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On the coefficients in the power series expansion of a rational function with an application on analytic continuation

By CHRISTER LECH

The coefficients of the power series

$$\frac{2}{2-x} = \frac{1}{1} + \frac{1}{2}x + \cdots + \frac{1}{2^n}x^n + \cdots = \sum_{n=0}^{\infty} \frac{\alpha_n}{\beta_n} x^n$$

have the property that the denominators β_n tend to infinity exponentially with the index n. Let the function $\frac{2}{2-x}$ be replaced by another *rational* function and the coefficients $\frac{\alpha_n}{\beta_n}$ $((\alpha_n, \beta_n) = 1)$* be altered accordingly. Might it then occur that simultaneously $\overline{\lim\limits_{n=\infty}}\, \beta_n = \infty$ and $\beta_n = O(n^k)$? We shall give an answer in the negative in proving the following theorem:

Let $r(x)$ be a rational function, which in the neighbourhood of $x = 0$ is represented by a power series with rational coefficients, whose reduced forms are $\frac{\alpha_n}{\beta_n}$,

$$r(x) = \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1}x + \cdots + \frac{\alpha_n}{\beta_n}x^n + \cdots$$

$$\alpha_n,\ \beta_n \text{ integers},\ (\alpha_n, \beta_n) = 1 \qquad n = 1, 2, 3, \ldots$$

$$\beta_n = 1, \text{ when } \alpha_n = 0.$$

*Then, either the sequence $|\beta_n|$ $(n = 1, 2, 3, \ldots)$ is bounded or $\overline{\lim\limits_{n=\infty}} \sqrt[n]{|\beta_n|} > 1$.***

The proof is given in the sections 1 and 2 below. In section 3 there is an application on analytic continuation in connection with a paper[1] by Professor F. CARLSON. The results of this note were also suggested by him.

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* (a, b) means the highest common divisor of a and b.

** We put $\beta_n = 1$, when $\alpha_n = 0$, for the sake of brevity. In reality, only those values of n are considered for which $\alpha_n \neq 0$. A remark of this kind is relevant sometimes also in the sequel.

[1] CARLSON, Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Z., 9 (1921), p. 1–13.

1. We begin with the case where $r(x)$ is of the form

$$(1)\qquad \frac{b_0}{b_0 + b_1 x + \cdots + b_m x^m} = a_0 + a_1 x + \cdots + a_n x^n + \cdots,$$

in which the numbers b_ν are rational integers, not all with a common factor. Apparently $a_0 = 1$. Multiplying (1) by $b_0 + b_1 x + \cdots + b_m x^m$ and equating the coefficients for x^n in both members, we get a recursion formula for a_n:

$$(2)\qquad a_n = -\frac{b_1}{b_0} a_{n-1} - \frac{b_2}{b_0} a_{n-2} - \cdots - \frac{b_m}{b_0} a_{n-m} \qquad n \geq 1$$

$$(a_\nu = 0 \text{ for } \nu < 0).$$

From this formula it appears that all the a_ν are rational numbers and, if $b_0 = \pm 1$, integers. Next we assume that b_0 is a power of a prime p. If $a_n = \frac{\alpha_n}{\beta_n}$, $(\alpha_n, \beta_n) = 1$, we shall show that $\overline{\lim_{n=\infty}} \sqrt[n]{|\beta_n|} \geq \sqrt[m]{2}$. In this case, the formula (2) may be written

$$(3)\qquad a_n = \frac{c_1}{p^{\mu_1}} a_{n-1} + \frac{c_2}{p^{\mu_2}} a_{n-2} + \cdots + \frac{c_m}{p^{\mu_m}} a_{n-m} \qquad n \geq 1,$$

where c_ν and μ_ν are integers, $(c_\nu, p) = 1$ and, for one ν at least, $\mu_\nu > 0$. If $c_\nu = 0$ for certain values of ν, we also put the corresponding $\mu_\nu = 0$.

Let the greatest of the numbers $\frac{\mu_\nu}{\nu}$, $\nu = 1, 2, \ldots, m$ be $\frac{r}{q}$, where $q > 0$, $(r, q) = 1$ and consequently $r > 0$, $1 \leq q \leq m$. We introduce $\varkappa$ as a q:th root of p, thus $\varkappa^q = p$, and make the substitution

$$a_n = \frac{a_n'}{\varkappa^{rn}}.$$

We obtain $a_0' = 1$ and a recursion formula:

$$(3)' \qquad a_n' = k_1 a_{n-1}' + k_2 a_{n-2}' + \cdots + k_m a_{n-m}' \qquad n \geq 1.$$

Denote by S the class of numbers which can be written in the form

$$(4)\qquad H \varkappa^s,$$

where H is an integer and s a non-negative integer. Since $p = \varkappa^q$, we may assume that in (4) H is not divisible by p. A number in S is then said to be divisible by $\varkappa$, if and only if $s > 0$. Evidently, for a rational integer, divisibility by $\varkappa$ is equivalent to divisibility by p.

The numbers k_ν and a_ν' in the formula (3)' belong to S, and when these numbers are written in the form (4), the exponents of $\varkappa$ become congruent to $r \cdot \nu$ to modulus q; at least one of the k_ν is not divisible by $\varkappa$. This is seen directly for the k_ν, and by induction for the a_ν'.

We assert that if one a'_n is an integer, not divisible by p, then there is another such number with higher index. Since $a'_0 = 1$, it follows then that there is an infinity of such numbers.

Let $k_{\nu'}$ be the last of the k_ν which is not divisible by $\varkappa$. Suppose that $\varkappa$ does not divide $a'_{n'}$ but all the following a'_n $(n > n')$. In particular $\varkappa$ would divide the numbers $a'_{n'+1}, a'_{n'+2}, \ldots, a'_{n'+\nu'-1}$. But then the formula (3)′ with $n = n' + \nu'$ implies that $a'_{n'+\nu'}$ is not divisible by $\varkappa$, and the statement follows.

$$a'_{n'+\nu'} = \underbrace{k_1 a'_{n'+\nu'-1} + \cdots + k_{\nu'-1} a'_{n'+1}}_{\varkappa \text{ divides every } a'_n} + k_{\nu'} a'_{n'} + \underbrace{k_{\nu'+1} a'_{n'-1} + \cdots + k_m a'_{n'+\nu'-m}}_{\varkappa \text{ divides every } k_\nu}$$

It is seen that $a'_{n'+\nu'}$ is equal to a sum of integers (the exponent of $\varkappa$ in every term being congruent to zero to modulus q), which are all except one divisible by p.

If an infinity of the a'_ν are integers, not divisible by p, it follows that infinitely many $a_n = \dfrac{a'_n}{\varkappa^{rn}}$ have a reduced denominator $\beta_n = \varkappa^{rn} \geq \varkappa^n \geq p^{\frac{n}{m}} \geq 2^{\frac{n}{m}}$, that is

$$\overline{\lim_{n=\infty}} \sqrt[n]{|\beta_n|} \geq \sqrt[m]{2}.$$

When $b_0 \neq \pm 1$ is not a power of a prime, we may write $b_0 = b'_0 \cdot b''_0$, where b'_0 is a power of a prime, and $(b'_0, b''_0) = 1$. Substituting

$$a_n = \frac{a''_n}{(b''_0)^n}, \tag{5}$$

we get from (2) a recursion formula for a''_n of the same kind as the formula (3) for a_n:

$$a''_n = -\frac{b_1}{b'_0} a''_{n-1} - \frac{b_2 b''_0}{b'_0} a''_{n-2} - \cdots - \frac{b_m (b''_0)^{m-1}}{b'_0} a''_{n-m}.$$

When $a''_n = \dfrac{\alpha''_n}{\beta''_n}$, $(\alpha''_n, \beta''_n) = 1$, we have just proved $\overline{\lim\limits_{n=\infty}} \sqrt[n]{|\beta''_n|} \geq \sqrt[m]{2}$. With $a_n = \dfrac{\alpha_n}{\beta_n}$, $(\alpha_n, \beta_n) = 1$ it also follows from (5) that $\overline{\lim\limits_{n=\infty}} \sqrt[n]{|\beta_n|} \geq \sqrt[m]{2}$. Thus the proof is completed in the case where $r(x)$ is of the form (1).

2. In the more general case, when we have

$$r(x) = \frac{g(x)}{f(x)},$$

$g(x)$ and $f(x)$ being polynomials whose coefficients are rational integers, we may assume, without loss of generality, that the coefficients b_ν in the polynomial

$$f(x) = b_0 + b_1 x + \cdots + b_m x^m$$

have no common factors. This amounts to a possible multiplication of $r(x)$ by an integer, which is of no importance for what is to be proved. If $b_0 = \pm 1$ we have seen before that the coefficients in the power series for the function $\frac{1}{f(x)}$ are integers, and this must obviously hold also for the function $\frac{g(x)}{f(x)}$. If $b_0 \neq \pm 1$, it is true for the function $\frac{1}{f(x)}$ that, with the notations used, $\lim_{n=\infty} \sqrt[n]{|\beta_n|} > 1$. We shall now prove the corresponding inequality for $\frac{g(x)}{f(x)}$ under the non-restrictive assumption that $(f(x), g(x)) = 1$.

For the sake of brevity we introduce the notation $\beta_n\{F(x)\}$ as follows. $F(x)$ is to be an arbitrary function which can be expanded about the origin in a power series with rational coefficients,

$$F(x) = \frac{\alpha_0'}{\beta_0'} + \frac{\alpha_1'}{\beta_1'} x + \cdots + \frac{\alpha_n'}{\beta_n'} x^n + \cdots$$

$$\alpha_n', \beta_n' \text{ integers, } (\alpha_n', \beta_n') = 1 \qquad n = 1, 2, 3, \ldots$$

$$\beta_n' = 1, \text{ when } \alpha_n' = 0.$$

Then, by definition, $\beta_n\{F(x)\} = |\beta_n'|$. Our statement will be:

If $f(x)$ and $g(x)$ are polynomials whose coefficients are rational integers, and $(f(x), g(x)) = 1$, then

$$\overline{\lim_{n=\infty}} \sqrt[n]{\beta_n\left\{\frac{1}{f(x)}\right\}} > 1 \text{ implies } \overline{\lim_{n=\infty}} \sqrt[n]{\beta_n\left\{\frac{g(x)}{f(x)}\right\}} > 1. \tag{6}$$

By means of Euclid's algorithm it is possible to find polynomials $p_1(x)$ and $p_2(x)$ whose coefficients are rational integers, and a rational integer K, so that

$$p_1(x) f(x) + p_2(x) g(x) = K$$

or

$$\frac{K}{f(x)} = p_2(x) \frac{g(x)}{f(x)} + p_1(x). \tag{7}$$

We put

$$\frac{1}{f(x)} = c_0 + c_1 x + \cdots + c_n x^n + \cdots,$$

$$\frac{g(x)}{f(x)} = d_0 + d_1 x + \cdots + d_n x^n + \cdots,$$

$$p_2(x) = e_0 + e_1 x + \cdots + e_k x^k.$$

Then, according to (7), for large values of n,

$$K c_n = e_0 d_n + e_1 d_{n-1} + \cdots + e_k d_{n-k},$$

from which follows

$$\frac{\beta_n\left\{\frac{1}{f(x)}\right\}}{K} \leq \beta_n\left\{\frac{g(x)}{f(x)}\right\}\beta_{n-1}\left\{\frac{g(x)}{f(x)}\right\}\cdots\beta_{n-k}\left\{\frac{g(x)}{f(x)}\right\}.$$

Thus

$$\overline{\lim_{n=\infty}}\sqrt[n]{\beta_n\left\{\frac{1}{f(x)}\right\}} \leq \left[\overline{\lim_{n=\infty}}\sqrt[n]{\beta_n\left\{\frac{g(x)}{f(x)}\right\}}\right]^{k+1},$$

of which (6) is an immediate consequence.

Our theorem is now completely proved by the following lemma[1], which we shall not demonstrate here:

If, in the power series expansion of a rational function about the origin, all the coefficients are rational numbers, then the function can be written as the quotient of two polynomials with integral coefficients.

3. *Suppose that $f(x)$ is an analytic function, regular and one-valued inside the unit circle, except for a finite number of singularities, and that, in the neighbourhood of the origin, it is represented by a power series with rational coefficients,*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

$$a_n = \frac{\alpha_n}{\beta_n}$$

$$\alpha_n,\ \beta_n \text{ integers, } (\alpha_n, \beta_n) = 1 \qquad n = 1, 2, 3, \ldots$$

$$\beta_n = 1, \text{ when } \alpha_n = 0.$$

Then, if

$$\overline{\lim_{n=\infty}}\,|\beta_n| = \infty,\ \lim_{n=\infty}\frac{\beta_n}{n} = 0, \tag{8}$$

$f(x)$ cannot be continued across the unit circle.

The proof of this theorem depends wholly upon CARLSON's work, mentioned in the introduction, which states a necessary condition for us to be able to continue $f(x)$ across the unit circle: with the notation

$$\Delta_p^{(q)} = \begin{vmatrix} a_p, & a_{p+1}, & \ldots, & a_{p+q-1} \\ a_{p+1}, & a_{p+2}, & \ldots, & a_{p+q} \\ \cdot & \cdot & \cdot & \cdot \\ a_{p+q-1}, & a_{p+q}, & \ldots, & a_{p+2q-2} \end{vmatrix}$$

the inequality

$$\overline{\lim_{p=\infty}}\,|\Delta_p^{(q)}|^{\frac{1}{p^2}} < 1 \tag{9}$$

must hold for $q = p, p+1$.

[1] HEINE, Kugelfunktionen I, in the second edition p. 52–53. The lemma is stated there more generally, for algebraic functions.

As we have shown previously, the relations (8) involve that $f(x)$ cannot be a rational function. It follows, according to BOREL[1] and PÓLYA[2], that there exists no number p_0 such that $p > p_0$ implies that $\Delta_p^{(p)} = \Delta_p^{(p+1)} = 0$. Thus, either $\Delta_p^{(p)} \neq 0$ for infinitely many p or $\Delta_p^{(p+1)} \neq 0$ for infinitely many p.

We write

$$\max_{p \leq \nu \leq 3p} |\beta_\nu| = N = p\varphi(p). \tag{10}$$

Then, by the second of the relations (8),

$$\lim_{p=\infty} \varphi(p) = 0. \tag{11}$$

Let $p_1, p_2, \ldots, p_{\pi(N)}$ be the primes $\leq N$, $\pi(x)$ the prime number function. If we multiply every column in the determinant $\Delta_p^{(p)}$ by*

$$T = p_1^{\left[\frac{\log N}{\log p_1}\right]} p_2^{\left[\frac{\log N}{\log p_2}\right]} \cdots p_{\pi(N)}^{\left[\frac{\log N}{\log p_{\pi(N)}}\right]} \leq N^{\pi(N)}, \tag{12}$$

all the elements in the new determinant become integers. Thus, if $\Delta_p^{(p)} \neq 0$,

$$T^p \, | \Delta_p^{(p)} | \geq 1. \tag{13}$$

By the prime number theorem, $\frac{\pi(N)\log N}{N}$ is a bounded function of N. Hence, on account of (10), (11) and (12)

$$\overline{\lim_{p=\infty}}\, T^{\frac{1}{p}} \leq \overline{\lim_{p=\infty}}\, e^{\frac{\pi(N)\log N}{N}\varphi(p)} = 1. \tag{14}$$

If $\Delta_p^{(p)} \neq 0$ for an infinity of p, it follows from (13) and (14) that

$$\overline{\lim_{p=\infty}}\, | \Delta_p^{(p)} |^{\frac{1}{p^2}} \geq 1.$$

In a similar way, it may be shown that, if $\Delta_p^{(p+1)} \neq 0$ for an infinity of p,

$$\overline{\lim_{p=\infty}}\, | \Delta_p^{(p+1)} |^{\frac{1}{p^2}} \geq 1.$$

The inequality (9) is therefore contradicted either for $q = p$ or $q = p + 1$, which proves the theorem.

[1] BOREL, Sur une application d'un théorème de M. Hadamard. Bull. Sciences Math., (2), XVIII (1894), p. 22–25.

[2] PÓLYA, Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Ann., 4 (1919), p. 497–513. See also Carlson's paper.

* $[x]$ is the integral part of x.

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