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A remark on a theorem by Frostman

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The origin of this remark is a lecture held by Frostman in Helsinki 1957 [2].

Let us introduce some definitions and notations.

Let K be an arbitrary compact set in the euclidean space R^n and let α be a number such that $0<\alpha<n$. Put

$$\|\mu\|_\alpha^2=\int\int\frac{d\mu(x)\,d\mu(y)}{|x-y|^{n-\alpha}}$$

where μ is a distribution of mass in R^n and where x and y denote points in R^n, $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$.

A is the set of all positive distributions of unit mass on K, that is,

$$\mu\geqslant 0,\quad \mu(K)=1,\quad \mu(R^n-K)=0.$$

Let $C_\alpha(K)$ be the capacity of K of order α

$$C_\alpha(K)=\frac{1}{\inf\limits_{\mu\in A}\|\mu\|_\alpha}.$$

It is well known that if $C_\alpha(K)>0$ then there exists a uniquely determined distribution μ_α in A that satisfies

$$\|\mu_\alpha\|_\alpha=\inf_{\mu\in A}\|\mu\|_\alpha.$$

μ_α is called the equilibrium distribution of order α on K. Frostman [2] has set the problem whether these equilibrium distributions vary continuously with α or not. Or, if $\alpha\searrow\beta$ ($\searrow$ means "tends non-increasingly to"), is it then true that μ_α converges towards a uniquely determined limit? (Convergence here in the weak sense, that is, $\mu_\alpha\to\mu$ is equivalent to $\int f\,d\mu_\alpha\to\int f\,d\mu$ for all continuous functions f with compact supports.)

If $C_\beta(K)>0$, the answer is yes. The limit in this case is μ_β which is easy to prove [2]. On the other hand, if $C_\beta(K)=0$, $C_\alpha(K)>0$ for $\alpha>\beta$, then the problem is not solved but for special cases. Frostman treats such a special case in [2] namely the case that $\alpha\searrow 1$ and that K is a curve in the plane ($n=2$) which is rectifiable. He proves that in this case $\mu_\alpha\to\mu_0$ where μ_0 is the distribution in A for which the mass which is situated on an arc is proportional to the lenght of that arc.

The aim of this remark is to prove another special case which does not contain the case of Frostman but which in a certain sense extends his result.

Let us suppose that the measure of K is greater than zero. We shall prove

Theorem. *If* $\alpha \searrow 0$ *then* $\mu_\alpha \to \mu_0$ *where* μ_0 *is the distribution in* A *which has constant density.*

Proof. It is evidently sufficient to prove that if μ_{α_n}, $\alpha_n \searrow 0$, is an arbitrary convergent sequence, then its limit is μ_0. Put

$$K_\alpha(x) = C\alpha\, 1/r^{n-\alpha}, \quad r = \sqrt{x_1^2 + \cdots + x_n^2},$$

where

$$C = \frac{1}{2}\Gamma\left(\frac{n}{2}\right) \Big/ \pi^{n/2}.$$

$\hat{f}$ denotes the Fourier transform of f, that is,

$$\hat{f}(x) = \int_{R^n} e^{-2\pi i x\xi} f(\xi)\, d\xi, \quad x\xi = \sum_{i=1}^{n} x_i \xi_i.$$

Hence

$$\hat{K}_\alpha(x) = C\alpha\pi^{n/2-\alpha} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} r^{-\alpha} \to 1 \text{ when } \alpha \searrow 0.$$

This implies that $K_\alpha \to \delta$ in the distribution sense. δ is the Dirac function. Put

$$\|\mu\|_{K_\alpha}^2 = C\alpha \|\mu\|_\alpha^2, \quad \|\mu\|_0^2 = \int_{R^n} |\hat{\mu}(x)|^2\, dx.$$

Then we have

$$\|\mu\|_{K_\alpha}^2 = \int_{R^n} \hat{K}_\alpha(x)\, |\hat{\mu}(x)|^2\, dx$$

which is easily seen by Parseval's formula.

Now let μ_{α_n} be a sequence which converges towards a distribution μ' (necessarily in A). Then we have

$$\hat{\mu}_{\alpha_n}(x) \to \hat{\mu}'(x) \text{ everywhere.}$$

By Fatou's lemma we conclude

$$\varliminf_{n\to\infty} \|\mu_{\alpha_n}\|_{K_{\alpha_n}}^2 = \varliminf_{n\to\infty} \int_{R^n} \hat{K}_{\alpha_n}(x)\, |\hat{\mu}_{\alpha_n}(x)|^2\, dx \geq \int_{R^n} |\hat{\mu}'(x)|^2\, dx = \|\mu'\|_0^2$$

Observing that μ_{α_n} is a minimal distribution also with respect to the kernel K_{α_n} we get

$$\|\mu'\|_0^2 \leqslant \varliminf_{n\to\infty} \|\mu_{\alpha_n}\|_{K_{\alpha_n}}^2 \leqslant \varliminf_{n\to\infty} \|\mu_0\|_{K_{\alpha_n}}^2 =$$

$$= \varliminf_{n\to\infty} \int_{R^n} \hat{K}_{\alpha_n}(x)\,|\hat{\mu}_0(x)|^2\,dx = \int_{R^n} |\hat{\mu}_0(x)|^2\,dx = \|\mu_0\|_0^2.$$

Or,

$$\|\mu'\|_0^2 \leqslant \|\mu_0\|_0^2.$$

But μ_0 is the uniquely determined equlibrium distribution of K with respect to the kernel δ. This implies

$$\|\mu_0\|_0 = \inf_{\mu\in A} \|\mu\|_0 \text{ (see [1])}.$$

Hence $\mu' = \mu_0$ and the theorem is proved.

REFERENCES

1. DENY, J.: Les potentiels d'énergie finie. Acta Mathematica 82 (1950).
2. FROSTMAN, O.: Suites convergentes de distributions d'équilibre. Treizième Congrès des Math. Scandinaves (Helsinki) 1957.

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