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Fractional categories

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Introduction

In this paper we study the "localization" of a category C with respect to certain subcategories $\mathcal{S}$. This is done by a category of "right fractions", $\mathsf{C}\mathcal{S}^{-1}$ and a functor ϕ: $\mathsf{C}\to\mathsf{C}\mathcal{S}^{-1}$. In § 1 conditions for the existence of $\mathsf{C}\mathcal{S}^{-1}$ are given and it turns out that ϕ is left exact.

In § 2 the existence of left adjoint $^*\phi$ of ϕ is discussed. If $^*\phi$ exists, a full subcategory $\check{\mathsf{C}}$ of C is defined. $\check{\mathsf{C}}$ consists of those objects A such that $^*\phi\cdot\phi(A)\approx A$. It follows that $\check{\mathsf{C}}$ is equivalent ot $\mathsf{C}\mathcal{S}^{-1}$.

If in the *dual* case (i.e. the right adjoint ϕ^* exists) ϕ is exact and C is a category of set-valued presheaves then $\check{\mathsf{C}}$ is a category of sheaves for some Grothendieck topology. Furthermore the imbedding functor $\check{\mathsf{C}}\to\mathsf{C}$ has a left adjoint which is the associated sheaf functor.

Section 3 is devoted to a study of the functional properties of $\mathsf{C}\mathcal{S}^{-1}$, i.e. its behaviour under functors and under change of $\mathcal{S}$.

In § 4 properties inherited from C to $\mathsf{C}\mathcal{S}^{-1}$ are studied under various conditions on $\mathcal{S}$. If ϕ^* exists, it follows that if C is abelian (a topos), then $\mathsf{C}\mathcal{S}^{-1}$ is also.

In § 5 some examples are given when C is a special abelian category, in particular when C is the category of modules over a commutative ring.

Notation and generalities

All categories in this paper are sets. Let $\mathcal{U}$ be a fixed universe (see [10], Exp. VI). If nothing else is stated a category C will mean a *$\mathcal{U}$-category*, i.e. for any pair of objects A, B in C there is a bijection from $\mathrm{Hom}_{\mathsf{C}}(A, B)$ onto a set belonging to $\mathcal{U}$. $\mathcal{E}ns$ is the category of sets of cardinality less than Card $(\mathcal{U})$. A category C is *small* if the set underlying C is in $\mathcal{E}ns$.

If C is a category we denote the set of objects of C by C_0 and identify C_0 with the identities of C. If F: $\mathsf{C}\to\mathcal{D}$ is a functor, then a functor $^*F$: $\mathcal{D}\to\mathsf{C}$ is a *left adjoint* of F (and F is a *right adjoint* of $^*F$) if there is a functor isomorphism $\mathrm{Hom}_{\mathsf{C}}(^*F\cdot,\cdot)\approx \mathrm{Hom}_{\mathcal{D}}(\cdot, F\cdot)$. Let $\mathcal{J}$ be a small category and let C: $\mathsf{C}\to Hom\ (\mathcal{J}, \mathsf{C})$ denote the functor that maps each $A\in\mathsf{C}_0$ onto the constant functor C_A defined by $C_A(\alpha)=1_A$ for all $\alpha\in\mathcal{J}$. C maps the morphisms of C in the obvious way.

If C has a right adjoint, $C^*=\lim_{\leftarrow}$: $Hom\ (\mathcal{J}, \mathsf{C})\to\mathsf{C}$ then C is said to have $\mathcal{J}$-$\lim_{\leftarrow}$. In particular if C has $\mathcal{J}$-$\lim_{\leftarrow}$ for all small (finite) $\mathcal{J}$, then C has (finite) $\lim_{\leftarrow}$. $^*C=\lim_{\rightarrow}$ is defined dually.

If a category has both $\lim_{\leftarrow}$ and $\lim_{\rightarrow}$ we say *it has limits.*

A functor having a left (right) adjoint commutes with $\lim_{\leftarrow}$ ($\lim_{\rightarrow}$).

In particular $\lim_{\leftarrow}$ commutes with $\lim_{\leftarrow}$. A functor commuting with finite $\lim_{\leftarrow}$ ($\lim_{\rightarrow}$) is called *left* (*right*) *exact*. It is *exact* if it is both left and right exact.

If $\mathfrak{J}$ is a discrete small category (i.e. $\mathfrak{J}_0 = \mathfrak{J}$), then $\lim_{\leftarrow}$ ($\lim_{\rightarrow}$) coincides with the *product* $\prod_{\mathfrak{J}}$ (*sum* $\oplus_{\mathfrak{J}}$). The sum in $\mathcal{E}ns$ is the disjoint union and is denoted by $\amalg$.

Let $A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B$ be a pair of morphisms in $\mathbf{C}$. Then the *kernel* (or equalizer), $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \xrightarrow{i} A$ is defined as follows:

$$\begin{tikzcd} \operatorname{Ker}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \arrow[r, "i"] & A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & B \\ \cdot \arrow[u, dashed, "\exists_1 g"] \arrow[ur, "\forall f"'] & & \end{tikzcd}$$

(1) $a_1 \cdot i = a_2 \cdot i$.

(2) For all f such that $a_1 \cdot f = a_2 \cdot f$ there exists a unique g such that $f = i \cdot g$.

$\operatorname{Ker}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a special case of finite $\lim_{\leftarrow}$.

$\operatorname{Coker}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is defined dually. A diagram of the form $\operatorname{Ker}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \xrightarrow{i} \cdot \overset{a_1}{\underset{a_2}{\rightrightarrows}} \cdot$ is said to be exact.

A functor commutes with (finite) $\lim_{\leftarrow}$ if and only if it commutes with Ker and (finite) products.

Given two morphisms s: $A \to B$ and a: $C \to B$ we define the *fiber product* (pull back)

$$\begin{tikzcd} A \times_B C \arrow[r, "s_1"] \arrow[d, "a_1"'] & C \arrow[d, "a"] \\ A \arrow[r, "s"'] & B \end{tikzcd}$$

of a and s by the universal property:

(1) $a \cdot s_1 = s \cdot a_1$.

(2) Given any f, g such that $a \cdot f = s \cdot g$ then there exists a unique h such that $f = s_1 \cdot h$ and $g = a_1 \cdot h$.

The fiber product is also a special case of finite $\lim_{\leftarrow}$.

A functor F is *faithful* if $F(a_1) = F(a_2)$ implies $a_1 = a_2$. An object G is a *generator* if Hom $(G, \cdot)$ is faithful. *Cogenerator* is defined dually.

Two categories C_1 and C_2 are *equivalent* if there exist functors $C_1 \underset{F_2}{\overset{F_1}{\rightleftarrows}} C_2$ such that

$$F_2 \cdot F_1 \approx I_{C_1} \quad \text{and} \quad F_1 \cdot F_2 \approx I_{C_2}.$$

(I_C is the identity functor of C.)

$\mathcal{C}^0$ denotes the dual category of $\mathcal{C}$.

If (S) is any statement, then (S^0) is the dual statement.

An object A in $\mathcal{C}$ is *initial* (*final*) if Hom (A, B) (Hom (B, A)) consists of exactly one element for all $B \in \mathcal{C}_0$.

A is a *zero-object* if it is both initial and final.

1. Existence theorems

Let $\mathcal{C}$ be a $\mathcal{U}$-category and let $\mathcal{S} \subset \mathcal{C}$ be a subcategory containing all the isomorphisms of $\mathcal{C}$ (so in particular $\mathcal{S}_0 = \mathcal{C}_0$). We say that $\mathcal{S}$ is *nice* if for every object $A \in \mathcal{C}_0$ there is a set of objects, $\mathcal{J}_A \in \mathcal{U}$ such that for every $s: C \to A$ in $\mathcal{S}$ there exists $u: D \to C$ with $D \in \mathcal{J}_A$ and $s \cdot u \in \mathcal{S}$.

Conice is defined dually.

If $\mathcal{C}$ is small than every $\mathcal{S}$ is nice.

Definition 1.1. A functor $\phi: \mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ from $\mathcal{C}$ to a category $\mathcal{C}\mathcal{S}^{-1}$ defines a *right fractional category of* $\mathcal{C}$ *with respect to* $\mathcal{S}$ if the following axioms are satisfied:

F_1: $\phi(s)$ is an isomorphism for all $s \in \mathcal{S}$.

F_2: Every morphism $\alpha \in \mathcal{C}\mathcal{S}^{-1}$ can be written as $\alpha = \phi(a) \cdot \phi(s)^{-1}$ with $a \in \mathcal{C}$ and $s \in \mathcal{S}$.

F_3: $\phi(a_1) = \phi(a_2)$ if and only if there exists $s \in \mathcal{S}$ such that $a_1 \cdot s = a_2 \cdot s$.

The *left* fractional category $\mathcal{S}^{-1}\mathcal{C}$ is defined dually.

Theorem 1.2. Let $\mathcal{S} \subset \mathcal{C}$ be a nice subcategory containing the isomorphisms of $\mathcal{C}$. Then $\mathcal{C}\mathcal{S}^{-1}$ exists if and only if the following two conditions are satisfied:

S_1: For every $a \in \mathcal{C}$, $s \in \mathcal{S}$ with common terminal, there exist $b \in \mathcal{C}$ and $t \in \mathcal{S}$ such that $s \cdot b = a \cdot t$.

S_2: If $s \cdot a_1 = s \cdot a_2$ where $s \in \mathcal{S}$ and $a_1, a_2 \in \mathcal{C}$, then there exists $t \in \mathcal{S}$ such that $a_1 \cdot t = a_2 \cdot t$.

Proof. Assume first that (S_1) and (S_2) hold.

We construct $\mathcal{C}\mathcal{S}^{-1}$ in the following way. We first set $(\mathcal{C}\mathcal{S}^{-1})_0 = \mathcal{C}_0$. Given two objects A and B we consider the set of all pairs (a, s) where $a \in \mathcal{C}$ and $s \in \mathcal{S}$ such that there is a diagram $A \xleftarrow{s} \cdot \xrightarrow{a} B$. On this set we define a relation, $\sim$, as follows: $(a_1, s_1) \sim (a_2, s_2)$ if and only if there exist $u_1, u_2 \in \mathcal{C}$ such that $s_1 \cdot u_1 = s_2 \cdot u_2 \in \mathcal{S}$ and $a_1 \cdot u_1 = a_2 \cdot u_2$. A computation shows that $\sim$ is an equivalence relation. We now define the set

$$\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(A, B) = [\coprod_{C \in \mathcal{J}_A} (\mathrm{Hom}_{\mathcal{S}}(C, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B))]/\sim .$$

$\mathcal{C}\mathcal{S}^{-1}$ clearly becomes a $\mathcal{U}$-category.

Since $\mathcal{S}$ is nice every pair $A \xleftarrow{s} E \xrightarrow{a} B$ is in the same equivalence class as some $A \xleftarrow{s_1} C \xrightarrow{a_1} B$ with $C \in \mathcal{J}_A$. For obvious reasons we denote by as^{-1} the equivalence class represented by (a, s).

Let as^{-1} and bt^{-1} be two morphisms in $\mathcal{C}\mathcal{S}^{-1}$ suitable for composition. By (S_1) there exist $t_1 \in \mathcal{S}$ and $a_1 \in \mathcal{C}$ such that $a \cdot t_1 = t \cdot a_1$, i.e. we have a diagram:

$$\begin{tikzcd} & & \cdot \arrow[dl, "t_1"'] \arrow[dr, "a_1"] & & \\ \cdot & \cdot \arrow[l, "s"'] \arrow[r, "a"] & \cdot & \cdot \arrow[l, "t"'] \arrow[r, "b"] & \cdot \end{tikzcd}$$

A rather long but straightforward computation shows that $(b \cdot a_1)(s \cdot t_1)^{-1}$ does not depend on the various choices made. For the details see [1].

We now define the composition $\mathcal{C}\mathcal{S}^{-1}$ by:

$$(bt^{-1}) \cdot (as^{-1}) = (b \cdot a_1)(s \cdot t_1)^{-1}.$$

One verifies that the composition is associative.

We define ϕ: $\mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ by $\phi(A) = A$ if $A \in \mathcal{C}_0$ and if a: $A \to B$ is a morphism in $\mathcal{C}$ we set $\phi(a) = a 1_A^{-1}$.

Then clearly ϕ is a functor and $as^{-1} = \phi(a) \cdot \phi(s)^{-1}$ so F_2 is satisfied.

F_1 and F_3 are also trivial to check.

To prove the converse let $a \in \mathcal{C}$ and $s \in \mathcal{S}$ with common terminal be given. Using (F_2) we can find $b_1 \in \mathcal{C}$, $t_1 \in \mathcal{S}$ such that $\phi(s)^{-1} \cdot \phi(a) = \phi(b_1) \cdot \phi(t_1)^{-1}$ or $\phi(a \cdot t_1) = \phi(s \cdot b_1)$. By (F_3) there exists $s_0 \in \mathcal{S}$ such that $a \cdot t_1 \cdot s_0 = s \cdot b_1 \cdot s_0$. Put $t = t_1 \cdot s_0 \in \mathcal{S}$ and $b = b_1 \cdot s_0$ and we get $a \cdot t = s \cdot b$ which is (S_1).

If $s \cdot a_1 = s \cdot a_2$ with $s \in \mathcal{S}$, then $\phi(s) \cdot \phi(a_1) = \phi(s) \cdot \phi(a_2)$ and hence $\phi(a_1) = \phi(a_2)$ by (F_1). From (F_3) we conclude that there exists $t \in \mathcal{S}$ such that $a_1 \cdot t = a_2 \cdot t$.

This ends the proof of the theorem.

Proposition 1.3. The functor ϕ: $\mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ solves the following universal problem: Given any functor F: $\mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, then there exists a unique functor G: $\mathcal{C}\mathcal{S}^{-1} \to \mathcal{D}$ such that $F = G \cdot \phi$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\forall F"'] & \mathcal{C}\mathcal{S}^{-1} \arrow[dl, dashed, "\exists_1 G"] \\ \mathcal{D} & \end{tikzcd}$$

Proof. Putting $G(as^{-1}) = F(a) \cdot F(s)^{-1}$ one verifies that G is a well-defined, unique functor such that $F = G \cdot \phi$.

Corollary. If $\mathcal{C}\mathcal{S}^{-1}$ exists, then it is unique up to an isomorphism. In particular if $\mathcal{S}^{-1}\mathcal{C}$ also exists then $\mathcal{S}^{-1}\mathcal{C} \approx \mathcal{C}\mathcal{S}^{-1}$.

Remark.

$$\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(A, B) = \lim_{\substack{(C \to A) \\ \text{in } \mathcal{S}_{1/A}}} \mathrm{Hom}_{\mathcal{C}}(C, B),$$

where $\mathcal{S}_{1/A}$ is the category with objects $C \xrightarrow{s} A$ where $s \in \mathcal{S}$ and $C \in \mathcal{J}_A$ and whose morphisms are the commutative diagrams

$$\begin{tikzcd}
C_1 \arrow[rr, "u"] \arrow[dr, "s_1"'] & & C_2 \arrow[dl, "s_2"] \\
& A &
\end{tikzcd}$$

with u a morphism of $\mathcal{C}$.

Proof. From S_1 and S_2 it is easily seen that $(\mathcal{S}_{1/A})^0$ satisfies the axioms of a "pseudofiltered category" (see [10], def. 2.7). Now the computation of $\lim_{\rightarrow}$ can be made in the oldfashioned way for directed sets and we immediately get the result.

Lemma 1.4. Let $s_i\colon A_i \to B$ be in $\mathcal{S}$ for $i=1, 2, \dots, n$. Then there exist $u_i\colon C \to A$, $i=1, 2, \dots, n$, such that $s_i \cdot u_i = s \in \mathcal{S}$ for $i=1, 2, \dots, n$.

Proof. For $n=2$ the lemma is (S_1). For larger n it follows by induction.

Lemma 1.5. Assume that $s \in \mathcal{S}$ and $s \cdot u_1 = s \cdot u_2 = \dots = s \cdot u_n$. Then there exists $t \in \mathcal{S}$ such that

$$u_1 \cdot t = u_2 \cdot t = \dots = u_n \cdot t.$$

Proof. For $n=2$ it is (S_2). For larger n it follows from Lemma 1.4 by induction.

Lemma 1.6. ("putting fractions on common denominator"): Let $\alpha_i\colon A \to B_i$ be morphisms in $\mathcal{C}\mathcal{S}^{-1}$ $(i=1, 2, \dots, n)$. Then there exist $s \in \mathcal{S}$ and $a_1, a_2, \dots, a_n \in \mathcal{S}$ such that $\alpha_i = a_i s^{-1}$ for $i=1, 2, \dots, n$.

Proof. Assume that α_i can be represented as $\alpha_i = b_i t_i^{-1}$. Then we use Lemma 1.4 to find u_i such that $s = t_i \cdot u_i \in \mathcal{S}$ $(i=1, 2, \dots, n)$. Finally we put $a_i = b_i \cdot u_i$ and get $b_i t_i^{-1} = (b_i \cdot u_i)(t_i \cdot u_i)^{-1} = a_i s^{-1}$ for $i=1, 2, \dots, n$.

Lemma 1.7. Assume that $s \in \mathcal{S}$ and $s \cdot u_i \in \mathcal{S}$ for $i=1, 2, \dots, n$. Then there exist $v_1, v_2, \dots, v_n$ such that $u_1 \cdot v_1 = u_2 \cdot v_2 = \dots = u_n \cdot v_n = u$ and $s \cdot u \in \mathcal{S}$.

Proof. First use Lemma 1.4 and then Lemma 1.5.

Proposition 1.8. If $\mathcal{C}$ has kernels, then $\mathcal{C}\mathcal{S}^{-1}$ has kernels and ϕ commutes with kernels.

Proof. By Lemma 1.6 it is sufficient to consider pairs of the form $\begin{pmatrix} a_1 s^{-1} \\ a_2 s^{-1} \end{pmatrix}$ in $\mathcal{C}\mathcal{S}^{-1}$. Hence all we have to do is to show that

$$\operatorname{Ker} \begin{pmatrix} \phi(a_1) \\ \phi(a_1) \end{pmatrix} = \phi\left(\operatorname{Ker} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right)$$

Consider the diagram:

$$\begin{tikzcd}
{} & \cdot \arrow[r, "k"] & \cdot \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & \cdot \\
{} & \cdot \arrow[u, "b_1"'] \arrow[r, "u"] & \cdot \arrow[u, "b"'] \arrow[dd, "t"] & \\
\cdot \arrow[ur, "v_1"'] \arrow[d, "v_2"] & & & \\
\cdot \arrow[uuur, "c"] \arrow[rr, "s"'] & & \cdot &
\end{tikzcd}$$

$$\text{where } k = \operatorname{Ker}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\cdot \text{Assume } a_1\cdot(bt^{-1}) = a_2\cdot(bt^{-1}).$$

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "t"] \arrow[dr, "a_1\cdot b"'] & \cdot \arrow[l, "u_1"'] \arrow[r, "u_2"] & \cdot \arrow[ul, "t"'] \arrow[dl, "a_2\cdot b"] \\ & \cdot & \end{tikzcd}$$

Hence there exist $u_1, u_2 \in \mathcal{C}$ such that $t\cdot u_1 = t\cdot u_2$ and $a_1\cdot b\cdot u_1 = a_2\cdot b\cdot u_2$. Using (S_2) there exists $s_0 \in \mathcal{S}$ such that $u_1\cdot s_0 = u_2\cdot s_0 = u$ say.

Hence $t\cdot u \in \mathcal{S}$ and $a_1\cdot(b\cdot u) = a_2\cdot(b\cdot u)$. By the definition of kernel there exists a unique b_1 such that $k\cdot b_1 = b\cdot u$. Then clearly $\phi(k)\cdot(b_1(t\cdot u)^{-1}) = bt^{-1}$ and we only need to show uniqueness.

Assume that $\phi(k)\cdot(cs^{-1}) = bt^{-1} = \phi(k)\cdot(b_1(t\cdot u)^{-1}$, i.e. there exist v_1, v_2 such that $t\cdot u\cdot v_1 = s\cdot v_2 \in \mathcal{S}$ and $k\cdot b_1\cdot v_1 = k\cdot c\cdot v_2$. But k is a monomorphism and it follows that $b_1\cdot v_1 = c\cdot v_2$ and hence $cs^{-1} = b_1(t\cdot u)^{-1}$.

Proposition 1.9. If $\mathcal{C}$ has finite products so has $\mathcal{C}\mathcal{S}^{-1}$ and ϕ commutes with finite products.

Proof. We show that $\phi(A_1\times A_2) = \phi(A_1)\times\phi(A_2)$. Given two arbitrary morphisms α_1: $B\to A_1$ and α_2: $B\to A_2$ we may assume (by Lemma 1.6) that $\alpha_1 = a_1 s^{-1}$ and $\alpha_2 = a_2 s^{-1}$. The definition of product in $\mathcal{C}$ then implies that there exists a unique a such that $p_1\cdot a = a_1$ and $p_2\cdot a = a_2$ (p_1, p_2 are the canonical projections). Clearly

$$a_1 s^{-1} = \phi(p_1)\cdot(as^{-1}),$$

$$a_2 s^{-1} = \phi(p_2)\cdot(as^{-1}).$$

To show the uniqueness of as^{-1} assume that $\phi(p_i)\cdot(b_1 s^{-1}) = \phi(p_i)\cdot(b_2 s^{-1})$ for $i=1,2$. The there exist u_1, u_2 such that $s\cdot u_i \in \mathcal{S}$ and $p_i\cdot b_1\cdot u_i = p_i\cdot b_2\cdot u_i$, $i=1,2$. Lemma 1.7 implies that there exist v_1, v_2 such that $u = u_1\cdot v_1 = u_2\cdot v_2$ and $s\cdot u \in \mathcal{S}$. Hence we get $p_i\cdot b_1\cdot u = p_i\cdot b_2\cdot u$, $i=1,2$, and then $b_1\cdot u = b_2\cdot u$ that together with $s\cdot u\in\mathcal{S}$ says that $b_1 s^{-1} = b_2 s^{-1}$.

Corollary 1.10. If $\mathcal{C}$ has finite $\lim_{\leftarrow}$ so has $\mathcal{C}\mathcal{S}^{-1}$ and ϕ is left exact. If $\mathcal{C}$ also has finite $\lim_{\rightarrow}$ and $\mathcal{C}$ is S_1, S_2, S_1^0, and S_2^0 then $\mathcal{C}\mathcal{S}^{-1}$ has finite limits and ϕ is exact.

Definition 1.11. Let F: $\mathcal{C}\to\mathcal{D}$ be a functor. We set $\mathcal{S}_F = \{s\in\mathcal{C} \mid F(s) \text{ is an isomorphism}\}$. A subcategory $\mathcal{S}$ of $\mathcal{C}$ is called *saturated* if $\mathcal{S}$ satisfies (S_1) and (S_2), and $\mathcal{S} = \mathcal{S}_\phi$ for the corresponding ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$.

Proposition 1.12. Assume that $\mathcal{C}$ is small and has finite $\lim_{\leftarrow}$ and let $\mathcal{S}\subset\mathcal{C}$ be a subcategory. Then $\mathcal{S}$ is saturated if and only if $\mathcal{S} = \mathcal{S}_F$ for some left exact functor F: $\mathcal{C}\to\mathcal{D}$.

Proof. ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ is left exact and $\mathcal{S} = \mathcal{S}_\phi$ if $\mathcal{S}$ is saturated. Conversely if F: $\mathcal{C}\to\mathcal{D}$ is a left exact functor, then F commutes with kernels and fiber products.

The morphisms in $\mathcal{S}_F$, the existence of which is required in (S_1) and (S_2), are easily constructed by fiber products and kernels, respectively. By the definition of $\mathcal{C}\mathcal{S}_F^{-1}$ we find a factorization $F=G\cdot\phi$ where ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$. Hence if $\phi(s)$ is an isomorphism then $F(s)=G(\phi(s))$ is an isomorphism and $\mathcal{S}_F=\mathcal{S}_\phi$. Thus $\mathcal{S}_F$ is saturated.

Proposition 1.13. Let $\mathcal{S}_i\subset\mathcal{C}$ be saturated for all $i\in J$ and assume that $\mathcal{C}$ is small and has finite $\lim_{\leftarrow}$. Then $\mathcal{S}=\bigcap_{i\in J}\mathcal{S}_i$ is saturated.

Proof. First we show that $\mathcal{S}$ satisfies (S_1) and (S_2):

S_1: Let $a\in\mathcal{C}$ and $s\in\mathcal{S}$, i.e. $s\in\mathcal{S}_i=\mathcal{S}_{\phi_i}$ for all $i\in J$, where ϕ_i: $\mathcal{C}\to\mathcal{C}\mathcal{S}_i^{-1}$. We form the fiber product and it is clear that

$$\begin{tikzcd} \cdot \arrow[r, "s_1"] \arrow[d, "a_1"'] & \cdot \arrow[d, "a"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

$s_1\in\mathcal{S}_{\phi_i}$ for all $i\in J$. Hence $s_1\in\mathcal{S}$ and $a\cdot s_1=s\cdot a_1$.

S_2: If $\cdot\overset{a_1}{\underset{a_2}{\rightrightarrows}}\cdot\overset{s}{\longrightarrow}$ is given with $s\cdot a_1=s\cdot a_2$ and $s\in\mathcal{S}$, then $\operatorname{Ker}\begin{pmatrix}a_1\\a_2\end{pmatrix}\in\mathcal{S}$ satisfies the requirements.

Consider the diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[dr, "\phi_i"'] & \mathcal{C}\mathcal{S}^{-1} \arrow[d, dashed, "U_i"] \\ & \mathcal{C}\mathcal{S}_i^{-1} \end{tikzcd}$$

Since $\mathcal{S}\subset\mathcal{S}_i$, $\phi_i(s)$ is an isomorphism for all $s\in\mathcal{S}$. Hence there exists a functor U_i such that $\phi_i=U_i\cdot\phi$. If $\phi(s)$ is an isomorphism, i.e. $s\in\mathcal{S}_\phi$, then $\phi_i(s)=U_i(\phi(s))$ is an isomorphism for all $i\in J$.

Thus $s\in\mathcal{S}_{\phi_i}=\mathcal{S}_i$ for all $i\in J$ and $s\in\bigcap_J\mathcal{S}_i=\mathcal{S}$ which proves that $\mathcal{S}$ is saturated.

Denote by $\bar{\mathcal{S}}$ the smallest saturated subcategory of $\mathcal{C}$ that contains $\mathcal{S}$ (if it exists).

Proposition 1.14. Let $\mathcal{S}\subset\mathcal{C}$ be nice and satisfy (S_1) and (S_2). Then $\bar{\mathcal{S}}=\mathcal{S}_\phi=\{a\in\mathcal{C}\mid\exists c,$ $d\in\mathcal{C}$ such that $a\cdot c\in\mathcal{S}$ and $c\cdot d\in\mathcal{S}\}$ is nice.

Proof. $\bar{\mathcal{S}}=\mathcal{S}_\phi$ follows from the universal properties of ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$. Call the right-hand side $\mathcal{J}$.

$\mathcal{J}\subset\mathcal{S}_\phi$: Assume that $a\in\mathcal{J}$, i.e. there exist c, $d\in\mathcal{C}$ such that $a\cdot c=s\in\mathcal{S}$ and $c\cdot d=t\in\mathcal{S}$. Then $\phi(a)\cdot\phi(c)=\phi(s)$ and $\phi(c)\cdot\phi(d)=\phi(t)$.

Thus $(\phi(s)^{-1}\cdot\phi(s))\cdot\phi(c)=1$ and $\phi(c)(\phi(d)\cdot\phi(t)^{-1})=1$. Hence $\phi(c)$ has both left and right inverse. It follows that $\phi(a)=\phi(s)\cdot\phi(c)^{-1}$, is invertible and $a\in\mathcal{S}_\phi$.

$\mathcal{S}_\phi \subset \mathfrak{J}$: Let $a \in \mathcal{S}_\phi$ with $\phi(a)^{-1} = bt^{-1}$, where $b \in \mathcal{C}$ and $t \in \mathcal{S}$. Then we have a commutative diagram:

$$\begin{tikzcd}
& & \bullet & & \\
\bullet \arrow[urr, "t"] \arrow[drr, "a\cdot b"'] & & \bullet \arrow[ll, "u_1"'] \arrow[rr, "u_2"] & & \bullet \arrow[ull, "1"'] \arrow[dll, "1"] \\
& & \bullet & &
\end{tikzcd}$$

$\phi(a)\cdot bt^{-1} = 1$ gives $a\cdot b\cdot u_1 = u_2 = t\cdot u_1 \in \mathcal{S}$. We set $c = b\cdot u_1$ and get $a\cdot c = u_2 \in \mathcal{S}$. Now $bt^{-1} = cu_2^{-1} = \phi(a)^{-1}$ and hence $(c\cdot u_2^{-1})\cdot\phi(a) = 1$ and we have the following diagram:

$$\begin{tikzcd}
& & \bullet \arrow[r, "a"] & \bullet \\
& & \bullet \arrow[u, "u_3"'] \arrow[r, "a_1"'] & \bullet \arrow[u, "u_2"'] \arrow[dd, "c"] \\
& \bullet \arrow[ur, "v_1"] \arrow[dl, "v_2"] & & \\
\bullet \arrow[uuurr, "1"] \arrow[rrr, "1"'] & & & \bullet
\end{tikzcd}$$

(S_1) implies that there exist $u_3 \in \mathcal{S}$, $a_1 \in \mathcal{C}$ such that $u_2\cdot a_1 = a\cdot u_3$. Thus $c\cdot a_1\cdot v_1 = v_2 = u_3\cdot v_1 \in \mathcal{S}$ and $d = a_1\cdot v_1$ will do.

One verifies that $\bar{\mathcal{S}}$ is nice.

2. Existence of adjoints of ϕ: $\mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$

In general when a functor F: $\mathcal{C} \to \mathcal{D}$ is given, one can ask if F has adjoints, commutes with limits or if F is left or right exact. So far we only know that ϕ: $\mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ is left exact. We will find conditions for the existence of left and right adjoints of ϕ.

Definition 2.1. Let $\mathcal{C}$ have $\lim_{\leftarrow}$. Let F, G: $\mathfrak{J} \to \mathcal{C}$ be two functors from a small ~~category $\mathfrak{J}$ and φ: $F \to G$ a functor morphism~~ such that $\varphi(i) \in \mathcal{S}$ for all $i \in \mathfrak{J}_0$. We say that $\mathcal{S}$ *satisfies* (S_3) if under the above assumptions $\lim_{\leftarrow}\varphi$: $\lim_{\leftarrow} F \to \lim_{\leftarrow} G$ is in $\mathcal{S}$. If the above is true for all finite $\mathfrak{J}$, we say that $\mathcal{S}$ satisfies (S_3^{fin}).

Remark 2.2. (S_3) is equivalent to the following:

(*a*) If in the following commutative diagram, where the rows are exact, $s_1, s_2 \in \mathcal{S}$, then the induced morphism $s \in \mathcal{S}$.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "s"'] & \bullet \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "s_2"] & \bullet \arrow[d, "s_1"] \\
\bullet \arrow[r] & \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet
\end{tikzcd}$$

and

(*b*) If s_i: $A_i \to B_i$ is in $\mathcal{S}$ for all $i \in \mathfrak{J}$, then $\prod_{\mathfrak{J}} s_i$: $\prod_{\mathfrak{J}} A_i \to \prod_{\mathfrak{J}} B_i$ is also in $\mathcal{S}$. (In case of (S_3^{fin}) we take $\mathfrak{J}$ finite in (*b*).)

Proposition 2.3. Assume that $\mathcal{C}$ has finite $\lim_{\leftarrow}$.

(*a*) If $\mathcal{S}$ is (S_3^{fin}) then $\mathcal{S}$ is (S_1) and (S_2).

(*b*) If $\mathcal{S}$ is (S_1) and (S_2) then $\bar{\mathcal{S}}$ is (S_3^{fin}).

Proof. (*a*) Assume that $\mathcal{S}$ is (S_3^{fin}).
S_1: Let $s\colon A\to C$ in $\mathcal{S}$ and $a\colon B\to C$ be given.
Consider the diagram (where everything commutes).

$$\begin{tikzcd}
B\times_C C = B \arrow[rrr, "1_B"] \arrow[ddd, "a"'] & & & B \arrow[ddd, "a"] \\
& A\times_C B \arrow[ul, "s_1"'] \arrow[r, "s_1"] \arrow[d, "a_1"'] & B \arrow[ur, "1_B"'] \arrow[d, "a"] & \\
& A \arrow[r, "s"] \arrow[dl, "s"] & C \arrow[dr, "1_C"] & \\
C & & & C \arrow[lll, "1_C"']
\end{tikzcd}$$

Here $A\times_C B$ and $B=B\times_C C$ are finite $\lim_{\leftarrow}$. Since 1_B, 1_C and $s\in\mathcal{S}$, (S_3^{fin}) implies that $s_1\in\mathcal{C}$.
S_2: Consider the diagram, where $s\cdot a_1=s\cdot a_2$ with $s\in\mathcal{S}$.

$$\begin{tikzcd}
\operatorname{Ker}\begin{pmatrix}a_1\\ a_2\end{pmatrix} \arrow[r, "i"] \arrow[d, "i"'] & A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[d, "1_A"] & B \arrow[d, "s"] \\
A=\operatorname{Ker}\begin{pmatrix}s\cdot a_1\\ s\cdot a_2\end{pmatrix} \arrow[r, "1_A"] & A \arrow[r, shift left, "s\cdot a_1"] \arrow[r, shift right, "s\cdot a_2"'] & C
\end{tikzcd}$$

Since 1_A, $s\in\mathcal{S}$, (S_3^{fin}) implies that $i\in\mathcal{S}$ and $a_1\cdot i=a_2\cdot i$.

(*b*) This follows directly from Corollary 1.10 and Proposition 1.12.

Remark. In particular (S_3) implies (S_1) and (S_2).

Theorem 2.4. Let $\mathcal{C}$ have $\lim_{\leftarrow}$. Assume that $\mathcal{S}$ is nice, saturated and (S_3). Then for every $A\in\mathcal{C}_0$ there exists a morphism $s_0\colon \bar{A}\to A$ in $\mathcal{S}$ such that for every $s\colon C\to A$ there exists a *unique* $u\colon \bar{A}\to C$ such that $s_0=s\cdot u$. Furthermore, ϕ has a left adjoint ${}^*\phi\colon \mathcal{C}\mathcal{S}^{-1}\to\mathcal{C}$.

Proof. $\mathcal{J}=\{s\colon C\to A \mid s\in\mathcal{S}$ and $C\in\mathcal{J}_A\}\in\mathcal{U}$ since $\mathcal{S}$ is nice. We form the fiber product B of all $s\colon C\to A$ in $\mathcal{J}$. Then every $t\colon D\to A$ in $\mathcal{S}$ can be "extended" to B (see the diagram) and $x\in\mathcal{S}$ since $\mathcal{S}$ is (S_3).

$$\begin{tikzcd}
& B \arrow[dl, "p_s"'] \arrow[dd, "x"] \\
C_s \arrow[dr, "s"] \arrow[d, dashed, "c"'] & \\
D \arrow[r, "t"'] & A
\end{tikzcd}$$

Set $H=\{b\colon B\to B \mid x\cdot b=x\}$. Every $b\in H$ is in $\mathcal{S}$ because $\mathcal{S}$ is saturated. Let $y\colon \bar{A}\to B$ be the projective limit of all $b\in H$ (i.e. the kernel of perhaps infinitely many morphisms). Then y is a monomorphism and $b\cdot y=b'\cdot y$ for all $b, b'\in H$. Let $\phi\colon C\to\mathcal{C}\mathcal{S}^{-1}$

be the usual functor. Then $\phi(x)$ is an isomorphism and $\phi(x)=\phi(x)\cdot\phi(b)$ implies that $\phi(b)=1_{\phi(B)}$ for all $b\in H$. Hence $\phi(y)=1_{\phi(B)}$ since ϕ commutes with $\lim_{\leftarrow}$ and it follows that $y\in \mathcal{S}$. Then $s_0=x\cdot y\in \mathcal{S}$ and there exists h: $A\to\bar{A}$ such that $x=s_0\cdot h=x\cdot y\cdot h$. Hence $y\cdot h\in H$ and since $1_B\in H$ we get $y\cdot h\cdot y=1_B\cdot y=y$. But y is a monomorphism and so

$$h\cdot y=1_{\bar{A}}.$$

To prove the uniqueness of the extension assume that t: $D\to A$ is in $\mathcal{S}$ and that there are two morphisms $u, v:\bar{A}\to D$ such that $s_0=t\cdot u=t\cdot v$. By (S_2) there exists $t_1\in \mathcal{S}$ such that $u\cdot t_1=v\cdot t_1$. Now $s_0\cdot t_1\in \mathcal{S}$ so there exists t_2 such that $s_0\cdot t_1\cdot t_2=s_0$. Furthermore, $t_2\in \mathcal{S}$ since $\mathcal{S}$ is saturated and setting $d=y\cdot t_1\cdot t_2\cdot h$ we find $x\cdot d=x$. Hence $d\in H$ and $d\cdot y=y$ or $y\cdot t_1\cdot t_2\cdot h\cdot y=y$. Then t_1 is an epimorphism since y is a monomorphism and finally we get $u=v$.

We define two maps

$$\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(A),\phi(D)) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{C}}(\bar{A},D)$$

by the following: $\varphi(as^{-1})=a\cdot u$ where u is the unique morphism such that $s_0=s\cdot u$,

$$\psi(c)=cs_0^{-1}.$$

One verifies that φ is welldefined and that φ and ψ behave functorially with respect to A and D and that they are each others' inverses. Hence $^*\phi$ exists and $\bar{A}={}^*\phi\cdot\phi(A)$.

Remark 2.5. If $\mathcal{S}$ is (S_1) and (S_2) but we do not know that $\mathcal{S}$ is nice, then we can construct $\mathcal{C}\mathcal{S}^{-1}$ as in Theorem 1.2 but $\mathcal{C}\mathcal{S}^{-1}$ is not necessarily a $\mathcal{U}$-category. However, if ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ has a left (or right) adjoint, $^*\phi$, then

$$\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(A),\phi(B))\approx \mathrm{Hom}_{\mathcal{C}}({}^*\phi\cdot\phi(A),B)$$

and $\mathcal{C}\mathcal{S}^{-1}$ is a $\mathcal{U}$-category.

Proposition 2.6. Assume that $\mathcal{C}$ has $\lim_{\leftarrow}$ and that $\mathcal{S}$ is (S_1) and (S_2). If ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ has a left adjoint $^*\phi$: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$, then

(*a*) $\mathcal{C}\mathcal{S}^{-1}$ has $\lim_{\leftarrow}$ and ϕ commutes with $\lim_{\leftarrow}$,

(*b*) $\bar{\mathcal{S}}$ is nice and (S_3).

Proof. (*a*) We only need to show that $\mathcal{C}\mathcal{S}^{-1}$ has products. Indeed $\prod_{\mathcal{J}}\phi(A_i)=\phi(\prod_{\mathcal{J}}(A_i))$ will work which follows from the isomorphisms below, valid for any $B\prod(\mathcal{C}\mathcal{S}^{-1})_0$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(B,\prod_{\mathcal{J}}\phi(A_i)) &= \mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(B,\phi(\prod_{\mathcal{J}}A_i))\\ &\approx \mathrm{Hom}_{\mathcal{C}}({}^*\phi(B),\prod_{\mathcal{J}}A_i)\approx\prod_{\mathcal{J}}\mathrm{Hom}_{\mathcal{C}}({}^*\phi(B),A_i)\\ &\approx\prod_{\mathcal{J}}\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(B,\phi(A_i)).\end{aligned}$$

Clearly ϕ commutes with $\lim_{\leftarrow}$.

(*b*) An argument similar to the proof of Proposition 1.12 shows that $\bar{\mathcal{S}}=\mathcal{S}_\phi$ is (S_3).

Let a: $^*\phi\cdot\phi\rightarrow I_C$ and b: $I_{CS^{-1}}\rightarrow\phi\cdot{}^*\phi$ be the adjunction morphisms. Then we have (see [3]) $\phi(a_A)\cdot b_{\phi(A)}=1_{\phi(A)}$ for every $A\in C_0$. Assume that $b_{\phi(A)}=ct^{-1}$ with $c\in C$, $t\in S$. Then $(a_A\cdot c)t^{-1}=1_{\phi(A)}$ and there exists $u\in S$ such that $a_A\cdot c\cdot u=t\cdot u\in S$. Let u start at A_1. Hence a_A can be extended to a morphism in S starting at A_1. But every morphism s: $D\rightarrow A$ in $\bar{S}$ can be extended to a_A. This follows immediately from the diagram below:

$$\begin{array}{ccc} B & \xleftarrow{\quad a_B \quad} & {}^*\phi\cdot\phi(A) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \approx} \\ A & \xleftarrow{\quad a_A \quad} & {}^*\phi\cdot\phi(A) \end{array}$$

Remark 2.7. Theorem 2.4 and Proposition 2.6 imply that the adjunction morphism a_A: $^*\phi\cdot\phi(A)\rightarrow A$ is in $\bar{S}$ for every $A\in C_0$.

Furthermore, we can take $\bar{A}={}^*\phi\cdot\phi(A)$.

All this can be formulated in the following way:

Theorem 2.8. Let C be a $\mathcal{U}$-category with $\lim_{\leftarrow}$ and let $S\subset C$ be saturated. Then the following statements are equivalent:

(*a*) ϕ: $C\rightarrow CS^{-1}$ has a left adjoint $^*\phi$;

(*b*) S is nice and (S_3);

(*c*) For every $A\in C_0$ there exists s_0: $\bar{A}\rightarrow A$ in S such that for every s: $C\rightarrow A$ in S there exists a unique u: $\bar{A}\rightarrow C$ such that $s_0=s\cdot u$.

Remark 2.9. It is also true that if S is (S_3) (but not necessarily saturated) then ϕ commutes with $\lim_{\leftarrow}$ (see [1]).

For the rest of this chapter we assume, unless something else is stated, that S satisfies the conditions in Theorem 2.8.

Proposition 2.10. (*a*) The adjunction morphism b: $I_{CS^{-1}}\rightarrow\phi\cdot{}^*\phi$ is an isomorphism.
(*b*) $^*\phi$ is fully faithful.

Proof. (*a*) In the proof of Proposition 2.6 we had the following: $\phi(a_A)\cdot b_{\phi(A)}=1_{\phi(A)}$. But $\phi(a_A)$ is an isomorphism (Remark 2.7) and so is $b_{\phi(A)}$ for every $A\in C_0$. It follows that b is an isomorphism.

(*b*) We must show that the canonical map:

$$\varphi:\ \mathrm{Hom}_{CS^{-1}}(\phi A,\phi B)\rightarrow\mathrm{Hom}_C\,({}^*\phi\cdot\phi A,{}^*\phi\cdot\phi B)$$

is a bijection for all A, $B\in C_0$.

Let cs^{-1}: $\phi(A)\rightarrow\phi(B)$ be given. Writing $\bar{A}$ for $^*\phi\cdot\phi(A)$ we get a commutative diagram where $\bar{c}={}^*\phi\cdot\phi(c)$.

$$\begin{tikzcd} \bar{A} \arrow[r, "\exists_1 v"] \arrow[d, "a_A"'] & \bar{C} \arrow[r, "\bar{c}"] \arrow[d, "a_C"] \arrow[dl, "\exists_1 u"] & \bar{B} \arrow[d, "a_B"] \\ A \arrow[r, "s"'] & C \arrow[r, "c"'] & B \end{tikzcd}$$

φ is given by $\varphi(cs^{-1}) = \bar{c}\cdot v$. One verifies that φ is well defined and has an inverse ψ where $\psi(f) = (a_B \cdot f)a_A^{-1}$ for a given f: $\bar{A} \to \bar{B}$.

Definition 2.11. (*a*) Let $\mathcal{S}$ be any subset of $\mathcal{C}$. We define a subset of $\mathcal{C}_0$ in the following way:

$\mathcal{D}(\mathcal{S}) = \{P \in \mathcal{C}_0 |$ for all s: $A \to B$ in $\mathcal{S}$ and all b: $P \to B$ in $\mathcal{C}$ there exists a unique a: $P \to A$ such that $b = s \cdot a\}$.
$\mathcal{J}(\mathcal{S})$ is defined dually.

(*b*) If $\mathcal{O} \subset \mathcal{C}_0$, then we define a subset of $\mathcal{C}$ as follows:

$\mathcal{E}(\mathcal{O}) = \{s$: $A \to B|$ for all $P \in \mathcal{O}$ and all b: $P \to B$ in $\mathcal{C}$ there exists a unique a: $P \to A$ such that $b = s \cdot a\}$.

$\mathcal{M}(\mathcal{O})$ is defined dually.

Remark. If $\mathcal{S}$ = all epimorphisms (monomorphisms) in $\mathcal{C}$, then $\mathcal{D}(\mathcal{S})$ ($\mathcal{J}(\mathcal{S})$) is a subset of the projective (injective) objects in $\mathcal{C}_0$. The definitions look similar to those of projective (injective) classes (see [8] p. 135 and [7]), but we require uniqueness for the morphism a.

Proposition 2.12. Let $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{O} \subset \mathcal{C}_0$. Then

(*a*) $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{E}(\mathcal{O}_1) \supset \mathcal{E}(\mathcal{O}_2)$ and $\mathcal{M}(\mathcal{O}_1) \supset \mathcal{M}(\mathcal{O}_2)$;
(*b*) $\mathcal{S}_1 \subset \mathcal{S}_2$ implies $\mathcal{D}(\mathcal{S}_1) \supset \mathcal{D}(\mathcal{S}_2)$ and $\mathcal{J}(\mathcal{S}_1) \supset \mathcal{J}(\mathcal{S}_2)$;
(*c*) $\mathcal{S} \subset \mathcal{E}\mathcal{D}(\mathcal{S})$, $\mathcal{S} \subset \mathcal{M}\mathcal{J}(\mathcal{S})$, $\mathcal{O} \subset \mathcal{D}\mathcal{E}(\mathcal{O})$ and $\mathcal{O} \subset \mathcal{J}\mathcal{M}(\mathcal{O})$;
(*d*) $\mathcal{D}\mathcal{E}\mathcal{D}(\mathcal{S}) = \mathcal{D}(\mathcal{S})$, etc.;
(*e*) If $\mathcal{C}$ has $\lim_{\leftarrow}$ ($\lim_{\rightarrow}$), then $\mathcal{E}(\mathcal{O})$ is (S_3) ($\mathcal{M}(\mathcal{O})$ is $(S_3{}^0)$ for any $\mathcal{O} \subset \mathcal{C}_0$;
(*f*) The embedding functor $\mathcal{D}(\mathcal{S}) \to \mathcal{S}(\mathcal{J}(\mathcal{C}) \subset \mathcal{C})$ of the full subcategory $\mathcal{D}(\mathcal{S})(\mathcal{J}(\mathcal{S}))$ into $\mathcal{C}$ commutes with $\lim_{\rightarrow}$ ($\lim_{\leftarrow}$) for any $\mathcal{S} \subset \mathcal{C}$.

Proof. (*a*)–(*d*) follow from the usual lattice arguments.
(*e*) The functor $H^P = \mathrm{Hom}_{\mathcal{C}}(P, \cdot)$: $\mathcal{S} \to \mathcal{E}ns$ commutes with $\lim_{\leftarrow}$. But then $\mathcal{E}(\mathcal{O}) = \bigcap_{P \in \mathcal{O}} \mathcal{S}_{H^P}$ is (S_3) since each $\mathcal{S}_{H^P}$ is.

(*f*) We show that $\mathcal{D}(\mathcal{S}) \to \mathcal{C}$ commutes with sums and cokernels. Let $P_i \in \mathcal{D}(\mathcal{S})$ for all $i \in I$. We want $\oplus_I P_i \in \mathcal{D}(\mathcal{S})$. Let s: $A \to B$ be in $\mathcal{S}$ and let a: $\oplus_I P_i \to B$ be in $\mathcal{C}$. Let u_i: $P_i \to \oplus_I P_i$ be the canonical morphism.
Since $P_i \in \mathcal{D}(\mathcal{S})$, there exists a unique b_i: $P_i \to A$ such that $s \cdot b_i = a \cdot u_i$ (for all $i \in I$).

$$\begin{tikzcd} P_i \arrow[r, "u_i"] \arrow[d, dashed, "b_i"'] & \bigoplus_I P_i \arrow[d, "a"] \arrow[dl, dashed, "b"'] \\ A \arrow[r, "s"'] & B \end{tikzcd}$$

By the definition of sum there exists a unique $b\colon \oplus_I P_i \to A$ such that $b_i = b\cdot u_i$ for all $i\in I$. But then $s\cdot b\cdot u_i = s\cdot b_i = a\cdot u_i$ for all $i\in I$ and by the uniqueness it follows that $a = s\cdot b$. Hence $\mathcal{D}(\mathcal{S})\to\mathcal{C}$ commutes with sums.

Next let $A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B$ be given with $A, B\in\mathcal{D}(\mathcal{S})$. Let $p\colon B\to C$ be Coker $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. We show that $C\in\mathcal{D}(\mathcal{S})$. Let $s\colon E\to F$ be in $\mathcal{S}$ and $c\colon C\to F$ in $\mathcal{C}$ (see diagram below). Since $B\in\mathcal{D}(\mathcal{S})$ there exists a unique $d\colon B\to E$ such that $s\cdot d = c\cdot p$.

$$\begin{tikzcd} A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[dr, dashed, "e"'] & B \arrow[r, "p"] \arrow[d, "d"] & C \arrow[d, "c"] \arrow[dl, dashed, "f"] \\ & E \arrow[r, "s"'] & F \end{tikzcd}$$

But $A\in\mathcal{D}(\mathcal{S})$ and $c\cdot p\cdot a_1 = c\cdot p\cdot a_2 = s\cdot d\cdot a_1 = s\cdot d\cdot a_2$ implies that there exists a unique $e\colon A\to E$ such that $s\cdot e = s\cdot d\cdot a_1 = s\cdot d\cdot a_2$. Hence $d\cdot a_1 = d\cdot a_2 = e$ and by the definition of Coker $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ there exists a unique $f\colon C\to E$ such that $d = f\cdot p$. But then $c\cdot p = s\cdot d = s\cdot f\cdot p$ and $c = s\cdot f$ since p is an epimorphism. Hence $C\in\mathcal{D}(\mathcal{S})$ and we are done.

Now let $\mathcal{S}$ satisfy the conditions of Theorem 2.8 again.

Definition 2.13. $\check{\mathcal{C}}\subset\mathcal{C}$ is the full subcategory defined by

$$\check{\mathcal{C}}_0 = \{A\in\mathcal{C}_0 \mid a_A\colon {}^*\phi\cdot\phi(A)\to A \text{ is an isomorphism}\}.$$

Proposition 2.14. *(a)* $\check{\mathcal{C}}_0 = \mathcal{D}(\mathcal{S})$.
(b) $\mathcal{S} = \mathcal{E}(\check{\mathcal{C}}_0)$.

Proof. (a) Assume that $a_A\colon \bar{A}\to A$ is an isomorphism and let $f\colon A\to C$ in $\mathcal{C}$ and $s\colon B\to C$ in $\mathcal{S}$ be given. Consider the diagram where $\bar{f} = {}^*\phi\cdot\phi(f)$. It follows that $f = s\cdot(u\cdot\bar{f}\cdot a_A{}^{-1})$ and it is easily seen that this factorization is unique.

$$\begin{tikzcd} \bar{A} \arrow[r, "a_A", "\approx"'] \arrow[d, "\bar{f}"'] & A \arrow[d, "f"] \\ \bar{C} \arrow[r, "a_C"] \arrow[d, "\exists_1 u"'] & C \\ B \arrow[ur, "s"'] & \end{tikzcd}$$

Hence $A\in\mathcal{D}(\mathcal{S})$.

Conversely assume that $A\in\mathcal{D}(\mathcal{S})$. Since $a_C\in\mathcal{S}$ we get two diagrams:

$$\begin{tikzcd} & A \arrow[dl, "\exists_1 u"'] \arrow[d, equal] \\ \bar{A} \arrow[r, "a_A"'] & A \end{tikzcd}$$

$$\begin{tikzcd} & \bar{A} \arrow[d, "a_A"] \arrow[dl, "\exists_1"'] \\ \bar{A} \arrow[r, "a_A"'] & A \end{tikzcd}$$

From the first one we get $u\colon A \to \bar{A}$ such that $a_A \cdot u = 1_A$. But then $a_A \cdot (u \cdot a_A) = a_A \cdot 1$ and the uniqueness in the second diagram implies that a_A is an isomorphism and $A \in \check{\mathsf{C}}_0$.

(*b*) The proof of $\mathcal{S} \subset \mathcal{E}(\check{\mathsf{C}}_0)$ is identical with the first part of the proof of (*a*). Assume that $t\colon B \to C$ is in $\mathcal{E}(\check{\mathsf{C}}_0)$ and consider the diagram:

$$\begin{tikzcd} \bar{B} \arrow[r, "\bar{t}"] \arrow[d, "a_B"'] & \bar{C} \arrow[d, "a_C"] \arrow[dl, "\exists_1 u"'] \\ B \arrow[r, "t"'] & C \end{tikzcd}$$

Since $\bar{B}, \bar{C} \in \check{\mathsf{C}}_0$ there exists a unique u such that $a_C = t \cdot u$. Now $t \cdot a_B = a_C \cdot \bar{t} = t \cdot (u \cdot \bar{t})$ and uniqueness implies that $a_B = u \cdot \bar{t}$. But a_B, $a_C \in \mathcal{S}$ and from Proposition 1.14 we conclude that $u \in \mathcal{S} = \bar{\mathcal{S}}$ and hence $t \in \mathcal{S}$.

Proposition 2.15. (*a*) The imbedding functor $j\colon \check{\mathsf{C}} \to \mathsf{C}$ has a right adjoint $j^*\colon \mathsf{C} \to \check{\mathsf{C}}$.
(*b*) The categories $\check{\mathsf{C}}$ and $\mathsf{C}\mathcal{S}^{-1}$ are equivalent.

Proof. (*a*) follows immediately from Proposition 2.14 by setting $j^*(A) = {}^*\phi \cdot \phi(A) = \bar{A}$.
(*b*) First we observe that if $A, B \in \check{\mathsf{C}}_0$ then the canonical map

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{C}\mathcal{S}^{-1}}(\phi(A), \phi(B))$$

is bijective. Hence the functor $\phi \cdot j\colon \check{\mathsf{C}} \to \mathsf{C}\mathcal{S}^{-1}$ is fully faithful. Furthermore, $\phi(a_A)$: $\phi(\bar{A}) \to \phi(A)$ is an isomorphism and thus every object in $\mathsf{C}\mathcal{S}^{-1}$ is isomorphic to an object of the form $\phi \cdot j(B)$ with $B \in \check{\mathsf{C}}_0$. This implies that $\phi \cdot j$ is an equivalence.

3. Functorial properties of $\mathsf{C}\mathcal{S}^{-1}$

In this section we study the behaviour of $\mathsf{C}\mathcal{S}^{-1}$ under functors and changes of $\mathcal{S}$.

Proposition 3.1. Let $F\colon \mathsf{C} \to \mathcal{D}$ be left exact and assume that C is small and has finite $\lim_{\leftarrow}$. Set $\mathcal{S} = \mathcal{S}_F$ and let $F = F_1 \cdot \phi$ be the canonical factorization of F. Then:

(*a*) ϕ is left exact and an epimorphism in *Cat*.

(*b*): F_1 is left exact and conservative, i.e. $F_1\,(a)$ is an isomorphism if and only if a is an isomorphism.

Proof. (*a*): We know that ϕ is left exact (Corollary 1.10). That ϕ is an epimorphism follows from the universal property of $\mathsf{C}\mathcal{S}^{-1}$.

(*b*): $F_1(as^{-1}) = F(a) \cdot F(s)^{-1}$ by definition. Assume that $F(a) \cdot F(a)^{-1}$ is an isomorphism. It follows that $F(a)$ is an isomorphism and hence $a \in \mathcal{S}_F$. Thus $\phi(a)$ and $as^{-1} = \phi(a)\cdot\phi(s)^{-1}$ are isomorphisms and F_1 is conservative. To prove that F_1 is left exact we first note that $F_1(\phi(A_1) \times \phi(A_2) \approx (F_1\cdot\phi)(A_1 \times A_2) = F(A_1 \times A_2) = F(A_1) \times F(A_2) \approx F_1(\phi(A_1)) \times F_1(\phi(A_2))$, so F_1 commutes with finite products.

By Lemma 1.6 it is sufficient to consider kernels of pairs of the form $\begin{pmatrix}\phi(a_1)\\ \phi(a_2)\end{pmatrix}$. Then

$$\operatorname{Ker}\begin{pmatrix}F_1(\phi(a_1))\\ F_1(\phi(a_2))\end{pmatrix} = \operatorname{Ker}\begin{pmatrix}F(a_1)\\ F(a_2)\end{pmatrix} \approx F\left(\operatorname{Ker}\begin{pmatrix}a_1\\ a_2\end{pmatrix}\right)$$

$$= F_1\left(\phi\left(\operatorname{Ker}\begin{pmatrix}a_1\\ a_2\end{pmatrix}\right)\right) \approx F_1\left(\operatorname{Ker}\begin{pmatrix}\phi(a_1)\\ \phi(a_2)\end{pmatrix}\right),$$

since ϕ commutes with kernels. Hence F_1 is left exact.

Definition 3.2. Let $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{S}_1 \subset \mathcal{C}_1$ be categories such that $\mathcal{S}$ and $\mathcal{S}_1$ are nice and satisfy (S_1) and (S_2). A functor G: $\mathcal{C}_1 \to \mathcal{C}$ is *continuous* (with respect to $\mathcal{S}_1$ and $\mathcal{S}$) if $G(\mathcal{S}_1) \subset \bar{\mathcal{S}}$. Assume furthermore that $^*\phi$ and $^*\phi_1$ exist. G is called *cocontinuous* if $G((\check{\mathcal{C}}_1)_0) \subset \check{\mathcal{C}}_0$.

Proposition 3.3. (*a*) G: $\mathcal{C}_1 \to \mathcal{C}$ is continuous if and only if there exists a functor $\tilde{G}$: $\mathcal{C}_1\mathcal{S}_1^{-1} \to \mathcal{C}\mathcal{S}^{-1}$ such that $\phi\cdot G = \tilde{G}\cdot\phi_1$.
(*b*) G: $\mathcal{C}_1 \to \mathcal{C}$ is cocontinuous if and only if there exists a functor $\bar{G}$: $\mathcal{C}_1\mathcal{S}_1^{-1} \to \mathcal{C}\mathcal{S}^{-1}$ such that $^*\phi\cdot\bar{G} \approx G\cdot{}^*\phi_1$.

Proof. (*a*) This follows directly from the definitions.
(*b*) Assume that G is cocontinuous. Then we define a functor $\check{G}$: $\check{\mathcal{C}}_1 \to \check{\mathcal{C}}$ by setting $\check{G}(a) = G(a)$ for all $a \in \check{\mathcal{C}}_1$. Clearly $G\cdot j_1 = j\cdot\check{G}$ and we have a diagram:

$$\begin{tikzcd}
\check{\mathcal{C}}_1 \arrow[rr, "\check{G}"] \arrow[d, shift right, "j_1"'] & & \check{\mathcal{C}} \arrow[d, shift right, "j"'] \\
\mathcal{C}_1 \arrow[rr, "G"] \arrow[u, shift right, "j_1^*"'] \arrow[d, shift right, "\phi_1"'] & & \mathcal{C} \arrow[u, shift right, "j^*"'] \arrow[d, shift right, "\phi"'] \\
\mathcal{C}_1\mathcal{S}_1^{-1} \arrow[rr, "\bar{G}"] \arrow[u, shift right, "^*\phi_1"'] & & \mathcal{C}\mathcal{S}^{-1} \arrow[u, shift right, "^*\phi"']
\end{tikzcd}$$

where $\qquad j_1^*\cdot j_1 \approx I_{\check{\mathcal{C}}},\ j^*\cdot j \approx I_{\check{\mathcal{C}}},\ \phi_1\cdot{}^*\phi_1 \approx I$ and $\phi\cdot{}^*\phi \approx I$.

Furhermore, $\qquad j_1\cdot j_1^* = {}^*\phi\cdot\phi_1$ and $j\cdot j^* = {}^*\phi\cdot\phi$. Set $\bar{G} = \phi\cdot G\cdot{}^*\phi_1$.

Then we get

$$\begin{aligned}
{}^*\phi\cdot\bar{G} &= {}^*\phi\cdot\phi\cdot G\cdot{}^*\phi_1 \approx j\cdot j^*\cdot G\cdot({}^*\phi_1\cdot\phi_1)\cdot{}^*\phi_1 \\
&= j\cdot j^*\cdot(G\cdot j_1)\cdot j_1^*\cdot{}^*\phi_1 = j\cdot(j^*\cdot j)\cdot\check{G}\cdot j_1^*\cdot{}^*\phi_1 \\
&\approx (j\cdot\check{G})\cdot j_1^*\cdot{}^*\phi_1 = G\cdot(j_1\cdot j_1^*)\cdot{}^*\phi_1 = G\cdot({}^*\phi_1\cdot(\phi_1\cdot{}^*\phi_1)) \approx G\cdot{}^*\phi_1.
\end{aligned}$$

Conversely, assume that there exists $\bar{G}$ such that $^*\phi\cdot\bar{G}\approx G\cdot{}^*\phi_1$. Let $B\in(\check{C}_1)_0$, i.e. $^*\phi_1\cdot\phi_1(B)\approx B$. Then we get $^*\phi\cdot\phi\cdot G(B)\approx{}^*\phi\cdot\phi\cdot(G\cdot{}^*\phi_1)\cdot\phi_1(B)\approx{}^*\phi\cdot(\phi\cdot{}^*\phi)\cdot\bar{G}\cdot\phi_1(B)\approx{}^*\phi\cdot\bar{G}\cdot\phi_1(B)\approx G\cdot{}^*\phi_1\cdot\phi_1(B)\approx G(B)$, i.e. $G(B)\in C_0$ and G is cocontinuous.

Remark. (*a*) Since ϕ_1 is an epimorphism, it follows that G is unique with property $\tilde{G}\cdot\phi_1=\phi\cdot G$.

(*b*) If G is both continuous and cocontinuous, then $\tilde{G}\approx\bar{G}=\phi\cdot G\cdot{}^*\phi_1$.

Proposition 3.4. (*a*) Assume that G: $C_1\to C$ has a right adjoint G^*. If G is cocontinuous then G^* is continuous and $\bar{G}$ has a right adjoint $(\bar{G})^*=\phi_1\cdot G^*\cdot{}^*\phi$.

(*b*) Assume that G has a left adjoint $^*G$. Then $^*G$ is cocontinuous if and only if G is continuous and $\tilde{G}$ has a left adjoint $^*(\tilde{G})$.

Proof. (*a*) We have $^*\phi\cdot\bar{G}\approx G\cdot{}^*\phi_1$ since G is cocontinuous. Using the fact that $^*\phi$ is fully faithful we obtain:

$$\mathrm{Hom}_{CS^{-1}}(\bar{G}(A),B)\approx\mathrm{Hom}_C({}^*\phi\cdot\bar{G}(A),{}^*\phi(B))\approx\mathrm{Hom}_C(G\cdot{}^*\phi(A),{}^*\phi(B))$$

$$\approx\mathrm{Hom}_{C_1}({}^*\phi_1(A),G^*\cdot{}^*\phi(B))\approx\mathrm{Hom}_{C_1S_1^{-1}}(A,\phi_1\cdot G^*\cdot{}^*\phi(B)),$$

which shows that $(\bar{G})^*=\phi_1\cdot G^*\cdot{}^*\phi$.

Now taking right adjoints of both sides of $G\cdot{}^*\phi_1\approx{}^*\phi\cdot\bar{G}$ gives $\phi_1\cdot G^*\approx(\bar{G})^*\cdot\phi$, i.e. G^* is continuous.

(*b*) If G is continuous then $\phi\cdot G=\tilde{G}\cdot\phi_1$. Assuming that $^*(\tilde{G})$ exists we conclude that $^*G\cdot{}^*\phi={}^*\phi_1\cdot{}^*(\tilde{G})$ and $^*G$ is cocontinuous. Conversely if $^*G$ is cocontinuous, then (*a*) implies that $(^*G)^*=G$ is continuous and so $\tilde{G}\approx\phi\cdot G.\ {}^*\phi_1$ exists. Hence

$$\mathrm{Hom}_{CS^{-1}}(A,\tilde{G}(B))\approx\mathrm{Hom}_{CS^{-1}}(A,\phi\cdot G\cdot{}^*\phi_1(B))\approx\mathrm{Hom}_{C_1}({}^*G\cdot{}^*\phi(A),{}^*\phi_1(B))$$

$$\approx\mathrm{Hom}_{C_1}({}^*\phi_1\cdot\overline{({}^*G)}(A),{}^*\phi_1(B))\approx\mathrm{Hom}_{C_1S_1^{-1}}(\overline{({}^*G)}(A),B)$$

since $^*G$ is cocontinuous, and ϕ_1 is fully faithful. This shows that $^*(\tilde{G})=\overline{({}^*G)}$.

Remark. Let G: $C_1\to C$ be left exact with C_1 small and let $S\subset C$ be nice and satisfy (S_1) and (S_2). Then $S_1=\{a\in C_1\,|\,G(a)\in\bar{S}\}\subset C_1$ satisfies (S_1) and $S_2)$ and makes G continuous.

Proof. $S_1=S_{\phi\cdot G}$ and $\phi\cdot G$ is left exact.

Proposition 3.5. Let $S_i\subset C$ for $i\in\mathcal{J}$ be a set of nice subcategories of C, each of them satisfying (S_1) and (S_2). Let S be the subcategory of C generated by $\bigcup_{\mathcal{J}}S_i$. Then S is nice and satisfies (S_1) and (S_2).

Proof. A morphism s in S is a finite composition $s_i\cdot s_j\cdot\ldots\cdot s_k=s$ with $s_i\in S_i$, $s_j\in S_j$, etc. Given $a\in C$ with the same terminal as s there exist $t_i\in S_i$ and $a_i\in C$ such that $a\cdot t_i=s_i\cdot a_i$ since S_i is (S_1). Similarly there exist $t_j\in S_j$ and $a_j\in C$ such that $a_i\cdot t_j=s_j\cdot a_j$. Continuing in this way a finite number of steps we prove that S is (S_1). In the same way we verify (S_2).

Let $A\in C_0$ and s: $C\to A$ in S be given. Assume that $s=s_{i_1}\cdot s_{i_2}\cdot\ldots\cdot s_{i_n}$ with $s_{i_k}\in S_{i_k}$ for $k=1,2,\ldots,n$.

Denote by $J_B^{(i)}$ the set of objects to which every $\cdot \xrightarrow{s_i} B$ in $\mathcal{S}_i$ can be extended in $\mathcal{S}_i$ (saying that $\mathcal{S}_i$ is nice). Considering the following commutative diagram we find $t_{i_k}\colon A_{i_k} \to A_{i_{k-1}}$ in $\mathcal{S}_{i_k}$ with $A_{i_k} \in J_{A_{i_{k-1}}}$ for $k = 1, 2, \ldots, n$ and $A_{i_0} = A$. The existence of $t' \in \mathcal{S}_{i_2}$ etc. follows from (S_1). Hence there exists $u \in \mathcal{C}$ such that

$$\begin{tikzcd}
C \arrow[d, dashed] & & & A_{i_n} \arrow[lll, "u"'] \arrow[dl, dashed] \\
\cdot \arrow[d, "s_{i_2}"'] & \cdot \arrow[l] \arrow[d, "t'"'] & A_{i_2} \arrow[l] \arrow[dl, "t_{i_2}"] & \\
\cdot \arrow[d, "s_{i_1}"'] & A_{i_1} \arrow[l] \arrow[dl, "t_{i_1}"] & & \\
A & & &
\end{tikzcd}$$

$$s \cdot u = t_{i_1} \cdot t_{i_2} \cdot \ldots \cdot t_{i_n} \in \mathcal{S}.$$

Now define $K_{i_1} = J_A^{(i_1)}$ and $K_{i_{k+1}} = \bigcup_{B \in K_{i_k}} J_B^{(i_{k+1})}$ for $k = 1, 2, \ldots, n-1$. Then $A_{i_n} \in K_{i_n}$ and K_{i_n} is an object in $\mathcal{E}ns$. Finally we set $K_A = \bigcup_{(i_1, i_2, \ldots, i_n)} K_{i_n}$ where the union is taken over all finite subsets of J. Clearly K_A is in $\mathcal{E}ns$ and every morphism in $\mathcal{S}$ ending in A can be extended in $\mathcal{S}$ to an object in K_A. Thus $\mathcal{S}$ is nice and we are done.

Proposition 3.6. Let $\mathcal{S}_i$ and $\mathcal{S}$ be defined as in Proposition 3.5. Then $\phi\colon \mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ is the fiber sum (in *Cat*) of $\phi_i\colon \mathcal{C} \to \mathcal{C}\mathcal{S}_i^{-1}$, $i \in \mathfrak{J}$.

Proof. Consider the diagram: Since $\mathcal{S}_i \subset \mathcal{S}$ then exist U_i such that $\phi = U_i \cdot \phi_i$ for all $i \in \mathfrak{J}$. Given functors $F_i\colon \mathcal{C}\mathcal{S}^{-1} \to \mathcal{D}$ such that $F_i \cdot \phi_i = F_j \cdot \phi_j = G$ for all $i, j \in \mathfrak{J}$ then $G(s)$ is an isomorphism for every $s \in \mathcal{S}$. Hence there exists a unique functor H: $\mathcal{C}\mathcal{S}^{-1} \to \mathcal{D}$ such that $G = H \cdot \phi$. Then $F_i \cdot \phi_i = G = H \cdot \phi = H \cdot U_i \cdot \phi_i$ and it follows $F_i = H \cdot U_i$ for all $i \in \mathfrak{J}$ since ϕ_i is an epimorphism. It is easily seen that H is unique with this property and the proof is finished.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\phi_i"] \arrow[d, "\phi"] & \mathcal{C}\mathcal{S}_i^{-1} \arrow[dl, "U_i"] \arrow[d, "F_i"] \\
\mathcal{C}\mathcal{S}^{-1} \arrow[r, "H"'] & D
\end{tikzcd}$$

4. Hereditary properties

In this section we consider properties of $\mathcal{C}$ which are inherited by $\mathcal{C}\mathcal{S}^{-1}$ under various conditions on $\mathcal{S}$.

Proposition 4.1. Let $\mathcal{S} \subset \mathcal{C}$ satisfy (S_1) and (S_2). Then

(*a*) If $\mathcal{C}$ has finite $\lim_{\leftarrow}$ so has $\mathcal{C}\mathcal{S}^{-1}$.

(*b*) If $\mathcal{C}$ is additive so is $\mathcal{C}\mathcal{S}^{-1}$.

Proof. (*a*) is Corollary 1.10.

(*b*) From proposition 1.9 we know that $\mathcal{C}\mathcal{S}^{-1}$ has finite products. Let O be a zero-object in $\mathcal{C}$. Then clearly $\phi(O)$ is a final object in $\mathcal{C}\mathcal{S}^{-1}$. Consider the following commutative diagram:

$$\begin{tikzcd} O & A \arrow[l, "s"'] \arrow[r, "a"] & B \\ & O \arrow[ul, "1_O"] \arrow[u, "O"'] \arrow[ur, "O_{OB}"'] \arrow[ul, phantom, shift left=3, "\approx"] & \end{tikzcd}$$

Here we read off that $as^{-1} = O_{OB}1_O{}^{-1}$ for all $a \in \mathcal{C}$, $s \in \mathcal{S}$.

Hence $\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(O), \phi(B))$ consists of exactly one element and $\phi(O)$ is a zero-object in $\mathcal{C}\mathcal{S}^{-1}$.

Given two morphisms in $\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(A, B)$, they can be represented as a_1s^{-1} and a_2s^{-1}, and we define addition by $a_1s^{-1} + a_2s^{-1} = (a_1 + a_2)s^{-1}$. This defines an abelian group structure on $\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(A, B)$, as is easily checked.

Let $\mathcal{C}$ be a category with finite limits. Then, given any morphism f: $A \to B$ in $\mathcal{C}$, there exists a canonical factorization.

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & B +_A B \\ & \mathrm{Coim}\, f \arrow[r, "\bar{f}"] & \mathrm{Im}\, f \arrow[u, "j"'] & \end{tikzcd}$$

Here

$$\begin{tikzcd} A \times_B A \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is the fiber product of f by itself ("the square of f"). $B +_A B$ is defined dually and $q = \mathrm{Coker} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $j = \mathrm{Ker} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$. It is easy to check that there exists a unique $\bar{f}$ such that $f = j \cdot \bar{f} \cdot q$.

Definition 4.2. A category $\mathcal{C}$ is called *regular* if $\mathcal{C}$ has finite limits and $\bar{f}$ (defined above) is an isomorphism for every $f \in \mathcal{C}$.

Example. $\mathcal{E}ns$, any topos, any abelian category, are all regular categories.

Definition 4.3. An epimorphism (monomorphism) f: $A \to B$ is *effective* if

$$\mathrm{Coker} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = f \qquad \left(\mathrm{Ker} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = f \right).$$

Proposition 4.4. Let $\mathcal{C}$ be a regular category.

(*a*) The factorization, $f = j \cdot q$, of a morphism f into an epimorphism q and a monomorphism j is unique up to an isomorphism.

(*b*) Every epimorphism (monomorphism) is effective.

(*c*) If f is both a monomorphism and an epimorphism, then f is an isomorphism.

Proof. (*a*) Since kernels and cokernels are only defined up to isomorphism we may as well assume that $\operatorname{Coim} f = \operatorname{Im} f$. Assume that $f = j \cdot q = v \cdot u$ where v is a monomorphism and u an epimorphism. We look at the following diagram:

$$\begin{tikzcd}
A \times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "q"] \arrow[dr, "u"'] & \operatorname{Im} f \arrow[r, "j"] \arrow[d, dashed, "\exists_1 \, a"] & B \\
& & \cdot \arrow[ur, "v"'] &
\end{tikzcd}$$

$f \cdot p_1 = f \cdot p_2$ implies that $v \cdot u \cdot p_1 = v \cdot u \cdot p_2$ and $u \cdot p_1 = u \cdot p_2$ since v is a monomorphism. Hence there exists a unique a such that $u = a \cdot q$ and a must be an epimorphism since u is. Furthermore, $v \cdot a \cdot q = v \cdot u = f = j \cdot q$ and $v \cdot a = j$, since q is an epimorphism. From the fact that j is a monomorphism it follows that a is both a monomorphism and an epimorphism. (*c*) implies that a is an isomorphism.

(b) If $f: A \to B$ is an epimorphism, then $i_1 = i_2$ is an isomorphism and $j: \operatorname{Im} f \to B$ is an isomorphism. Hence $f = \operatorname{Coker} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and f is an effective epimorphism.

(*c*) Since f is both an effective epimorphism and a monomorphism we know from the proof of (*b*) that j and q and hence f are all isomorphisms.

Remark 4.5: It is easily verified that a morphism f is an effective epimorphism if and only if there exists a pair $\cdot \overset{a_1}{\underset{a_2}{\rightrightarrows}} \cdot$ such that $f = \operatorname{Coker} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

For the rest of this chapter we assume that $S \subset \mathsf{C}$ is a nice subcategory containing the isomorphisms of C.

Proposition 4.6. Let C be regular and let $S \subset \mathsf{C}$ be such that $\phi: \mathsf{C} \to \mathsf{C}S^{-1}$ is exact (e.g. satisfies S_1, S_2, S_1^0 and S_2^0). Then $\mathsf{C}S^{-1}$ is regular.

Proof. Let $\alpha = fs^{-1}$ be a morphism in $\mathsf{C}S^{-1}$. We first show that it suffices to consider morphisms of the form $\phi(a)$ with $a \in \mathsf{C}$. Indeed we have $\alpha = \phi(f) \cdot \beta$, where $\beta = \phi(s)^{-1}$ is an isomorphism in $\mathsf{C}S^{-1}$. Consider the following diagram (in $\mathsf{C}S^{-1}$), where everything commutes:

$$\begin{tikzcd}
A \times_B A \arrow[r, "\approx"] \arrow[rr, bend left, "p_1"] \arrow[dd, "p_2"'] \arrow[dr, "\gamma"', "\approx"] & C \times_B A \arrow[r] \arrow[d, "\approx"] & A \arrow[d, "\approx"', "\beta"] \\
& C \times_B C \arrow[r, "q_1"] \arrow[d, "q_2"'] & C \arrow[d, "\phi(f)"] \\
A \arrow[r, "\approx", "\beta"'] & C \arrow[r, "\phi(f)"'] & B
\end{tikzcd}$$

Here we used the fact $A\times_C(C\times_B A)\approx A\times_B A$. Hence $p_i=\beta^{-1}\cdot q_i\cdot\gamma$ $(i=1, 2)$ where β and γ are isomorphisms and it follows: $\operatorname{Coim}(\phi(f)\cdot\beta)=\operatorname{Coker}\begin{pmatrix}p_1\\p_2\end{pmatrix}=\operatorname{Coker}\begin{pmatrix}\beta^{-1}\cdot q_1\cdot\gamma\\\beta^{-1}\cdot q_1\cdot\gamma\end{pmatrix}$ $\approx\operatorname{Coker}\begin{pmatrix}q_1\\q_2\end{pmatrix}=\operatorname{Coim}(f)$. Similarly we have $\operatorname{Im}(\phi(f)\cdot\beta)\approx\operatorname{Im} f$.

Now let $\alpha=\phi(f)$ be a morphism in $\mathcal{C}\mathcal{S}^{-1}$ where $f\colon A\to B$ is in $\mathcal{C}$. We take ϕ on the diagram preceding Definition 4.2 and use the fact that ϕ commutes with finite limits. Hence $\operatorname{Coim}\phi(f)=\phi(\operatorname{Coim} f)$ and $\operatorname{Im}\phi(f)=\phi(\operatorname{Im} f)$. Furthermore $\phi(f)=\phi(j)\cdot\phi(\bar f)\cdot\phi(q)$ and $\bar f$ is an isomorphism. It follows that $\phi(\bar f)$ is an isomorphism (since ϕ is a functor) and $\mathcal{C}\mathcal{S}^{-1}$ is regular.

Proposition 4.7. Let $\mathcal{S}\subset\mathcal{C}$ satisfy the assumptions of Proposition 4.6. Let $f=j\cdot q$ be the canonical factorization of a morphism in $\mathcal{C}$. If $f\in\mathcal{S}$ then $j, q\in\mathcal{S}$.

Proof. $\phi(f)=\phi(j)\cdot\phi(q)$ is an isomorphism and hence $\phi(q)$ is a monomorphism. But q is an epimorphism and ϕ is exact, hence $\phi(q)$ is an epimorphism. It follows that $\phi(q)$ is an isomorphism since $\mathcal{C}\mathcal{S}^{-1}$ is regular. Thus $q\in\bar{\mathcal{S}}$ and similarly $j\in\bar{\mathcal{S}}$.

Definition 4.8. An epimorphism $f\colon A\to C$ is called *universal* if for every morphism $g\colon B\to C$ the lifted morphism f_1 is an epimorphism.

$$\begin{tikzcd}
A\times_C B \arrow[r, "f_1"] \arrow[d, "g_1"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

A *universal monomorphism* is defined dually.

Proposition 4.9. Let $\mathcal{C}$ be regular and ϕ exact. If every epimorphism (monomorphism) in $\mathcal{C}$ is universal, then the same is true for $\mathcal{C}\mathcal{S}^{-1}$.

Proof. It is clearly sufficient to consider epimorphisms of the form $\phi(f)$ with $f\in\mathcal{C}$. Let $f=j\cdot q$ be the canonical factorization of f in $\mathcal{C}$. Then $\phi(f)=\phi(j)\cdot\phi(q)$ is the canonical factorization of $\phi(f)$, since ϕ is exact. By uniqueness we conclude that $\phi(j)$ is an isomorphism if $\phi(f)$ is an epimorphism. Hence we can restrict to the case of $\phi(f)=\phi(q)$ where q is an epimorphism in $\mathcal{C}$. It is also sufficient to take fiber products with morphisms of the form $\phi(g)$ with $g\in\mathcal{C}$. Since ϕ is exact, it commutes with fiber products and the result follows.

Proposition 4.10. If $\mathcal{C}$ is abelian and ϕ is exact then $\mathcal{C}\mathcal{S}^{-1}$ is abelian.

Proof. Follows directly from Propositions 4.4 and 4.6.

Proposition 4.11. Assume that $\mathcal{C}$ has $\lim_{\leftarrow}$ and $^*\phi$ exists. If $\mathcal{C}$ has a cogenerator so has $\mathcal{C}\mathcal{S}^{-1}$.

Proof. If C is a cogenerator then $\phi(C)$ is a cogenerator of $\mathcal{C}\mathcal{S}^{-1}$. Indeed $\operatorname{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(\cdot,\phi(C))\approx\operatorname{Hom}_{\mathcal{C}}({}^*\phi(\cdot), C)$ is faithful since $^*\phi$ and $\operatorname{Hom}_{\mathcal{C}}(\cdot, C)$ are.

Definition 4.12: Set $A(X) = \mathrm{Hom}_{\mathsf{C}}(X, A)$. A pair $R \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ is an *equivalence relation* in C if for all $X \in \mathsf{C}_0$ the induced map $R(X) \to A(X) \times A(X)$ defines an equivalence relation (in the usual sense) in the set $A(X)$.

Assume that C has finite limits. Set further $A \xrightarrow{p} A/R = \mathrm{Coker}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. The equivalence relation R is *effective* if $R = A \times_{A/R} A =$ the square of p.

Lemma 4.13: Assume that ϕ: $\mathsf{C} \to \mathsf{C}S^{-1}$ has a right adjoint ϕ^*. If $R \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ is an equivalence relation in $\mathsf{C}S^{-1}$ then $\phi^*(R) \overset{\phi^*(f_1)}{\underset{\phi^*(f_2)}{\rightrightarrows}} \phi^*(A)$ is an equivalence relation in C.

Proof. Consider the following diagram (for an arbitrary $X \in \mathsf{C}_0$):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X \colon \phi^*(R)) \arrow[r, shift left, "\phi^*(f_1)\cdot"] \arrow[r, shift right, "\phi^*(f_2)\cdot"'] \arrow[d, "\approx"] & \mathrm{Hom}_{\mathsf{C}}(X, \phi^*(A)) \arrow[d, "\approx"] \\
\mathrm{Hom}_{\mathsf{C}S^{-1}}(\phi(X), R) \arrow[r, shift left, "f_1\cdot"] \arrow[r, shift right, "f_2\cdot"'] & \mathrm{Hom}_{\mathsf{C}S^{-1}}(\phi(X) \colon A)
\end{tikzcd}$$

The lower pair defines an equivalence relation in *Ens* for every $X \in \mathsf{C}_0$. By commutativity the same is true for the upper pair since the vertical maps are bijections. From this the Lemma follows.

Proposition 4.14. Assume that C has finite limits and that ϕ: $\mathsf{C} \to \mathsf{C}S^{-1}$ has a right adjoint ϕ^*. If every equivalence relation is effective in C, then the same is true in $\mathsf{C}S^{-1}$.

Proof: Let $R \underset{f_2}{\rightrightarrows} A$ be an equivalence relation in $\mathsf{C}S^{-1}$. Consider the diagram (in $\mathsf{C}S^{-1}$):

$$\begin{tikzcd}
R \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "\exists_1 h"'] & A \arrow[r, "p"] & A/R \\
A \times_{A/R} A \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & &
\end{tikzcd}$$

where $A \times_{A/R} A$ is the square of p. Since $p \cdot f_1 = p \cdot f_2$ there exists a unique h such that $f_1 = g_1 \cdot h$ and $f_2 = g_2 \cdot h$. Considering that ϕ^* is left exact and hence commutes with fiber products, we apply ϕ^* to the diagram above:

$$\begin{tikzcd}
\phi^*(A) \times_{\phi^*(A)/\phi^*(R)} \phi^*(A) \arrow[dd, "y"'] \arrow[drr, shift left, "m_1"] \arrow[drr, shift right, "m_2"'] & & & \\
& \phi^*(R) \arrow[ul, "x"'] \arrow[dl, "\phi^*(h)"] \arrow[r, shift left, "\phi^*(f_1)"] \arrow[r, shift right, "\phi^*(f_2)"'] & \phi^*(A) \arrow[r, "\phi^*(p)"] \arrow[dr, "q"'] & \phi^*(A/R) \\
\phi^*(A) \times_{\phi^*(A/R)} \phi^*(A) \arrow[urr, shift left, "\phi^*(g_1)"] \arrow[urr, shift right, "\phi^*(g_2)"'] & & & \phi^*(A)/\phi^*(R) = \mathrm{Coker}\begin{pmatrix} \phi^*(f_1) \\ \phi^*(f_2) \end{pmatrix} \arrow[u, "u"']
\end{tikzcd}$$

Here (m_1, m_2) is there square of $q = \operatorname{Coker}\begin{pmatrix}\phi^*(f_1)\\ \phi^*(f_2)\end{pmatrix}$. By Lemma 4.13, $\phi^*(R) \rightrightarrows \phi^*(A)$ is an equivalence relation in C and hence is effective. It follows that the canonical morphism x, such that $m_i \cdot x = \phi^*(f_i)$, $i=1, 2$, is an isomorphism. Since $\phi^*(p)\cdot\phi^*(f_1) = \phi^*(p)\cdot\phi^*(f_2)$ it follows from the definition of Coker that there exists a unique u such that $\phi^*(p) = u\cdot q$. Furthermore, $\phi^*(p)\cdot m_1 = u\cdot q\cdot m_1 = u\cdot q\cdot m_2 = \phi^*(p)\cdot m_2$ and thus there exists a unique y such that $m_i = \phi^*(g_i)\cdot y$, $i=1, 2$. Then we get $\phi^*(g_i)\cdot\phi^*(h) = \phi^*(f_i) = m_i\cdot x = \phi^*(g_i)\cdot y\cdot x$ and hence $\phi^*(h) = y\cdot x$ by uniqueness.

Now we take ϕ on the diagram. Using that ϕ is exact and that $\phi\cdot\phi^* \approx I_{\mathsf{C}S^{-1}}$ we see that $\phi(u)$: $A/R \to \phi(\phi^*(A)/\phi^*(R))$ is an isomorphism. But this implies that $\phi(y)$ is an isomorphism and from $(\phi\cdot\phi^*)(h) = \phi(y)\cdot\phi(x)$ we find that $(\phi\cdot\phi^*)(h)$ is an isomorphism. Since $\phi\cdot\phi^* \approx I_{\mathsf{C}S^{-1}}$ we finally conclude that h is an isomorphism and $R \rightrightarrows A$ is effective.

Due to a theorem of Linton ([6], Prop. 3) a category C is *varietal* if and only if there is a functor U: $\mathsf{C} \to \mathit{Ens}$ having a left adjoint $^*U$ and the following axioms are satisfied:

(0) C has kernels and cokernels,

(1) a morphism p in C is an effective epimorphism if and only if $U(p)$ is surjective.

(2) A pair $R \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ in C is a square of a morphism g if and only if

$$U(R) \overset{U(f_1)}{\underset{U(f_2)}{\rightrightarrows}} U(A)$$

is an equivalence relation in *Ens* (Def. 4.12).

Proposition 4.15. Let C be a varietal category and assume that ϕ: $\mathsf{C} \to \mathsf{C}S^{-1}$ has an *exact* right adjoint ϕ^*. Then $\mathsf{C}S^{-1}$ is varietal.

Proof. $V = U\cdot\phi^*$: $\mathsf{C}S^{-1} \to \mathit{Ens}$ has a left adjoint $^*V = \phi\cdot{}^*U$. We know that $\mathsf{C}S^{-1}$ has kernels and cokernels since C has.

(1): Let q: $A \to B$ be an effective epimorphism in $\mathsf{C}S^{-1}$. If $A\times_B A$ is the square of q then $A\times_B A \overset{p_1}{\underset{p_2}{\rightrightarrows}} A \xrightarrow{q} B$ is eaxct in $\mathsf{C}S^{-1}$. But since ϕ^* is exact we see that

$$\phi^*(A)\times_{\phi^*(B)}\phi^*(A) = \phi^*(A\times_B A) \overset{\phi^*(p_1)}{\underset{\phi^*(p_2)}{\rightrightarrows}} \phi^*(A) \xrightarrow{\phi^*(q)} \phi^*(B) \qquad (*)$$

is exact in C. Hence $\phi^*(q)$ is an effective epimorphism in C and $U\cdot\phi^*(q) = V(q)$ is surjective in *Ens*.

Conversely, assume that $V(q) = U\cdot\phi^*(q)$ is surjective in *Ens*. Then $\phi^*(q)$ is an effective epimorphism in C and (*) is exact. Now we take ϕ on (*) and use the fact that ϕ is exact and that $\phi\cdot\phi^* \approx I_{\mathsf{C}S^{-1}}$. We get a diagram:

$$\begin{tikzcd}
\phi\cdot\phi^*(A\times_B A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\approx"] & \phi\cdot\phi^*(A) \arrow[r] \arrow[d, "\approx"] & \phi\cdot\phi^*(B) \arrow[d, "\approx"] \\
A\times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "q"'] & B
\end{tikzcd}$$

The upper row is exact and the vertical morphisms are all isomorphisms. It follows that the lower row is exact and q is effective.

(2) Let $R = A \times_B A \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ be the square of $g: A \to B$ in $\mathcal{C}\mathcal{S}^{-1}$. Then

$$\begin{pmatrix} \phi^*(f_1) \\ \phi^*(f_2) \end{pmatrix}$$

is the square of $\phi^*(g)$ in $\mathcal{C}$. Since $\mathcal{C}$ is varietal

$$\begin{pmatrix} U \cdot \phi^*(f_1) \\ U \cdot \phi^*(f_2) \end{pmatrix} = \begin{pmatrix} V(f_1) \\ V(f_2) \end{pmatrix} \tag{**}$$

is an equivalence relation in *Ens*.

Conversely let $R \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ be given in $\mathcal{C}\mathcal{S}^{-1}$ and assume (**) is an equivalence relation in *Ens*. But then

$$\begin{pmatrix} \phi^*(f_1) \\ \phi^*(f_2) \end{pmatrix}$$

s a square in $\mathcal{C}$. Taking ϕ on it we get a commutative diagram in $\mathcal{C}\mathcal{S}^{-1}$:

$$\begin{tikzcd} \phi \cdot \phi^*(R) \arrow[r, shift left, "\phi \cdot \phi^*(f_1)"] \arrow[r, shift right, "\phi \cdot \phi^*(f_2)"'] \arrow[d, "\approx"] & \phi \cdot \phi^*(A) \arrow[d, "\approx"] \\ R \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A \end{tikzcd}$$

The first row is a square since ϕ is exact. It follows that also the second row is a square and this ends the proof.

Proposition 4.16. Let $\mathcal{C}$ be a category with finite limits and such that all epimorphisms and monomorphisms are universal. Set

$$\mathcal{S} = \{s \in \mathcal{C} \mid s \text{ is both an epimorphism and a monomorphism}\}.$$

Then $\mathcal{S}$ satisfies (S_1), (S_2), (S_1^0), (S_2^0) and hence $\phi: \mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ is exact. Furthermore, $\mathcal{C}\mathcal{S}^{-1}$ is regular.

Proof. Everything is clear except that $\mathcal{C}\mathcal{S}^{-1}$ is regular. It is sufficient to study the canonical factorization of a morphism of the form $\phi(f)$ with $f \in \mathcal{C}$. Let $f = j \cdot \bar{f} \cdot q$ (j a monomorphism, q an epimorphism) be the canonical factorization of f in $\mathcal{S}$. Since ϕ is exact it is sufficient to prove that $\bar{f} \in \mathcal{S}$, i.e. that $\bar{f}$ is both an epimorphism and a monomorphism.

We show that $\bar{f}$ is a monomorphism (then $\bar{f}$ is an epimorphism by duality). Assume that $\bar{f} \cdot u_1 = \bar{f} \cdot u_2$ and consider the following diagram:

$$\begin{tikzcd} A\times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "f"] \arrow[d, "q"] & B \\ \cdot \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] & \cdot \arrow[r, "\bar{f}"'] & \cdot \arrow[u, "j"'] \end{tikzcd}$$

Let (q_i, v_i) be the fiber product of (u_i, q) for $i=1, 2$. q_i is an epimorphism since q is a universal epimorphism. Let (r_1, r_2) be the fiber product of (q_1, q_2). As before r_1, r_2 are epimorphisms and $q_1 \cdot r_1 = q_2 \cdot r_2$. Set $w_i = v_i \cdot r_i$ for $i=1, 2$. Then $\bar{f} \cdot u_1 = \bar{f} \cdot u_2$ implies that $j \cdot \bar{f} \cdot u_1 \cdot q_1 \cdot r_1 = j \cdot \bar{f} \cdot u_2 \cdot q_2 \cdot r_2$ or $f \cdot w_1 = f \cdot w_2$.

From the definition of $A \times_B A$ we see that there exists w such that $w_1 = p_1 \cdot w$ and $w_2 = p_2 \cdot w$. It follows that $u_1 \cdot q_1 \cdot r_1 = q \cdot v_1 \cdot r_1 = q \cdot w_1 = q \cdot p_1 \cdot w = q \cdot p_2 \cdot w = q \cdot w_2 = q \cdot v_2 \cdot r_2 = u_2 \cdot q_2 \cdot r_2 = u_2 \cdot q_1 \cdot r_1$. Hence $u_1 = u_2$ since $q_1 \cdot r_1$ is an epimorphism. Thus $\bar{f}$ is a monomorphism and the proof is finished.

Corollary 4.18. If we furthermore assume that $\mathcal{C}$ is additive, then $\mathcal{C}S^{-1}$ is abelian.

5. Special categories and examples

In this section we consider special abelian categories (in particular the category of modules over a ring) and give some examples.

Proposition 5.1. Assume that $\mathcal{C}$ is abelian with $\varinjlim$ and that ϕ: $\mathcal{C} \to \mathcal{C}S^{-1}$ has a right adjoint ϕ^*. If $\varinjlim_J$ over filtered J is an exact functor in $\mathcal{C}$, then the same is true in $\mathcal{C}S^{-1}$.

Proof: We know that $\mathcal{C}S^{-1}$ has $\varinjlim$ (Corollary 2.7). Furthermore, since $\phi \cdot \phi^* \approx I_{\mathcal{C}S^{-1}}$ and ϕ commutes with $\varinjlim$, we get

$$\varinjlim_J \approx \varinjlim_J \cdot \phi \cdot \phi^* \approx \phi \cdot \varinjlim_J \cdot \phi^*.$$

The last $\varinjlim_J$ is taken in $\mathcal{C}$ and is exact. But ϕ^* and ϕ are left exact and so is $\varinjlim_J$ in $\mathcal{C}S^{-1}$. Since $\varinjlim$ always is right exact we are done.

Corollary 5.2: The same assumptions as above and assume further that $\mathcal{C}$ has a generator. Then $\mathcal{C}S^{-1}$ has exact $\varinjlim_J$ over filtered J and a generator. Hence $\mathcal{C}S^{-1}$ has injective envelopes.

Proof. This follows from Corollary 4.11° and [3] (p. 362).

Remark. Assume that $\mathcal{C}$ is abelian and that ϕ is exact and ϕ^* exists. If $B \in (\mathcal{C}S^{-1})_0$ is injective then $\phi^*(B)$ is injective in $\mathcal{C}$.

Proof. We have $\mathrm{Hom}_{\mathcal{C}}(\cdot, \phi^*(B)) \approx \mathrm{Hom}_{\mathcal{C}S^{-1}}(\phi(\cdot), B)$. This is an exact functor since it is the composition of ϕ and $\mathrm{Hom}_{\mathcal{C}S^{-1}}(\cdot, B)$ which both are exact since B is injective. Hence $\phi^*(B)$ is injective.

For convenience we set $\mathcal{C}^c = \mathfrak{I}(S) = \{A \in \mathcal{C} \mid \phi^* \cdot \phi(A) \approx A\}$.

Proposition 5.3. The same assumptions as in Corollary 5.2. Let $Q(A)$ denote the injective envelope in $\mathcal{C}$ of $A \in \mathcal{C}_0$. If $B \in \mathcal{C}_0^c$ then $Q(j(B)) \in \mathcal{C}_0^c$. Here $j\colon \mathcal{C}_0^c \to \mathcal{C}$ is the imbedding.

Proof. Since $\mathcal{C}^c$ is equivalent to $\mathcal{C}\mathcal{S}^{-1}$ it has the same properties. In particular $B \in \mathcal{C}_0^c$ has an injective envelope $u\colon B \to C$ in $\mathcal{C}_0^c$ (Corollary 5.2). If $v\colon j(B) \to Q(j(B))$ is the injective envelope of $j(B)$ in $\mathcal{C}$ we get a diagram in $\mathcal{C}$.

$$\begin{tikzcd} j(B) \arrow[r, "j(u)"] \arrow[d, "v"'] & j(C) \arrow[dl, "w"] \\ Q(j(B)) & \end{tikzcd}$$

$j(u)$ is a monomorphism since j is left exact. $Q(j(B))$ is injective and hence there exists $w\colon j(C) \to Q(j(B))$ such that $v = w \cdot j(u)$. Furthermore, w is a monomorphism since v is essential. But then also $j(u)$ and w are essential ([8], Lemma 2.4, p. 88). Since $j(C)$ is injective in $\mathcal{C}$, $j(C)$ is an injective envelope of $j(B)$ in $\mathcal{C}$. This implies that w is an isomorphism and finally $Q(j(B)) \in \mathcal{C}_0^c$.

Proposition 5.4 (Gabriel). Let $\mathcal{C}$ be abelian with generator and exact filtered $\varinjlim$. Then $\mathcal{S} \subset \mathcal{C}$ is saturated and satisfies (S_3^0) if and only if $\mathcal{S} = \mathcal{M}(\mathcal{O})$ where $\mathcal{O}$ is contained in the set of injective objects of $\mathcal{C}$.

Proof. Assume first that $\mathcal{O}$ consists only of injective objects. Then

$$H_Q = \mathrm{Hom}_{\mathcal{C}}(\cdot, Q)\colon \mathcal{C} \to Ab^0$$

is exact and hence $\mathcal{S}_{HQ}$ is (S_3^0) and saturated for all $Q \in \mathcal{O}$. Proposition 1.13 then implies that $\mathcal{M}(\mathcal{O}) = \bigcap_{Q \in \mathcal{O}} \mathcal{S}_{HQ}$ is saturated and satisfies (S_3^0).

Conversely assume that $\mathcal{S}$ is saturated and satisfies (S_3^0). We form $\mathcal{C}\mathcal{S}^{-1}$ and $\mathcal{C}^c$ and set $\mathcal{O} = \{Q \in \mathcal{C}_0^c \mid Q \text{ is injective}\}$. $\mathcal{O} \subset \mathcal{C}_0^c$ and hence $\mathcal{M}(\mathcal{O}) \supset \mathcal{M}(\mathcal{C}_0^c) = \bar{\mathcal{S}} = \mathcal{S}$ (Proposition 2.14^0 b). We must show that $\mathcal{M}(\mathcal{O}) \subset \mathcal{S}$. Let $u\colon A \to B$ be in $\mathcal{M}(\mathcal{O})$ and consider the exact sequence $O \to K \to A \to B \to C \to O$ where $K = \mathrm{Ker}\,(u)$ and $C = \mathrm{Coker}\,(u)$. We want to show that $\phi(u)$ is an isomorphism, that is $\phi(K) = O$ and $\phi(C) = O$ since ϕ is exact. Now $u \in \mathcal{M}(\mathcal{O})$ implies that $H_Q(u)$ is an isomorphism for every $Q \in \mathcal{O}$. But H_Q is exact since Q is injective and thus $H_Q(K) = O$ and $H_Q(C) = O$ for every $Q \in \mathcal{O}$. Let $\phi^* \cdot \phi(K) \to Q(\phi^* \cdot \phi(K))$ denote the injective envelope of $\phi^* \cdot \phi(K) \in \mathcal{C}_0^c$. By Proposition 5.3 $Q(\phi^* \cdot \phi(K)) \in \mathcal{C}_0^c$ and hence $\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(K), \phi(K)) \approx \mathrm{Hom}_{\mathcal{C}}(K, \phi^* \cdot \phi(K)) \to \mathrm{Hom}_{\mathcal{C}}(K, Q(\phi^* \cdot \phi(K))) = O$. We get $\mathrm{End}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(K)) = O$ and so $\phi(K) = O$. Similarly $\phi(C) = O$ and $\phi(u)$ is an isomorphism and $u \in \bar{\mathcal{S}} = \mathcal{S}$. This shows $\mathcal{M}(\mathcal{O}) \subset \mathcal{S}$ and finishes the proof.

Example 5.5. Let $\mathcal{A} \to \mathcal{C}$ be a Serre subcategory of the abelian category $\mathcal{C}$ i.e. if $O \to A \to B \to C \to O$ is exact in $\mathcal{C}$ then $B \in \mathcal{A}_0$ if and only if A and $C \in \mathcal{A}_0$ ([3], p. 365). Set $\mathcal{S} = \{s \in \mathcal{C} \mid \mathrm{Ker}\, s,\ \mathrm{Coker}\, s \in \mathcal{A}_0\}$. Then $\mathcal{S}$ is (S_1), (S_2), (S_1^0) and (S_2^0). $\mathcal{S}$ is nice if $\mathcal{C}$ has a generator. $\phi\colon \mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ is exact and $\mathcal{C}\mathcal{S}^{-1} = \mathcal{C}/\mathcal{A}$, the quotient category of $\mathcal{C}$ with respect to $\mathcal{S}$ ([3]). If ϕ^* exists, $\mathcal{A}$ is called a localizing subcategory. Conversely, if $\phi\colon \mathcal{C} \to \mathcal{C}\mathcal{S}^{-1}$ is exact, then $\mathrm{Ker}\,\phi = \{A \in \mathcal{C}_0 \mid \phi(A) = O\}$ is a Serre subcategory and $\mathcal{C}\mathcal{S}^{-1} = \mathcal{C}/\mathrm{Ker}\,\phi$.

Example 5.6. Let $\mathcal{C}$ be an abelian category and $\mathcal{S}=\{s\in\mathcal{C}\mid s$ is an essential monomorphism$\}$. $\mathcal{S}$ satisfies (S_1) and (S_2) and is nice if $\mathcal{C}$ has a generator. $\mathcal{C}\mathcal{S}^{-1}$ is the *spectral category* of $\mathcal{C}$ (see Gabriel–Oberst: Spektralkategorien, Math. Z. *92*, 1966, p. 389). If $\mathcal{C}$ is a Grothendieck category, then $\mathcal{C}\mathcal{S}^{-1}$ is abelian.

Example 5.7. Let $\mathcal{B}$ be a small category and set $\mathcal{C}=\hat{\mathcal{B}}=Hom\,(\mathcal{B}^0, Ens)$, the category of set-valued presheaves on $\mathcal{B}$. Let $\mathcal{S}\subset\mathcal{C}$ be the subcategory consisting of the bicovering morphisms corresponding to some Grothendieck topology on $\mathcal{B}$ (see [10]). Then $\mathcal{S}$ is conice and satisfies (S_1), (S_2) and (S_3^0). It follows that ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ is exact and has a right adjoint ϕ^* and $\mathcal{C}\mathcal{S}^{-1}$ is equivalent to $\mathcal{J}(\mathcal{S})=\tilde{\mathcal{B}}=$the category of sheaves for the given topology on $\mathcal{B}$ (Propositions $2.14^o a$ and $2.15^o b$).

Conversely given $\mathcal{C}=\hat{\mathcal{B}}$ and any $\mathcal{S}\subset\mathcal{C}$ such that ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ is exact and ϕ^* exists then $\mathcal{C}\mathcal{S}^{-1}$ is equivalent to a category of sheaves for some topology on $\mathcal{B}$. Indeed $\mathcal{C}\mathcal{S}^{-1}$ is equivalent to $\mathcal{J}(\mathcal{C})$ and the imbedding $\mathcal{J}(\mathcal{S})\to\mathcal{C}=\hat{\mathcal{B}}$ has an exact left adjoint since ϕ is exact. The result now follows from a theorem of Giraud ([10], Prop. II.3.19). A *topos* is a category equivalent to a category, $\tilde{\mathcal{B}}$, of setvalued sheaves ([10]).

Proposition 5.8. If $\mathcal{C}$ is a topos and ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ is exact and ϕ^* exists, then $\mathcal{C}\mathcal{S}^{-1}$ is a topos.

Proof. Proposition 4.9 and 4.15 that show two of the characteristic properties of a topos ([10] Def. II.4.12) are preserved under forming of $\mathcal{C}\mathcal{S}^{-1}$. One can verify that the same is true for the other axioms of a topos. However, it is easy to give a direct proof. The imbedding

$$\mathcal{J}(\mathcal{S})\to\mathcal{C}=\tilde{\mathcal{B}}\to\hat{\mathcal{B}}$$

has an exact left adjoint and the theorem of Giraud mentioned above implies that $\mathcal{J}(\mathcal{S})$ is equivalent to a category of sheaves. Hence $\mathcal{C}\mathcal{S}^{-1}$ is a topos.

A *radical* in an abelian category $\mathcal{C}$ is a monomorphism φ: $R\to I_{\mathcal{C}}$ in $Hom\,(\mathcal{C},\mathcal{C})$, such that $R\cdot\operatorname{Coker}\varphi=O$ ([7]).

Proposition 5.9. Let ϕ: $\mathcal{C}\to\mathcal{S}^{-1}\mathcal{C}$ be a *left* fractional category. Assume that ϕ^* exists. Set $R=\operatorname{Ker}\beta\xrightarrow{\varphi}I_{\mathcal{C}}$ where β: $I\to\phi^*\cdot\phi$ is the adjunction morphism. Then $R\xrightarrow{\varphi}I_{\mathcal{C}}$ is a radical in $\mathcal{C}$ and R is left exact.

Set $\mathcal{B}_0=\{B\in\mathcal{C}_0\mid R(B)=O\}$. Then the imbedding L: $\mathcal{B}\to\mathcal{C}$ of the full subcategory has a left adjoint $^*L$. Furthermore, $\mathcal{J}(\mathcal{S})\subset\mathcal{B}_0$.

Proof. Using Theorem $2.8^o c$ one verifies that $R(A)\approx\operatorname{Ker}s$ for any s: $A\to Q$ with $s\in\bar{\mathcal{S}}$ and $Q\in\mathcal{J}(\mathcal{S})$. Let β_A: $A\to\phi^*\cdot\phi(A)$ be factored through $\operatorname{Im}\beta_A$ as $\beta_A=k\cdot h$ with k a monomorphism and h an epimorphism. Proposition 4.7 implies that $k\in\bar{\mathcal{S}}$ and since $\phi^*\cdot\phi(A)\in\mathcal{J}(\mathcal{S})$ we get $R\,(\operatorname{Coker}\varphi_A)=R\,(\operatorname{Im}\beta_A)\approx\operatorname{Ker}k=O$. Hence R is a radical. Clearly R is left exact.

It is easily checked that $^*L=\operatorname{Coker}\varphi=\operatorname{Im}\beta$ is a left adjoint of L. Finally, if $A\in\mathcal{J}(\mathcal{C})$ then β_A is an isomorphism and $R(A)=\operatorname{Ker}\beta_A=O$.

Example 5.10. If $\mathcal{C}=Ens$ there are only two saturated subcategories: $\mathcal{C}$ itself and all isomorphisms in $\mathcal{C}$. For the proof one uses Proposition 4.7 and the fact that the subcategory is closed under fiber products.

In other categories there are in general many saturated subcategories. For the category of right modules, $\mathfrak{M}_R$, over a ring R, there is a one-to-one correspondence between certain sets of ideals in R (see [3] and [11]) and subcategories $\mathcal{S}$ of $\mathcal{C}=\mathfrak{M}_R$ such that the corresponding ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ is exact and has a right adjoint.

We collect some elementary observations in the following proposition (For a proof see [1]).

Proposition 5.11. Let R be a commutative ring. If ϕ: $\mathfrak{M}_R=\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ has a right adjoint ϕ^*, then the following is true:

(*a*) $\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(A, B)$ has a canonical R-module structure.

(*b*) The bijections h: $\mathrm{Hom}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(A), B)\xrightarrow{\approx}\mathrm{Hom}\,(A, \phi^*(B))$ are R-module isomorphisms for all $A\in\mathcal{C}_0$, $B\in(\mathcal{C}\mathcal{S}^{-1})_0$.

(*c*) ϕ^* is representable.

(*d*) $S=\phi^*\cdot\phi(R)$ is an R-algebra and the canonical map β_R: $R\to\phi^*\cdot\phi(R)$ is a ring-homomorphism preserving identity element.

(*e*) $S\approx\mathrm{End}_{\mathcal{C}\mathcal{S}^{-1}}(\phi(R))$ and β_R: $R\to S$ induces an isomorphism $\mathrm{End}_R\,S\xrightarrow{\approx}S$.

(*f*) If $B\in\mathfrak{J}(\mathcal{S})$ then β_R induces and isomorphism $\mathrm{Hom}_R\,(S, B)\to B$.

Remark. If ϕ has both left and right adjoints, then $\mathcal{S}$ can be characterized by one ideal A (satisfying $A^2=A$) in R (see [9]).

Example 5.12. Let $\mathcal{C}=\mathfrak{M}_R$ be the category of right R-modules over a ring R. Let $T\subset R$ be a multiplicative subset satisfying (S_1) and (S_2) when R is considered as a category with one object. Then RT^{-1}, the ring of right fractions of R with respect to T exists (see [3] p. 415). RT^{-1} is R-flat and thus the functor $F=\cdot\otimes_R RT^{-1}$ is exact. Clearly F commutes with $\lim_{\to}$. If we set $\mathcal{S}=\mathcal{S}_F$, then $\mathcal{S}$ is nice, (S_1), (S_2) and (S_3^0) and hence ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$ has a right adjoint ϕ^*. We have $F(A)=A\otimes_R RT^{-1}$ for all $A\in\mathcal{C}_0$. Hence $F\cdot F(A)=(AT^{-1})T^{-1}\approx AT^{-1}=F(A)$. One verifies that $\mathcal{C}\mathcal{S}^{-1}$ is equivalent to the category of right modules over the ring $S=\phi^*\cdot\phi(R)=RT^{-1}$.

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