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Spectral synthesis of bounded functions[1]

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This paper is intended as a sequel to the elegant paper of GODEMENT [4][2] on harmonic analysis. The fundamental problem, first posed by BEURLING, is this: when is a bounded measurable function φ defined on a locally compact abelian group G the weak limit of linear combinations of characters belonging to the spectral set Λ_φ of φ? In the dual language of L^1, the problem is to determine when a closed ideal of the group algebra is the intersection of regular maximal ideals.

SCHWARTZ [8] has given an example showing that the spectral approximation is not possible for all functions in Euclidean space of three dimensions. On the other hand the approximation is known to be possible if G is the real line and if Λ_φ is assumed to have denumerable (or reducible) boundary. Proofs of this theorem have been published by DITKIN [2], SEGAL [9], and MANDELBROJT and AGMON [6]. By using Segal's method and the structure theory of groups, KAPLANSKY [5] extended the theorem to a wide class of groups; and actually to arbitrary G if Λ_φ contains only one point.

This theorem of KAPLANSKY states that if Λ_φ consists of a single point, then φ is itself a multiple of a character. A proof based on the theory of distributions was found independently by JEAN RISS [7]. Our first objective is to give a new proof of this theorem using only simple analysis on the group itself. This seems desirable for aesthetic reasons, but more important, the structure theory is evidently not always enough to extend results from the real line to arbitrary groups.

Second, we extend an unpublished proof of BEURLING to obtain for arbitrary groups the theorem quoted above for the real line, thus accomplishing what KAPLANSKY set out to do by structure theory.

Finally, we modify a theorem of von Neumann and Dixmier about operators on Hilbert space to show that if the spectral approximation of φ is possible in the weak topology, then it is also possible in certain stronger topologies.

§ 1. Introduction

We begin with some definitions and theorems from GODEMENT [4] which will be used without reference hereafter.

[1] The content of § 2 and part of Theorem 2 appeared in my thesis. I wish to thank Professor LYNN LOOMIS most cordially for his direction. The other parts have roots in conversations with Professor ARNE BEURLING.

[2] Numbers in brackets refer to references at the end of the paper.

If f is summable on a group G (always assumed locally compact abelian), its Fourier transform $\hat{f}$ is a continuous bounded function on the dual group $\hat{G}$, given by the formula

$$\hat{f}(\hat{x}) = \int \overline{(x, \hat{x})}\, f(x)\, dx.$$

If $\hat{g}$ is summable on $\hat{G}$, its transform is defined similarly but the bar over the character is omitted. For later convenience we shall study a function $\hat{\varphi} \varepsilon L^{\infty}(\hat{G})$. If $\hat{f}$ is summable, $\hat{\varphi} * \hat{f}$ is bounded and continuous. The spectral set $\Lambda_{\hat{\varphi}}$ of $\hat{\varphi}$ is the set of $p \varepsilon G$ such that for all summable $\hat{f}$, $\hat{\varphi} * \hat{f} \equiv 0$ implies $f(p) = 0$, where f is the transform of $\hat{f}$. (Hereafter we shall use the same letter to denote a function and its transform without mentioning the relation.) Then $\Lambda_{\hat{\varphi}}$ is a closed subset of G, and the Wiener Tauberian theorem asserts that $\Lambda_{\hat{\varphi}}$ is not empty unless $\hat{\varphi}$ vanishes almost everywhere.

If U is any open set in G containing $\Lambda_{\hat{\varphi}}$, then $\hat{\varphi}$ is the weak limit of *trigonometric polynomials from* U, which is to say functions of the form

$$\sum_{i=1}^{n} c_i \overline{(x_i, \hat{x})}$$

with each $x_i \varepsilon U$. The problem is to prove the same fact replacing U by $\Lambda_{\hat{\varphi}}$ itself. The same theorem can be given this form: if $\hat{f}$ is summable and f vanishes on U, then $\hat{\varphi} * \hat{f} \equiv 0$. Here the problem is to prove the convolution vanishes assuming only that f vanishes on $\Lambda_{\hat{\varphi}}$.

Summable functions are dense in $L^2(G)$, and so the Fourier transform is defined on a dense subset of $L^2(G)$. The Plancherel theorem asserts that this transform can be extended to all $L^2(G)$ so as to be an isometry onto $L^2(\hat{G})$.

§ 2. Kaplansky's theorem

Let $\hat{\varphi}$ be essentially bounded and measurable on $\hat{G}$. If $\hat{g}$ is summable or square-summable, the same is true of the product $\hat{\varphi} \cdot \hat{g}$; so multiplication by $\hat{\varphi}$ defines operators in $L^1(\hat{G})$ and in $L^2(\hat{G})$ which are evidently continuous with bound $\|\hat{\varphi}\|_\infty$. Denote by F the ring of functions f which are transforms of summable functions $\hat{f}$ on $\hat{G}$, and introduce a norm in F by setting $\|f\| = \|\hat{f}\|_1$. If g belongs to F or to $L^2(G)$ (which we shall write simply L^2), let Tg be the transform of $\hat{\varphi} \cdot \hat{g}$. Using the Plancherel theorem, T is defined and has bound $\|\hat{\varphi}\|_\infty$ in F and in L^2. It is easy to verify that T commutes with translation. It is true conversely that every bounded operator in L^2 which commutes with translation is obtained from some function $\hat{\varphi}$, but we shall not use this fact.

Lemma 1. If a directed system of functions $\hat{\varphi}_\xi \varepsilon L^{\infty}(\hat{G})$ converges weakly to $\hat{\varphi}$, and if each $\hat{\varphi}_\xi$ determines the operator T_ξ in F, then $Tf(x) = \lim T_\xi f(x)$ for any $f \in F$ and any fixed x.

The hypothesis that $\hat{\varphi}_\xi$ converges weakly to $\hat{\varphi}$ means that

$$\int \hat{\varphi}_\xi(\hat{x}) \hat{f}(\hat{x})\, d\hat{x} \to \int \hat{\varphi}(\hat{x}) \hat{f}(\hat{x})\, d\hat{x}$$

for any summable $\hat{f}$. This implies

$$\int \hat{\varphi}_\xi(\hat{x}) \hat{f}(\hat{x})(x, \hat{x})\, d\hat{x} \to \int \hat{\varphi}(\hat{x}) \hat{f}(\hat{x})(x, \hat{x})\, d\hat{x},$$

which says exactly that $T_\xi f(x) \to T f(x)$.

Lemma 2. Suppose $\Lambda_{\hat{\varphi}} = \{p\}$. Given a neighborhood $V(0)$ (where 0 is the identity of G) there is a neighborhood $U(p)$ such that if $f \varepsilon F$ vanishes in $V(0)$, then Tf vanishes in $U(p)$.

Take any symmetric neighborhood $U(0)$ such that $U^2(0) < V(0)$. Let $\hat{\varphi}_\xi$ be a directed system of trigonometric polynomials from $U(p)$ converging weakly to $\hat{\varphi}$, and let T_ξ be the associated operators in F. Then T_ξ is the transform of a function of the form $\sum_{i=1}^{n} c_i \overline{(x_i, \hat{x})} \hat{f}(\hat{x})$, and so is a sum of translates of f: $\sum c_i f(x_i^{-1} x)$. Since each $x_i \varepsilon U(p)$, if $x \varepsilon U(p)$ we have $x_i^{-1} x \varepsilon U^2(0) < V(0)$. But f vanishes on $V(0)$ by hypothesis and so each term of the sum is zero. Hence $T_\xi f$ vanishes on $U(p)$ for each ξ, and by the first lemma Tf also vanishes on $U(p)$.

Since T commutes with translation, this lemma can be restated immediately to say that if f vanishes in a neighborhood $V(q)$ of any point q, then Tf vanishes in $U(pq)$. Suppose f is constant in $V(q)$ but not necessarily zero. Then for any fixed point $r \varepsilon V(0)$, $f(x) - f(rx)$ vanishes as a function of x in a neighborhood of q. By the lemma as just modified, $Tf(x) - Tf(rx)$ vanishes in some neighborhood of pq and in particular at the point pq itself, so that $Tf(pq) = Tf(rpq)$. Since r was arbitrary in $V(0)$, we have shown that Tf is constant on $V(pq)$ assuming that f is constant on $V(q)$.

Lemma 3. Let V be an open set with compact closure and g its characteristic function (belonging to L^2). Let T be the operator in F and in L^2 associated with $\hat{\varphi}$, where $\Lambda_{\hat{\varphi}} = \{p\}$. Then Tg is a constant k times the characteristic function of the set pV, and k depends only on T.

For every integer n choose an open set U_n whose closure is contained in V such that $\mu(V - U_n) < 1/n$, where μ is Haar measure on G. The construction can be made so $U_m < U_n$ for $m < n$. We can find $f_n \varepsilon F$ equal to one on U_n, vanishing outside V, and never exceeding one in absolute value. Evidently g is the limit of the f_n in the norm of L^2.

By what has been proved, Tf_n is constant on the set pU_n for each n. If $m < n$, $f_n - f_m$ vanishes on U_m and $T(f_n - f_m)$ vanishes on pU_m, so the constant k does not depend on n. A similar argument shows that the constant does not depend on V, and so depends only on T. Since T is continuous, $Tg = \lim Tf_n$; it follows that Tg is constant on $U_{n=1}^{\infty} p\, U_n$, or almost everywhere on pV. It remains to show that Tg vanishes almost everywhere outside pV.

Suppose a point s has a neighborhood $M(s)$ disjoint from V. Then each f_n vanishes in $M(s)$, and each $T f_n$ vanishes in $M(ps)$. It follows that $T g$ vanishes in the same neighborhood, and so $T g$ vanishes outside the closure of $p V$. If the boundary of $p V$ is of measure zero there is nothing more to prove. Otherwise apply the preceding argument to the characteristic functions g_n of U_n. Every $T g_n$ vanishes outside $p V$, since the closure of U_n is contained in V. It follows that $T g$ also vanishes almost everywhere outside $p V$, as we had to show.

Now Kaplansky's theorem follows easily.

Theorem 1. Suppose $\hat{\varphi} \varepsilon L^\infty(\hat{G})$ and $\Lambda_{\hat{\varphi}} = \{p\}$. Then $\hat{\varphi}(\hat{x}) = k\overline{(p, \hat{x})}$ for some constant k and almost all $\hat{x}$.

By the last lemma there is a constant k depending only on $\hat{\varphi}$ such that $T g(x) = k g(p^{-1}x)$ for almost all x, if g is the characteristic function of an open set having compact closure. The finite linear combinations of such functions are dense in L^2, and so the same formula holds for any $g \varepsilon L^2$. That is to say that $T g$ is the Fourier transform of $k\overline{(p, \hat{x})}\hat{g}(\hat{x})$, and it follows that $\hat{\varphi}(\hat{x}) = k\overline{(p, \hat{x})}$ almost everywhere.[1]

§ 3. Boundary of $\Lambda_{\hat{\varphi}}$ is reducible

Lemma. There is a directed system of summable functions $\hat{h}_\eta$ each of norm one such that

$$h_\eta(0) = 1,$$

$$\hat{h}_\eta * \hat{f} \to 0 \text{ if } \hat{f} \text{ is summable and } f(0) = 0.$$

To prove the lemma we need Kaplansky's theorem in this form: if $\hat{f}$ is summable and $f(0) = 0$, then $\hat{f} = \lim \hat{f}_n$ where each f_n vanishes in a neighborhood of 0. Indeed, if $\hat{\varphi} \varepsilon L^\infty$ is orthogonal to every $\hat{g}$ whose transform vanishes in a neighborhood of 0, then $\Lambda_{\hat{\varphi}}$ contains at most the point 0. Hence every such $\hat{\varphi}$ is a multiple of the constant character, and it follows that the closure of the set of $\hat{g}$ considered contains every function orthogonal to a constant. These are just the functions whose transforms vanish at 0, proving the assertion.

Now let V_η be a fundamental system of neighborhoods of 0 each having compact closure, directed by inclusion. Let g_η be the characteristic function of V_η and set $h_\eta = g_\eta * g_\eta^* / \| g_\eta \|_2^2$, where $g_\eta^*(x) = g_\eta(x^{-1})$. It can be verified that $h_\eta(0) = 1$ and h_η vanishes off the set $V_\eta V_\eta^{-1}$. Furthermore h_η is positive definite and belongs to F, and so has norm equal to $h_\eta(0) = 1$. Now if $\hat{f}$ is summable and $f(0) = 0$, find a sequence $\hat{f}_n$ converging to $\hat{f}$ such that every f_n vanishes in a neighborhood of 0. Then for every η and some n

$$\| \hat{h}_\eta * \hat{f} - \hat{h}_\eta * \hat{f}_n \|_1 \leq \| \hat{f} - \hat{f}_n \|_1 < \varepsilon.$$

[1] I am indebted to Professor GODEMENT for showing that this proof can be carried through in L^2, thus avoiding the space F and simplifying some of the lemmas.

But we can choose η so high that $h_\eta f_n$ vanishes, so $\| \hat{h}_\eta * \hat{f} \|_1 < \varepsilon$ for all sufficiently high η. Since ε was arbitrary, the lemma is proved.

Theorem 2. If $\hat{\varphi} \varepsilon L^\infty(\hat{G})$ and $\Lambda_{\hat{\varphi}}$ has reducible boundary, then $\hat{\varphi}$ is the weak limit of trigonometric polynomials from $\Lambda_{\hat{\varphi}}$.

Our proof is a modification of a proof of BEURLING for the same theorem on the real line, given in a course of lectures at Harvard in 1949.

It suffices to prove that $\hat{\varphi} * \hat{f} \equiv 0$ if $\hat{f}$ is summable and f vanishes on $\Lambda_{\hat{\varphi}}$. Anyway $\hat{\psi} = \hat{\varphi} * \hat{f}$ is a bounded continuous function. We assert that $\Lambda_{\hat{\psi}}$ has no isolated points.

Suppose p is isolated in $\Lambda_{\hat{\psi}}$. After multiplication by a character we can assume $p = 0$. Let $\hat{g}$ be a function whose transform is one in a neighborhood of 0 and vanishes on an open set containing $\Lambda_{\hat{\psi}} - \{0\}$. Then the spectral set of $\hat{\psi} * \hat{g}$ contains at most the point 0 and so $\hat{\psi} * \hat{g}$ is constant by Kaplansky's theorem. Since $h_\eta(0) = 1$, $\hat{\psi} * \hat{g} * \hat{h}_\eta$ is the same constant for every η. But $f(0) = 0$ and so $\hat{h}_\eta * \hat{f} \to 0$, showing the constant was zero. Since $\hat{\psi} * \hat{g} \equiv 0$ and $g(0) \neq 0$, p does not belong to $\Lambda_{\hat{\psi}}$.

Now $\Lambda_{\hat{\psi}}$ is closed without isolated points, and so is perfect. Furthermore it is contained in the boundary of $\Lambda_{\hat{\varphi}}$ and must be empty by hypothesis. This proves the theorem.

The theorem applies in particular if the boundary of $\Lambda_{\hat{\varphi}}$ is denumerable.

§ 4. Bounded functions as operators

Regarding $L^\infty(G)$ as a space of operators on $L^2(G)$ by multiplication, it is immediate that the weak topology in L^∞ as the conjugate space of L^1 is the same as the weak operator topology. DIXMIER [3] showed that a strongly closed manifold of operators in Hilbert space is weakly closed, removing von Neumann's condition that the manifold be a ring. This theorem can be translated into a theorem about spectral synthesis, and in this context the proof applies to any L^p space as well as to Hilbert space. Say that a directed system of functions $\varphi_\lambda \varepsilon L^\infty$ converges to φ in the *strong p-topology* $(1 \leq p < \infty)$ just if $\varphi_\lambda \cdot f$ converges to $\varphi \cdot f$ in norm for every $f \varepsilon L^p$.

Theorem 3. A linear manifold in L^∞ is weakly closed if it is closed in any strong p-topology.

Now if $\hat{\varphi}$ admits a spectral approximation in the weak topology, it must have a spectral approximation in each strong p-topology. That is, given any ε, p, and summable $\hat{f}$, there is a trigonometric polynomial $\hat{\varphi}'$ such that

$$\int |\hat{\varphi} - \hat{\varphi}'|^p \hat{f} \, d\hat{x} < \varepsilon.$$

This result was known to BEURLING at least for $p = 1$ and 2. If the spectral synthesis is possible for every function of L^∞ on some group G, then every summable function is a regular weight function in BEURLING's language [1]; this follows from the last formula with $p = 1$.

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