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# A study of certain Green's functions with applications in the theory of vibrating membranes

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**Introduction.** The first part of this paper contains estimations for $\varkappa \to +\infty$ of the Green's functions which satisfy the equation

$$\left(\frac{\partial}{\partial x^1}\right)^2 u + \left(\frac{\partial}{\partial x^2}\right)^2 u - \varkappa^2 u = 0 \tag{1}$$

and Dirichlet's or Neumann's boundary conditions. A similar investigation in the theory of Laplace's equation was recently carried out in collaboration with T. Ganelius. A previous paper on Green's functions of the biharmonic equation has been published in the proceedings of the Symposium on Spectral Theory and Differential Problems in Stillwater, Oklahoma, 1951.

In the second part of the paper, eigenvalue problems of vibrating membranes are studied by Carleman's methods [1]. By the help of the results of part I certain theorems on the asymptotic behaviour of the eigenvalues and eigenfunctions are obtained.

In order to simplify the exposition, only membranes with infinitely differentiable boundaries are being considered.

## Part I. Estimates for the Green's functions.

**1.** The equation (1), viz. $\Delta u - \varkappa^2 u = 0$, is considered in an open, bounded and simply connected domain $V$ of the cartesian $x^1 x^2$-plane. The boundary $S$ of $V$ is given by equations $x^i = y^i(s)$, $i = 1, 2$, in which $s$ is the arc-length of the boundary, measured in the counter-clock-wise sense, and $y^i(s)$ are infinitely differentiable functions. $S$ also denotes the total length of the boundary. The distance from a point $x = (x^1, x^2)$ to $S$ is $n$, this distance being positive when $x$ belongs to $V$. The letter $n$ also denotes the inner normal of $S$; $n_s = n_y$ is the normal at the point $y(s) = (y^1(s), y^2(s))$.

The equation (1) has the elementary solution $\frac{1}{2\pi} K_0(\varkappa r)$, where $K_0$ is the Bessel $K_0$-function and $r = r_{x_1 x_2}$ is the distance between $x_1 = (x_1^1, x_1^2)$ and $x_2 = (x_2^1, x_2^2)$. We assume $\varkappa > 0$.

Let

$$G(x_1, x_2; -\varkappa^2) = \frac{1}{2\pi} K_0(\varkappa r) - \gamma(x_1, x_2; \varkappa)$$

denote any of the two Green's functions of (1) which satisfy the boundary conditions

$$G = 0 \text{ when } x_1 \in S \text{ (Dirichlet's condition),}$$

or

$$\frac{\partial G}{\partial n} = 0 \text{ when } x_1 \in S \text{ (Neumann's condition).}$$

The compensating (or regular) parts $u(x) = \gamma(x, x_2; \varkappa)$ of these functions satisfy (1) and fulfil the inhomogeneous conditions

$$u(y) = \frac{1}{2\pi} K_0(\varkappa r_{y x_2}) \text{ when } y \in S, \tag{2}$$

$$\frac{\partial u}{\partial n} = \frac{1}{2\pi} \frac{\partial}{\partial n} K_0(\varkappa r_{y x_2}) \text{ when } y \in S. \tag{3}$$

The functions $u(x) = \gamma(x, x_2; \varkappa)$ are constructed in a well-known way by assuming in the different cases

$$u(x) = \frac{1}{\pi} \int_S \frac{\partial}{\partial n_s} K_0(\varkappa r_{xs}) \varphi(s)\, ds, \tag{4}$$

and

$$u(x) = \frac{1}{\pi} \int_S K_0(\varkappa r_{xs}) \varphi(s)\, ds, \tag{5}$$

where $r_{xs} = r_{xy}$ is the distance from $x$ to $y = y(s)$. The conditions that (4), (5) satisfy (2) and (3) lead to integral equations of Fredholm's type for the functions $\varphi(s)$. When the unique solutions of these equations are introduced in (4), (5) we get formulæ for the $\gamma(x_1, x_2; \varkappa)$ suitable for our study of these functions.

If $\gamma(x_1, x_2; \varkappa)$ is the compensating part of the Green's function of Dirichlet's condition, then

$$\begin{aligned} \gamma(x_1, x_2; \varkappa) = {} & \frac{1}{2\pi^2} \int_S \frac{\partial}{\partial n_s} K_0(\varkappa r_{x_1 s}) \cdot K_0(\varkappa r_{x_2 s})\, ds \\ & - \frac{1}{2\pi^2} \int_S \int_S \frac{\partial}{\partial n_s} K_0(\varkappa r_{x_1 s}) \cdot L(s, s') \cdot K_0(\varkappa r_{x_2 s'})\, ds\, ds', \end{aligned} \tag{6}$$

where $L(s, s')$ is defined by the relations

$$K(s, s') = \frac{1}{\pi}\frac{\partial}{\partial n_{s'}} K_0(\varkappa r_{ss'}),$$

$$K^{(n)}(s, s') = \int_S \cdots \int_S K(s, s_1) K(s_1, s_2) \cdots K(s_n, s')\, ds_1 \cdots ds_n,$$

$$L(s, s') = K(s, s') - K^{(1)}(s, s') + K^{(2)}(s, s') - \cdots,$$

$r_{ss'}$ being the distance between $y(s)$ and $y(s')$.

Similarly, for the compensating part of the Green's function of Neumann's condition,

$$(7) \quad \gamma(x_1, x_2; \varkappa) = -\frac{1}{2\pi^2}\int_S K_0(\varkappa r_{x_1 s}) \cdot \frac{\partial}{\partial n_s} K_0(\varkappa r_{x_2 s})\, ds$$

$$-\frac{1}{2\pi^2}\int_S\int_S K_0(\varkappa r_{x_1 s}) \cdot M(s, s') \cdot \frac{\partial}{\partial n_{s'}} K_0(\varkappa r_{x_2 s'})\, ds\, ds'$$

where

$$M(s, s') = K(s', s) + K^{(1)}(s', s) + K^{(2)}(s', s) + \cdots.$$

On the basis of (6), (7) the functions $\gamma(x_1, x_2; \varkappa)$ will be estimated for large values of $\varkappa$. The case when $x_1$ and $x_2$ lie in the neighbourhood of the same boundary point is particularly interesting. We first examine the properties of

$$K_0(\varkappa r_{xs}), \quad \frac{\partial}{\partial n_s} K_0(\varkappa r_{xs}) \quad \text{and} \quad K(s, s')$$

when the distances between $x$ and $y(s)$ and between $y(s)$ and $y(s')$ are small.

**2. Coordinates in the neighbourhood of the boundary.** Let $h > 0$ be sufficiently small. Then $\xi^1 = s$ and $\xi^2 = n$ can be taken as new coordinates in the strip along $S$ where $0 \le n \le h$. If $\xi^1$ and $\xi^2$ are plotted in a cartesian $\xi^1\xi^2$-plane, the image of the strip $0 \le n \le h$ will be a strip along the $\xi^1$-axis in which $0 \le \xi^2 \le h$. We denote by $I$ a closed interval of the $\xi^1$-axis and by $C(I)$ a rectangular (closed) domain $\xi^1 \in I$, $0 \le \xi^2 \le \delta$, where $\delta < h$ and the length of $I$ is less than $S$. We also denote the inverse images of $I$ and of $C(I)$ in the $x^1x^2$-plane by the same symbols $I$ and $C(I)$.

In what follows, the letter $y$ is used for points $(y^1, y^2)$ of the boundary, and the images of these points on the $\xi^1$-axis are written $\eta = (\eta^1, 0)$. The distance between two points $\xi_1, \xi_2$ in the $\xi^1\xi^2$-plane is

$$\varrho_{\xi_1\xi_2} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 - \xi_2^2)^2}.$$

Let $x$ be a point in a domain $C(I)$ and let $y \in I$. The images of $x$ and $y$ in the $\xi^1\xi^2$-plane are $\xi$ and $\eta$. If $r = r_{xy}$ and $\varrho = \varrho_{\xi\eta}$, the relation $r = \sqrt{\varrho^2 + \Phi(\xi, \eta)}$ holds true with $\Phi(\xi, \eta) = O(\varrho^3)$ when $\varrho$ tends to zero. More precisely, if $c(\xi^1)$ is the curvature of $S$ in $y = y(\xi^1)$,

$$\Phi(\xi, \eta) = -c(\xi^1)(\xi^1-\eta^1)^2\xi^2 - \frac{1}{12}(c(\xi^1))^2(\xi^1-\eta^1)^4$$
$$+\frac{1}{3}c'(\xi^1)(\xi^1-\eta^1)^3\xi^2 + O(\varrho^5).$$

If $n$ is the inner normal of $S$ in $y(\eta^1)$, then

$$\frac{1}{2}\frac{\partial r^2}{\partial n} = -\xi^2 - \Psi(\xi, \eta),$$

where for $\varrho$ tending to zero, $\Psi(\xi, \eta) = O(\varrho^2)$ or

$$\Psi(\xi, \eta) = \frac{1}{2}c(\xi^1)(\xi^1-\eta^1)^2 - \frac{1}{2}(c(\xi^1))^2(\xi^1-\eta^1)^2\xi^2$$
$$-\frac{1}{3}c'(\xi^1)(\xi^1-\eta^1)^3 + O(\varrho^4).$$

On account of the regularity of the boundary, the functions $\Phi(\xi, \eta)$ and $\Psi(\xi, \eta)$ are infinitely differentiable with respect to $\xi^1$, $\xi^2$ and $\eta^1$.

We finally note the identity

$$\frac{\partial(x^1, x^2)}{\partial(\xi^1, \xi^2)} = 1 - c(\xi^1)\xi^2. \tag{8}$$

**3. Local expansions.** Let $x$ and $y$ be points in $C(I)$ and $I$ having the images $\xi$ and $\eta$ in the $\xi^1\xi^2$-plane. We consider, with $r = r_{xy}$ and $\varrho = \varrho_{\xi\eta}$, the function $K_0(\varkappa r) = K_0(x\sqrt{\varrho^2 + \Phi})$. According to Taylor's theorem this function is written

$$K_0(\varkappa r) = \sum_{\nu=0}^{N} \frac{(\Phi)^\nu}{2^\nu \nu!}\left(\frac{1}{\varrho}\frac{d}{d\varrho}\right)^\nu K_0(\varkappa\varrho) + \text{remainder}.$$

If $(\Phi(\xi, \eta))^\nu$, $\nu = 1, 2, \ldots N$, are expanded in finite Taylor series of powers of $\xi^1 - \eta^1$ and of $\xi^2$ and with coefficients depending on $\xi^1$, the function $K_0(\varkappa r)$ assumes the form

$$K_0(\varkappa r) = \sum{}^* f(\xi^1)(\xi^1-\eta^1)^\alpha(\xi^2)^\beta\left(\frac{1}{\varrho}\frac{d}{d\varrho}\right)^\nu K_0(\varkappa\varrho) + R^\Lambda(\xi, \eta, \varkappa)\cdot \tag{9}$$

Here $\sum^*$ denotes a sum of a finite number of terms, in which $f(\xi^1)$ are infinitely differentiable functions and $\alpha, \beta, \nu$ are non-negative integers. Since $\Phi(\xi, \eta) = O(\varrho^3)$, the inequality $\alpha + \beta \geq 3\nu$ holds true.

If $C(I)$ is sufficiently small it is easily seen that for any given positive integer $\Lambda$, the function $K_0(\varkappa r)$ has a local development (9) in which the remainder has continuous derivatives of orders $k \leq \Lambda$ satisfying the relations

$$D^k R^\Lambda(\xi, \eta, \varkappa) = O(\varkappa^{-\Lambda} e^{-A\varkappa\varrho}),\ A = \text{constant} > 0,$$

when $\varkappa$ tends to $+\infty$ and $\xi \in C(I)$, $\eta \in I$.

By definition, the integers $\alpha+\beta-2\nu$ are called the *degrees* of the terms in (9). Since $\alpha+\beta\geq 3\nu$, these degrees are non-negative. We also observe that $K_0(\varkappa\varrho)$ is the term in the expansion which has degree zero.

Similarly

$$(10)\qquad \frac{\partial}{\partial n}K_0(\varkappa r)=\sum^{*} f(\xi^1)(\xi^1-\eta^1)^{\alpha}(\xi^2)^{\beta}\left(\frac{1}{\varrho}\frac{d}{d\varrho}\right)^{\nu}K_0(\varkappa\varrho)+R^{A}(\xi,\eta,\varkappa),$$

where the remainder has the same properties as in (9). In (10) the integers $\alpha$, $\beta$, $\nu$ fulfil the inequalities $\alpha+\beta\geq 3\nu-2$, $\nu\geq 1$. Thus the degrees of the terms in (10) are $\geq -1$; the term of degree $-1$ is $-\dfrac{\partial}{\partial\xi^2}K_0(\varkappa\varrho)$.

4. **Transformation of the local expansions.** We define the inverse differential operator $\left(\dfrac{\partial}{\partial v}\right)^{-1}$ by the equation

$$(11)\qquad \left(\frac{\partial}{\partial v}\right)^{-1}F(v)=\int_{+\infty}^{v}F(t)\,dt.$$

With the factor $f(\xi^1)$ omitted, let

$$(12)\qquad (\xi^1-\eta^1)^{\alpha}(\xi^2)^{\beta}\left(\frac{1}{\varrho}\frac{d}{d\varrho}\right)^{\nu}K_0(\varkappa\varrho)$$

be a term in (9) or (10). By simple calculations it follows that (12) is a linear combination of a finite number of functions

$$(13)\qquad (\xi^2)^{p}\left(\frac{\partial}{\partial\xi^1}\right)^{l}\left(\frac{\partial}{\partial\xi^2}\right)^{m}K_0(\varkappa\varrho),$$

where $p$, $l$ and $m$ are integers, $p\geq 0$, $l\geq 0$. The degree of the expression (13), viz. $p-l-m$, equals the degree $\alpha+\beta-2\nu$ of (12).

To abbreviate, we introduce the symbol $\partial^{q}$ for the differential operator in (13), whereupon this function takes the form $(\xi^2)^{p}\partial^{q}K_0(\varkappa\varrho)$ with $q=l+m$. (The symbol $D^{q}$ is reserved for monomials containing only non-negative powers of simple differential operators.)

From now on the expansions (9) and (10) are written

$$(14)\qquad K_0(\varkappa r_{xs})=\sum^{*} f(\xi^1)(\xi^2)^{p}\partial^{q}K_0(\varkappa\varrho)+R^{A}(\xi,\eta,\varkappa),$$

and

$$(15)\qquad \frac{\partial}{\partial n_s}K_0(\varkappa r_{xs})=\sum^{*} f(\xi^1)(\xi^2)^{p}\partial^{q}K_0(\varkappa\varrho)+R^{A}(\xi,\eta,\varkappa).$$

In (14), $\min(p-q)=0$ and in (15) this value equals $-1$. The terms of minimum degree are $K_0(\varkappa\varrho)$ and $-\dfrac{\partial}{\partial\xi^2}K_0(\varkappa\varrho)$ respectively.

**5. $e^\lambda$-functions.** Let $\xi_1, \xi_2$ be points in the half-plane $\xi^2 > 0$, and let $\varrho_1$ and $\varrho_2$ be their distances from a variable point $\eta$ of the $\xi^1$-axis. The minimum of the sum $\varrho_1 + \varrho_2$ is

$$\hat{\varrho}_{12} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 + \xi_2^2)^2}.$$

By $\partial_1^q$ and $\partial_2^q$ we understand operators of the type $\partial^q = \left(\dfrac{\partial}{\partial \xi^1}\right)^l \left(\dfrac{\partial}{\partial \xi^2}\right)^m$, $l \geq 0$, but which here contain differential operators with respect to $\xi_1$ and $\xi_2$ instead of $\xi$.

We consider certain functions $e^\lambda(\xi_1, \xi_2, \varkappa)$ which occur when the expressions (14), (15) for $K_0(\varkappa r_{xs})$ and for $\dfrac{\partial}{\partial n_s} K_0(\varkappa r_{xs})$ are inserted in (6) and (7).

*An $e^\lambda(\xi_1, \xi_2, \varkappa)$-function is defined for points $\xi_1, \xi_2$ belonging to a sufficiently small domain $C(I)$, except when $\xi_1 = \xi_2 \in I$. It has the following properties. For every positive integer $\Lambda$ it has an expansion*

$$e^\lambda(\xi_1, \xi_2, \varkappa) = \sum{}^* f(\xi_1^1)(\xi_1^2)^{p_1}(\xi_2^2)^{p_2} \partial_1^q K_0(\varkappa \hat{\varrho}_{12}) + R^\Lambda(\xi_1, \xi_2, \varkappa) \tag{16}$$

*in which the functions $f(\xi_1^1)$ are infinitely differentiable, the integers $p_1$, $p_2$ are non-negative and $\lambda$ is the minimum of the integers $p_1 + p_2 - q$. The remainder has continuous derivatives $D^k R^\Lambda$ of orders $k \leq \Lambda$ with respect to $\xi_1$ and $\xi_2$. For the remainder and its derivatives relations*

$$D^k R^\Lambda = O(\varkappa^{-\Lambda} e^{-A \varkappa \hat{\varrho}_{12}}), \quad A = \text{constant} > 0, \tag{17}$$

*hold true when $\varkappa$ tends to infinity. We may suppose that $\lambda$ cannot be increased by transforming the right hand side of the expansion; its value is then called the degree of the $e^\lambda$-function.*

It is easy to see, that a function which for every $\Lambda$ has an expansion

$$\sum{}^* f(\xi_1^1, \xi_2^1)(\xi_1^2)^{p_1}(\xi_2^2)^{p_2} \partial_1^q K_0(\varkappa \hat{\varrho}_{12}) + R^\Lambda(\xi_1, \xi_2, \varkappa)$$

with infinitely differentiable coefficients $f(\xi_1^1, \xi_2^1)$ is an $e^\lambda(\xi_1, \xi_2, \varkappa)$-function. The proof follows by expanding the functions $f(\xi_1^1, \xi_2^1)$ in Taylor series. It is similarly seen that $F(\xi_1, \xi_2) e^\lambda(\xi_1, \xi_2, \varkappa)$ is an $e^\mu(\xi_1, \xi_2, \varkappa)$-function with $\mu \geq \lambda$, provided that $F(\xi_1, \xi_2)$ is infinitely differentiable. If $F = \xi_1^1 - \xi_2^1$ or $F = \xi_1^2$ or $F = \xi_2^2$, the value of $\mu$ equals $\lambda + 1$. Clearly a derivative of an $e^\lambda(\xi_1, \xi_2, \varkappa)$-function with respect to $\xi_1^1$, $\xi_1^2$, $\xi_2^1$ or $\xi_2^2$ is an $e^{\lambda-1}(\xi_1, \xi_2, \varkappa)$-function.

*Functions $e^\lambda(\xi, \eta, \varkappa)$ and $e^\lambda(\eta_1, \eta_2, \varkappa)$ are defined in a similar way as the $e^\lambda(\xi_1, \xi_2, \varkappa)$-functions when $\xi \in C(I)$, $\eta \in I$, $\xi \neq \eta$ and when $\eta_1, \eta_2 \in I$, $\eta_1 \neq \eta_2$. Thus, these functions have expansions*

$$e^\lambda(\xi, \eta, \varkappa) = \sum{}^* f(\xi^1)(\xi^2)^p \partial^q K_0(\varkappa \varrho_{\xi\eta}) + R^\Lambda(\xi, \eta, \varkappa), \tag{18}$$

$$e^\lambda(\eta_1, \eta_2, \varkappa) = \sum{}^* f(\eta_1^1)[\partial^q K_0(\varkappa \varrho_{\xi\eta_2})]_{\xi \to \eta_1} + R^\Lambda(\eta_1, \eta_2, \varkappa), \tag{19}$$

*in which $f(\xi_1)$, $f(\eta_1^1)$ are infinitely differentiable and the remainders satisfy* (17) *with $\hat{\varrho}_{12}$ replaced by $\varrho_{\xi\eta}$ and $\varrho_{\eta_1\eta_2}$ respectively. In* (18) *$\lambda$ is the minimum of*

*$p-q$ and in* (19) *it equals the minimum of $-q$. It is assumed that $\lambda$ cannot be replaced by a larger value in the expansions* (18), (19).

The properties of the $e^\lambda(\xi, \eta, \varkappa)$- and $e^\lambda(\eta_1, \eta_2, \varkappa)$-functions are essentially similar to those of the functions $e^\lambda(\xi_1, \xi_2, \varkappa)$ and need not be listed. We observe that

$$\lim_{\xi_2\to\eta_2} e^\lambda(\xi_1, \xi_2, \varkappa), \quad \lim_{\xi_1\to\eta_1,\, \xi_2\to\eta_2} e^\lambda(\xi_1, \xi_2, \varkappa)$$

are $e^\mu(\xi_1, \eta_2, \varkappa)$- and $e^\mu(\eta_1, \eta_2, \varkappa)$-functions with $\mu \geq \lambda$.

The formulæ (14), (15) show, that with respect to a sufficiently small domain $C(I)$ the functions $K_0(\varkappa r)$ and $\dfrac{\partial}{\partial n} K_0(\varkappa r)$ are $e^\lambda$-functions. Thus,

$$K_0(\varkappa r_{xy}) = e^0(\xi, \eta, \varkappa), \quad \frac{\partial}{\partial n_y} K_0(\varkappa r_{xy}) = e^{-1}(\xi, \eta, \varkappa)$$

when $x \in C(I)$, $y \in I$ and $x \neq y$. The only term of degree $-1$ in the $e^{-1}$-expansion of $\dfrac{\partial}{\partial n_y} K_0(\varkappa r_{xy})$ is $-\dfrac{\xi^2}{\varrho_{\xi\eta}} K_0'(\varkappa \varrho_{\xi\eta})$, which is zero when $\xi^2 = 0$. It follows that $K(s, s') = \dfrac{1}{\pi} \dfrac{\partial}{\partial n_{s'}} K_0(\varkappa r_{ss'})$ is locally an $e^0(\eta_1, \eta_2, \varkappa)$-function if we put $s = \eta_1^1$, $s' = \eta_2^1$.

**6. A theorem on $e^\lambda$-functions.** *If the functions $e^{\lambda_1}(\xi_1, \eta, \varkappa)$ and $e^{\lambda_2}(\xi_2\, \eta, \varkappa)$ are defined in $C(I_1)$ and $C(I_2)$ and if $I$ is an interior part of $I_1 \cap I_2$, then the integral*

$$\int_{I_1 \cap I_2} e^{\lambda_1}(\xi_1, \eta, \varkappa)\, e^{\lambda_2}(\xi_2, \eta, \varkappa)\, d\eta^1 = e^{\lambda_1+\lambda_2+1}(\xi_1, \xi_2, \varkappa) \tag{20}$$

*is an $e^{\lambda_1+\lambda_2+1}(\xi_1, \xi_2, \varkappa)$-function in every domain $C(I) \subset C(I_1) \cap C(I_2)$. This is true, provided that when one or both points $\xi_1, \xi_2$ belong to $I$, the value of the integral is defined as its limit value when $\xi_1$ and $\xi_2$ approach $I$ from the interior of $C(I)$.*

The proof of this theorem depends on the study of the expressions

$$\int_{I_1 \cap I_2} R^{A_1}(\xi_1, \eta, \varkappa)\, R^{A_2}(\xi_2, \eta, \varkappa)\, d\eta^1, \tag{21}$$

$$(\xi_1^2)^{p_1} \int_{I_1 \cap I_2} \partial_1^{q_1} K_0(\varkappa \varrho_1)\, R^{A_2}(\xi_2, \eta, \varkappa)\, d\eta^1, \tag{22}$$

$$(\xi_2^2)^{p_2} \int_{I_1 \cap I_2} R^{A_1}(\xi_1, \eta, \varkappa)\, \partial_2^{q_2} K_0(\varkappa \varrho_2)\, d\eta^1, \tag{23}$$

$$(\xi_1^2)^{p_1} (\xi_2^2)^{p_2} \int_{I_1 \cap I_2} \partial_1^{q_1} K_0(\varkappa \varrho_1)\, \partial_2^{q_2} K_0(\varkappa \varrho_2)\, d\eta^1, \tag{24}$$

where $\varrho_1 = \varrho_{\xi_1 \eta}$, $\varrho_2 = \varrho_{\xi_2 \eta}$ and where the integers $p_1, p_2, q_1, q_2$ satisfy the inequalities $p_1 \geq 0$, $p_2 \geq 0$, $p_1 - q_1 \geq \lambda_1$, $p_2 - q_2 \geq \lambda_2$.

Let $A$ be an arbitrary positive integer. Then if $A_1$ and $A_2$ are taken sufficiently large, it is clear that (21) has the same properties as the remainder in (16). The same is true also for (22) and (23). This is easily seen when $p_1 - q_1$ and $p_2 - q_2$ are large. In other cases it follows after partial integrations, if one observes that, when applied to $K_0(\varkappa \varrho_{\xi\eta})$, the operators $\frac{\partial}{\partial \xi_1}$, $\left(\frac{\partial}{\partial \xi^2}\right)^2$ may be replaced by $-\frac{\partial}{\partial \eta^1}$ and $\varkappa^2 - \left(\frac{\partial}{\partial \eta^1}\right)^2$. Surpassing the details of this discussion we examine functions of the type (24), which give the principal part of the expansion of $e^{\lambda_1+\lambda_2+1}(\xi_1, \xi_2, \varkappa)$. To do this we first deduce an integral theorem.

**7. An integral theorem.** Let $\xi_1, \xi_2$ be points in $\xi^2 > 0$ and put $\varrho_1 = \varrho_{\xi_1 \xi}$, $\varrho_2 = \varrho_{\xi_2 \xi}$, $\hat{\varrho}_2 = \varrho_{\hat{\xi}_2 \xi}$, where $\hat{\xi}_2 = (\xi_2^1, -\xi_2^2)$. For $\xi = \xi_1$ we have $\hat{\varrho}_2 = \hat{\varrho}_{12}$ (see section 5).

By Green's theorem for the upper half-plane one obtains the formulæ

$$(25) \qquad \int_{(\xi^2=0)} \left[K_0(\varkappa \varrho_1) \cdot \frac{\partial}{\partial n} K_0(\varkappa \varrho_2) - K_0(\varkappa \varrho_2) \cdot \frac{\partial}{\partial n} K_0(\varkappa \varrho_1)\right] d\eta^1 = 0,$$

$$(26) \qquad \int_{(\xi^2=0)} \left[K_0(\varkappa \varrho_1) \cdot \frac{\partial}{\partial n} K_0(\varkappa \hat{\varrho}_2) - K_0(\varkappa \hat{\varrho}_2) \cdot \frac{\partial}{\partial n} K_0(\varkappa \varrho_1)\right] d\eta^1 + 2\pi K_0(\varkappa \hat{\varrho}_{12}) = 0;$$

$\Big($since $\xi$ is here a point on the $\xi^1$-axis it is replaced by $\eta = (\eta^1, 0)$ in accordance with our earlier conventions; $n$ is the normal of the $\xi^1$-axis so that $\frac{\partial}{\partial n} = \frac{\partial}{\partial \xi^2}\Big)$.

For $\xi = \eta$ (point on the $\xi^1$-axis) it is clear that $K_0(\varkappa \hat{\varrho}_2) = K_0(\varkappa \varrho_2)$ and that $\frac{\partial}{\partial n} K_0(\varkappa \hat{\varrho}_2) = -\frac{\partial}{\partial n} K_0(\varkappa \varrho_2) = \frac{\partial}{\partial \xi_2^2} K_0(\varkappa \varrho_2)$. Hence, from (25) and (26) it follows that

$$\int_{(\xi^2=0)} K_0(\varkappa \varrho_1) \cdot \frac{\partial}{\partial \xi_2^2} K_0(\varkappa \varrho_2)\, d\eta^1 = -\pi K_0(\varkappa \hat{\varrho}_{12}).$$

Let $I$ and $I'$ be closed intervals of the $\xi^1$-axis of which $I$ is an interior subset of $I'$. According to the last equation

$$(27) \qquad \int_{I'} K_0(\varkappa \varrho_1) \cdot \frac{\partial}{\partial \xi_2^2} K_0(\varkappa \varrho_2)\, d\eta^1 = -\pi K_0(\varkappa \hat{\varrho}_{12}) + R(\xi_1, \xi_2, \varkappa).$$

It is easy to verify that $R(\xi_1, \xi_2, \varkappa)$ has continuous derivatives of all orders with respect to $\xi_1$ and $\xi_2$ when $\xi_1^1, \xi_2^1 \in I$ and that, with positive constants $c$

the relations

$$D^k R(\xi_1, \xi_2, \varkappa) = O(e^{-c\varkappa}) \tag{28}$$

are fulfilled when $\varkappa$ tends to infinity.

**8. The expression (24).** Let $I$ be an interior subset of $I' = I_1 \cap I_2$. The result of applying the operator $\partial_1^{q_1} \partial_2^{q_2} \left(\frac{\partial}{\partial \xi_2^2}\right)^{-1}$ to formula (27) is

$$\int\limits_{I_1 \cap I_2} \partial_1^{q_1} K_0(\varkappa \varrho_1) \cdot \partial_2^{q_2} K_0(\varkappa \varrho_1)\, d\eta^1 = -\pi \partial_1^{q_1} \partial_2^{q_2} \left(\frac{\partial}{\partial \xi_2^2}\right)^{-1} K_0(\varkappa \hat{\varrho}_{12}) + \partial_1^{q_1} \partial_2^{q_2} \left(\frac{\partial}{\partial \xi_2^2}\right)^{-1} R(\xi_1, \xi_2, \varkappa). \tag{29}$$

For a function $F(t)$ which tends exponentially to zero when $t$ tends to $+\infty$

$$\left(\frac{\partial}{\partial \xi_1^2}\right)^{-1} F(\xi_1^2 + \xi_2^2) = \left(\frac{\partial}{\partial \xi_2^2}\right)^{-1} F(\xi_1^2 + \xi_2^2). \tag{30}$$

Thus, in the first term of the right hand side of (29), the operator $\partial_1^{q_1} \partial_2^{q_2} \left(\frac{\partial}{\partial \xi_2^2}\right)^{-1}$ may be replaced by a similar monomial containing exclusively differentiations with respect to $\xi_1$. According to (28) the last term in (29) and its derivatives are of orders $O(e^{-c\varkappa})$ when $\xi_1, \xi_2 \in C(I)$ and $\varkappa$ tends to infinity.

Our study of the expression (24) completes the proof of the theorem which was announced in section 6.

**9.** We alse consider integrals of the type

$$\int\limits_{I_1 \cap I_2} e^{\lambda_1}(\xi_1, \eta, \varkappa)\, e^{\lambda_2}(\eta_2, \eta, \varkappa)\, d\eta^1 \tag{31}$$

where $e^{\lambda_2}(\eta_2, \eta, \varkappa)$ is defined when $\eta_2, \eta \in I_2$. We suppose that $C(I)$ is part of $C(I_1)$ and that $I$ is contained in the interior of $I_1 \cap I_2$.

The examination of (31) depends on the study of expressions similar to (21), (22), (23), (24) but with $p_2 = 0$ and with $\partial_2^{q_2} K_0(\varkappa \varrho_2)$ and $R^{A_2}(\xi_2, \eta, \varkappa)$ replaced by $\lim \partial_2^{q_2} K_0(\varkappa \varrho_2)$ and $R^{A_2}(\eta_2, \eta, \varkappa)$. The investigation of the expressions corresponding to (21), (22) is done in the way indicated in section 6. If $\lambda_2 \geq 0$, i. e. if $q_2 \geq 0$, the limit values of the integrals in (23), (24) when $\xi_2$ tends to $\eta_2$ are obtained by performing the transition to the limit under the integral signs. Therefore, if $\lambda_2 \geq 0$, the results obtained for (23), (24) are immediately transferred into similar results for the corresponding expressions occurring in the study of (31). *It follows that, if $\lambda_2 \geq 0$, the integral*

$$\int\limits_{I_1 \cap I_2} e^{\lambda_1}(\xi_1, \eta, \varkappa)\, e^{\lambda_2}(\eta_2\, \eta, \varkappa)\, d\eta^1 = e^{\lambda_1 + \lambda_2 + 1}(\xi_1, \eta_2, \varkappa) \tag{32}$$

*is an* $e^{\lambda_1+\lambda_2+1}(\xi_1, \eta_2, \varkappa)$*-function when* $\xi_1 \in C(I)$, $\eta \in I$. *When* $\xi_1 \in I$, $\xi_1 \neq \eta_2$ *the value of* (32) *is defined as the limit of the integral when* $\xi_1^2$ *tends to zero.*

*Similarly, when* $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ *and when* $I$ *is an interior part of* $I_1 \cap I_2$ *the integral*

$$\int_{I_1 \cap I_2} e^{\lambda_1}(\eta_1, \eta, \varkappa)\, e^{\lambda_2}(\eta_2\, \eta, \varkappa)\, d\eta^1 = e^{\lambda_1+\lambda_2+1}(\eta_1, \eta_2, \varkappa) \tag{33}$$

*is an* $e^{\lambda_1+\lambda_2+1}(\eta_1, \eta_2, \varkappa)$*-function, defined when* $\eta_1, \eta_2 \in I$, $\eta_1 \neq \eta_2$.

**10. $E^\lambda$-functions.** Let $x_1$, $x_2$ be points in $V+S$ and let $\hat{r}_{12}$ be the minimum of the sum $r_{x_1 y} + r_{x_2 y}$ when $y$ varies in $S$.

Functions defined in the large and having the local properties of $e^\lambda$-functions are called $E^\lambda$-functions. More precisely: *An* $E^\lambda(x_1, x_2, \varkappa)$*-function is defined and infinitely differentiable with respect to* $x_1$ *and* $x_2$, *when these points belong to* $V+S$ *except when* $x_1 = x_2 \in S$. *If* $D^k E^\lambda$ *is a derivative of order* $k$ $(D^0 E^\lambda = E^\lambda)$, *and if* $\delta$ *is an arbitrary positive number, a relation* $D^k E^\lambda = O(e^{-c\varkappa})$, *where* $c = c(\delta)$ *is a positive constant, holds true for* $\hat{r}_{12} \geq \delta$ *when* $\varkappa$ *tends to infinity. With respect to every sufficiently small domain* $C(I)$ *the function is an* $e^\mu(\xi_1, \xi_2, \varkappa)$*-function with* $\mu \geq \lambda$.

*Similarly* $E^\lambda(x_1, y_2, \varkappa)$*- and* $E^\lambda(y_1, y_2, \varkappa)$*-functions are defined when* $x_1 \in V+S$, $y_2 \in S$, $x_1 \neq y_2$ *and when* $y_1, y_2 \in S$, $y_1 \neq y_2$. *They are locally equal to* $e^\mu(\xi_1, \xi_2, \varkappa)$*- and* $e^\mu(\eta_1, \eta_2, \varkappa)$*-functions with* $\mu \geq \lambda$. *When* $r_{x_1 y_2} \geq \delta > 0$ *and when* $r_{y_1 y_2} \geq \delta > 0$ *the functions* $E^\lambda(x_1, y_2, \varkappa)$ *and* $E^\lambda(y_1, y_2, \varkappa)$ *and their derivatives tend exponentially to zero in the same way as the* $E^\lambda(x_1, x_2, \varkappa)$*-functions and their derivatives.*

*In an* $E^\lambda$*-function it is assumed that* $\lambda$ *cannot be replaced by a larger value;* $\lambda$ *is called the degree of the function.*

We observe that the functions of section 1, viz. $K_0(\varkappa r_{xy})$, $\frac{\partial}{\partial n} K_0(\varkappa r_{xy})$ and $K(s, s')$ are $E^0(x, y, \varkappa)$-, $E^{-1}(x, y, \varkappa)$- and $E^0(y_1, y_2, \varkappa)$-functions $(y_1 = y(s)$, $y_2 = y(s'))$.

**11. Integral relations for $E^\lambda$-functions.** From (20), (32), (33) one easily deduces analogous properties of the $E^\lambda$-functions. Thus, the integral

$$\int_S E^{\lambda_1}(x_1, y(s), \varkappa)\, E^{\lambda_2}(x_2, y(s), \varkappa)\, ds = E^{\lambda_1+\lambda_2+1}(x_1, x_2, \varkappa) \tag{34}$$

is an $E^{\lambda_1+\lambda_2+1}(x_1, x_2, \varkappa)$-function, provided that when one or both of the points $x_1$, $x_2$ belong to $S$, the value of the integral be defined as a limit value. Similarly, if $\lambda_2 \geq 0$

$$\int_S E^{\lambda_1}(x_1, y(s), \varkappa)\, E^{\lambda_2}(y_2, y(s), \varkappa)\, ds = E^{\lambda_1+\lambda_2+1}(x_1, y_2, \varkappa), \tag{35}$$

and if $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, the relation

$$\int_S E^{\lambda_1}(y_1, y(s), \varkappa)\, E^{\lambda_2}(y_2, y(s), \varkappa)\, ds = E^{\lambda_1+\lambda_2+1}(y_1, y_2, \varkappa) \tag{36}$$

holds true.

**12. Estimates for $E^\lambda$-functions.** We insert a remark on the asymptotic behaviour of the functions (we make a momentary exception from our conventions regarding the notation) $\left(\frac{\partial}{\partial x}\right)^l \left(\frac{\partial}{\partial y}\right)^m K_0(\varkappa r)$ where $l$, $m$ are integers, $l \geq 0$, $l+m=q$. The operator $\left(\frac{\partial}{\partial y}\right)^{-1}$ is defined according to (11) and $r=\sqrt{x^2+y^2}$. Then, with $A=$ constant $>0$, the relations

$$\left(\frac{\partial}{\partial x}\right)^l \left(\frac{\partial}{\partial y}\right)^m K_0(\varkappa r) = \begin{cases} O(\varkappa^q e^{-A\varkappa r}) \text{ when } q<0, \\ O([1+|\log \varkappa r|] e^{-A\varkappa r}) \text{ when } q=0, \\ O(r^{-q} e^{-A\varkappa r}) \text{ when } q>0, \end{cases} \tag{37}$$

are valid.

Since it is readily seen that $\lim \frac{\hat{r}_{12}}{\hat{\varrho}_{12}}=1$ when $\hat{r}_{12}$ (or $\hat{\varrho}_{12}$) tends to zero, it follows from (37) that

$$E^\lambda(x_1, x_2, \varkappa) = \begin{cases} O(\hat{r}_{12}^{\lambda} e^{-A\varkappa \hat{r}_{12}}) \text{ when } \lambda<0, \\ O([1+|\log \varkappa \hat{r}_{12}|] e^{-A\varkappa \hat{r}_{12}} \text{ when } \lambda=0, \\ O(\varkappa^{-\lambda} e^{-A\varkappa \hat{r}_{12}}) \text{ when } \lambda>0. \end{cases}$$

Similar relations hold true for $E^\lambda(x_1, y_2, \varkappa)$- and $E^\lambda(y_1, y_2, \varkappa)$-functions.

A derivative of an $E^\lambda$-function is an $E^{\lambda-1}$-function. Hence, it is superfluous to assign special estimates for the derivatives of $E^\lambda$-functions.

**13. Consequences for Green's functions.** In the case of Dirichlet's boundary condition the compensating part of Green's function is given by the formula (6). Since $\frac{\partial}{\partial n_s} K_0(\varkappa r_{x_1 y})$ and $K_0(\varkappa r_{x_1 s})$ are $E^\lambda$-functions with $\lambda=-1$ and $\lambda=0$, it follows from (34) that the first integral in (6) represents an $E^0(x_1, x_2, \varkappa)$-function.

The kernel $K(s, s')$ is an $E^0(y_1, y_2, \varkappa)$-function if we put $y_1=y(s)$, $y_2=y(s')$. Therefore, according to (36) the iterated kernel $K^{(n)}(s, s')$ is an $E^n(y_1, y_2, \varkappa)$-function. Thus, for $n>k$ and with $r=r_{y_1 y_2}$, we obtain the inequalities

$$|D^k K^{(n)}(s, s')| \leq \text{constant } \varkappa^{-n+k} e^{-A\varkappa r}.$$

By the help of these inequalities it is readily seen that

$$L_N(s, s') = \sum_{\nu=0}^{\infty} (-1)^\nu K^{(\nu N)}(s, s')$$

is an $E^N(y_1, y_2, \varkappa)$-function. Hence,

$$L(s, s') = \sum_{\nu=0}^{N} (-1)^\nu K^{(\nu)}(s, s') + \int_S \sum_{\nu=0}^{N} (-1)^\nu K^{(\nu)}(s, s'') L_N(s'' s') ds''$$

is an $E^0(y_1, y_2, \varkappa)$-function. It finally follows that the second integral in (6) is an $E^1(x_1, x_2, \varkappa)$-function. *Thus, $\gamma(x_1, x_2, \varkappa)$ is an $E^0(x_1, x_2, \varkappa)$-function. This is also true for the compensating part of Green's function in the case of Neumann's boundary condition, as can be seen by a similar investigation of the formula* (7).

As immediate consequences we obtain the relations

$$\gamma(x_1, x_2, \varkappa) = O([1 + |\log \varkappa \hat{r}_{12}|] e^{-A\varkappa \hat{r}_{12}})$$

and

$$D^k \gamma(x_1, x_2, \varkappa) = O(\hat{r}_{12}^{-k} e^{-A\varkappa \hat{r}_{12}}), \; k = 1, 2, \ldots .$$

**14. More precise estimates.** The $e^\lambda$-expansions of $\frac{\partial}{\partial n_s} K_0(\varkappa r_{x_1 s})$ and of $K_0(\varkappa r_{x_1 s})$ can be written

$$\frac{\partial}{\partial n_s} K_0(\varkappa r_{x_1 s}) = -\frac{\partial}{\partial \xi_1^2} K_0(\varkappa \varrho_{\xi_1 \eta}) + e^0(\xi_1, \eta, \varkappa),$$

$$K_0(\varkappa r_{x_2 s}) = K_0(\varkappa \varrho_{\xi_2 \eta}) + e^1(\xi_2, \eta, \varkappa).$$

Thus, on account of (27) we obtain the relation

$$\frac{1}{2\pi^2} \int\limits_S \frac{\partial}{\partial n_s} K_0(\varkappa r_{x_1 s}) \cdot K_0(\varkappa r_{x_2 s})\, ds = \frac{1}{2\pi} K_0(\varkappa \hat{\varrho}_{12}) + e^1(\xi_1, \xi_2, \varkappa), \tag{38}$$

which is valid if $x_1$, $x_2$ belong to a sufficiently small domain $C(I)$. The $e^1(\xi_1, \xi_2, \varkappa)$-function can be estimated by the help of formulæ (37). If (38) is inserted in (6), we see, that in the case of Dirichlet's boundary condition,

$$\gamma(x_1, x_2, \varkappa) = \frac{1}{2\pi} K_0(\varkappa \hat{\varrho}_{12}) + O(\varkappa^{-1} e^{-A\varkappa \hat{\varrho}_{12}}), \; A > 0.$$

This gives the local behaviour of the compensating part when $\hat{r}_{12}$ is small and $\varkappa$ tends to infinity. When $\hat{r}_{12} \geq \delta > 0$ the function $\gamma(x_1, x_2, \varkappa)$ is of the order $O(e^{-c\varkappa})$, $c > 0$. Thus, since $\lim \frac{\hat{r}_{12}}{\hat{\varrho}_{12}} = 1$,

$$\gamma(x_1, x_2, \varkappa) = \frac{1}{2\pi} K_0(\varkappa \hat{r}_{12}) + O(\varkappa^{-1} e^{-A\varkappa \hat{r}_{12}}), \; A > 0.$$ [1]

In the same way the local relation

$$\gamma(x_1, x_2, \varkappa) = -\frac{1}{2\varkappa} K_0(\varkappa \hat{\varrho}_{12}) + O(\varkappa^{-1} e^{-A\varkappa \hat{\varrho}_{12}}), \; A > 0, \tag{39}$$

as well as the formula in the large

[1] Estimates of Green's functions in terms of the light-distance $\hat{r}_{12}$ were first given by H. Weyl (see [3]).

$$\gamma(x_1, x_2, \varkappa) = -\frac{1}{2\pi} K_0(\varkappa \hat{r}_{12}) + O(\varkappa^{-1} e^{-A\varkappa \hat{r}_{12}}),\ A > 0,$$

are deduced in the case of Neumann's boundary condition.

By equally evaluating terms of higher orders in the $e^{\lambda}$-expansions, better approximations are obtained, which give remainders of orders $O(\varkappa^{-k} e^{-A\varkappa \hat{\varrho}_{12}})$, in which the integer $k$ can be made arbitrarily large.

**15. Integrals of the compensating parts over $V$.** Since for $\xi^2 \geq h > 0$ the function $\gamma(x, x, \varkappa)$ is of the order $O(e^{-c\varkappa})$, the integral over $V$ of this function can be approximated in the following way (see (8))

$$J = \int_V\int \gamma(x, x, \varkappa)\, dx^1\, dx^2 = \int_0^h\int_0^S \gamma(x, x, \varkappa)\,[1 - c(\xi^1)\,\xi^2]\, d\xi^1\, d\xi^2 + O(e^{-c\varkappa}).$$

If the $e^{\lambda}$-expansion of $\gamma(x, x, \varkappa)$ is introduced in the last integral, one obtains an asymptotic series of the form

$$J = \sum_{\nu=1}^{k} a_\nu \varkappa^{-\nu} + O(\varkappa^{-k-1}). \tag{40}$$

In the cases of the two different boundary conditions, the constants $a_\nu$ are calculated from the $e^{\lambda}$-expansions by the help of the formula ($q = l + m$; when $l$ is odd the integral vanishes)

$$\int_0^h (\xi^2)^p \left[\left(\frac{\partial}{\partial \xi_1^1}\right)^l \left(\frac{\partial}{\partial \xi_1^2}\right)^m K_0(\varkappa \hat{\varrho}_{12})\right]_{\substack{\xi_1^1 = \xi_2^1 = 0,\\ \xi_1^2 = \xi_2^2 = \xi^2}} d\xi^2$$

$$\sim (-1)^{\frac{l}{2} - q}\, 2^{-q-2}\, \frac{\Gamma(l+1)}{\Gamma\left(\frac{l}{2}+1\right)} \cdot \frac{\Gamma\left(\frac{p-q+l+1}{2}\right)}{\Gamma(p-q+l+1)} \cdot \Gamma(p+1)\, \Gamma\left(\frac{p-q+1}{2}\right) \varkappa^{-p+q-1},$$

which is valid for $\varkappa$ tending to infinity provided that $p - q \geq 0$.

The values of the three first coefficients $a_\nu$ are:

*in the case of Dirichlet's condition*

$$a_1 = \frac{S}{8},\ a_2 = -\frac{1}{6},\ a_3 = \frac{1}{512}\int_S (c(s))^2\, ds,$$

*and in the case of Neumann's condition*

$$a_1 = -\frac{S}{8},\ a_2 = -\frac{1}{6},\ a_3 = -\frac{7}{512}\int_S (c(s))^2\, ds.$$

Here $c(s)$ denotes the curvature of the boundary.

## Part II. Applications to eigenvalue problems

**16. Carleman's method.** We consider the problems to seek solution of the equation

$$\Delta u + \lambda u = 0 \text{ in } V, \tag{41}$$

satisfying Dirichlet's or Neumann's boundary conditions, viz.

$$u = 0 \text{ on } S. \tag{42}$$

or

$$\frac{\partial u}{\partial n} = 0 \text{ on } S. \tag{43}$$

In his study of vibrating membranes [1] Carleman made use of the formula

$$G(x_1, x_2; -\varkappa^2) - G(x_1, x_2; -\varkappa_0^2) = -(\varkappa^2 - \varkappa_0^2) \sum_{\nu=0}^{\infty} \frac{\varphi_\nu(x_1)\,\varphi_\nu(x_2)}{(\lambda_\nu + \varkappa^2)\,(\lambda_\nu + \varkappa_0^2)} \tag{44}$$

where $\lambda_\nu$ are the eigenvalues, and $\varphi_\nu(x)$ the eigenfunctions of one of these problems. The eigenfunctions are supposed to be orthonormalized on the domain $V$. In the problem $\{(41), (43)\}$ the smallest eigenvalue $\lambda_0$ is zero, in the problem $\{(41), (42)\}$ it is positive.

When $x_1, x_2$ coincide, (44) assumes the form

$$\frac{1}{2\pi}\log\varkappa + \gamma(x, x, \varkappa) - \frac{1}{2\pi}\log\varkappa_0 - \gamma(x, x, \varkappa_0) = \tag{45}$$

$$= (\varkappa^2 - \varkappa_0^2) \sum_{\nu=0}^{\infty} \frac{(\varphi_\nu(x))^2}{(\lambda_\nu + \varkappa^2)\,(\lambda_\nu + \varkappa_0^2)}.$$

For $\varkappa$ tending to infinity, the term $\frac{1}{2\pi}\log\varkappa$ is the dominating part of the left hand side, so that

$$\sum_{\nu=0}^{\infty} \frac{(\varphi_\nu(x))^2}{(\lambda + \varkappa^2)\;(\lambda_\nu + \varkappa_0^2)} \sim \frac{\log\varkappa}{2\pi\varkappa^2}. \tag{46}$$

On applying a tauberian theorem to this relation, Carleman proved the the asymtotic formula

$$\sum_{\lambda_\nu < t} (\varphi_\nu(x))^2 \sim \frac{t}{4\pi} \text{ when } t \to +\infty. \tag{47}$$

By integrating (45) over $V$ before applying the tauberian theorem, he was also able to deduce Weyl's law for the eigenvalue-distribution viz.

$$N(t) \sim \frac{V}{4\pi} t \text{ when } t \to +\infty,$$

where $N(t)$ is the number of eigenvalues less than $t$, and $V$ denotes the area of the domain $V$.

The formulæ (46), (47) are valid *provided that $x$ be an inner point of $V$.*

**17. Asymptotic behaviour of eigenfunctions on the boundary.** In this section we consider exclusively the membrane problem with Neumann's boundary condition.

For $x_1 = x_2 = x$ the equation (39) becomes

$$\gamma(x, x, \varkappa) = -\frac{1}{2\pi} K_0(2\varkappa\xi^2) + O(\varkappa^{-1} e^{-2A\varkappa\xi^2}).$$

If, according to this equation, the values of $\gamma(x, x, \varkappa)$ and $\gamma(x, x, \varkappa_0)$ are inserted in (45), we get a relation in which we can let $x$ tend to a boundary point. The result is

$$(\varkappa^2 - \varkappa_0^2) \sum_{\nu=0}^{\infty} \frac{(\varphi_\nu(y))^2}{(\lambda_\nu + \varkappa^2)(\lambda_\nu + \varkappa_0^2)} = \frac{1}{\pi} \log \frac{\varkappa}{\varkappa_0} + O(\varkappa^{-1}) + O(\varkappa_0^{-1}).$$

Hence, for $\varkappa$ tending to infinity, one obtains the formula

$$\sum_{\nu=0}^{\infty} \frac{(\varphi_\nu(y))^2}{(\lambda_\nu + \varkappa^2)(\lambda_\nu + \varkappa_0^2)} \sim \frac{\log \varkappa}{\pi \varkappa^2}.$$

From this relation it follows, in the same way as (47) follows from (46), that *the asymptotic formula*

$$\sum_{\lambda_\nu < t} (\varphi_\nu(y))^2 \sim \frac{t}{2\pi} \quad \text{when } t \to +\infty \text{ and } y \in S$$

*holds true for the eigenfunctions of the problem.*

**18. On certain Dirichlet's series.** In the following investigation we write $\omega$ and $\omega_0$ instead of $\varkappa^2$ and $\varkappa_0^2$.

According to (40) an integration of (45) over $V$ gives the result

$$(48) \quad \frac{V}{4\pi} \log \omega + \sum_{\nu=0}^{k} a_\nu \omega^{-\frac{\nu}{2}} + F(\omega) - \frac{V}{4\pi} \log \omega_0 - \sum_{\nu=1}^{k} a_\nu \omega_0^{-\frac{\nu}{2}} - F(\omega_0)$$

$$= (\omega - \omega_0) \sum_{\nu=0}^{\infty} \frac{1}{(\lambda_\nu + \omega)(\lambda_\nu + \omega_0)}$$

where $F(\omega) = O\left(\omega^{-\frac{k+1}{2}}\right)$ when $\omega$ tends to $+\infty$

We consider the problem with Dirichlet's boundary condition. Since in this problem the least eigenvalue $\lambda_0$ is positive, we can let $\omega_0$ tend to zero in (48) thus obtaining the formula

$$\frac{V}{4\pi}\log\omega+\sum_{\nu=1}^{k}a_\nu\omega^{-\frac{\nu}{2}}+F(\omega)+C=\omega\sum_{\nu=0}^{\infty}\frac{1}{\lambda_\nu(\lambda_\nu+\omega)} \tag{49}$$

in which $C$ is a constant.

[When the problem with Neumann's boundary condition is considered, we subtract $\frac{1}{\omega_0}-\frac{1}{\omega}$ from both sides of (48). The transition to the limit $\omega_0\to 0$ then leads to the formula

$$\frac{V}{4\pi}\log\omega+\sum_{\nu=1}^{k}a_\nu\omega^{-\frac{\nu}{2}}+\omega^{-1}+F(\omega)+C=\omega\sum_{\nu=0}^{\infty}\frac{1}{\lambda_\nu(\lambda_\nu+\omega)}.$$

Here, of course, the constants $a_\nu$ and $C$ are generally different from the constants $a_\nu$ and $C$ in (49).]

Let $H(\omega)$ be the function on the right hand side of (49). This function is multiplied by

$$\frac{1}{2\pi i}(-\omega)^{-z}=\frac{1}{2\pi i}|\omega|^{-z}e^{iz(\pi-\arg\omega)},\quad 0\le\arg\omega\le 2\pi,$$

and the product is integrated along a curve in the complex $\omega$-plane. This curve is taken from $\omega=+\infty$ to $\omega=a>0$ along the real axis (with $\arg\omega=2\pi$), then around the circle $|\omega|=a$ and at last from $\omega=a$ back to $\omega=+\infty$ along the real axis (with $\arg\omega=0$). According to the calculus of residues, the integral equals the sum of the Dirichlet's series $\sum_{\nu=0}^{\infty}\lambda_\nu^{-z}$. By integrating along the real axis, the function $H(\omega)$ may be replaced by the left hand side of (49). In this way we obtain the formula

$$\begin{aligned}\sum_{\nu=0}^{\infty}\lambda_\nu^{-z}=&-\frac{V}{4\pi}\frac{\sin\pi z}{\pi}\int_a^\infty\frac{\log\omega}{\omega^z}d\omega-\frac{\sin\pi z}{\pi}\sum_{\nu=1}^{k}a_\nu\int_a^\infty\frac{d\omega}{\omega^{z+\frac{\nu}{2}}}\\&-C\frac{\sin\pi z}{\pi}\int_a^\infty\frac{d\omega}{\omega^z}-\frac{\sin\pi z}{\pi}\int_a^\infty\frac{E(\omega)}{\omega^2}d\omega\\&+\frac{a^{1-z}e^{i\pi z}}{2\pi}\int_a^{2\pi}H(ae^{i\theta})e^{i\theta(1-z)}d\theta.\end{aligned} \tag{50}$$

The last integral in this formula represents an integral function of $z$, and since $F(\omega)=O\left(\omega^{-\frac{k+1}{2}}\right)$ the last but one is analytic for $Re(z)>-\frac{k-1}{2}$ ($Re$ means real part of). The other integrals in (5) can be explicitly evaluated. *One obtains the results*

$$\sum_{\nu=0}^{\infty} \lambda_\nu^{-z} = \frac{V}{4\pi} \cdot \frac{1}{z-1} + \sum_{\nu=1}^{k} \frac{a_{2\nu-1}}{\pi\left(z + \frac{2\nu-3}{2}\right)} + \chi(z) \tag{51}$$

*where the function* $\chi(z)$ *is analytic when* $Re(z) > -\frac{2k-1}{2}$.

*In the case of Neuman's boundary condition it is similarly seen that*

$$\sum_{\nu=1}^{\infty} \lambda_\nu^{-z} = \frac{V}{4\pi} \cdot \frac{1}{z-1} + \sum_{\nu=1}^{k} \frac{a_{2\nu-1}}{\pi\left(z + \frac{2\nu-3}{2}\right)} + \chi(z) \tag{52}$$

*with a function* $\chi(z)$ *which is analytic for* $Re(z) > -\frac{2k-1}{2}$.

The values of the coefficients $a_1$, $a_3$ in (51) and (51) are given in section 15.

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