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A nullstellensatz for ordered fields

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For an ordered field k, a *realzero* of an ideal P in the polynomial ring $k[X]=k[X_1, \ldots, X_n]$ in n variables is a zero in $\bar{k}^{(n)}$, where $\bar{k}$ is the realclosure of k, the *realvariety* $\mathcal{V}_R(P)$ is the set of all realzeros of P, and, as usual, $\mathcal{I}(G)$, for any subset G of $\bar{k}^{(n)}$ is the ideal of all members of $k[X]$ that vanish all over G. Our nullstellensatz asserts:

$$\mathcal{I}\mathcal{V}_R(P)=\sqrt[R]{P}=\textit{realradical of } P,$$

where $\sqrt[R]{P}$ is the set of all $f(X)$ such that for some exponent m, some *rational* functions $u_i(X)$ in $k(X)$, and positive $p_i \in k$

$$f(X)^m(1+\Sigma\, p_i u_i(X)^2) \in P.$$

The proof, which uses Artin's solution of Hilbert's 17th problem, and which grew out of an attempt to find an easier solution to the problem, is straight-forward, inspired in large part by Lang's elegant formulation of various extension theorems, especially Theorem 5, p. 278 [2]. We give a new proof of this theorem, and a generalization to finitely generated formally real rings over k (Theorem 1).

Throughout, k will be an ordered field. For any ordered field K, $\bar{K}$ is its real closure.

A simple consequence of Artin's work (see Theorem 13 and Lemma 1 of Jacobson, Chapter VI [1]) is:

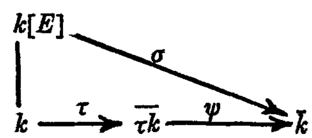
Artin's Theorem. *Let k be an ordered field, let $K=k(T)\equiv k(T_1, \ldots, T_n)$ be a pure transcendental ordered extension of k, with T_i algebraically independent. Let $f(Y) \in k[T][Y]$ have a root in $\bar{K}$, let $u_1, \ldots, u_m$ be a finite set of nonzero elements of $k[T]$. There exists a homomorphism σ over k from $k[T]$ to $\bar{k}$ satisfying*

(i) $\sigma(u_i) \neq 0$, $1 \leqslant i \leqslant m$.

(ii) *$f^\sigma(Y)$ has a root in $\bar{k}$.*

Lang's Theorem (Lang, Theorem 5, p. 278 [2]). *Let k be an ordered field, let $k \xrightarrow{\tau} R$ be an order-embedding of k into a realclosed field R. Let K be a field containing k and admitting an order extending the order of k. Then for every finite subset E of K there exists a homomorphism ψ: $k[E] \to R$ extending τ.*

Proof. Suppose the theorem is known for the case where τ is the inclusion map $k \subset \bar{k}$. For general τ, the algebraic closure $\overline{\tau k}$ in R is a real closure of τk and also of k, so by the uniqueness theorem for real closures there exists ψ: $\overline{\tau k} \cong \bar{k}$ such that ψ is order preserving and $\psi\tau$ is the inclusion $k \subset \bar{k}$. By supposition there exists σ: $k[E] \to \bar{k}$. Then $\psi^{-1}\sigma$: $k[E] \to R$ extends τ.



Hence we consider the case of τ equal to the inclusion of k in $\bar{k}$. It is obviously enough to prove the claim under the hypothesis that $K|k$ is finitely generated. If K is a pure transcendental extension of k then our claim is Artin's Theorem. Suppose $K = k(T)(u)$, where $k(T) = k(T_1, ..., T_n)$, $T_1, ..., T_n$ is a transcendence base and u is algebraic over $k(T)$. Let u be taken as integral over $k[T]$ with monic minimal polynomial $m(X)$ in $k[T][X]$. Let E be a finite subset of K and let D consist of all denominators of coefficients of u^i appearing in expressions for members of E as polynomials in u of degree less than the degree of $m(X)$. Let K be given a fixed order extending the order of k. Then $m(X)$ has a root in the real closure of $k(T)$. By Artin's Theorem there exists a homomorphism σ over k from $k[T]$ to $\bar{k}$ such that

(i) If $d \in D$ then $\sigma(d) \neq 0$.
(ii) $m^{\sigma}(X)$ has a root, say z, in $\bar{k}$.

Suppose $f(x) \in k[T][X]$ vanishes at u. Then $f(X) = m(X)q(X)$, $m(X)$ is primitive, so by Gauss's Theorem, $q(X) \in k[T][X]$. Hence $f(X)$ belongs to the kernel of the homomorphism

$$g(X) \to g^{\sigma}(z), \quad k[T][X] \to \bar{k}.$$

Induced is a homomorphism over k

$$\psi : f(u) \to f^{\sigma}(z), \quad k[T][u] \to \bar{k}.$$

For d in $D \subset k[T]$, $\psi(d) = \sigma(d) \neq 0$. Thus ψ extends to $\psi' : k[T][E] \to \bar{k}$.

Lang's Corollary [2]. *If $u_1 < ... < u_m$ are arbitrary members of $k[E]$ then the ψ of the theorem can be chosen so that $\psi(u_1) < ... < \psi(u_m)$.*

Definitions. Let k be an ordered field, let A be a unitary commutative ring containing k.

$$S(A) \equiv S(A \mid k) = \{1 + \Sigma\, p_i a_i^2;\ a_i \in A,\ 0 < p_i \in k\}.$$

"$A \mid k$ is *formally real*", "*A is formally real over k*" mean that if $\Sigma\, p_i a_i^2 = 0$, $0 < p_i \in k$, $a_i \in A$, then $a_i = 0$ for all i.

Examples of formally real rings over k: If $K \mid k$ is a field extension then $K \mid k$ is formally real if and only if the order of k extends to K. If $A \mid k$ is formally real so is $A[X] \mid k$, where $A[X] = A[X_1, ..., X_n]$ is the polynomial ring.

Proposition 1. *If $A \mid k$ is formally real then $S(A \mid k)$ is a multiplicative set containing no zerodivisors. The total ring of fractions is formally real over k. Thus we have*

$$k \subset A \subset S^{-1}A \subset A_1 = \textit{total ring of fractions},$$

each formally real over k.

The proof is routine and is omitted.

Definition. Let A be formally real over k, let A_1 be the total ring of fractions of A. We set

$$S_1(A) \equiv S_1(A \mid k) = A \cap S(A_1).$$

Clearly $S_1(A)$ *is a multiplicative subset of* A *containing no zero divisors, since* A_1 *is also formally real over* k.

Lemma. *Let* $A \mid k$ *be formally real, let* P *be an ideal of* A *which is maximally disjoint from* $S_1(A)$. *Then* A/P *is a formally real integral domain over* k, *and the order of* k *extends to the field of quotients of* A/P.

Proof. Since $S_1(A)$ is multiplicative, P is prime and A/P is an integral domain. Let P be any ideal disjoint from $S_1(A)$ and suppose A/P is not formally real over k. Then there exist a_i in A, $p_i>0$ in k, a_1 not in P, such that

$$a=\sum_{i=1}^{n} p_i a_i^2 \text{ belongs to } P. \tag{1}$$

Now we shall show that $P+a_1A$ is also disjoint from $S_1(A)$ from which follows the Lemma. Suppose $P+a_1A$ meets $S_1(A)$. Then there exist u in P, d in A, b_i in A_1, $q_i>0$ in k, such that

$$u+dp_1a_1 = 1+\Sigma\, q_i b_i^2 \in S_1(A).$$

Squaring both sides gives ($r_i>0$ in k, c_i in A_1):

$$u(u+2dp_1a_1) = 1+\sum r_i c_i^2 + d^2 p_1\left(-a+\sum_{i=2}^{n} p_i a_i^2\right), \tag{2}$$

where we have substituted from (1) for $p_1a_1^2$. Now a belongs to P, so after transposing d^2p_1a, we have a member of P on the left side of (2) and a member of $S_1(A)$ on the right side, contradicting our hypothesis that P is disjoint from $S_1(A)$.

Theorem 1. *If* $A \mid k$ *is a finitely generated formally real ring then any order-embedding* ψ *of* k *into a real closed field* F *extends to a homomorphism of* $A \mid k$ *into* F.

Proof. Let P be an ideal of A maximally disjoint from $S_1(A)$. Write $A=k[x]=k[x_1, ..., x_n]$ let σ be the canonical map $A\rightarrow A/P$:

$$k[x]\xrightarrow{\sigma} k[\sigma x_1, ..., \sigma x_n] = k[\sigma x].$$

Since A/P is finitely generated, the Lemma allows application of Lang's Theorem to yield a map $\bar{\psi}: k[\sigma x]\rightarrow F$ extending ψ. Then $\bar{\psi}\sigma$ also extends ψ.

Corollary. *Let* A *be formally real over* k, *let* $u_1, ..., u_n$ *be elements of* A *which are not zero-divisors. Then there exists a homomorphism* $\psi: k[u_1, ..., u_n]\rightarrow \bar{k}$ *over* k *with* $\psi(u_i)\neq 0$, $i=1, ..., n$.

Proof. Apply the theorem to the finitely generated formally real subring $k[u_1, ..., u_n, u_1^{-1}, ..., u_n^{-1}] \mid k$ of the total fraction ring of A.

Let $A=k[X]=k[X_1, ..., X_n]$ be the ring of all polynomials in n variables over the ordered field k. Let P be an ideal of A. If $f(X)$ is a polynomial and if there exist $m>0$, $0<p_i\in k$, and polynomials $g_i(X)$, $h_i(X)$ such that $f(X)^m(1+\Sigma p_i g_i(X)^2 h_i(X)^{-2})\in P$, then $f(X)$ clearly vanishes at every *realzero* of the ideal P. Thus

$$\sqrt[R]{P} \subset \mathscr{I}\mathscr{V}_R(P).$$

It is easy to verify that $P \cap S_1(A) = \phi \Leftrightarrow \sqrt[R]{P} \neq A.$

Nullstellensatz. *For an ordered field k, $\mathscr{I}\mathscr{V}_R(P) = \sqrt[R]{P}$ for every P in $k[X_1, ..., X_n]$.*

Proof. Following the argument of Zariski-Samuel vol. II, p. 164 [3], we first show: *If $\sqrt[R]{P}$ is a proper subset of $k[X]$ then $\mathscr{V}_R(P) \neq \phi$.* If $P \subset M$, where M is an ideal, then obviously $\mathscr{V}_R(M) \subset \mathscr{V}_R(P)$, so it is quite enough to verify the claim when P is maximally disjoint from S_1. According to the Lemma, $k[X]/P$ is a finitely generated formally real integral domain which admits a homomorphism ψ over k (by Lang's Theorem) into $\bar{k}$. Denoting the coset of X_i in $k[X]/P$ by x_i, the point $(\psi(x_1), ..., \psi(x_n))$ is a member of $\mathscr{V}_R(P)$, since if $f(X)$ belongs to P then

$$0 = \psi(f(X) + P) = f(\psi(x_1), ..., \psi(x_n)).$$

This proves the italicized assertion.

Now to prove the theorem, let P be any ideal, say $P = (f_1(X), ..., f_q(X))$, and suppose $f(X)$ belongs to $\mathscr{I}\mathscr{V}_R(P)$. The ideal in $k[X][T]$ generated by $\{f_1(X), ..., f_q(X), 1 - Tf(X)\}$ has no realzeros. By the italicized claim there exist polynomials $h(X, T)$, $h_i(X, T)$, and rational functions $g_i(X, T)$ such that

$$1 + \Sigma\, p_i g_i(X, T)^2 = h(X, T)(1 - Tf(X)) + \Sigma\, h_i(X, T) f_i(X),\ 0 < p_i \in k). \quad (1)$$

Suppose $f(X)^{-1}$ can be substituted for T on the left side. The right side has only powers of $f(X)$ in the denominators so for some $m > 0$, we get an expression, with polynomials for $u_i(X)$,

$$f(X)^m(1 + \Sigma\, p_i g_i(X, f(X)^{-1})^2) = \Sigma\, u_i(X) f_i(X),$$

and the right side belongs to P. Hence $f(X) \in \sqrt[R]{P}$. The proof will be completed by showing that $f(X)^{-1}$ can be substituted for T on the left side of (1). Observe that

$$k(X, T) = k(X_1, .., X_n, 1 - Tf(X)).$$

Set $Y = 1 - Tf(X)$. Extend the order of k to $K(X, T)$ so that $1 - Tf(X)$ is infinitesimal relative to $k(X)$. Suppose $f(X)^{-1}$ cannot be substituted for T in (1). Then for at least one $g_i(X)$, the denominator is divisible by $1 - Tf(X) = Y$, hence the left side is infinitely large relative to $k(X_1, ..., X_n)$ while the right side is not, since it is a polynomial in $T = f(X)^{-1}(1 - Y)$ (coefficients in $k(X)$) which has the same order of magnitude as $f(X)^{-1} \in k(X)$. The contradiction completes the proof.

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