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On singular monotonic functions whose spectrum has a given Hausdorff dimension

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1. This paper deals exclusively with continuous monotonic functions which are singular and of the Cantor type, that is to say which are constant in each interval contiguous to a perfect set of measure zero. This perfect set will be called the spectrum of the function.

We shall first prove the following results:

Theorem I. *Given any number* α, $0 < \alpha < 1$, *and a positive* ε, *arbitrarily small, but fixed, there exists a perfect set* E, *with Hausdorff dimension* α, *and a non-decreasing function* $F(x)$, *singular, with spectrum* E, *such that the Fourier Stieltjes transform of* dF *belongs to* L^q *for every* $q \geq \frac{2}{\alpha} + \varepsilon$.

Theorem II. *Given any number* α, $0 < \alpha < 1$, *and a positive* ε, *arbitrarily small, but fixed, there exists a perfect set* E, *with Hausdorff dimension* α, *and a non-decreasing function* $F(x)$, *singular, with spectrum* E, *such that the Fourier-Stieltjes coefficients of* dF *are of order* $1/n^{\frac{\alpha}{2}-\varepsilon}$.

Remarks.

1). Theorem I could be deduced from Theorem II, but since the method of the proof is the same, we prove both theorems.

2). Theorem I has been proved in an earlier paper[1] for the case $\alpha = 1$ (the Lebesgue measure of the set being of course zero), even in the stronger form, that the Fourier Stieltjes transform of the singular function belongs to L^q for every $q > 2$. The argument is the same as in the present paper, although much simpler.

We next prove:

Theorem III. *No singular function (except constant) exists having as spectrum a perfect set of Hausdorff dimension* $\alpha > 0$, *and whose Fourier-Stieltjes transform belongs to* L^q *for some* $q < \frac{2}{\alpha}$.

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[1] R. SALEM. On sets of multiplicity for trigonometrical series. American Journal of Mathematics, Vol. 64 (1942), pp. 531–538.

Likewise, no singular function (except constant) can have as spectrum a perfect set of Hausdorff dimension $\alpha > 0$, *and have Fourier Stieltjes coefficients of order* $n^{-\frac{\alpha}{2}-\varepsilon}$ $\varepsilon > 0$ *(no matter how small* ε *is).*

Remark. The results of theorem III are trivial for $\alpha = 1$. So we prove them for $0 < \alpha < 1$.

Finally, we show that the results of theorem I and II can be sharpened so as to obtain:

Theorem IV. *There exist singular monotonic functions having as spectrum a perfect set of Hausdorff dimension* α $(0 < \alpha < 1)$ *and whose Fourier-Stieltjes transform belongs to* L^q *for every* $q > \frac{2}{\alpha}$.

Theorem V. *There exist singular monotonic functions having as spectrum a perfect set of Hausdorff dimension* α $(0 < \alpha < 1)$ *and whose Fourier-Stieltjes coefficients are of order* $\frac{\Omega(n)}{n^{\frac{\alpha}{2}}}$, $\Omega(n)$ *increasing less rapidly than any positive power of* n.

Remark. As we have said, Theorem IV is known for $\alpha = 1$. Theorem V, in case $\alpha = 1$, has also been proved in an earlier paper.[1]

2. Preliminary constructions. Let OA be a segment of length L whose end points have abscissae 0 and L respectively. Let d be an integer ≥ 2. Let $a_1, a_2, \ldots a_d$ be d distinct numbers such that

$$0 < a_1 < a_2 \cdots < a_d < 1.$$

Let each of the points $L a_j$ be the origin of an interval $(L a_j,\ L a_j + L\eta)$ of length $L\eta$, the number η satisfying the conditions

$$(1) \quad \eta > 0, \quad \eta < a_2 - a_1, \quad \eta < a_3 - a_2, \ldots \eta < a_d - a_{d-1}, \quad \eta < 1 - a_d.$$

The d disjoint intervals thus obtained will be called "white" intervals, the $d + 1$ complementary intervals with respect to OA will be called "black" intervals, and the dissection of OA will be said to be of the type $(d, a_1, a_2, \ldots \ldots \alpha_d, \eta)$.

Starting from the interval (0, 1) and fixing the numbers $d, \alpha_1, \alpha_2, \ldots \alpha_d$, we operate a dissection of the type $(d, \alpha_1, \ldots \alpha_d, \eta_1)$ and we remove the black intervals. On each white interval left we operate a dissection of the type $(d, \alpha_1, \ldots \alpha_d, \eta_2)$ and we remove the black intervals, — and so on. After p operations we have d^p white intervals, each of length $\eta_1 \eta_2 \ldots \eta_p$. When $p \to \infty$ we obtain a perfect set E nowhere dense, which is of measure zero if

[1] R. SALEM. On singular monotonic functions of the Cantor type. Journal of Mathematics and Physics, Vol. 21 (1942), pp. 69—82.

$d_p \eta_1 \eta_2 \ldots \eta_p \to 0$. This will be always the case throughout the paper. The sequence $\eta_1, \eta_2, \ldots$ is arbitrary, provided each η_k satisfies the inequalities (1). The abscissæ of the points of the set are given by the formula

$$x = \alpha(\varepsilon_0) + \eta_1 \alpha(\varepsilon_1) + \eta_1 \eta_2 \alpha(\varepsilon_2) + \eta_1 \eta_2 \eta_3 \alpha(\varepsilon_3) + \cdots$$

where $\alpha(j)$ stands for α_j and each ε_k takes all values $1, 2, \ldots d$.

Let now $F_p(x)$ be a continuous non-decreasing function such that $F_p(0) = 0$, $F_p(1) = 1$, F_p increases linearly by $\frac{1}{d^p}$ on each of the d^p white intervals obtained on the p^{th} step of the dissection, F_p is constant in every black interval. The limit $F(x)$ of $F_p(x)$ as $p \to \infty$ is a continuous non decreasing function, singular, having the perfect set E as spectrum, and such that $F(0) = 0$, $F(1) = 1$. We extend the function so as to have $F(x) = 0$ for $x \leq 0$, $F(x) = 1$ for $x \geq 1$. The Fourier-Stieltjes transform $\gamma(u) = \int_{-\infty}^{\infty} e^{iux}\, dF(x)$ is the limit, for $k = \infty$, of

$$\sum \frac{1}{d^k} \exp\left[iu\alpha(\varepsilon_0) + iu\eta_1 \alpha(\varepsilon_1) + \cdots + iu\eta_1 \ldots \eta_{k-1} \alpha(\varepsilon_{k-1})\right]$$

the sum being extended to the d^k possible combinations of $\varepsilon_0, \varepsilon_1, \ldots \varepsilon_{k-1}$. Thus, writing

$$Q(\varphi) = \frac{1}{d}\left(e^{i\alpha_1 \varphi} + e^{i\alpha_2 \varphi} + \cdots + e^{i\alpha_d \varphi}\right)$$

we have

$$\gamma(u) = Q(u) \prod_{k=1}^{\infty} Q(u \eta_1 \eta_2 \ldots \eta_k).$$

The Fourier Stieltjes coefficient

$$\int_0^1 e^{2\pi i n x}\, dF = \gamma(2\pi n)$$

will be denoted by c_n or $c(n)$.

The above construction of a set E and of the corresponding function F will be used for the proof of Theorems I and II. For the proof of Theorems IV and V it is necessary to use more complicated sets in which not only the number η, but also $d, \alpha_1, \ldots \alpha_d$ change from one dissection to another (the type of dissection being, however, always the same for each white interval at a given step). If the successive dissections are of the type $(d^{(k)}, \alpha_1^{(k)}, \ldots \alpha_{d^{(k)}}^{(k)}, \eta_k)$ where

(2) $$\eta_k > 0, \; \eta_k < \alpha_j^{(k)} - \alpha_{j-1}^{(k)} \; (j = 2, 3, \ldots d^{(k)}), \; \eta_k < 1 - \alpha_{d^{(k)}}$$

and if

$$Q^{(k)}(\varphi) = \frac{1}{d^{(k)}} \sum_{j=1}^{d^{(k)}} e^{i \alpha_j^{(k)} \varphi}$$

then

$$\gamma(u) = Q^{(1)}(u) \prod_{k=1}^{\infty} Q^{(k+1)}(u \eta_1 \dots \eta_k).$$

3. **Lemma.** Let $P(\varphi) = \lambda_1 e^{i\alpha_1 \varphi} + \cdots + \lambda_d e^{i\alpha_d \varphi}$, where the α_j are linearly independent. Let $r > 0$. There exists a positive T_0 (depending on r, d, the λ_j, the α_j) such that for $T \geq T_0$, *and for all values of* a,

$$\frac{1}{T} \int_a^{T+a} |P(\varphi)|^r d\varphi < 2 \left(\frac{r}{2} + 1\right)^{\frac{r}{2}} (\Sigma \lambda_j^2)^{\frac{r}{2}}.$$

The proof is immediate. Let $2q$ be the even integer such that $r \leq 2q < r + 2$. Then

$$|P(\varphi)|^{2q} = \sum_{h_1 + \cdots + h_d = q} \lambda_1^{2h_1} \dots \lambda_d^{2h_d} \left(\frac{q!}{h_1! \dots h_d!}\right)^2 + R$$

R being a sum of terms of the form $A e^{i\mu\varphi}$ with non-vanishing μ. Then obviously

$$\begin{aligned} \lim_{T=\infty} \frac{1}{T} \int_a^{T+a} |P(\varphi)|^{2q} d\varphi &= \sum \lambda_1^{2h_1} \dots \lambda_d^{2h_d} \left(\frac{q!}{h_1! \dots h_d!}\right)^2 \\ &\leq q!\, (\lambda_1^2 + \cdots + \lambda_d^2)^q \\ &\leq q^q\, (\lambda_1^2 + \cdots + \lambda_d^2)^q \end{aligned}$$

uniformly in a. Hence for $T \geq T_0$, where T_0 is independent of a,

$$\begin{aligned} \left\{\frac{1}{T} \int_a^{T+a} |P(\varphi)|^{2q} d\varphi\right\}^{\frac{1}{2q}} &< 2^{\frac{1}{2q}} q^{\frac{1}{2}} (\lambda_1^2 + \cdots + \lambda_d^2)^{\frac{1}{2}} \\ &< 2^{\frac{1}{2q}} \left(\frac{r}{2} + 1\right)^{\frac{1}{2}} (\lambda_1^2 + \cdots + \lambda_d^2)^{\frac{1}{2}} \end{aligned}$$

hence

$$\frac{1}{T} \int_a^{T+a} |P(\varphi)|^r d\varphi < 2 \left(\frac{r}{2} + 1\right)^{\frac{r}{2}} (\lambda_1^2 + \cdots + \lambda_d^2)^{\frac{r}{2}}.$$

4. **Proof of Theorem I.** The numbers α, $(0 < \alpha < 1)$, and $\varepsilon > 0$ are given. Let $s = \frac{2}{\alpha} + \varepsilon$. Take for d the smallest integer ≥ 2 such that

$$d^{\frac{s}{10}} \geq 2\left(\frac{s}{2}+1\right)^{\frac{s}{2}}. \tag{3}$$

Having thus fixed d, determine the number ξ by the condition

$$\frac{\log d}{\log 1/\xi} = \alpha, \tag{4}$$

so that $0 < \xi < \frac{1}{d}$. Fix now d numbers $\alpha_1, \ldots \alpha_d$, linearly independent, satisfying the conditions

$$0 < \alpha_1 < \frac{1}{d} - \xi$$

$$\xi < \alpha_2 - \alpha_1 < \frac{1}{d}$$

$$\xi < \alpha_3 - \alpha_2 < \frac{1}{d}$$

$$\cdots\cdots\cdots$$

$$\xi < \alpha_d - \alpha_{d-1} < \frac{1}{d}$$

from which it follows that $\alpha_d < 1 - \xi$, that is to say all inequalities (1) are satisfied for $\eta = \xi$. A fortiori, all inequalities (1) are satisfied when η is any positive number $< \xi$. Consider a sequence of numbers $\xi_1, \xi_2, \ldots \xi_k \ldots$ satisfying the following conditions

$$\left\{\begin{array}{l} a_1 = \xi\left(1 - \frac{1}{2^2}\right) \leq \xi_1 \leq \xi \\ a_2 = \xi\left(1 - \frac{1}{3^2}\right) \leq \xi_2 \leq \xi \\ \cdots\cdots\cdots\cdots\cdots \\ a_k = \xi\left(1 - \frac{1}{(k+1)^2}\right) \leq \xi_k \leq \xi \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{5}$$

Denote by $E(\xi_1, \ldots \xi_k \ldots)$ the perfect set obtained by the successive dissections $(d, \alpha_1, \ldots \alpha_d, \xi_k)$, where the ξ_k satisfy the inequalities (5). To every sequence $\xi_1, \ldots \xi_k, \ldots$ satisfying (5) corresponds a set E. It is clear from (4) and (5) that all such sets have Hausdorff dimension α. To every set E we associate the corresponding function F described above and having E as spectrum. Writing

$$\xi_k = a_k + (\xi - a_k)\,\zeta_k,$$

we have $0 \leq \zeta_k \leq 1$ and by the Steinhaus method, we map the interval $0 \leq t \leq 1$ on the "cube" $0 \leq \zeta_k \leq 1$ $(k = 1, 2, \ldots)$ of infinitely many dimensions. If

$$t = \cdot \beta_1 \beta_2 \beta_3 \ldots$$

is the dyadic expansion of t we put

$$\begin{aligned} \zeta_1(t) &= \cdot \beta_1 \beta_3 \beta_6 \ldots \\ \zeta_2(t) &= \cdot \beta_2 \beta_5 \beta_9 \ldots \\ \zeta_3(t) &= \cdot \beta_4 \beta_8 \beta_{13} \ldots \\ &\ldots\ldots\ldots \end{aligned}$$

The correspondence is one–one, except for sets of measure zero. Moreover it is well known that for any measurable function $\Phi(\zeta_1 \ldots \zeta_p)$ one has

$$\int_0^1 \Phi[\zeta_1(t) \ldots \zeta_p(t)]\, dt = \int_0^1 \int_0^1 \cdots \int_0^1 \Phi(\zeta_1, \ldots \zeta_p)\, d\zeta_1 \ldots d\zeta_p$$

whenever either side exists.

The set $E(\xi_1, \xi_2, \ldots \xi_k \ldots) = E_t$ depends now on the variable t and we shall show that for almost all t the Fourier-Stieltjes transform of the function $F_t(x)$ corresponding to E_t belongs to L^q for $q \geq \frac{2}{\alpha} + \varepsilon = s$. Writing, with the notations of § 2,

$$\gamma_t(u) = Q(u) \prod_{k=1}^{\infty} Q(u\, \xi_1 \ldots \xi_k)$$

it will be enough to show that $\gamma_t(u)$ belongs to L^s for almost all t, and to this end it will be sufficient to prove that

$$\int_0^\infty du \int_0^1 |\gamma_t(u)|^s\, dt < \infty. \tag{6}$$

First, fix a T_0 such that

$$\frac{1}{T}\int_a^{T+a} |Q(\varphi)|^s\, d\varphi \leq 2\left(\frac{s}{2}+1\right)^{s/2} \frac{1}{d^{s/2}}$$

for all a, and $T \geq T_0$, as is clearly possible by the lemma. Then one has, by (3)

$$\frac{1}{T}\int_a^{T+a} |Q(\varphi)|^s\, d\varphi \leq \frac{1}{d^{\frac{s}{2}-\frac{\varepsilon}{10}}} = \varrho \qquad (T \geq T_0).$$

Now

$$|\gamma_t(u)|^s \leq \prod_{k=1}^{p} |Q(u\,\xi_1 \ldots \xi_k)|^s = f(u, p),$$

say, p being any positive integer. But

$$\int_0^1 f(u, p)\,dt = \int_0^1 \cdots \int_0^1 f(u, p)\,d\zeta_1 \ldots d\zeta_p =$$

$$= \int_0^1 \cdots \int_0^1 f(u, p-1)\,d\zeta_1 \ldots d\zeta_{p-1} \int_0^1 |Q(u\,\xi_1 \ldots \xi_p)|^s\,d\zeta_p.$$

The last integral with respect to the variable ζ_p is equal to

$$\int_0^1 |Q(u\,\xi_1 \ldots \xi_{p-1}[a_p + (\xi - a_p)\zeta_p])|^s\,d\zeta_p = \int_0^1 |Q(l\,\zeta_p + m)|^s\,d\zeta_p$$

$$= \frac{1}{l} \int_m^{l+m} |Q(\varphi)|^s\,d\varphi$$

where $l = u\,\xi_1 \ldots \xi_{p-1}(\xi - a_p) > b\,u\,\xi^p \frac{1}{(p+1)^2}$, b being an absolute constant. Choose p in function of u, such that

$$b\,u\,\xi^p \frac{1}{(p+1)^2} \geq T_0$$

that is to say

$$\log u - p \log 1/\xi - 2 \log (p+1) + \log b \geq \log T_0. \tag{7}$$

It is sufficient to take, when u is large enough

$$p = p(u) = \left[\theta \frac{\log u}{\log 1/\xi}\right] + 1$$

the brackets denoting the integral part, $\theta < 1$ being fixed, but arbitrarily close to one if u is sufficiently large. Having thus chosen $p = p(u)$, the inequality

$$b\,u\,\xi^q \frac{1}{(q+1)^2} \geq T_0$$

is satisfied a fortiori for every $q < p$; hence, successive integrations give us

$$\int_0^1 |\gamma_t(u)|^s\,dt \leq \varrho^p, \qquad u > u_0(\theta).$$

Now

$$\varrho^p = \frac{1}{d^{\left(\frac{s}{2}-\frac{\varepsilon}{10}\right)p}} < \frac{1}{d^{\left(\frac{s}{2}-\frac{\varepsilon}{10}\right)\theta\frac{\log u}{\log 1/\xi}}} = \frac{1}{u^{\theta\frac{\log d}{\log 1/\xi}\left(\frac{s}{2}-\frac{\varepsilon}{10}\right)}}.$$

Fix now θ such that $\theta\left(\frac{s}{2}-\frac{\varepsilon}{10}\right)=\frac{s}{2}-\frac{\varepsilon}{5}$. Then, for u larger than a fixed number,

$$\int_0^1 |\gamma_t(u)|^s\,dt \leq \frac{1}{u^{\alpha\left(\frac{s}{2}-\frac{\varepsilon}{5}\right)}}.$$

But

$$\frac{s}{2}-\frac{\varepsilon}{5}=\frac{1}{\alpha}+\frac{3\varepsilon}{10},\quad \alpha\left(\frac{s}{2}-\frac{\varepsilon}{5}\right)=1+\frac{3\alpha\varepsilon}{10}>1.$$

This proves (6) and, consequently, Theorem I.

5. Proof of Theorem II. Take here, α and ε being given, $(0<\alpha<1$, $\varepsilon>0)$, $s=\frac{2+\alpha}{\varepsilon}$. Determine d as the smallest integer ≥ 2 such that

$$d^{\frac{1}{2}} \geq 2\left(\frac{s}{2}+1\right)^{\frac{s}{2}}.$$

Then determine ξ, the α_j, the ξ_k as before. Next determine T_0 and $p=p(u)$ in the same way as in the proof of Theorem I. Then

$$\int_0^1 |c_t(n)|^s\,dt=\int_0^1 |\gamma_t(2\pi n)|^s\,dt \leq \frac{1}{d^{\left(\frac{s-1}{2}\right)p}} \leq \frac{1}{n^{\theta\alpha\left(\frac{s-1}{2}\right)}}.$$

Fixing θ such that $\theta\left(\frac{s-1}{2}\right)=\frac{s}{2}-1$, one has

$$\int_0^1 |c_t(n)|^s\,dt \leq \frac{1}{n^{\alpha\left(\frac{s}{2}-1\right)}},\qquad n\geq n_0.$$

Writing $\alpha\left(\frac{s}{2}-1\right)=2+\gamma$, we have

$$\sum n^\gamma \int_0^1 |c_t(n)|^s\,dt<\infty$$

and so $n^{\gamma} |c_n|^s \to 0$ for almost every set. A fortiori, for such sets and the corresponding functions

$$|c_n| < \frac{1}{n^{\gamma/s}},$$

but

$$\frac{\gamma}{s} = \frac{\alpha}{2} - \frac{\alpha}{s} - \frac{2}{s} = \frac{\alpha}{2} - \varepsilon$$

which proves the theorem.

Remark. Taking $\alpha = 1 - \delta$, $\varepsilon = \frac{\delta}{2}$, δ arbitrarily small, one proves the existence of a singular monotonic function of the Cantor type with Fourier Stieltjes coefficients of order $\frac{1}{n^{1/2 - \delta}}$. This result has been obtained in an earlier paper[1] by a quite different method. This method is inapplicable to the proof of the general result of Theorem II.

6. Proof of Theorem III. This is reduced immediately to a known result. Suppose the existence of F non-constant having as spectrum a perfect set E of Hausdorff dimension α, and Fourier Stieltjes cofficients $c_n = O\left(\frac{1}{n^{\alpha/2 + \varepsilon}}\right)$. One has $\sum \frac{|c_n|^2}{n^{1-\alpha-\varepsilon}} < \infty$. This proves, by classical results on capacity of sets, that the $(\alpha + \varepsilon)$ capacity of E is positive (the terms of the series being asymptotically of the same order as the terms of the series representing the energy-integral with respect to the distribution dF, and the generalized potential with kernel $r^{-(\alpha+\varepsilon)}$). But the Hausdorff measure of order $\alpha + \varepsilon$ being zero, the $(\alpha + \varepsilon)$ capacity is also zero, by a well known theorem, and this contradiction proves the second part of theorem III.

The first part is proved in a similar fashion, using the fact that if $\gamma(u) \in L^q$ where $q = \frac{2}{\alpha + 2\varepsilon}$, $(\varepsilon > 0$, arbitrarily small, $\alpha + 2\varepsilon < 1)$, one has

$$\int_1^\infty \frac{|\gamma(u)|^2 \, du}{u^{1-\alpha-\varepsilon}} < \left\{ \int_1^\infty |\gamma(u)|^{\frac{2}{\alpha+2\varepsilon}} \, du \right\}^{\alpha+2\varepsilon} \left\{ \int_1^\infty \frac{du}{u^{\frac{1-\alpha-\varepsilon}{1-\alpha-2\varepsilon}}} \right\}^{1-\alpha-2\varepsilon}.$$

7. Proof of Theorem IV. We only sketch the proof which is more complicated than the proof of Theorem I, without involving essentially new ideas.

Fix first an $s_0 > \frac{2}{\alpha}$, taking, for instance, $s_0 = \frac{2}{\alpha} + 1$. Let $r = r(\alpha)$ be the integer such that $s_0 \le 2r < s_0 + 2$. Take an increasing sequence $d^{(1)}, d^{(2)} \ldots d^{(k)} \ldots$

[1] R. Salem. On singular monotonic functions of the Cantor type. Journal of Mathematics and Physics, Vol. 21 (1942), pp. 69–82.

For each $d^{(k)}$ determine a $\xi^{(k)}$ such that $\log d^{(k)} = \alpha \log 1/\xi^{(k)}$. We construct the polynomial $Q^{(k)}(\phi)$ of § 2 by choosing the numbers $\alpha_j^{(k)}$, $(j = 1, 2, \dots d^{(k)})$ such that

$$0 < \alpha_1^{(k)} < \frac{1}{d^{(k)}} - \xi^{(k)}$$

$$\xi^{(k)} < \alpha_j^{(k)} - \alpha_{j-1}^{(k)} < \frac{1}{d^{(k)}} \quad (j = 2, 3, \dots d^{(k)}$$

(so that all inequalities (2) are satisfied for $\eta_k = \xi^{(k)}$), and we further submit the $\alpha_j^{(k)}$ to the condition that $\left|\sum h_j \alpha_j^{(k)}\right|$ has a *positive* lower bound $\mu^{(k)}$ when the integers h_j take all possible values, not all zero, such that $|h_j| \leq r$. It is easily seen that the conditions for the $\alpha_j^{(k)}$ are compatible if $\mu^{(k)} = (C r)^{-2d^{(k)}}$, C being an absolute constant. This number $\mu^{(k)}$ is relevant in the determination of $T_0^{(k)}$, which replaces T_0 at each step, and which is such that

$$\left\{\frac{1}{T}\int_a^{T+a} |Q^{(k)}(\varphi)|^s \, d\varphi\right\}^{\frac{1}{s}} \leq \left\{\frac{1}{T}\int_a^{T+a} |Q^{(k)}(\varphi)|^{2r} \, d\varphi\right\}^{\frac{1}{2r}} \leq \frac{A_r}{\sqrt{d^{(k)}}}$$

if $T \geq T_0^{(k)}$, $s \leq 2r$, A_r depending on r only. One finds by easy calculation that $T_0^{(k)}$ may be taken equal to $(C r)^{3d^{(k)}}$.

Now, the sequence ξ_k which will, as before, constitute our set of infinitely many variables of integration, will satisfy the conditions

$$\xi^{(k)}\left(1 - \frac{1}{(k+1)^2}\right) \leq \xi_k \leq \xi^{(k)}.$$

The condition (7) will be replaced by

$$\log u - \sum_1^p \log \frac{1}{\xi^{(k)}} - 2\log(p+1) + \log C \geq \log T_0^{(p+1)}$$
$$= 3\, d^{(p+1)} \log C r$$

where $\sum_1^p \log 1/\xi^{(k)}$ can be replaced by $\frac{1}{\alpha}\sum_1^p \log d^{(k)}$. Let us assume, as we may, that $d^{(k)}$ increases in such a way that

$$d^{(p+1)} = o\left\{\sum_1^p \log d^{(k)}\right\}.$$

We can take, e.g., $d^{(k)} = k + 1$. Then $p = p(u)$ can be determined so as to have

$$\sum_{1}^{p} \log d^{(k)} = \theta \alpha \log u \qquad u > u_0(\theta)$$

where $\theta < 1$ can be chosen as close to 1 as we wish. Let now s be a fixed exponent such that $s_0 > s > \frac{2}{\alpha}$, and fix θ such that $\theta s > 2/\alpha$. Then

$$\int_0^1 |\gamma_t(u)|^s \, dt \leq \frac{A_r^{sp}}{[d^{(2)} \ldots d^{(p+1)}]^{s/2}} < \frac{A_r^{sp}}{[d^{(1)} \ldots d^{(p)}]^{s/2}} = \frac{A_r^{sp}}{u^{\theta \alpha \frac{s}{2}}}.$$

But $\log u$ is of order $p \log p$, and thus p is of order $\frac{\log u}{\log \log u}$. Hence

$$\int_0^\infty du \int_0^1 |\gamma_t(u)|^s \, dt < \infty,$$

which proves that given s such that $s_0 > s > \frac{2}{\alpha}$, there exists a set G_s of measure 1 in $(0, 1)$ such that for $t \in G_s$, the corresponding $\gamma_t(u)$ belongs to L^s. It is enough now to take a sequence $s_0 > s_1 > s_2 \ldots > s_m \ldots$, $s_m \to \frac{2}{\alpha}$, to prove the existence of a Fourier Stieltjes transform $\gamma_t(u)$ belonging to L^q for every $q > \frac{2}{\alpha}$.

8. Proof of Theorem V. Again we only sketch the proof, inasmuch as the result is apparently not the best possible one. We consider a sequence of integers r_k increasing infinitely with k. Following the method of the preceding paragraph we determine the polynomials $Q^{(k)}(\phi)$, for an increasing sequence $d^{(k)}$, in such a way that

$$(8) \qquad \frac{1}{T} \int_a^{T+a} |Q^{(k)}(\varphi)|^{2 r_k} \, d\varphi \leq \frac{2 r_k^{r_k}}{[d^{(k)}]^{r_k}}$$

for $T \geq T_0^{(k)}$. It is not difficult to see that $T_0^{(k)}$ can be taken equal to $(C r_k)^{3 d^{(k)}}$, provided $r_k = o(d^{(k)})$, which we shall suppose. We remark that (8) implies, for $p < k$,

$$\left\{ \frac{1}{T} \int_a^{T+a} |Q^{(k)}(\varphi)|^{2 r_p} \, d\varphi \right\}^{\frac{1}{2 r_p}} \leq \frac{2^{\frac{1}{2 r_k}} \sqrt{r_k}}{\sqrt{d^{(k)}}}$$

$$(9) \qquad \frac{1}{T} \int_a^{T+a} |Q^{(k)}(\varphi)|^{2 r_p} \, d\varphi \leq \frac{2 r_k^{r_p}}{[d^{(k)}]^{r_p}}.$$

We write now here, changing slightly the method of Theorem I:

$$|\gamma_t(u)|^{2r_p} \le \prod_{k=p+1}^{2p} |Q^{(k)}(u\,\xi_1 \dots \xi_{k-1})|^{2r_p}$$

and we integrate successively with respect to $\zeta_{2p-1}, \zeta_{2p-2}, \dots \zeta_p$, that is to say p times. Provided that $p = p(u)$ satisfies the condition

$$\text{(10)} \qquad \log u - \sum_{p+1}^{2p} \log \frac{1}{\xi^{(k-1)}} - 2\log(2p) + \log b \ge \log T_0^{(2p)} = 3\,d^{(2p)} \log C r_p$$

one has, using (9),

$$\int_0^1 |\gamma_t(u)|^{2r_p}\,dt \le \frac{2^p\,(r_{p+1}\,r_{p+2}\dots r_{2p})^{r_p}}{[d^{(p+1)}\dots d^{(2p)}]^{r_p}} \le \frac{\left(2\,r_{2p}^{r_p}\right)^p}{[d^{(p)}\dots d^{(2p-1)}]^{r_p}}.$$

Now, taking again $d^{(k)} = k+1$ and $\log r_k = o(\log d_k)$ one sees that (10) is satisfied by taking $p = p(u)$ such that:

$$\sum_{p+1}^{2p} \log d^{(k-1)} = \theta_p\,\alpha \log u$$

where θ_p is a certain function of p with the property $\theta_p < 1$, $\theta_p \to 1$ as u, and so $p = p(u)$, increase infinitely. Then

$$\int_0^1 |\gamma_t(2\pi n)|^{2r_p}\,dt \le \frac{\left(2\,r_{2p}^{r_p}\right)^p}{n^{\theta_p \alpha r_p}}.$$

Now $\log n$ is of order $p \log p$, and so p is of order $\dfrac{\log n}{\log\log n}$. Take, now, e.g., $r_p \sim \sqrt{\log p}$. Then

$$p[\log 2 + r_p \log r_{2p}] = O\left(p\sqrt{\log p}\,\log\log p\right) < \frac{p\log p}{2} < \log n$$

for large n, so that for $n > n_0$,

$$\int_0^1 |c_t(n)|^{2r_p}\,dt \le \frac{1}{n^{\theta_p \alpha r_p - 1}}.$$

Taking $\nu_n = \theta_p \alpha r_p - 3$, one has

$$\sum n^{\nu_n} \int_0^1 |c_t(n)|^{2r_p} dt < \infty$$

and thus, for almost all t,

$$n^{\nu_n} |c_n|^{2r_p} \leq 1$$

for n large enough. Hence the existence of a function such that

$$|c_n| \leq \frac{1}{n^{\nu_n/2r_p}} \qquad (n > n_0)$$

but

$$\frac{\nu_n}{2r_p} = \frac{\theta_p \alpha}{2} - \frac{3}{2r_p} = \frac{\alpha}{2} - \varepsilon_n$$

with $\varepsilon_n = o(1)$, which proves the theorem.

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