

## Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part

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### 0. Introduction

Let  $P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$  be a complex polynomial of degree  $m$  in the complex variables  $\xi = (\xi_1, \dots, \xi_{d+1})$ , and let  $P_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha$  be its principal part. Let  $(x_1, \dots, x_{d+1})$  be real variables, and put  $D_k = \partial / i \partial x_k$ . A distribution  $E(x)$  on  $R^{d+1}$  is said to be a fundamental solution of the differential operator  $P(D)$  if  $P(D)E(x) = \delta(x)$ , the Dirac distribution. The operator  $P(D)$  is said to be hyperbolic if it has a fundamental solution  $E$  with support in a proper cone  $K$  having its vertex at the origin (Gårding [5]). Let  $N \in R^{d+1}$  be such that the halfspace  $\langle x, N \rangle = x_1 N_1 + x_2 N_2 + \dots + x_{d+1} N_{d+1} > 0$  contains  $\dot{K} = K - \{0\}$ . Then

$$P_m(N) \neq 0, P(\xi + i\tau N) \neq 0 \quad \text{if} \quad \xi \in R^{d+1}, \tau \in R, |\tau| > \tau_0 \quad (0.1)$$

for some  $\tau_0$ . Conversely, this condition implies that  $P(D)$  has a fundamental solution with support in some  $K$  such that  $\langle x, N \rangle > 0$  on  $\dot{K}$  (Gårding [5], [4]).

When (0.1) holds, we say that  $P$  is hyperbolic with respect to  $N$  and denote by  $\text{Hyp } N$  the corresponding class of polynomials.

It follows that  $P_m$  is in  $\text{Hyp } N$  if  $P$  is, and that a homogeneous hyperbolic polynomial has only real characteristics. We shall, conversely, consider the problem of characterizing the lower order terms one may add to a homogeneous hyperbolic polynomial without loss of the hyperbolicity. In the case  $d=1$ , this problem has been solved completely by A. Lax [8]. A generalization of A. Lax's condition was given by Hörmander in [6]. His generalized condition is necessary but not sufficient when  $d > 1$ .

A sufficient condition by Gårding [4] for a polynomial  $P$  to belong to  $\text{Hyp } N$ , if its principal part  $P_m$  does, is that the roots  $\sigma$  of  $P(\sigma(\tau N + i\xi)) = 0$  tend to zero, uniformly in  $\xi \in R^{d+1}$ , when  $\tau \rightarrow +\infty$ . Gårding conjectured that this condition would be necessary too. (See footnote, page 50 in Gårding [4].)

In section 1 of this paper we shall prove Gårding's conjecture. We use a sufficient condition by Hörmander [6], which can be shown to be equivalent to that of Gårding, namely that  $P$  is weaker than  $P_m$ , i.e. that for some constant  $C$  we have

$$|P(\xi)| \leq C \tilde{P}_m(\xi), \quad \xi \in R^{d+1}.$$

Here, when  $Q$  is a polynomial, we put

$$\tilde{Q}(\xi) = \left( \sum_{\alpha} |\partial^\alpha Q(\xi)|^2 \right)^{\frac{1}{2}}, \quad \partial = (\partial / \partial \xi_1, \dots, \partial / \partial \xi_{d+1}).$$

Our proof consists of essentially two steps. First, by use of the Puiseux series expansion and the Newton algorithm, we prove that if  $P \in \text{Hyp } N$  and if  $r \rightarrow \eta(r)$  is a real curve, meromorphic in a neighborhood of  $r=0$ , then we have

$$P(\eta(r)) = O(1)\tilde{P}_m(\eta(r)) \quad \text{when } r \rightarrow 0. \quad (0.2)$$

Then Seidenberg's lemma enables us to prove that if  $P$  is weaker than  $P_m$  along any curve  $\eta(r)$ , meromorphic in a neighborhood of  $r=0$  in the sense of (0.2), then  $P$  is weaker than  $P_m$ .

Several, mutually equivalent, sufficient conditions for hyperbolicity were given by McCarthy and Pederson in [8]. In section 2 we give a brief discussion of these conditions which are, in fact, equivalent to those of Gårding and Hörmander.

In section 3 we consider Hörmander's generalization of A. Lax's condition.

Section 4—which was added on November 7th, 1968—consists of an application to hyperbolic systems of the results of section 1.

I am deeply grateful to J. Friberg and L. Gårding for valuable advice and kind interest in my work. The subject of this paper was suggested to me by J. Friberg. The starting point of the investigation was an idea of his that in the case  $d=2$  one would get enough information to solve the problem by use of the Puiseux series expansion. An idea by L. Gårding inspired me to the proof of Lemma 1.2. I also want to thank L. Hörmander who has read the manuscript and suggested valuable improvements.

### 1. The necessity of Gårding's condition

Our main tool in this section is the Puiseux series expansion of the zeros of polynomials  $\sum_{0 \leq j \leq m} c_j(r)\tau^j$ , where the  $c_j$  are Puiseux series of the real variable  $r$ . We shall also make use of the Newton algorithm to compute the first non-vanishing term in such expansions. For an account of these matters we refer to e.g. Friberg [2]. When we use the notation  $r^{1/p}$ , where  $p$  is a positive integer, we shall always mean the value taken by the branch of the function  $r \rightarrow r^{1/p}$  with  $0 \leq \arg r^{1/p} < 2\pi/p$ . By the lower Newton polygon of a polynomial  $\sum_{\lambda, \mu} a_{\lambda\mu} \tau^\lambda r^\mu$  in  $\tau$  whose coefficients are of the type described above, we shall mean the set of all  $(\lambda, \mu)$  for which there is a  $\mu' \leq \mu$  such that  $(\lambda, \mu')$  belongs to the convex hull (in  $R^2$ ) of  $\{(\lambda, \mu) \mid a_{\lambda\mu} \neq 0\}$ .

**Lemma 1.1.** *Let  $P_m \in \text{Hyp } N$  be homogeneous of degree  $m$  and let  $\eta(r) = \sum_{v \geq v_0} \eta_v r^v$ , where the  $\eta_v \in R^{d+1}$ , be meromorphic in a neighborhood of  $r=0$ . Then we can write*

$$P_m(\eta(r) + \tau N) = P_m(N) \prod_{i=1}^m \left( \tau - \sum_{j \geq j_i} c_{i,j} r^j \right),$$

where  $\sum_{j \geq j_i} c_{i,j} r^j$  are meromorphic in a neighborhood of  $r=0$  and  $c_{i,j} \in R$ ,  $j \geq j_i$ ,  $1 \leq i \leq m$ .

*Proof.* Since, by the hyperbolicity,  $P_m(N) \neq 0$ , we can write

$$P_m(\eta(r) + \tau N) = P_m(N) \prod_{i=1}^m (\tau - \tau_i(r)).$$

The zeros  $\tau_i$  can be represented by Puiseux series expansions

$$\tau_i(r) = \sum_{v \geq v_i} \gamma_{i,v} r^{v/n}, \quad 1 \leq i \leq m,$$

in a neighborhood of  $r=0$ . Hence with this representation the  $\tau_i$  are meromorphic

functions of  $r^{1/n}$ , for some positive integer  $n$ , in a neighborhood of  $r^{1/n}=0$ . Since  $P_m \in \text{Hyp } N$  and is homogeneous, it follows that  $\tau_i(r)$ ,  $1 \leq i \leq m$ , are real for real  $r$ . Let  $1 \leq i \leq m$  and assume that  $\gamma_{i,p_0} r^{p_0/n}$  is the first term in  $\tau_i(r)$  which takes non-real values in every real neighborhood of  $r=0$ . Then we can choose  $\arg r = q\pi$  in such a way that  $\arg \gamma_{i,p_0} r^{p_0/n} = \arg \gamma_{i,p_0} + qp_0\pi/n \neq k\pi$  for any integer  $k$ . Since we have

$$\tau_i(r) = \sum_{p_i \leq p < p_0} \gamma_{i,p} r^{p/n} + \gamma_{i,p_0} r^{p_0/n} (1 + o(1)) \quad \text{when } r \rightarrow 0,$$

it follows that if we choose  $\arg r$  as above and  $|r|$  sufficiently small, we have  $\text{Im } \tau_i(r) \neq 0$  which is a contradiction. Hence all the terms  $\gamma_{i,p} r^{p/n}$  must be real, and this gives eventually that  $\gamma_{i,p} = 0$  if  $n$  is not a divisor of  $p$  and that  $\gamma_{i,nj} = c_{i,j} \in R$ ,  $nj \geq p_i$ .

**Theorem 1.1.** *Let  $P = \sum_{0 \leq k \leq m} P_k \in \text{Hyp } N$ , where  $P_k(\xi) = \sum_{|\alpha|=k} c_\alpha \xi^\alpha$ , and let  $\eta(r) = \sum_{p \geq p_0} \eta_p r^p$ , where  $\eta_p \in R_{d+1}$ , be meromorphic in a neighborhood of  $r=0$ . Then the lower Newton polygon of  $P_m(\eta(r) + \tau N)$  contains the lower Newton polygons of  $\tau^k P_{m-k}(\eta(r) + \tau N)$ ,  $0 \leq k \leq m$ .*

*Proof.* Let  $P_{m-k}(\eta(r) + \tau N) = \sum_{\lambda, \mu} a_{k\lambda\mu} \tau^\lambda r^\mu$ ,  $0 \leq k \leq m$ .

Since  $P_m(N) \neq 0$ , it follows that the point  $(m, 0)$  belongs to the lower Newton polygon of  $P_m(\eta(r) + \tau N)$ . For every integer  $j$ , let  $n_j$  be the uniquely determined integer for which

$$\begin{aligned} a_{0\lambda\mu} &= 0 \quad \text{if } \mu < n_j - \lambda j, \\ a_{0\lambda\mu} &\neq 0 \quad \text{for some } (\lambda, \mu) \quad \text{with } \mu = n_j - \lambda j. \end{aligned} \tag{1.1}$$

Now, in view of Lemma 1.1, the non-vertical line segments of the boundary of the lower Newton polygon of  $P_m(\eta(r) + \tau N)$  have slopes given by integers. Hence the lines  $\mu = n_j - \lambda j$  constitute in an obvious way the lower Newton polygon of  $P_m(\eta(r) + \tau N)$ . It is further clear that what we shall prove is that

$$a_{k\lambda\mu} = 0 \quad \text{if } \mu < n_j - (\lambda + k)j \quad \text{for some } j, 0 \leq k \leq m. \tag{1.2}$$

We assume that (1.2) is false. Then, since  $(m, 0)$  belongs to the Newton polygon of  $P_m(\eta(r) + \tau N)$ , it is clear that there is a smallest integer  $p$  such that

$$a_{k\lambda\mu} \neq 0 \quad \text{for some } (k, \lambda, \mu) \quad \text{with } \mu < n_p - (\lambda + k)p.$$

Since this means that  $a_{k\lambda\mu} \neq 0$  for some  $(k, \lambda, \mu)$  with  $\mu' = \mu + pk < n_p - \lambda p$ , we can choose a real  $c \neq 0$  so that

$$\sum_{\mu + pk = \mu'} c^k a_{k\lambda\mu} \neq 0 \quad \text{for some } (\lambda, \mu') \quad \text{with } \mu' < n_p - \lambda p. \tag{1.3}$$

With this  $c$  we write

$$Q(\tau, r) = c^m r^{2m} P(c^{-1} r^{-p} (\eta(r) + \tau N)). \tag{1.4}$$

For reasons of homogeneity we get

$$Q(\tau, r) = \sum_{0 \leq k \leq m} c^k r^{pk} P_{m-k}(\eta(r) + \tau N)$$

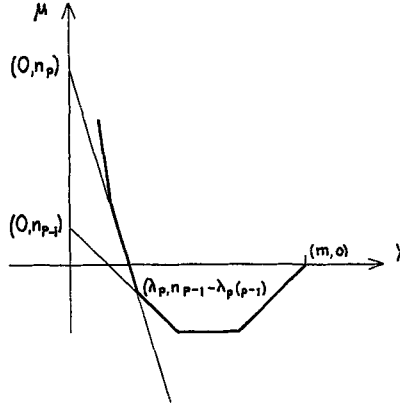


Fig. 1. The Newton polygon belonging to  $P_m(\eta(r) + \tau N)$ .

Hence we have, by a simple computation,

$$Q(\tau, r) = \sum_{\lambda, \mu'} \tau^\lambda r^{\mu'} \left( \sum_{\mu + p k = \mu'} c^k a_{k\lambda\mu} \right). \quad (1.5)$$

In view of (1.4), the hyperbolicity of  $P$  gives that the imaginary parts of the zeros  $\tau$  of  $Q(\tau, r)$  are  $O(r^p)$  when  $r \rightarrow 0$ . In order to get a contradiction, we shall study the Newton polygon of  $Q(\tau, r)$ .

By the definition of  $p$  we have that  $a_{k\lambda\mu} = 0$  if  $\mu < n_{p-1} - (\lambda + k)(p-1)$ , i.e. if  $\mu + p k < n_{p-1} - \lambda(p-1) + k$ . Hence we have

$$\sum_{\mu + p k = \mu'} c^k a_{k\lambda\mu} = 0 \quad \text{if } \mu' < n_{p-1} - \lambda(p-1), \quad (1.6)$$

$$\sum_{\mu + p k = \mu'} c^k a_{k\lambda\mu} = a_{0\lambda\mu'} \quad \text{if } \mu' = n_{p-1} - \lambda(p-1). \quad (1.7)$$

Let  $\lambda = \lambda_p$  be the smallest integer such that  $a_{0\lambda\mu} \neq 0$  for  $\mu = n_{p-1} - \lambda(p-1)$ . By (1.5) it is clear that (1.3), (1.6), and (1.7) give direct information about the lower Newton polygon of  $Q(\tau, r)$ . We put  $A_1 = \{(\lambda, \mu) \mid \mu < n_{p-1} - \lambda(p-1)\}$ ,  $A_2 = \{(\lambda, \mu) \mid \lambda < \lambda_p, \mu = n_{p-1} - \lambda(p-1)\}$ , and  $A_3 = \{(\lambda, \mu) \mid \lambda < \lambda_p, n_{p-1} - \lambda(p-1) < \mu < n_p - \lambda p\}$ . It is clear, by the definition of  $\lambda_p$  and  $n_p$  that the point  $(\lambda_p, n_{p-1} - \lambda_p(p-1))$  is the intersection between the lines  $\mu = n_{p-1} - \lambda(p-1)$  and  $\mu = n_p - \lambda p$ . Hence it follows that

$$\{(\lambda, \mu) \mid \mu < n_p - \lambda p\} \subseteq A_1 \cup A_2 \cup A_3. \quad (\text{See Fig. 1.})$$

Now it is clear by (1.6) that no points of the Newton polygon of  $Q(\tau, r)$  lie in  $A_1$ . By (1.7) and the definition of  $\lambda_p$  no such points belong to  $A_2$  either. But by (1.3) at least one point of the lower Newton polygon of  $Q(\tau, r)$  is in  $\{(\lambda, \mu) \mid \mu < n_p - \lambda p\}$  and thus in  $A_3$ . Therefore there must be a line segment of the boundary of the lower Newton polygon of  $Q(\tau, r)$  starting in a point in  $A_3$  and ending in  $(\lambda_p, n_{p-1} - \lambda_p(p-1))$ . It is then clear that this line segment will have slope  $-q$ , where  $p-1 < q < p$ . But this means that there is a root  $\tau(r)$  of  $Q(\tau, r) = 0$  such that  $\tau(r) = br^q(1 + o(1))$  when  $r \rightarrow 0$  for some  $b \neq 0$ . We have  $r^{-p} \operatorname{Im} \tau(r) = \operatorname{Im} r^{-p} \tau(r) = \operatorname{Im} (br^{q-p})(1 + o(1))$  when  $r \rightarrow 0$  through real values. Since  $p-1 < q < p$ , it follows that  $\operatorname{Im} br^{q-p}$ —and consequently  $r^{-p} \operatorname{Im} \tau(r)$ —is not bounded in any neighborhood of  $r = 0$ . Hence we have reached a contradiction, and the theorem is proved.

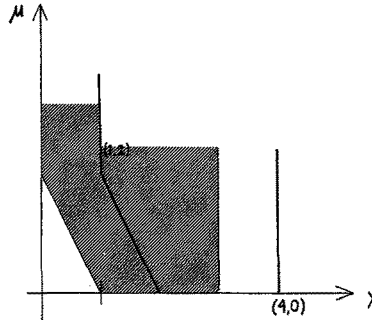


Fig. 2. The Newton polygon belonging to  $\tau^4 + 4(1+r^2)^{\frac{1}{2}}\tau^3 + (4+3r^2)\tau^2 - 2r^2(1+r^2)^{\frac{1}{2}}\tau$ . The Newton polygon belonging to the lower order term of degree 3 must lie in the shaded region if we shall not lose the hyperbolicity.

*Example.* Consider the polynomial

$$P((\xi_1, \xi_2, \xi_3)) = (\xi_1^2 - \xi_2^2 - \xi_3^2)(\xi_1^2 - \xi_2^2 - 2\xi_3^2) + \xi_2^2 \xi_3.$$

(Due to P. D. Lax; see Courant and A. Lax [1].)

We put  $\eta(r) = ((1+r^2)^{\frac{1}{2}}, 1, r)$ . Then we have  $P(\eta(r) + \tau(1, 0, 0)) = \tau^4 + 4(1+r^2)^{\frac{1}{2}}\tau^3 + (4+3r^2)\tau^2 - 2r^2(1+r^2)^{\frac{1}{2}}\tau + r$ . We see that the lower order term contributes the point  $(0, 1)$ . By the figure and Theorem 1.1 we see that  $P$  is not hyperbolic with respect to  $(1, 0, 0)$  although its principal part is.

**Theorem 1.2.** Let  $P \in \text{Hyp } N$ , let  $P_m$  be the principal part of  $P$ , and let  $\eta(r) = \sum_{\nu \geq \nu_0} \eta_\nu r^\nu$ , where  $\eta_\nu \in R^{d+1}$ , be meromorphic in a neighborhood of  $r=0$ . Then we have

$$P(\eta(r)) = O(1)\tilde{P}_m(\eta(r)) \quad \text{when } r \rightarrow 0. \tag{1.8}$$

*Proof.* Let  $\mu_0$  be the least integer such that  $(\lambda, \mu_0)$  belongs to the Newton polygon of  $P_m(\eta(r) + \tau N)$  for some  $\lambda$ . (The existence of  $\mu_0$  is clear, since  $P_m(N) \neq 0$  and since  $\eta$  is meromorphic.) It is obvious that  $(\lambda_0, \mu_0)$  is a vertex of the lower Newton polygon of  $P_m(\eta(r) + \tau N)$  for some  $\lambda_0$ . Put

$$\langle \partial, N \rangle = \sum_{1 \leq \nu \leq d+1} N_\nu \partial / \partial \xi_\nu, \quad N = (N_1, \dots, N_{d+1}).$$

By Taylor's formula and the chain rule we have

$$P_m(\eta(r) + \tau N) = \sum_{0 \leq j \leq m} \langle \partial, N \rangle^j P_m(\eta(r)) \tau^j / j!$$

Thus, by the definition of  $(\lambda_0, \mu_0)$ , we have with some  $b_0 \neq 0$ ,

$$\langle \partial, N \rangle^{\lambda_0} P_m(\eta(r)) = r^{\mu_0}(b_0 + o(1)) \quad \text{when } r \rightarrow 0. \tag{1.9}$$

We write  $P = \sum_{0 \leq k \leq m} P_k$ , where  $P_k(\xi) = \sum_{|\alpha| = k} c_\alpha \xi^\alpha$ . Let  $0 \leq k \leq m$  and assume that for some  $b'_k \neq 0$  and some integer  $\mu'_k$  we have

$$P_{m-k}(\eta(r)) = r^{\mu'_k}(b'_k + o(1)) \quad \text{when } r \rightarrow 0. \tag{1.10}$$

It is clear then that  $(0, \mu'_k)$  belongs to the Newton polygon of  $P_{m-k}(\eta(r) + \tau N)$  so that, by Theorem 1.1,  $\mu'_k \geq \mu_0$ . Hence we have, by (1.9) and (1.10),

$$P_{m-k}(\eta(r)) = O(1)\langle \partial, N \rangle^{\lambda_0} P_m(\eta(r)) \quad \text{when } r \rightarrow 0. \tag{1.11}$$

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If  $P_{m-k}(\eta(r))$  is identically zero, (1.11) is trivial. Since, by definition,  $P_m(\eta(r)) = (\sum_{\alpha} |\partial^{\alpha} P_m(\eta(r))|^2)^{\frac{1}{2}}$ , it is obvious that  $\langle \partial, N \rangle^{\lambda_0} P_m(\eta(r)) = O(1) \tilde{P}_m(\eta(r))$  when  $r \rightarrow 0$ . Hence we have

$$P_{m-k}(\eta(r)) = O(1) \tilde{P}_m(\eta(r)) \quad \text{when } r \rightarrow 0, 0 \leq k \leq m.$$

By the triangle inequality we get (1.8), and the proof is complete.

*Remark.* It may seem that we have used only a small part of Theorem 1.1 in the proof. We have only used what we know about the Newton polygons of  $P_{m-k}(\eta(r) + \tau N)$  in relation to the line  $\mu = n_0$  of the proof of Theorem 1.1. But it is clear that in order to get any information about this we must first examine the relations between the Newton polygons of  $P_{m-k}(\eta(r) + \tau N)$  and the lines  $\mu = n_j - \lambda_j$ , for  $j < 0$ .

**Lemma 1.2.** *Let  $Q_1$  and  $Q_2$  be complex polynomials in  $d+1$  variables,  $Q_2$  not identically zero. Assume that for any curve  $\eta(r) = \sum_{\nu \geq \nu_0} \eta_{\nu} r^{\nu}$ , where  $\eta_{\nu} \in R^{d+1}$ , meromorphic in a neighborhood of  $r=0$ , we have*

$$Q_1(\eta(r)) = O(1) Q_2(\eta(r)) \quad \text{when } r \rightarrow 0. \quad (1.12)$$

Then we have with a constant  $C$

$$|Q_1(\xi)| \leq C |Q_2(\xi)|, \quad \xi \in R^{d+1}. \quad (1.13)$$

*Proof.* Denote by  $B$  the set  $\{\xi \in R^{d+1} | Q_2(\xi) \neq 0\}$ . Since  $Q_2$  is not identically zero, it follows that  $B$  is a dense subset of  $R^{d+1}$ . Assume that

$$\sup_{\xi \in B} |Q_1(\xi)/Q_2(\xi)| = +\infty. \quad (1.14)$$

Consider the system

$$|Q_1(\xi)|^2 - s |Q_2(\xi)|^2 = 0, \quad |Q_2(\xi)|^2 > 0 \quad (1.15)$$

We observe that  $|Q_1(\xi)|^2 - s |Q_2(\xi)|^2$  and  $|Q_2(\xi)|^2$  are real polynomials in  $\xi \in R^{d+1}$ ,  $s \in R$ . Seidenberg's theorem (see e.g. Gorin [3]) asserts then for every  $j$ ,  $1 \leq j \leq d+1$ , the existence of a condition  $H_j$ , consisting of a finite number of systems of polynomial equations  $h_{k,j}(\xi_1, \dots, \xi_j, s) = 0$ ,  $1 \leq k \leq k'_j$ , and polynomial inequalities  $h_{k,j}(\xi_1, \dots, \xi_j, s) > 0$ ,  $k'_j < k \leq k_j$ , such that for every  $(\xi_1, \dots, \xi_j, s) \in R^{j+1}$  the following conditions are equivalent:

- I. There exist real  $\xi_{j+1}, \dots, \xi_{d+1}$  so that  $(\xi, s)$ ,  $\xi = (\xi_1, \dots, \xi_{d+1})$ , is a solution of the system (1.15).
- II. The condition  $H_j$  is satisfied by  $(\xi_1, \dots, \xi_j, s)$ ; i.e.  $(\xi_1, \dots, \xi_j, s)$  satisfies at least one of the systems in the condition.

Assume that for some  $j$ ,  $1 \leq j \leq d+1$ , we have found Puiseux series  $\gamma_1(s), \dots, \gamma_{j-1}(s)$ , convergent and real for all large real  $s$ , such that the system (1.15) has real solutions  $\xi = (\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, \dots, \xi_{d+1})$  for some arbitrarily large  $s$ . If  $j=1$ , we mean by this that the system (1.15) has real solutions  $\xi$  for some arbitrarily large real  $s$ . Hence, in view of (1.14), the assumption is correct when  $j=1$ . We study the Puiseux series expansions of the roots  $\xi_j$  of the equations  $h_{k,j}(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s) = 0$ ,  $1 \leq k \leq k_j$ , for large real  $s$ . Everyone of these expansions is a meromorphic function of  $s^{1/p}$

in a neighborhood of  $s^{1/p} = \infty$ , for some positive integer  $p$ . In particular it is either real or non-real for all sufficiently large real  $s$ . Let  $\vartheta_1(s), \dots, \vartheta_j(s)$  be the different real expansions, continuous and arranged so that  $\vartheta_1(s) < \vartheta_2(s) < \dots < \vartheta_j(s)$  for  $s_0 < s$ . We may assume  $s_0$  so large that these are the only possible real roots of the equations  $h_{k,j}(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s) = 0$  if  $s_0 < s$ . We put  $\vartheta_0 = -\infty$  and  $\vartheta_{j+1} = +\infty$ . We observe that if the condition  $H_j$  is satisfied by some  $(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s)$  with  $s > s_0$  and  $\vartheta_{l-1}(s) < \xi_j < \vartheta_l(s)$ , for some  $l, 1 \leq l \leq j+1$ , then it is satisfied by all such  $(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s)$ . If the condition  $H_j$  is satisfied by some  $(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s)$  with  $s > s_0$  and  $\xi_j = \vartheta_l$  for some  $l, 1 \leq l \leq j$ , then it is satisfied by all such  $(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s)$ . (Cf. the proof of Lemma 2.1 in the appendix of Hörmander [6].) Further it is clear that if  $1 \leq l \leq j+1$ , we can always find a Puiseux series  $\varphi_l$  so that  $\vartheta_{l-1}(s) < \varphi_l(s) < \vartheta_l(s)$  for  $s > s_0$ . (Take e.g.  $(\vartheta_{l-1} + \vartheta_l)/2$  if  $1 < l \leq j, \vartheta_l - 1$  and  $\vartheta_j + 1$ .)

Now it follows from the assumption that the condition  $H_j$  is satisfied by some  $(\gamma_1(s), \dots, \gamma_{j-1}(s), \xi_j, s)$  with  $s > s_0$  and real  $\xi_j$ . Hence it follows from the discussion above that there exists a Puiseux series  $\gamma_j$ , convergent and real for  $s > s_0$ , so that  $(\gamma_1(s), \dots, \gamma_j(s), s)$  satisfies the condition  $H_j$  for  $s > s_0$ . This means that the system (1.15) has real solutions  $(\gamma_1(s), \dots, \gamma_j(s), \xi_{j+1}, \dots, \xi_{d+1})$  for  $s > s_0$ . Since the assumption is correct if  $j=1$ , we can thus in a finite number of steps prove the existence of a function  $\gamma(s) = (\gamma_1(s), \dots, \gamma_{d+1}(s))$ , meromorphic of  $s^{1/q}$  in a neighborhood of  $s^{1/q} = \infty$  for some positive integer  $q$ , and real for all large real  $s$ , so that  $\xi = \gamma(s)$  solves the system (1.15) for all sufficiently large  $s$ . We put  $s = r^{-2q}$  and  $\eta(r) = \gamma(r^{-2q})$ . Then  $\eta$  becomes meromorphic in a neighborhood of  $r=0$ , real for real  $r$ , and

$$|Q_1(\eta(r))| / |Q_2(\eta(r))| = |r|^{-a}$$

in a deleted neighborhood of  $r=0$ . But this contradicts (1.12). Hence we must have with a constant  $C$

$$|Q_1(\xi)| \leq C |Q_2(\xi)|, \xi \in B.$$

But, since  $B$  is a dense subset of  $R^{d+1}$ , it follows by continuity that this inequality is valid for all  $\xi \in R^{d+1}$ . The proof is complete.

**Theorem 1.3.** *Let  $P$  be a polynomial with principal part  $P_m \in \text{Hyp } N$ . Each of the following conditions is necessary and sufficient for  $P$  to belong to  $\text{Hyp } N$ .*

I. (Gårding [4])

*The roots  $\sigma$  of the equation  $P(\sigma(\tau N + i\xi)) = 0$  tend to zero, uniformly in  $\xi \in R^{d+1}$ , when  $\tau \rightarrow +\infty$ .*

II. (Hörmander [6] Theorem 5.5.7)

*$P$  is weaker than  $P_m$ .*

*Proof.* The necessity of II is immediate from Theorem 1.2 and Lemma 1.2.

To see that II implies I, we write  $P = \sum_{0 \leq k \leq m} P_k$ , where  $P_k(\xi) = \sum_{|\alpha| = k} c_\alpha \xi^\alpha$ , and observe that by the proof of Theorem 5.5.7 in Hörmander [6] it follows that II implies that there exists a number  $C$  so that

$$\text{II}' \quad |\tau^k P_{m-k}(i\tau N + \xi) / P_m(i\tau N + \xi)| \leq C \text{ if } \tau \geq 1 \text{ and } \xi \in R^{d+1}, 0 \leq k \leq m.$$

(Cf. Theorem 1.1.) We consider the polynomial in  $\rho$

$$\sum_{0 \leq k \leq m} \rho^{m-k} (\tau^k P_{m-k}(i\tau N - \xi) / P_m(i\tau N - \xi)). \tag{1.16}$$

The coefficient of the leading term in (1.16) is 1. Hence, in view of  $\Pi'$ , the zeros of (1.16) are bounded for  $\tau \geq 1$ ,  $\xi \in R^{d+1}$ . But since, for  $\tau \geq 1$ , and  $\xi \in R^{d+1}$ ,

$$P(\sigma(\tau N + i\xi)) = \tau^{-m} P_m(i\tau N - \xi) \sum_{0 \leq k \leq m} (-i\tau\sigma)^{m-k} (\tau^k P_{m-k}(i\tau N - \xi) / P_m(i\tau N - \xi)),$$

it follows that

I' the roots  $\sigma$  of  $P(\sigma(\tau N + i\xi))$  are  $O(\tau^{-1})$ , uniformly in  $\xi \in R^{d+1}$ , when  $\tau \rightarrow +\infty$ .

That I' implies I is trivial.

Assume now that  $P$  fulfills condition I. Take  $\tau_0$  so that the least upper bound of the absolute values of the roots  $\sigma$  of  $P(\sigma(\tau N + i\xi))$  is  $< 1$  for  $\tau > \tau_0$ ,  $\xi \in R^{d+1}$ . Then  $P(i(\tau N + i\xi)) = P(i\tau N - \xi) \neq 0$  for  $\tau > \tau_0$ ,  $\xi \in R^{d+1}$ . Since it is sufficient for hyperbolicity, that the imaginary parts of the characteristics are bounded from above (see e.g. Hörmander [6]), it follows that  $P \in \text{Hyp } N$ . The proof is complete.

*Remark.* I' and II' are of course also necessary and sufficient conditions for hyperbolicity.

## 2. Further necessary and sufficient conditions

Let  $Q(\tau) = A \prod_{i=1}^m (\tau - \tau_i)$  be a complex polynomial. We consider the Lagrange interpolation polynomials  $Q_I(\tau) = Q(\tau) / \prod_{i \in I} (\tau - \tau_i)$  where  $i$  runs through a subset  $I$  of  $\{1, \dots, m\}$ . When  $I = \{i\}$ , we write  $Q_I = Q_i$ ,  $1 \leq i \leq m$ . We shall also need the polynomials (McCarthy and Pederson [9])

$$L_k(Q, \tau) = \sum_{I_k} |Q_{I_k}(\tau)|^2,$$

where the summation goes over all  $I = I_k$  with  $k$  elements,  $0 \leq k \leq m$ .

Let  $P_m \in \text{Hyp } N$  be homogeneous of degree  $m$ . Denote by  $N^\perp$  the plane perpendicular to  $N$ . We consider for each  $\xi \in N^\perp$  the polynomials in  $\tau$ ,  $P_m^{(j)}(\tau; \xi) = (\partial/\partial\tau)^j P_m(\xi + \tau N)$ . These polynomials have only real roots, in view of the hyperbolicity of  $P_m$ . We define in the natural way for each  $\xi \in N^\perp$ ,  $(P_m^{(j)})_{I_k}(\tau; \xi)$  and  $L_k(P_m^{(j)}; \tau, \xi)$ ,  $0 \leq k \leq m - j$ .

We shall need the simple fact (McCarthy and Pederson [9]) that if  $Q(\tau)$  is a complex polynomial of degree  $m$  with  $m$  real zeros, then we have

$$|Q(\tau + i\sigma)|^2 = \sum_{0 \leq k \leq m} L_k(Q; \tau) \sigma^{2k}, \quad \tau \in R, \sigma \in R.$$

This is easily proved, e.g. by induction with respect to the degree of  $Q$ .

We shall also need the following lemma which is due to McCarthy and Pederson [9].

**Lemma 2.1.** *Let  $Q(\tau)$  be a polynomial of degree  $m$  with  $m$  real zeros. Then we have*

$$\frac{(m-r)!(k-r)!}{m!k!} \leq \frac{L_k(Q; \tau)}{L_{k-r}(Q^{(r)}; \tau)} \leq \frac{(k-1-r)!(k-r)!}{(k-1)!k!}, \quad r < k.$$

*Proof.* It suffices to assume  $Q$  real. We have

$$|Q(\tau + i\sigma)|^2 = \sum_{0 \leq k \leq m} L_k(Q; \tau) \sigma^{2k}, \quad \tau \in R, \sigma \in R. \quad (2.1)$$



We apply  $\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \sigma^2}$  to both sides of (2.1) and get

$$4|Q'(\tau + i\sigma)|^2 = \sum_{0 \leq k \leq m} (L_k(Q; \tau)'' \sigma^{2k} + 2k(2k-1)L_k(Q; \tau) \sigma^{2k-2}). \quad (2.2)$$

But  $Q'$  too has only real zeros, and therefore

$$|Q'(\tau + i\sigma)|^2 = \sum_{0 \leq k \leq m-1} L_k(Q'; \tau) \sigma^{2k}. \quad (2.3)$$

From (2.2) and (2.3) we get

$$4L_k(Q'; \tau) = L_k(Q; \tau)'' + (2k+2)(2k+1)L_{k+1}(Q; \tau). \quad (2.4)$$

Now it follows immediately from the definition of  $L_k$  that

$$L_k(Q; \tau)'' = 2 \sum_{I_k} ((Q'_{I_k}(\tau))^2 + Q''_{I_k}(\tau) Q_{I_k}(\tau)).$$

Differentiating  $Q_{I_k}$  yields

$$Q'_{I_k}(\tau) = Q_{I_k} \sum_{i \in I_k} (\tau - \tau_i)^{-1}$$

and

$$Q''_{I_k}(\tau) = Q_{I_k}(\tau) [(\sum_{i \in I_k} (\tau - \tau_i)^{-1})^2 - \sum_{i \in I_k} (\tau - \tau_i)^{-2}].$$

Hence we have

$$L_k(Q; \tau)'' = \sum_{I_k} (Q_{I_k}(\tau))^2 (4(\sum_{i \in I_k} (\tau - \tau_i)^{-1})^2 - 2 \sum_{i \in I_k} (\tau - \tau_i)^{-2}). \quad (2.5)$$

Since there are  $k+1$  subsets of each  $I_{k+1}$  with  $k$  elements, it follows that

$$L_{k+1}(Q; \tau) = 1/(k+1) \sum_{I_k} \sum_{i \in I_k} (Q_{I_k}(\tau))^2 / (\tau - \tau_i)^2. \quad (2.6)$$

Substitution of (2.5) and (2.6) into (2.4) gives

$$L_k(Q'; \tau) = \sum_{I_k} (Q_{I_k}(\tau))^2 [(\sum_{i \in I_k} (\tau - \tau_i)^{-1})^2 + k \sum_{i \in I_k} (\tau - \tau_i)^{-2}].$$

From Schwarz's inequality and (2.6) it follows that

$$\begin{aligned} k(k+1)L_{k+1}(Q; \tau) &= k \sum_{I_k} (Q_{I_k}(\tau))^2 \sum_{i \in I_k} (\tau - \tau_i)^{-2} \\ &\leq L_k(Q'; \tau) \\ &\leq \sum_{I_k} (Q_{I_k}(\tau))^2 [(m-k) \sum_{i \in I_k} (\tau - \tau_i)^{-2} + k \sum_{i \in I_k} (\tau - \tau_i)^{-2}] \\ &\leq m(k+1)L_{k+1}(Q; \tau). \end{aligned}$$

Hence we have

$$1/mk \leq \frac{L_k(Q; \tau)}{L_{k-1}(Q'; \tau)} \leq 1/k(k-1).$$

If we replace  $Q$  with  $Q^{(j)}$ ,  $j=1, 2, \dots, r-1$  and multiply the inequalities thus obtained, we get the wanted inequality.

**Theorem 2.1.** *Let  $P = \sum_{0 \leq k \leq m} P_k$ , where  $P_k(\xi) = \sum_{|\alpha|=k} c_\alpha \xi^\alpha$ , be a polynomial with principal part  $P_m \in \text{Hyp } N$ . Then the following conditions are equivalent.*

I  $P$  is weaker than  $P_m$

I'  $P_{m-k}$  is weaker than  $\langle \partial, N \rangle^{k-1} P_m = P_m^{(k-1)}$ ,  $1 \leq k \leq m$ ,

II (Peysers [12])

We can write

$$P_{m-k}(\tau N + \xi) = \sum_{1 \leq j \leq m-k+1} b_{kj}(\xi) (P_m^{(k-1)})_j(\tau, \xi), \tau \in R, \xi \in N^\perp, 0 \leq k \leq m,$$

where the  $b_{kj}$  are bounded for  $\xi \in N^\perp$ .

III (McCarthy and Pederson [9])

There exists a number  $C$  such that

$$|P_{m-k}(\tau N + \xi)|^2 \leq CL_k(P_m; \tau, \xi), \tau \in R, \xi \in N^\perp, 0 \leq k \leq m.$$

*Remark.* II and III are the two main conditions among the several equivalent conditions of McCarthy and Pederson [9].

*Proof.* We observe that it follows from the proof of Theorem 5.5.7 in Hörmander [6] that  $P$  is weaker than  $P_m$ , where  $P_m \in \text{Hyp } N$ , if and only if there exists a number  $C$  such that

$$|P_{m-k}(\tau N + \xi)| \leq C |P_m((\tau + i)N + \xi)|, \tau \in R, \xi \in N^\perp, 0 \leq k \leq m.$$

Hence, by (2.1),  $P$  is weaker than  $P_m$  if and only if there is a  $C$  such that

$$|P_{m-k}(\tau N + \xi)|^2 \leq C \sum_{0 \leq k \leq m} L_k(P_m; \tau, \xi), \tau \in R, \xi \in N^\perp, 0 \leq k \leq m. \quad (2.7)$$

Assume that  $P$  is weaker than  $P_m$ , hence fulfills the condition (2.7). We observe that  $L_q(P_m; \tau, \xi)$  is homogeneous in  $\tau$  and  $\xi$  of degree  $2(m-q)$ . Let  $0 \leq k \leq m$ . The homogeneity of  $P_{m-k}$  and  $L_q(P_m; \tau, \xi)$  gives

$$\begin{aligned} |P_{m-k}(\tau N + \xi)|^2 &= |r|^{2(m-k)} |P_{m-k}(r^{-1}(\tau N + \xi))|^2 \\ &\leq C |r|^{2(m-k)} \sum_{0 \leq q \leq m} L_q(P_m; r^{-1}\tau, r^{-1}\xi) \\ &= C \sum_{0 \leq q \leq m} |r|^{2(q-k)} L_q(P_m; \tau, \xi), \tau \in R, \xi \in N^\perp, r \neq 0. \end{aligned} \quad (2.8)$$

We write

$$P_m((\sigma + \tau)N + \xi) = P_m(N) \prod_{i=1}^m (\sigma - \sigma_i), \tau \in R, \sigma \in R, \xi \in N^\perp,$$

where  $|\sigma_1| \leq |\sigma_2| \leq \dots \leq |\sigma_m|$ , and observe that the largest term in  $L_q(P_m; \tau, \xi)$  is  $|P_m(N)|^2 \prod_{i=q+1}^m |\sigma_i|^2$ ,  $0 \leq q \leq m$ . We separate two cases. If  $\tau \in R$  and  $\xi \in N^\perp$  are such that  $\sigma_{k+1} \neq 0$ , it follows trivially that

$$|\sigma_{k+1}|^{2(q-k)} \prod_{i=q+1}^m |\sigma_i|^2 \leq \prod_{i=k+1}^m |\sigma_i|^2, 0 \leq q \leq m.$$

Hence we have with some constant  $C'$ , independent of  $\tau$  and  $\xi$ ,

$$|\sigma_{k+1}|^{2(q-k)} L_q(P_m; \tau, \xi) \leq C' L_k(P_m; \tau, \xi), \quad 0 \leq q \leq m.$$

We put  $r = \sigma_{k+1}$  in (2.8) and get

$$\begin{aligned} |P_{m-k}(\tau N + \xi)|^2 &\leq C \sum_{0 \leq q \leq m} |\sigma_{k+1}|^{2(q-k)} L_q(P_m; \tau, \xi) \\ &\leq CC'(m+1) L_k(P_m; \tau, \xi). \end{aligned}$$

If  $\tau \in R$  and  $\xi \in N^\perp$  are such that  $\sigma_{k+1} = 0$ , it follows that  $L_q(P_m; \tau, \xi) = 0$ ,  $0 \leq q \leq k$ . Hence we have in this case

$$|P_{m-k}(\tau N + \xi)|^2 \leq C \sum_{k+1 \leq q \leq m} |r|^{2(q-k)} L_q(P_m; \tau, \xi), \quad r \neq 0.$$

We let  $r \rightarrow 0$ , and get  $P_{m-k}(\tau N + \xi) = 0$ .

Hence we have in both cases

$$|P_{m-k}(\tau N + \xi)|^2 \leq CC'(m+1) L_k(P_m; \tau, \xi), \quad \tau \in R, \xi \in N^\perp,$$

and we have proved that I implies III.

That III and II are equivalent has been proved by McCarthy and Pederson [9]. We indicate the proof. By III and Lemma 2.1 we get

$$\begin{aligned} |P_{m-k}(\tau N + \xi)|^2 &\leq CL_1(P_m^{(k-1)}; \tau, \xi), \\ &= C \sum_{1 \leq j \leq m-k+1} ((P_m^{(k-1)})_j(\tau; \xi))^2, \quad \xi \in N^\perp, \tau \in R, 1 \leq k \leq m. \end{aligned}$$

But then it follows easily that for each  $\xi \in N^\perp$  we can write

$$P_{m-k}(\tau N + \xi) = \sum_{1 \leq j \leq m-k+1} b_{kj}(\xi) (P_m^{(k-1)})_j(\tau; \xi),$$

with  $|b_{kj}(\xi)|^2 \leq C$ ,  $1 \leq k \leq m$ .

Assume, now, that  $P$  fulfills the condition II. Then we get

$$|P_{m-k}(\tau N + \xi)|^2 \leq CL_1(P_m^{(k-1)}; \tau, \xi), \quad \xi \in N^\perp, \tau \in R, 1 \leq k \leq m.$$

But since  $L_1(P_m^{(k-1)}; \tau, \xi) \leq |P_m^{(k-1)}((\tau + i)N + \xi)|^2$ , I' follows immediately.

That I' implies I is trivial.

### 3. A necessary condition for hyperbolicity

**Theorem 3.1.** (Hörmander [6] Theorem 5.5.8.) *Let  $P \in \text{Hyp } N$  and let  $P_m$  be the principal part of  $P$ . Then the degree of  $P(\tau\xi + \eta)$  with respect to  $\tau$  for a fixed real  $\xi$  and indeterminate  $\eta$  never exceeds that of  $P_m(\tau\xi + N)$ .*

*Proof.* Immediate consequence of Theorem 1.1.

A condition equivalent to the one given in Theorem 3.1 is given by the following theorem of R. N. Pederson [10].

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**Theorem 3.2.** *Let  $P = \sum_{0 \leq k \leq m} P_k \in \text{Hyp } N$ , where  $P_k(\xi) = \sum_{|\alpha|=k} c_\alpha \xi^\alpha$ . Then we have, for every  $\xi \in R^{d+1}$ , that if*

$$\langle \partial, N \rangle^j P_m(\xi) = 0 \text{ for } j < \nu$$

then also

$$\partial^\alpha P_{m-k}(\xi) = 0 \text{ for } |\alpha| < \nu - k, 0 \leq k \leq m.$$

*Proof of the equivalence.* Let  $\xi \in R^{d+1}$ . We observe that

$$\begin{aligned} P_m(\tau\xi + N) &= \sum_{0 \leq j \leq m} \langle \partial, N \rangle^j P_m(\tau\xi)/j! \\ &= \sum_{0 \leq j \leq m} \langle \partial, N \rangle^j P_m(\xi) \tau^{m-j}/j!, \tau \in R. \end{aligned}$$

Hence the degree of  $P_m(\tau\xi + N)$  with respect to  $\tau$  is less than or equal to  $m - \nu$  if and only if  $\langle \partial, N \rangle^j P_m(\xi) = 0, 0 \leq j \leq \nu$ . On the other hand we have

$$\begin{aligned} P(\tau\xi + \eta) &= \sum_{\alpha} \partial^\alpha P(\tau\xi) \eta^\alpha / \alpha! \\ &= \sum_{\alpha} \sum_{0 \leq k \leq m - |\alpha|} \partial^\alpha P_{m-k}(\xi) \tau^{m-k-|\alpha|} \eta^\alpha / \alpha! \\ &= \sum_{0 \leq j \leq m} \tau^j \left( \sum_{0 \leq |\alpha| \leq m-j} \partial^\alpha P_{j+|\alpha|}(\xi) \eta^\alpha / \alpha! \right), \tau \in R, \eta \in R^{d+1}. \end{aligned}$$

Hence the degree of  $P(\tau\xi + \eta)$  with respect to  $\tau$  is less than or equal to  $m - \nu$  for all  $\eta \in R^{d+1}$  if and only if

$$\sum_{0 \leq |\alpha| \leq m-j} \partial^\alpha P_{j+|\alpha|}(\xi) \eta^\alpha / \alpha! = 0, \eta \in R^{d+1}, 0 \leq m - j < \nu.$$

But this is equivalent to  $\partial^\alpha P_{m-k}(\xi) = 0$  if  $|\alpha| < \nu - k$ .

The conditions of Theorem 3.1 and 3.2 are, however, not sufficient for hyperbolicity. We consider once more the polynomial

$$P((\tau, \xi_1, \xi_2)) = (\tau^2 - \xi_1^2 - \xi_2^2)(\tau^2 - \xi_1^2 - 2\xi_2^2) + \xi_1^2 \xi_2.$$

The principal part is clearly hyperbolic with respect to  $(1, 0, 0)$ , and has simple characteristics everywhere except for  $\xi_2 = 0$  where it has double characteristics. The lower order term is zero when  $\xi_2 = 0$  so the condition of Theorem 2.2 is fulfilled. However, we can see by the example after Theorem 1.1 that the polynomial  $P$  is not hyperbolic with respect to  $(1, 0, 0)$ .

#### 4. An application to hyperbolic systems<sup>1</sup>

We consider  $r \times r$  matrices  $Q(\xi) = (q_{jk}(\xi))$  where the elements  $q_{jk}$  of  $Q$  are polynomials in  $\xi = (\xi_1, \dots, \xi_{d+1})$ . We let  $I$  denote the  $r \times r$  unit matrix. The operator  $Q(D)$  is hyperbolic if it has a fundamental solution  $E$  with support in a proper cone  $K$ , that is, if there is a matrix  $E = (E_{jk})$  where the  $E_{jk}$  are distributions with support in  $K$  such that

$$Q(D)\delta * E = \delta I.$$

<sup>1</sup> This section was added to the paper on November 7th, 1968.

The matrix  $Q$  is hyperbolic if and only if the polynomial  $\det Q$  is hyperbolic. In fact, if  $\det Q$  is hyperbolic and if  $F$  is a fundamental solution of  $\det Q(D)$  with support in some proper cone  $K$ , then we have

$$Q(D)\delta * {}^{\circ}Q(D)\delta * FI = ((\det Q(D))\delta * F)I = \delta I.$$

But this means that  ${}^{\circ}Q(D)\delta * FI$  is a fundamental solution of  $Q(D)$  with support in  $K$ . Assume on the other hand that  $Q$  is hyperbolic and let  $E$  be a fundamental solution of  $Q(D)$  with support in some proper cone  $K$ . We observe that all scalar distributions with support in  $K$  constitute an associative and commutative convolution algebra. In view of this fact it follows that  $Q(D)\delta * E = \delta I$  implies that  $(\det Q(D))\delta * (\det E) = \delta$  where  $\det E$  means the convolution determinant. Since the support of  $\det E$  lies in  $K$ , it follows that  $\det Q(D)$  is hyperbolic.<sup>1</sup>

By this discussion it is clear that we should call  $Q$  hyperbolic with respect to  $N \in R^{d+1}$  if and only if the polynomial  $\det Q$  is in  $\text{Hyp } N$ . We define  $\text{Hyp}_r N$  to be the set of all polynomial matrices  $Q$  of type  $r \times r$  such that  $\det Q$  is in  $\text{Hyp } N$ .

Let

$$Q(\xi) = A(\xi) + B(\xi) \tag{4.1}$$

be  $r \times r$ -matrices where the elements  $a_{jk}$  of  $A$  are homogeneous polynomials in  $\xi$  of degree  $m_j + n_k$ , and where the elements  $b_{jk}$  of  $B$  are polynomials of degree  $< m_j + n_k$ ,  $j, k = 1, \dots, r$ . All the  $m_j$  and  $n_k$  are integers, not necessarily  $\geq 0$ . We shall say that the zero polynomial is a polynomial of any degree (even negative). We call  $Q = A + B$  strongly hyperbolic with respect to  $N \in R^{d+1}$  if  $A + B'$  is in  $\text{Hyp}_r N$  for any choice of the lower order matrix  $B'$  (Yamaguti and Kasahara [14], Strang [13]).

Assume that the matrix  $A$  of (4.1) is in  $\text{Hyp}_r N$ . In particular this implies that  $\det A$  is not identically zero, and it follows easily that the principal part of  $\det Q$  is  $\det A$ . But then we get immediately from Theorem 1.3 the following theorem.

**Theorem 4.1.** *Let  $Q = A + B$  be a matrix of the type (4.1), and assume that  $A$  is in  $\text{Hyp}_r N$ . Then  $Q$  is in  $\text{Hyp}_r N$  if and only if*

$$\det Q(\xi + iN) / \det A(\xi + iN) = \det(I + B(\xi + iN)A^{-1}(\xi + iN))$$

*is bounded for real  $\xi$ .*

The condition of Theorem 4.1 means that the product of all the eigenvalues of  $Q(\xi + iN)A^{-1}(\xi + iN)$  is bounded for real  $\xi$ . When all the  $m_j + n_k$  of (4.1) are equal to 1, it is easy to prove that even the individual eigenvalues must be bounded. For the proof we shall need the following lemma.

**Lemma 4.1.** *Let  $P = \sum_{0 \leq k \leq m} P_k \in \text{Hyp } N$ , where the  $P_k$  are homogeneous polynomials in  $\xi \in R^{d+1}$  of degree  $k$ . Then there is a number  $C$  such that*

$$P_m(\xi + iN) + \sum_{0 \leq k \leq m-1} z_k P_k(\xi + iN) \neq 0 \quad \text{if} \quad \xi \in R^{d+1} \quad \text{and} \quad \sum_{0 \leq k \leq m-1} |z_k| < C.$$

*Proof.* By Theorem 1.3 it follows that there is a number  $C_1 > 0$  such that

$$|P_k(\xi + iN)| / |P_m(\xi + iN)| \leq C_1 \quad \text{if} \quad \xi \in R^{d+1}, 0 \leq k \leq m.$$

<sup>1</sup> This very short proof is due to L. Gårding.

By the triangle inequality we get

$$\begin{aligned} |P_m(\xi + iN) + \sum_{0 \leq k \leq m-1} z_k P_k(\xi + iN)| &\geq |P_m(\xi + iN)| - \sum_{0 \leq k \leq m-1} |z_k P_k(\xi + iN)| \\ &\geq |P_m(\xi + iN)| (1 - C_1 \sum_{0 \leq k \leq m-1} |z_k|) \\ &> 0 \quad \text{if} \quad \sum_{0 \leq k \leq m-1} |z_k| < 1/C_1 = C. \end{aligned}$$

**Theorem 4.2.** *Let  $Q(\xi) = A(\xi) + B$  be a  $r \times r$  matrix, where the elements  $a_{jk}$  of  $A$  are homogeneous polynomials of degree one in  $\xi = (\xi_1, \dots, \xi_{d+1})$ , and where the elements  $b_{jk}$  of  $B$  are complex numbers. Assume that  $A$  is in  $\text{Hyp}_r N$ . Then  $Q$  is in  $\text{Hyp}_r N$  if and only if the spectral radius of  $BA^{-1}(\xi + iN)$  is bounded for real  $\xi$ .*

*Proof.* If the spectral radius of  $BA^{-1}(\xi + iN)$  is bounded, then the same is true of the spectral radius of  $I + BA^{-1}(\xi + iN)$ . Hence  $\det(I + BA^{-1}(\xi + iN))$  is then bounded for real  $\xi$  which in view of Theorem 4.1 means that  $Q$  is hyperbolic.

Assume, on the other hand, that  $Q$  is in  $\text{Hyp}_r N$ . Then we have, by Lemma 4.1,

$$\det(\lambda I + BA^{-1}(\xi + iN)) = \lambda^r \det(A(\xi + iN) + \lambda^{-1}B) / \det A^{-1}(\xi + iN) \neq 0$$

for all real  $\xi$  if  $|\lambda|^{-1}$  is sufficiently small. But this means that the eigenvalues of  $BA^{-1}(\xi + iN)$  are bounded for real  $\xi$ .

*Remark.* It is easy to see that if  $r > 1$ , the necessary and sufficient condition on  $B$  given by Theorem 4.2 is strictly weaker than the sufficient condition used by Kopáček and Suchá [7] to define a class of first-order hyperbolic systems of the type (4.1). Their condition is that if  $({}^{\circ}A(\xi))B = (\tilde{b}_{jk}(\xi))$ , then (see Theorem 2.1, II)

$$\tilde{b}_{jk}(\xi + \tau N) = \sum_{1 \leq \nu \leq m} \gamma_{\nu}^{jk}(\xi) (\det A)_{\nu}(\tau; \xi)$$

with bounded functions  $\gamma_{\nu}^{jk}$ ,  $\tau \in R$ ,  $\xi \in N^{\perp}$ ,  $j, k = 1, \dots, n$ . In view of Theorem 2.1 it follows that this condition is equivalent to the condition that

$$\|A^{-1}(\xi + iN)B\| = \|(\det(A(\xi + iN)))^{-1}({}^{\circ}A(\xi + iN))B\|$$

is bounded for real  $\xi$ . This implies of course that the spectral radius of  $BA^{-1}(\xi + iN)$  is bounded for real  $\xi$ .

*Example 1.* The following example of a non-hyperbolic matrix is due to Petrowsky [11]. The matrix

$$Q(\xi) = \begin{pmatrix} -\xi_1 + \xi_2 & -\xi_3 & 0 \\ -\xi_3 & -\xi_1 & -\xi_3 \\ 0 & -1 & -\xi_1 \end{pmatrix}$$

is not hyperbolic with respect to  $(1, 0, 0)$  although the corresponding matrix  $A$  is. In fact a simple computation shows that the only non-zero eigenvalue of  $BA^{-1}(\xi + iN)$  is

$$-\xi_3(\xi_1 + i - \xi_2) / ((\xi_1 + i)((\xi_1 + i)^2 - (\xi_1 + i)\xi_2 - \xi_3^2)).$$

If we put  $\xi_1=0$  and  $\xi_2=\xi_3^2$  we get

$$-\xi_3(i-\xi_3^2)/i(-1-i\xi_3^2-\xi_3^2)$$

which is clearly not bounded.

*Example 2.* Consider the matrix

$$A(\xi) = \begin{pmatrix} \xi_1 - \xi_2 & 0 & 0 \\ 0 & \xi_1 - \xi_2 & 0 \\ \xi_2 & 0 & \xi_1 - \xi_2 \end{pmatrix}.$$

The matrix  $A$  is clearly hyperbolic with respect to  $N=(1, 0)$ .

Let further

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A$  simple computation yields

$$A^{-1}(\xi + iN) = \begin{pmatrix} (\xi_1 + i - \xi_2)^{-1} & 0 & 0 \\ 0 & (\xi_1 + i - \xi_2)^{-1} & 0 \\ -\xi_2(\xi_1 + i - \xi_2)^{-2} & 0 & (\xi_1 + i - \xi_2)^{-1} \end{pmatrix}.$$

We get

$$BA^{-1}(\xi + iN) = \begin{pmatrix} 0 & (\xi_1 + i - \xi_2)^{-1} & 0 \\ 0 & 0 & 0 \\ -\xi_2(\xi_1 + i - \xi_2)^{-2} & 0 & (\xi_1 + i - \xi_2)^{-1} \end{pmatrix},$$

and

$$A^{-1}(\xi + iN)B = \begin{pmatrix} 0 & (\xi_1 + i - \xi_2)^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & -\xi_2(\xi_1 + i - \xi_2)^{-2} & (\xi_1 + i - \xi_2)^{-1} \end{pmatrix}.$$

Since  $\xi_2(\xi_1 + i - \xi_2)^{-2}$  is not bounded for real  $\xi$ , as is seen by putting  $\xi_1 = \xi_2$ , it follows that neither  $\|BA^{-1}(\xi + iN)\|$  nor  $\|A^{-1}(\xi + iN)B\|$  is bounded for real  $\xi$ . But the eigenvalues of  $BA^{-1}(\xi + iN)$  are 0 and  $(\xi_1 + i - \xi_2)^{-1}$ . It follows that the spectral radius of  $BA^{-1}(\xi + iN)$  is bounded for real  $\xi$ , and hence that  $A + B$  is in  $\text{Hyp}_3 N$ .

*Example 3.* The condition of Theorem 4.2 is of course always sufficient for hyperbolicity, but in general not necessary if some  $m_j + n_k$  is different from one. Consider e.g. the matrix

$$A(\xi) = \begin{pmatrix} (\xi_1 - \xi_2)^2 & \xi_2 \\ 0 & \xi_1 - \xi_2 \end{pmatrix}.$$

$A$  is hyperbolic with respect to  $N=(1, 0)$ , and we may put  $n_1=2$ ,  $n_2=1$ , and  $m_1=m_2=0$ . Set

$$B(\xi) = \begin{pmatrix} \xi_2 & 0 \\ \xi_1 - \xi_2 + 1 & 1 \end{pmatrix}.$$

We put  $\eta = (\xi_1 + i - \xi_2)^{-1}$ . A computation yields immediately

$$B(\xi + iN) A^{-1}(\xi + iN) = \begin{pmatrix} \xi_2 \eta^2 & -\xi_2^2 \eta^3 \\ \eta + \eta^2 & -\xi_2 \eta^2 - \xi_2 \eta^3 + \eta \end{pmatrix}.$$

Further we get

$$\det(\lambda I + B(\xi + iN) A^{-1}(\xi + iN)) = \lambda^2 + \lambda(\eta - \xi_2 \eta^3) + \xi_2 \eta^3.$$

Now it is clear, by putting  $\xi_1 = \xi_2$ , that  $\xi_2 \eta^3$  is not bounded for real  $\xi$ . Since the coefficients of the polynomial in  $\lambda$  are not bounded for real  $\xi$ , we see that the zeros cannot be bounded either. Thus the spectral radius of  $B(\xi + iN) A^{-1}(\xi + iN)$  is not bounded. But  $\det(I + B(\xi + iN) A^{-1}(\xi + iN)) = 1 + \eta$  is bounded for real  $\xi$ , and thus  $A + B$  is in  $\text{Hyp}_2 N$ .

Theorem 4.1 makes it easy to derive necessary and sufficient conditions for the matrix  $A$  of (4.1) to be strongly hyperbolic.

**Theorem 4.3.** *Let  $A$  be the matrix of (4.1). If  $A$  is strongly hyperbolic with respect to  $N \in R^{d+1}$ , it follows that the matrix  $A^{-1}(\xi + iN) = (c_{jk}(\xi))$  exists for all real  $\xi$ , and that there is a number  $C$  such that*

$$|c_{jk}(\xi)| \leq C(1 + |\xi|)^{1-m_k-n_j} \quad (4.2)$$

if  $\xi \in R^{d+1}$  and  $m_k + n_j > 0$ ,  $j, k = 1, \dots, r$ . On the other hand, if (4.2) is valid for real  $\xi$  and all  $j, k = 1, \dots, r$ , it follows that  $A$  is strongly hyperbolic.

*Proof.* Assume first that  $A$  is strongly hyperbolic with respect to  $N$ . Then, in particular,  $A$  is in  $\text{Hyp}_r N$ , and it follows that  $A^{-1}(\xi + iN)$  exists for all real  $\xi$ . Further,  $A + B$  is in  $\text{Hyp}_r N$  for any choice of the lower order matrix  $B$ . We choose  $B$  with only one non-zero element, say  $b_{pq}$ ,  $1 \leq p \leq r$ ,  $1 \leq q \leq r$ . It is easy to see that the condition of Theorem 4.1 for our choice of  $B$  means that  $|1 + b_{pq}(\xi + iN)c_{qp}(\xi)|$  is bounded for real  $\xi$ , and this implies that  $b_{pq}(\xi + iN)c_{qp}(\xi)$  is bounded for real  $\xi$ . But we may choose any polynomial of degree  $\leq m_p + n_q - 1$  for  $b_{pq}$ . It follows that

$$|c_{qp}(\xi)| \leq C(1 + |\xi|)^{1-m_p-n_q} \quad \text{if } \xi \in R^{d+1} \quad \text{and} \quad m_p + n_q > 0.$$

Assume, on the other hand, that  $A^{-1}(\xi + iN)$  exists for all real  $\xi$  and that  $|c_{jk}(\xi)| \leq C(1 + |\xi|)^{1-m_k-n_j}$  if  $\xi \in R^{d+1}$ ,  $j, k = 1, \dots, r$ . Let  $B$  be any lower order matrix. Since the elements  $b_{jk}$  of  $B$  are polynomials of degree  $\leq m_j + n_k - 1$ , it follows that for some constant  $C_1$

$$|b_{jk}(\xi + iN)| \leq C_1(1 + |\xi|)^{m_j+n_k-1}, \quad \xi \in R^{d+1}, \quad j, k = 1, \dots, r.$$

Hence we have for the elements of  $B(\xi + iN) A^{-1}(\xi + iN)$

$$\left| \sum_{1 \leq j \leq r} b_{j\nu}(\xi + iN) c_{\nu k}(\xi) \right| \leq C_2(1 + |\xi|)^{m_j-m_k}, \quad \xi \in R^{d+1}, \quad j, k = 1, \dots, r.$$

In particular the elements on the main diagonal ( $j = k$ ) are bounded by a constant, and this property is not altered by adding a constant to those elements. Hence, if  $I + B(\xi + iN) A^{-1}(\xi + iN) = (d_{jk}(\xi))$ , we have

$$|d_{jk}(\xi)| \leq C_3(1 + |\xi|)^{m_j-m_k}, \quad \xi \in R^{d+1}, \quad j, k = 1, \dots, r.$$



But then it follows, quite trivially, that  $\det(I + B(\xi + iN)A^{-1}(\xi + iN))$  is bounded for real  $\xi$ .

It remains only to prove that  $A$  is in  $\text{Hyp}_r N$ . But this is clear since the existence of  $A^{-1}(\xi + iN)$  means that  $\det A(\xi + iN) \neq 0$  for real  $\xi$ . From the homogeneity it follows that  $\det A(\xi + i\tau N) \neq 0$  if  $\xi \in R^{d+1}$  and  $\tau \in R - \{0\}$ . In particular we have  $\det A(iN) \neq 0$ , and it follows that  $\det A(N) \neq 0$ . Thus  $A$  is in  $\text{Hyp}_r N$ . By Theorem 4.1 it follows that  $A + B$  is in  $\text{Hyp}_r N$  too. Hence  $A$  is strongly hyperbolic with respect to  $N$ . The proof is complete.

When  $A$  is a  $1 \times 1$ -matrix, i.e. a polynomial of degree  $m$ , the condition of the theorem is simply

$$|A(\xi + iN)| \geq C(1 + |\xi|)^{m-1}, \quad \xi \in R^{d+1}.$$

If all the  $m_j + n_k$  of (4.1) are equal to a common integer  $m_0$ , the condition can be expressed as

$$\|A^{-1}(\xi + iN)\| \leq C(1 + |\xi|)^{1-m_0}, \quad \xi \in R^{d+1},$$

or if  $m_0 = 1$ , simply

$$\|A^{-1}(\xi + iN)\| \leq C, \quad \xi \in R^{d+1}.$$

Because  $A(\xi + i\tau N)$  is homogeneous in  $\xi$  and  $\tau$ , the last inequality is equivalent to

$$\tau \|A^{-1}(\xi + i\tau N)\| \leq C, \quad \xi \in R^{d+1}, \quad \tau > 0,$$

which is essentially the condition for strong hyperbolicity, derived by Strang in [13].

A couple of examples will show that it is not necessary for strong hyperbolicity that (4.2) is valid for all  $j, k = 1, \dots, r$  in the case when some  $m_k + n_j \leq 0$ , and that it is not sufficient that (4.2) is valid for all  $j, k$  with  $m_k + n_j > 0$ .

*Example 1.* Put

$$A(\xi) = \begin{pmatrix} \xi_1 - \xi_2 & \xi_2^2 \\ 0 & \xi_1 - \xi_2 \end{pmatrix}.$$

Then  $A$  is in  $\text{Hyp}_2 N$  where  $N = (1, 0)$ . We may take  $m_1 = 1, m_2 = 0, n_1 = 0,$  and  $n_2 = 1$ . It is easy to see that  $A + B$  is in  $\text{Hyp}_2 N$  for any choice of the lower order matrix

$$B(\xi) = \begin{pmatrix} a & b\xi_1 + c\xi_2 + d \\ 0 & e \end{pmatrix}.$$

This means that  $A$  is strongly hyperbolic with respect to  $N$ . But if we compute the  $c_{12}$ , corresponding to  $A$ , we find that  $c_{12}(\xi) = -\xi_2^2(\xi_1 + i - \xi_2)^{-2}$ . In particular we have that  $c_{12}((\xi_2, \xi_2)) = \xi_2^2$ . We see that  $c_{12}$  does not fulfill the condition (4.2).

*Example 2.* Consider the matrix

$$A(\xi) = \begin{pmatrix} (\xi_1 - \xi_2)\xi_1 & 0 & 0 \\ 0 & (\xi_1 - \xi_2) & 0 \\ 0 & \xi_2^2 & (\xi_1 - \xi_2) \end{pmatrix}.$$

$A$  is clearly hyperbolic with respect to  $N = (1, 0)$ , and we may put  $m_1 = 1, m_2 = 0, m_3 = 1, n_1 = 1, n_2 = 1,$  and  $n_3 = 0$ . We have  $m_k + n_j > 0$ , except for the case  $k = 2, j = 3$ . We compute  $A^{-1}(\xi + iN)$ , and get, with  $\eta = (\xi_1 + i - \xi_2)^{-1}$ ,

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$$A^{-1}(\xi + iN) = \begin{pmatrix} \eta(\xi_1 + i)^{-1} & 0 & 0 \\ 0 & \eta & 0 \\ 0 & -\xi_2^2 \eta^2 & \eta \end{pmatrix}$$

We see that all the elements  $c_{jk}$ , except  $c_{32}$ , fulfill the condition (4.2). However,  $A$  is not strongly hyperbolic, as is seen by choosing as lower order matrix

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We get

$$BA^{-1}(\xi + iN) = \begin{pmatrix} 0 & -\xi_2^2 \eta^2 & \eta \\ \eta(\xi_1 + i)^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence it follows that  $\det(I + BA^{-1}(\xi + iN)) = 1 + \xi_2^2 \eta^3 (\xi_1 + i)^{-1}$ , and this is clearly not bounded for real  $\xi$ , so by Theorem 4.1  $A + B$  is not hyperbolic.

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