C such that $P_m(N) \neq 0$ and $P_m(\xi + i\tau N) \neq 0$ when $\xi \in R^n$ and $\operatorname{Re}\tau \leq -C(1 + |\xi|^{1/d})$.

Theorem 12. *P is d-hyperbolic with respect to $-N$ if P is d-hyperbolic with respect to N.*

Proof. The homogeneity of the principal part P_m gives that $P_m(-N) = (-1)^m P_m(N) \neq 0$. All the roots of $P(\xi + i\tau N) = 0$ satisfy $\operatorname{Re}\tau \geq -C(1 + |\xi|^{1/d})$ for some fixed C when $\xi \in R^n$. We know that the coefficients of τ^m and τ^{m-1} are $i^m P_m(N) \neq 0$ respectively a linear function of ξ. Denoting the zeros of $P(\xi + i\tau N)$ by τ_j, $\sum_{j=1}^{\infty} \tau_j$ is thus a linear function of ξ. This implies that $\sum_{j=1}^{m} \operatorname{Re}\tau_j$ is a linear function of $\xi \in R^n$ bounded from below by $-C(1 + |\xi|^{1/d})$. But then $\sum_{j=1}^{m} \operatorname{Re}\tau_j$ must be a constant l since $d > 1$. This gives

$$\operatorname{Re}\tau_k = l - \sum_{j \neq k} \operatorname{Re}\tau_j \leq l + C(1 + |\xi|^{1/d}).$$

Consequently, $P(\xi + i\tau N) \neq 0$ when $\xi \in R^n$ and $\tau > l + C(1 + |\xi|^{1/d})$. The proof is complete.

The theorem can also be written in the following form.

Corollary. *If P is d-hyperbolic with respect to N, there is a constant $C > 0$ such that*

$$|\operatorname{Re}\tau| \leq C(1 + |\xi|^{1/d}) \text{ when } \xi \in R^n \text{ and } P(\xi + i\tau N) = 0.$$

Theorem 13. *If P is d-hyperbolic with respect to N, then P_m is ∞-hyperbolic with respect to N.*

Proof. Let $\sigma > 0$. According to the corollary of Theorem 12 we have a constant $C > 0$ such that $\sigma|\operatorname{Re}\tau| \leq C(1 + |\sigma\xi|^{1/d})$ when $\xi \in R^n$ and $P(\sigma\xi + i\sigma\tau N) = 0$. Further,

$$P_m(\xi + i\tau N) = \lim_{\sigma \to +\infty} \sigma^{-m} P(\sigma\xi + i\sigma\tau N).$$

Since $P_m(N) \neq 0$, the zeros τ of $\sigma^{-m} P(\sigma\xi + i\sigma\tau N)$ depend continuously on σ^{-1} in a neighborhood of $\sigma^{-1}=0$. Hence $|\operatorname{Re}\tau|=0$ if $P_m(\xi+i\tau N)=0$ and $\xi \in R^n$. The proof is complete.

Theorem 13 and the definition of d-hyperbolicity give immediately

Theorem 14. *A homogeneous polynomial P is d-hyperbolic with respect to N if and only if* $P(N) \neq 0$ *and the zeros* τ *of* $P(\xi+\tau N)$ *are real when* $\xi \in R^n$.

As in the special case of ∞-hyperbolicity, we make the following definition.

Definition 5. If P is d-hyperbolic with respect to N, we define $\Gamma(P, N) = \Gamma(P_m, N)$ as the set of all real vectors ϑ such that $P_m(\vartheta+\tau N)$ has only negative zeros τ.

Then the following theorem is well known.

Theorem 15. $\Gamma(P, N)$ *is the N-component of the open set* $\{\vartheta; P_m(\vartheta) \neq 0\}$.

Proof. We refer to the proof of Lemma 5.5.1, p. 133, in Hörmander [1].

Next theorem will make it possible to prove that P is d-hyperbolic with respect to every $\vartheta \in \Gamma(P, N)$ if it is d-hyperbolic with respect to N.

Theorem 16. *Let P be d-hyperbolic with respect to N and let* $\vartheta \in \Gamma(P,N)$. *Then there is a constant C such that* $P(\xi+i\tau N+i\sigma\vartheta) \neq 0$ *when* $\xi \in R^n$, $\operatorname{Re}\sigma \leqslant 0$ *and* $\tau \leqslant -C(1+|\xi|^{1/d})$.

Proof. We consider first the case $\operatorname{Re}\sigma=0$. The corollary of Theorem 12 gives a constant C such that $|\tau| \leqslant C(1+|\xi|^{1/d}+|\sigma|^{1/d})$ when $\tau \in R$, $\xi \in R^n$ and $P(\xi+i\tau N+i\sigma\vartheta)=0$. Further, since $P_m(\vartheta) \neq 0$, we have according to Lemma 1 a fixed $D>0$ so that

$$|\sigma| \leqslant D(1+|\xi|+|\tau|) \quad \text{when} \quad P(\xi+i\tau N+i\sigma\vartheta)=0.$$

Hence, with a suitable $C>0$, $|\tau| \leqslant C(1+|\xi|^{1/d}+|\tau|^{1/d})$ when $\tau \in R$, $\xi \in R^n$ and $P(\xi+i\tau N+i\sigma\vartheta)=0$. Because $d>1$, this gives the existence of still another constant $C_0>0$ such that $P(\xi+i\tau N+i\sigma\vartheta)=0$ implies $|\tau| \leqslant C_0(1+|\xi|^{1/d})$ when $\tau \in R$ and $\xi \in R^n$. This completes the proof in the special case $\operatorname{Re}\sigma=0$.

For the general proof we study $P(\xi+i\tau N+i\sigma\vartheta)$ as a polynomial in σ when ξ is an arbitrary vector in R^n and τ varies in $\tau \leqslant -C_0(1+|\xi|^{1/d})$. Here C_0 is the constant obtained above. The zeros σ of this polynomial vary continuously with τ since the coefficient $i^m P_m(\vartheta)$ of σ^m is unequal to zero. As $P(\xi+i\tau N+i\sigma\vartheta)$ has no zeros when $\xi \in R^n$, $\operatorname{Re}\sigma=0$ and $\tau \leqslant -C_0(1+|\xi|^{1/d})$, it follows that the number of zeros σ with negative real part is constant when $\tau \leqslant -C_0(1+|\xi|^{1/d})$. It is thus enough to prove that there are no zeros σ when $\operatorname{Re}\sigma<0$ and τ is large negative. We set $\sigma=\mu\tau$. Then the equation $P(\xi+i\tau N+i\sigma\vartheta)=0$ can be written $i^{-m}\tau^{-m}P(\xi+i\tau(N+\mu\vartheta))=0$. When $\tau \to -\infty$, this equation converges to $P_m(N+\mu\vartheta)=0$ which has only negative roots. Since $P_m(\vartheta) \neq 0$ is the coefficient of μ^m in our equation, the roots μ depend continuously on τ^{-1}. Hence, all zeros σ of $P(\xi+i\tau N+i\sigma\vartheta)$ must have a positive real part when $\xi \in R^n$ and $\tau \leqslant -C_0(1+|\xi|^{1/d})$. The proof of the theorem is complete.

Theorem 17. *P is d-hyperbolic with respect to every* $\vartheta \in \Gamma(P, N)$ *if P is d-hyperbolic with respect to N.*

Proof. Let $\vartheta \in \Gamma(P, N)$ and consider real σ and τ such that $\tau = \varepsilon\sigma$. According to Theorem 16, P is d-hyperbolic with respect to $\vartheta + \varepsilon N$ for every $\varepsilon > 0$. Since $\Gamma(P, N)$ is open, $\vartheta - \varepsilon N \in \Gamma(P, N)$ for small $|\varepsilon|$. Hence, for small $\varepsilon > 0$, P is d-hyperbolic with respect to $(\vartheta - \varepsilon N) + \varepsilon N = \vartheta$.

Theorem 18. *The cone $\Gamma(P, N)$ is convex.*

Proof. See the proof of Theorem 5.5.6, p. 134, in Hörmander [1].

We now need the following definitions.

Definitions. Let P_m be a homogeneous polynomial of order m. We set

$$\nabla^k P_m(\xi) = \sum_{|\alpha| = k} |P_m^{(\alpha)}(\xi)|^2$$

and $V_k = \{\xi;\, \xi \in R^n \text{ and } \nabla^k P_m(\xi) = 0\}$.

Euler's theorem for homogeneous polynomials gives that

$$V_0 \supset V_1 \supset \ldots \supset V_m = \phi.$$

Further, $V_k \supset \{0\}$ when $k < m$. We set

$$s = \inf\,(j,\ V_j = \{0\})$$

and call P_m s-singular or singular of order s.

Theorem 19. *Let P_m be a homogeneous polynomial of order m which is s-singular and hyperbolic with respect to N. Let further Q be a polynomial of order $l < m$. Then $P_m + Q$ is d-hyperbolic with respect to N where $1/d + (m-l)/s = 1$ with the convention that $d = \infty$ when $1/d \leqslant 0$.*

Proof. We define $|\tilde{P}_m(\zeta)| = (\sum_\alpha |P_m^{(\alpha)}(\zeta)|^2)^{1/2}$ and prove first that

$$|\tilde{P}_m(\xi + iN)| \leqslant C|P_m(\xi + iN)| \tag{19.1}$$

for some constant C when $\xi \in R^n$. Since $\Gamma(P_m, N)$ is open, the Theorems 17 and 14 imply $P_m(\xi + iN + i\zeta) \neq 0$ for all ξ in R^n when $|\zeta|$ is smaller than a suitable constant $\varepsilon > 0$. This gives

$$|P_m(\xi + iN + i\zeta)| \leqslant 2^m |P_m(\xi + iN)|$$

when $\xi \in R^n$ and $|\zeta| < \varepsilon$, so by the Cauchy integral formula we have a constant C such that

$$|P_m^{(\alpha)}(\xi + iN)| \leqslant C|P_m(\xi + iN)|$$

when $\xi \in R^n$ (cf. Lemma 4.1.1, p. 99, in Hörmander [1]). This proves the above inequality.

We write $Q = \sum_{j=0}^{l} Q_j$ where Q_j is homogeneous of order j. $|\tilde{P}_m(\xi)|^2$ contains $\nabla^s P_m(\xi)$ which is of order $2(m-s)$ and elliptic since P_m is s-singular. Hence,

$$|\tilde{Q}_j(\xi)|^2 \leqslant C|\tilde{P}_m(\xi)|^2(1 + |\xi|^2)^{j+s-m}, \quad \xi \in R^n, \tag{19.2}$$

for a fixed $C > 0$. Applying (19.1), (19.2) and the Taylor formula we obtain the existence of two constants C and C' such that

$$|Q_j(\xi+iN)|^2 \leqslant C' |\tilde{P}_m(\xi+iN)|^2 (1+|\xi+iN|^2)^{j+s-m}$$
$$\leqslant C|P_m(\xi+iN)|^2 |\xi+iN|^{2(j+s-m)}$$

when $\xi \in R^n$. The homogeneity implies

$$|\tau|^{-j}|Q_j(\tau\xi+i\tau N)| = |Q_j(\xi+iN)| \leqslant C|P_m(\xi+iN)| |\xi+iN|^{j+s-m}$$
$$= C|\tau|^{-j-s}|P_m(\tau\xi+i\xi N)| |\tau\xi+i\tau N|^{j+s-m}.$$

Hence, $$|Q_j(\xi+i\tau N)| \leqslant C|\tau|^{-s}|P_m(\xi+i\tau N)| |\xi+i\tau N|^{j+s-m}$$

when $\xi \in R^n$ and $0 \neq \tau \in R$. This gives

$$|P(\xi+i\tau N) - P_m(\xi+i\tau N)| \leqslant \sum_{j=0}^{l} |Q_j(\xi+i\tau N)|$$
$$\leqslant C|\tau|^{-s}|P_m(\xi+i\tau N)| \sum_{j=0}^{l} |\xi+i\tau N|^{j+s-m}.$$

If $|\tau| \geqslant D(1+|\xi|^{1/d})$ where $1/d+(m-l)/s=1$ and D is a sufficiently large constant, we have

$$C|\tau|^{-s}|\xi+i\tau N|^{j+s-m} \leqslant \frac{1}{2(l+1)}.$$

Hence $$\tfrac{1}{2}|P_m(\xi+i\tau N)| \leqslant |P(\xi+i\tau N)| \leqslant 2|P_m(\xi+i\tau N)|$$

for all such τ in R. Since $P_m(\xi+i\tau N) \neq 0$ for $\tau \in R$, the proof is complete.

To be able to prove the converse of this theorem we need the following result. We let $[x]$ stand for the integral part of x.

Theorem 20. *Let P be d-hyperbolic with respect to N and set for fixed ξ and ϑ in R^n*

$$\deg_\tau P(\tau\xi+\vartheta) = l \quad \textit{and} \quad \deg_\tau P_m(\tau\xi+N) = g.$$

Then $$l \leqslant g + \left[\frac{m-g}{d}\right].$$

Proof. We consider $P(\tau\xi+\vartheta+\sigma N)$ and give an estimate of $\deg_\tau P(\tau\xi+\vartheta+\sigma N)$ from above for every fixed ϑ in R^n. We study the zeros σ as functions of τ. If we set $\sigma=\omega\tau$, the equation $P(\tau\xi+\vartheta+\sigma N)=0$ can be written

$$\tau^{-m}P(\tau\xi+\vartheta+\omega\tau N) = P_m(\xi+\omega N) + Q(\tau^{-1}, \omega) = 0,$$

where $Q(\tau^{-1}, \omega)$ is a polynomial in τ^{-1} and ω which vanishes for $\tau^{-1}=0$. The polynomial $P_m(\xi+\omega N) = \omega^m P_m(\omega^{-1}\xi+N)$ has, according to the assumption, a $(m-g)$-fold zero $\omega=0$. Since $P_m(N) \neq 0$, the zeros ω of $P_m(\xi+\omega N)+Q(\tau^{-1},\omega)$ are bounded when $\tau^{-1} \to 0$, and $m-g$ of them converge to zero. The Puiseux series expansion of these $(m-g)$ zeros around $\tau^{-1}=0$ can thus be written

$$\omega(\tau) = \sum_{j=1}^{\infty} c_j \tau^{-j/p}.$$

Let c_r be the first non-vanishing coefficient. The corresponding zeros $\sigma=\tau\omega$ of $P(\tau\xi+\vartheta+\sigma N)$ then behave asymptotically as $c_r\tau^{(p-r)/p}$ when $\tau^{-1}\to 0$. In particular, the argument of σ tends to $\arg c_r+((p-r)/p)\nu\pi$ when $\arg\tau=\nu\pi$ and $\tau^{-1}\to 0$. Since P is d-hyperbolic with respect to N, we also have $|\operatorname{Im}\sigma|\leqslant C(1+|\vartheta|^{1/d}+|\tau|^{1/d}|\xi|^{1/d})$ for a fixed C when $\tau\in R$. A suitable choice of ν then gives the condition

$$\frac{p-r}{p}\leqslant\frac{1}{d}.$$

Hence, $m-g$ zeros of $P(\tau\xi+\vartheta+\sigma N)$ are $O(|\tau|^{1/d})$ when $|\tau|\to\infty$. For the rest of the zeros we have $O(|\tau|)$ when $|\tau|\to\infty$. The connection between the coefficients and the zeros of our polynomial then implies that the coefficients satisfy $O(|\tau|^{g+(m-g)/d})$ when $|\tau|\to\infty$. Hence,

$$\deg_\tau P(\tau\xi+\vartheta+\sigma N)\leqslant g+\left[\frac{m-g}{d}\right].$$

The theorem is proved.

For fixed m and l we define d_s by

$$\frac{1}{d_s}+\frac{m-l}{s}=1$$

with the convention that $d_s=\infty$ when $m\geqslant l+s$.

Corollary. *Let P_m be a homogeneous polynomial of order m. If $l\geqslant m-s$ and P_m+Q is d_s-hyperbolic with respect to N for all Q of order $\leqslant l$, then $P_m(\xi+\tau N)$ cannot have more than s coinciding zeros τ for any ξ in R^n non-proportional to N.*

Proof. Assume that the corollary is not true. Then there is $t>s$ such that $P_m(\xi_0+\tau N)$ has a t-fold zero $\tau=0$ for some $\xi_0\neq 0$ in R^n non-proportional to N. This and $l\geqslant m-s$ gives $\deg_\tau P_m(\tau\,\xi_0+N)=\deg_\tau\tau^m P_m(\xi_0+\tau^{-1}N)=m-t<l$. Applying Theorem 20 with $g=m-t$ and $d=d_s$, we obtain

$$\deg_\tau(P_m(\tau\xi_0+N)+Q(\tau\xi_0+N))\leqslant\left[l-\frac{(t-s)(m-l)}{s}\right]\leqslant l-1$$

for every Q of order $\leqslant l$. Since $\deg_\tau P_m(\tau\xi_0+N)<l$, this implies that $\deg_\tau Q(\tau\xi_0+N)\leqslant l-1$ for all Q of order $\leqslant l$ which is a contradiction. The corollary is proved.

We can now give a theorem in the opposite direction to Theorem 19.

Theorem 21. *Let P_m be a homogeneous polynomial of order m such that P_m+Q is d_s-hyperbolic with respect to some N for every Q of order $\leqslant l$. Assume further that there is at least one such Q so that P_m+Q is not d_{s-1}-hyperbolic with respect to N. Then P_m must be s-singular.*

Proof. P_m+Q is not d_{s-1}-hyperbolic for every Q of order $\leqslant l$. Then, Theorem 19 implies that P_m is at least s-singular. But because of $d_s<\infty$, i.e. $l>m-s$, and the corollary of Theorem 20, P_m can at most be s-singular, so the proof is complete.

Fundamental solutions and the sufficiency of d-hyperbolicity

We shall now prove that d-hyperbolicity with respect to N is necessary and sufficient for the existence of a fundamental solution in $G_0'(d)$ if we require the support to be contained in a cone $(x, N) \geq \varepsilon |x|$, $\varepsilon > 0$. As above, let $H = \{x; (x, N) \geq 0\}$.

Theorem 22. *Assume that a differential operator $P(D)$ has a fundamental solution E in $G_0'(d)$ with the support in a cone $(x, N) \geq \varepsilon |x|$, $\varepsilon > 0$. If then $\psi \in G_0'(d)$ and* supp $\psi \subset H$, *the equation $P(D)\varphi = \psi$ has a unique solution φ with the same properties. When $\psi \in G(d)$, the solution $\varphi \in G(d)$.*

Proof. Supp $E \subset \{x; (x, N) \geq \varepsilon |x|\}$ for some $\varepsilon > 0$. Let ψ belong to $G_0'(d)$ or $G(d)$ with the support in H. Then, according to the section on convolutions (p. 3), $E * \psi$ exists in $G_0'(d)$ respectively $G(d)$ with its support in H. Further, $E * \psi$ solves the equation $P(D)\varphi = \psi$. This proves the existence. If $P(D)\varphi = 0$ with $\varphi \in G_0'(d)$ and supp $\varphi \subset H$, $\varphi = \varphi * P(D)E = P(D)\varphi * E = 0$. The proof is complete. This gives the uniqueness.

Theorem 23. *Let $P(D)$ be a differential operator with a fundamental solution E in $G_0'(d)$ such that the support is contained in a cone $(x, N) \geq \varepsilon |x|$, $\varepsilon > 0$. Then P is d-hyperbolic with respect to N.*

Proof. The theorem is an immediate consequence of the Theorems 11 (p. 10) and 22.

Theorem 24. *Let P be d-hyperbolic with respect to N. Then the operator $P(D)$ has one and only one fundamental solution E in $G_0'(d)$ with support in the closed half space H. More precisely, the support of E is contained in the convex cone*

$$\Gamma^*(P, N) = \{x; (x, \vartheta) \geq 0 \textit{ for every } \vartheta \in \Gamma(P, N)\}$$

but in no smaller convex cone with vertex at 0.

Proof. The uniqueness follows from Theorem 22 when the existence is proved.
Let $\vartheta \in \Gamma(P, N)$. Then P is d-hyperbolic with respect to ϑ. If we write

$$P(\xi + i\tau\vartheta) = i^m P_m(\vartheta) \prod_{k=1}^{m} (\tau - \tau_k(\xi, \vartheta)),$$

we thus have a constant $C(\vartheta) > 0$ such that

$$\operatorname{Re} \tau_k(\xi, \vartheta) \geq -C(\vartheta)\ (1 + |\xi|^{1/d}) \text{ when } \xi \in R^n.$$

Specializing τ to $t(1 + |\xi|^{1/d})$ with $t \leq -2\, C(\vartheta)$ we get

$$|P(\xi + i\tau\vartheta)| \geq |P_m(\vartheta)|\ |2^{-1} t|^m (1 + |\xi|^{1/d}).$$

For such τ we let $\sigma(\vartheta, t)$ be the surface

$$(\xi_1 + i\tau\vartheta_1,\ \xi_2 + i\tau\vartheta_2, \dots \xi_n + i\tau\vartheta_n) \text{ in } C^n.$$

Hence, $$|P(\zeta)| \geq |P_m(\vartheta)|\, |2^{-1} t|^m (1 + |\xi|^{1/d}) \text{ when } \zeta \in \sigma(\vartheta, t).$$

We define E on $G_0(d)$ by

$$\check{E}(\varphi)=(2\pi)^{-n}\int_{\sigma(\vartheta,t)}\frac{\hat{\varphi}(\zeta)}{P(\zeta)}\,d\zeta,$$

where we use the notations $\check{\varphi}(x)=\varphi(-x)$ and $\check{E}(\varphi)=E(\check{\varphi})$. Theorem 7 (7.3) gives to every compact set K in R^n a constant C such that

$$|\hat{\varphi}(\zeta)|\leqslant C|\varphi,K|_{d,l}\exp\left(t(1+|\xi|^{1/d})\,S'(\vartheta)-\lambda|\xi|^{1/d}\right)$$

when $\operatorname{supp}\varphi\subset K$ and $\zeta\in\sigma(\vartheta,t)$. Here $\lambda=(\mathrm{ne}\,l)^{-1/d}$ and $S'(\vartheta)=\inf_{x\in K}(x,\vartheta)$ since $t<0$. Our estimates of $\hat{\varphi}(\zeta)$ and $P(\zeta)$ imply the convergence of the integral and, for fixed t and ϑ, the inequality

$$|\check{E}(\varphi)|\leqslant C|\varphi,K|_{d,l},$$

where the constant C only depends on K and $\lambda>t\,S'(\vartheta)$. Hence, E belongs to $G'(d)$. Because of the estimates and the analyticity of $\hat{\varphi}(\zeta)$ and $1/P(\zeta)$ in the considered regions of C^n, we also have that the integral is independent of ϑ and $t\leqslant -2\,C(\vartheta)$ when $\vartheta\in\Gamma(P,N)$. Further,

$$\check{E}(P(D)\,\varphi)=(2\pi)^{-n}\int_{\sigma(\vartheta,t)}\frac{P(\zeta)\,\hat{\varphi}(\zeta)}{P(\zeta)}\,d\zeta=(2\pi)^{-n}\int_{R^n}\hat{\varphi}(\xi)\,d\xi=\varphi(0).$$

Consequently, $P(D)E=\delta$.

Now it only remains to localize the support of E. If $\operatorname{supp}\varphi\subset\{x;\,(x,\vartheta)>0\}$, we have $S'(\vartheta)>0$. The estimates of $P(\zeta)$ and $\hat{\varphi}(\zeta)$ then give for $l>0$

$$|\check{E}(\varphi)|\leqslant C|\varphi,K|_{d,l}|t|^{-m}e^{tS'(\vartheta)}\int_{\sigma(\vartheta,t)}\exp\left(-\lambda|\xi|^{1/d}\right)|d\zeta|\to 0$$

when $\vartheta\in\Gamma(P,N)$ and $t\to-\infty$. Hence, $\check{E}(\varphi)=0$ when $\operatorname{supp}\varphi\subset\{x;\,(x,\vartheta)>0\}$, i.e. $\operatorname{supp}E\subset\{x;\,(x,\vartheta)\geqslant 0\}$ when $\vartheta\in\Gamma(P,N)$. This proves that $\operatorname{supp}E\subset\Gamma^*(P,N)$. Let finally K be a closed convex cone with vertex at 0 and containing the support of the constructed fundamental solution. According to Theorem 23, all proper planes $(x,\theta)=0$ of support of K must then be non-characteristic, i.e. $P_m(\theta)\neq 0$. The open convex set

$$K^*=\{\vartheta;\,(x,\vartheta)>0,\text{ for every }x\neq 0\text{ in }K\},$$

containing N, is thus contained in $\{\vartheta;\,P_m(\vartheta)\neq 0\}$, which gives that $K^*\subset\Gamma(P,N)$. Hence $K\supset\Gamma^*(P,N)$ and the proof is complete.

(The rest of this paper from here on has been added to proof as a partly rewritten MS, presented to the academy on 16 August 1966. Editor.)

If P is d-hyperbolic with respect to N, we can, according to the Theorems 24 and 22, solve $P(D)\varphi=f$ uniquely in $\overline{G_0(d,H)}$ for every $f\in\overline{G_0(d,H)}$. Theorem 10 states the reverse implication, so d-hyperbolicity with respect to N is both necessary and sufficient for the unique solvability of $P(D)\varphi=f$ in $\overline{G_0(d,H)}$.

We can now go a step further and consider the following Cauchy problem where P is of order m and D_N denotes derivation along N:

$$\begin{cases} P(D)\,\varphi = f \\ D_N^j \varphi = g_j \text{ for } (x, N) = 0 \text{ and } 0 \leqslant j < m, \end{cases}$$

when f and $\{g_j\}_{j=0}^{m-1} \in G(d)$.

In order to solve this problem we first prove the following theorem (cf. Hörmander [1], p. 149). Choosing $N = (1, 0, \dots 0)$ we write

$$D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \dots \frac{1}{i}\frac{\partial}{\partial x_n}\right) = (D_1, D')$$

and $\zeta = (\zeta_1, \zeta_2, \dots \zeta_n) = (\xi_1 + i\eta_1, \xi_2 + i\eta_2, \dots \xi_n + i\eta_n) = (\zeta_1, \zeta') = (\xi_1 + i\eta_1, \xi' + i\eta')$.

Hence, $P(D) = P(D_1, D')$ and $P(\zeta) = P(\zeta_1, \zeta')$. Further, we set $T(\varphi) = (T, \varphi)$ when $T \in G_0'(d)$ and $\varphi \in G_0(d)$.

Theorem 25. *Let P be of order m and d-hyperbolic with respect to $N = (1, 0, \dots 0)$. Then, when $0 \leqslant k < m$ and $x_1 \in R$, there is a unique $H_k(x_1) \in G'(d, R^{n-1})$ such that*

$$D_1^j H_k(x_1) \in G'(d, R^{n-1}) \textit{ for every integer } j \geqslant 0,$$

$$P(D_1, D')H_k(x_1) = 0, \; D_1^j H_k(0) = 0 \textit{ when } k \neq j < m,$$

$$\textit{and } D_1^k H_k(0) = \delta \textit{ where } \delta \textit{ is the Dirac measure.}$$

Further, $(H_k(x_1), \varphi) \in G(d, R)$ *when* $\varphi \in G(d, R^{n-1})$, *and* $(x_1^0, \operatorname{supp} H_k(x_1^0)) \subset \operatorname{supp} E \cap \{x; x_1 = x_1^0\}$ *for* $x_1^0 \geqslant 0$ *where E is the fundamental solution in Theorem* 24.

Proof. We write
$$P(\zeta) = P(\zeta_1, \zeta') = \sum_{j=0}^{m} \zeta_1^{m-j} q_j(\zeta')$$

and define
$$p_k(\zeta_1, \zeta') = \sum_{j=0}^{k} \zeta_1^{k-j} q_j(\zeta').$$

Let Γ be a simple, positively oriented curve which for fixed ζ' surrounds the zeros ζ_1 of $P(\zeta_1, \zeta')$. We consider

$$\hat{H}_k(x_1, \zeta') = (2\pi i)^{-1} \int_\Gamma e^{i\zeta_1 x_1} p_{m-1-k}(\zeta_1, \zeta')/P(\zeta_1, \zeta')\, d\zeta_1.$$

Then
$$D_1^j \hat{H}_k(x_1, \zeta') = (2\pi i)^{-1} \int_\Gamma e^{i\zeta_1 x_1} (\zeta_1)^j p_{m-1-k}(\zeta_1, \zeta')/P(\zeta_1, \zeta')\, d\zeta_1$$

is an entire function of $\zeta' = (\zeta_2, \dots \zeta_n)$ for every $x_1 \in R$ and every integer $j \geqslant 0$. According to Lemma 1 and the Theorems 8 and 12, respectively,

$$|\zeta_1| \leqslant C(1 + |\zeta'|) \text{ and}$$

$$|\eta_1| \leqslant C(1 + |\eta'| + |\xi'|^{1/d} + |\xi_1|^{1/d})$$

for a constant C when $P(\zeta_1, \zeta') = 0$. In order to estimate $D_1^j H_k(x_1, \zeta')$ we can then choose Γ as the rectangle defined by

$$|\xi_1| = C(1 + |\zeta'|); \; |\eta_1| = C(1 + |\eta'| + |\xi'|^{1/d})$$

where C is a suitable constant. Since $|p_{m-1-k}(\zeta_1, \zeta')|$ is majorized by a constant times $(1+|\zeta'|)^{m-1-k}$, and both $|\zeta_1|$ and the length of Γ by a constant times $(1+|\zeta'|)$, we get

$$|D_1^j \hat{H}_k(x_1, \zeta')| \leqslant C^{j+1}(1+|\zeta'|)^{m-k+j} \exp\,(C|x_1|(1+|\eta'|+|\xi'|^{1/d}))$$

and
$$\sup_j j^{-jd}|D_1^j \hat{H}_k(x_1, \zeta')| \leqslant \exp C(1+|x_1|)(1+|\eta'|+|\xi'|^{1/d})$$

for some constants C. Hence, because of Theorem 7, $\hat{H}_k(x_1, \zeta')$ is the Fourier-Laplace transform of an element $H_k(x_1) \in G'(d, R^{n-1})$ given by

$$(H_k(x_1), \varphi) = (2\pi)^{-n+1}\int \hat{H}_k(x_1, \xi')\,\hat{\varphi}(-\xi')\,d\xi'$$

when $\varphi \in G_0(d, R^{n-1})$. We define $(D_1^j H_k(x_1), \varphi) = D_1^j(H_k(x_1), \varphi)$. Our estimates imply

$$D_1^j(H_k(x_1), \varphi) = (2\pi)^{-n+1}\int D_1^j \hat{H}_k(x_1, \xi')\,\hat{\varphi}(-\xi')\,d\xi'$$

and $(H_k(x_1), \varphi) \in G(d, R)$. Hence $D_1^j H_k(x_1) \in G'(d, R^{n-1})$ and $[D_1^j H_k(x_1)]^\wedge(\zeta') = D_1^j \hat{H}_k(x_1, \zeta')$. Further,

$$P(D_1, \xi')\,\hat{H}_k(x_1, \xi') = (2\pi i)^{-1}\int_\Gamma e^{i\zeta_1 x_1} p_{m-1-k}(\zeta_1, \xi')\,d\zeta_1 = 0$$

since the integrand is analytic. This means that $P(D_1, D')H_k(x_1) = 0$.

For the proof of $D_1^k H_k(0) = \delta$ and $D_1^j H_k(0) = 0$ when $k \neq j < m$, we use that

$$D_1^j \hat{H}_k(0, \zeta') = (2\pi i)^{-1}\int_\Gamma \zeta_1^j p_{m-1-k}(\zeta_1, \zeta')/P(\zeta_1, \zeta')\,d\zeta_1.$$

The integrand is

$$\zeta_1^j p_{m-1-k}(\zeta_1, \zeta')/P(\zeta_1, \zeta') = \zeta_1^{j-k-1} + \zeta_1^{j-k-1}(\zeta_1^{k+1} p_{m-1-k}(\zeta_1, \zeta') - P(\zeta_1, \zeta'))/P(\zeta_1, \zeta').$$

The degree of ζ_1 in the numerator of the second term is majorized by $j-k-1+k = j-1$, hence by $m-2$ when $j<m$. Since the degree of ζ_1 in the denominator $P(\zeta_1, \zeta')$ is m, we get

$$D_1^j \hat{H}_k(0, \zeta') = (2\pi i)^{-1}\int_\gamma \zeta_1^{j-k-1}\,d\zeta_1 \quad \text{for} \quad 0 \leqslant j < m,$$

where γ is a positively oriented circle surrounding the origin. Consequently, $D_1^k H_k(0) = \delta$ and $D_1^j H_k(0) = 0$ when $k \neq j < m$.

Finally we localize the support of $H_k(x_1^0)$. Let $\varphi \in G_0(d, R^{n-1})$ with $(x_1^0, \operatorname{supp} \varphi) \cap \operatorname{supp} E = \phi$ and take $\psi \in G_0(d, R)$ satisfying $\operatorname{supp} \psi \subset [-1, 1]$ and $\int \psi(x)dx = 1$. We set

$$\chi_\varepsilon(x_1, x_2, \ldots x_n) = \chi_\varepsilon(x_1, x') = \varepsilon^{-1}\psi(\varepsilon^{-1}(x_1 - x_1^0))\,\varphi(x').$$

Then, $\hat{\chi}_\varepsilon(\zeta) = \hat{\chi}_\varepsilon(\zeta_1, \zeta') = e^{-i\zeta_1 x_1^0}\hat{\psi}(\varepsilon\zeta_1)\hat{\varphi}(\zeta')$ and $\operatorname{supp} \chi_\varepsilon \cap \operatorname{supp} E = \phi$ when $\varepsilon > 0$ is small enough. Hence, for such ε

$$0 = E(p_{m-1-k}(-D_1, -D')\chi_\varepsilon)$$
$$= (2\pi)^{-n} \int_{\sigma(N,t)} e^{i\zeta_1 x_1^0} p_{m-1-k}(\zeta_1, \zeta')\,\hat{\psi}(-\varepsilon\zeta_1)\,\hat{\varphi}(-\zeta')/P(\zeta_1, \zeta')\,d\zeta$$

where $\sigma(N, t)$ is the surface

$$(\xi_1 + it(1 + |\xi_1|^{1/d} + |\xi'|^{1/d}), \xi_2, \dots \xi_n) \text{ with } t \leqslant -C(N) < 0$$

(see the definition of E in Theorem 24). From Theorem 7 we know that to every $\lambda > 0$ there is a constant C_λ such that

$$|e^{i\zeta_1 x_1^0}\hat{\psi}(-\varepsilon\zeta_1)| \leqslant C_\lambda \exp(-\eta_1 x_1^0 + \varepsilon|\eta_1| - \lambda|\varepsilon\xi_1|^{1/d}).$$

Integrating first with respect to ξ_1 for fixed ξ', this estimate and the analyticity of the integrand implies that the integration path

$$(\xi_1 + it(1 + |\xi_1|^{1/d} + |\xi'|^{1/d}), \xi_2, \dots \xi_n),\ t \leqslant -C(N) < 0,$$

can be deformed to a positively oriented circle Γ surrounding the zeros ζ_1 of $P(\zeta_1, \xi')$ when $0 < \varepsilon < x_1^0$. Then, letting $\varepsilon \to +0$ we get

$$0 = (2\pi)^{-n} \int_{R^{n-1}}\int_{\Gamma} e^{i\zeta_1 x_1^0} p_{m-1-k}(\zeta_1, \xi')\,\hat{\varphi}(-\xi')/P(\zeta_1, \xi')\,d\zeta_1\,d\xi'$$
$$= i\,(H_k(x_1^0), \varphi) \quad \text{for} \quad x_1^0 > 0.$$

Hence, $(x_1^0, \operatorname{supp} H_k(x_1^0)) \subset \operatorname{supp} E \cap \{x;\ x_1 = x_1^0\}$ when $x_1^0 > 0$. Since this is trivial for $x_1^0 = 0$, the proof of the existence is complete. The uniqueness is proved in the following theorem.

We can now turn to our general Cauchy problem.

Theorem 26. *Let P be of order m and d-hyperbolic with respect to $N = (1, 0, \dots 0)$. Then the Cauchy problem*

$$\begin{cases} P(D_1, D')\,\varphi(x_1, x') = f(x_1, x') \\ D_1^j \varphi(0, x') = g_j(x'), \quad 0 \leqslant j < m, \end{cases}$$

has a unique solution $\varphi \in G(d, R^n)$ when $f \in G(d, R^n)$ and $\{g_j\}_{j=0}^{m-1} \in G(d, R^{n-1})$.

Proof. Because of Theorem 24, $P(D) = P(D_1, D')$ has a unique fundamental solution E_1 with the support in $\{x;\ x_1 \geqslant 0\}$. Let E_2 be the corresponding fundamental solution supported by $\{x;\ x_1 \leqslant 0\}$ and write $f = f_1 + f_2$ where $\operatorname{supp} f_1 \subset \{x;\ x_1 \geqslant -1\}$, $\operatorname{supp} f_2 \subset \{x;\ x_1 \leqslant 1\}$ and $f_1, f_2 \in G(d, R^n)$. Set $(E_1 * f_1)(x_1, x') + (E_2 * f_2)(x_1, x') = v(x_1, x')$. We apply Theorem 25 and the notations there. Writing

$$(H_k(x_1), \psi) = \int_{R^{n-1}} H_k(x_1, x')\,\psi(x')\,dx'$$

we then have that

$$\varphi(x_1, x') = \sum_{k=0}^{m-1} \int H_k(x_1, y')\,(g_k(x'-y') - D_1^k v(0, x'-y'))\,dy' + v(x_1, x')$$

belongs to $G(d, R^n)$ and solves the given problem.

In order to prove the uniqueness let

$$\begin{cases} P(D_1, D')\,L(x_1) = 0 \\ \qquad D_1^j L(0) = 0, \quad 0 \leqslant j < m, \end{cases}$$

where $D_1^j L(x_1) \in G_0'(d, R^{n-1})$ and $(L(x_1), \varphi) \in G(d, R)$ for $\varphi \in G_0(d, R^{n-1})$.

Then,
$$\begin{cases} P(D_1, D')\,L(x_1) * \varphi = 0 \\ \qquad D_1^j L(0) * \varphi = 0, \quad 0 \leqslant j < m, \end{cases}$$

when $\varphi \in G_0(d, R^{n-1})$. Since $P_m(N) \neq 0$, this implies that $D_1^j L(0) * \varphi = 0$ for every integer $j \geqslant 0$. Hence, $L(x_1) * \varphi = g_1 + g_2$ where supp $g_1 \subset \{x; x_1 \geqslant 0\}$, supp $g_2 \subset \{x; x_1 \leqslant 0\}$ and $g_1, g_2 \in G(d, R^n)$. Then, $g_i = g_i * \delta = g_i * P(D) E_i = P(D)\, g_i * E_i = 0$, $i = 1, 2$. Consequently, $L(x_1) = 0$. The proof is complete.

According to Theorem 26 and the remark on p. 9, we know that a solution of the above Cauchy problem is unique if and only if the plane $(x, N) = 0$ carrying the data is non-characteristic, i.e. $P_m(N) \neq 0$. The following theorem shows that it is in this case rather natural to restrict oneself to the function spaces $G(d)$ where $d \geqslant 1$ is rational. However, some of the theorems can be refined when we have more precise estimates of the zeros τ of $P(\xi + i\tau N)$.

Theorem 27. *Let $P_m(N) \neq 0$ and let $\{\tau_j(\xi)\}_{j=1}^m$ be the zeros of $P(\xi + i\tau N)$ when $\xi \in R^n$. Define*

$$\pi(r) = \sup_{|\xi| = r} \max_{1 \leqslant j \leqslant m} \operatorname{Re} \tau_j(\xi).$$

Then the function π is piece-wise algebraic and there are rational and real constants, $h \leqslant 1$ and C respectively, such that

$$\pi(r) = Cr^h(1 + o(1)) \text{ when } r \to \infty.$$

Proof. We refer to the proof of Theorem 4.3, p. 114 in Gorin [1].

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