ARKIV FÖR MATEMATIK Band 3 nr 35

Read 13 February 1957

On linear recurrences with constant coefficients

By Trygve Nagell

1.—An arithmetical function $A(n)=A_n$ of n may be defined by *recursion* in the following way: The value of A_n is defined for $n=0, 1, 2, \ldots, m-1$, and there is given a rule indicating how the value of A_{m+n} may be determined when the values of A_μ are known for $\mu=n, n+1, n+2, \ldots, n+m-2, n+m-1$, n being an integer $\geqq 0$.

The infinite sequence

$$A_0, A_1, A_2, A_3, \ldots, A_n, \ldots$$

thus defined is said to be a *recurrent sequence*. We denote it by $\{A_n\}$. The rule of recursion has often the shape of a *recursive formula*.

For instance, the function $A_n = n!$ satisfies the recursive formula

$$A_{n+1} = (n+1)\, A_n \tag{1}$$

and the initial condition $A_0 = 1$.

The general solution of the recursive formula

$$A_{n+1} = 2\, A_n \tag{2}$$

is obviously

$$A_n = 2^n A_0.$$

The arithmetical function A_n satisfying the recursive formula

$$A_{n+2} = \sqrt{A_{n+1}\, A_n} \tag{3}$$

is a function of n, A_0 and A_1.

Another example is the function A_n defined by the recursive formula

$$A_{n+2} = A_{n+1} + A_n \tag{4}$$

and the initial conditions $A_0 = A_1 = 1$. In this case we get the following series:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

the so-called *Fibonacci numbers*.

2.—In Algebra and in Number Theory we often have to do with *linear recurrences*, that is to say recursive formulae of the type

$$A_{m+n} = a_1 A_{m+n-1} + a_2 A_{m+n-2} + \cdots + a_m A_n + b, \tag{5}$$

where the coefficients $a_1, a_2, \ldots, a_m$ and b are functions of n. In the sequel we shall only consider the case in which the coefficients are *constants*. We shall develop an elementary theory of this category of linear recurrences.

When $a_m \neq 0$, the recurrence (5) is said to be of the mth *order*. When $a_m = a_{m-1} = \ldots = a_{\mu+1} = 0$ and $a_\mu \neq 0$, the order of the recurrence is μ. It suffices to consider the case with $a_m \neq 0$.

The recurrent sequence

$$A_0, A_1, A_2, A_3, \ldots, A_n, \ldots \tag{6}$$

is said to be of the mth *order* if the numbers A_n satisfy a linear recurrence of order m but no recurrence of a lower order. A recurrent sequence of the mth order satisfies exactly one recurrence of the type (5). In fact, if it satisfied another recurrence

$$A_{m+n} = c_1 A_{m+n-1} + c_2 A_{m+n-2} + \cdots + c_m A_n + d,$$

we should have by elimination of A_{m+n}

$$(a_1 - c_1) A_{m+n-1} + (a_2 - c_2) A_{m+n-2} + \cdots + (a_m - c_m) A_n + (b - d) = 0.$$

But this recurrence is at most of order $m-1$.

The formula (2) is of the first order. The formula (4) is of the second order.

When $b = 0$, the recurrence (5) is *homogeneous*. The homogeneous recurrence

$$X_{m+n} = a_1 X_{m+n-1} + a_2 X_{m+n-2} + \cdots + a_m X_n \tag{7}$$

is said to have the *scale* $[a_1, a_2, \ldots, a_m]$.

As a direct consequence of the above definition we have

Theorem 1. *If* $\{A_n\}$ *and* $\{B_n\}$ *are two recurrent sequences satisfying the recurrence* (7), *then* $\{A_n + B_n\}$ *is also a recurrent sequence satisfying* (7).

3.—We shall prove

Theorem 2. *Suppose that the numbers* $a_1, a_2, a_3, \ldots, a_m$ *are given,* $a_m \neq 0$.

If $\{A_n\}$ *is a recurrent sequence such that the numbers* A_n *satisfy the homogeneous recurrence*

$$A_{m+n} = a_1 A_{m+n-1} + a_2 A_{m+n-2} + \cdots + a_m A_n \tag{8}$$

of order m, *we have the relation*

$$\sum_{n=0}^{\infty} A_n z^n = \frac{b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1}}{1 - a_1 z - a_2 z^2 - \cdots - a_m z^m}, \tag{9}$$

where $b_0, b_1, b_2, \ldots, b_{m-1}$ *are constants which are uniquely determined by* $A_0, A_1, \ldots, A_{m-1}, a_1, a_2, \ldots, a_m$.

Conversely, when $b_0, b_1, \ldots, b_{m-1}$ *are arbitrarily given constants, the coefficients* A_n *in* (9) *satisfy the recurrence* (8).

Proof. Given the recurrent sequence $\{A_n\}$ satisfying (8) it is easy to see that the infinite series

$$\sum_{n=0}^{\infty} A_n z^n \tag{10}$$

has a certain circle of convergence. In fact, we will show by induction that, for all $N \geqq 0$,

$$|A_N| \leqq Q^N Q_1, \tag{11}$$

where
$$Q = 1 + |a_1| + |a_2| + \cdots + |a_m|$$

and
$$Q_1 = \max(|A_0|, |A_1|, \ldots, |A_{m-1}|).$$

The relation (11) is clearly true for $N = 0, 1, 2, \ldots, m-1$. It follows from (8) that (11) is true for $N = m$. If we suppose that (11) is true for $N = m, m+1, m+2, \ldots, m+\nu$, we get from (8)

$$|A_{m+\nu+1}| \leqq Q \cdot \max(|A_{m+\nu}|, \ldots, |A_{\nu+1}|)$$

and, since (11) is true for all $N \leqq m+\nu$,

$$|A_{m+\nu+1}| \leqq Q\,Q_1 \cdot \max(Q^{m+\nu}, Q^{m+\nu-1}, \ldots, Q^{\nu+1}) \leqq Q^{m+\nu} Q_1.$$

This proves that (11) is true for all $N \geqq 0$. Hence the circle of convergence of the series (10) has a radius which is

$$= \limsup_{n\to\infty} \frac{1}{\sqrt[n]{|A_n|}} \geqq \lim_{n\to\infty} \frac{1}{\sqrt[n]{Q^n Q_1}} = \frac{1}{Q}.$$

Thus, multiplying the series (10) by the polynomial

$$1 - a_1 z - a_2 z^2 - \cdots - a_m z^m, \tag{12}$$

we get, since the convergence is absolute in the inner of the circle, the following product

$$\sum_{h=0}^{m-1} b_h z^h + \sum_{n=0}^{\infty} (A_{m+n} - a_1 A_{m+n-1} - a_2 A_{n+m-2} - \cdots - a_m A_n) z^{n+m}, \tag{13}$$

where the coefficients b_h are uniquely determined by the relations

$$\left.\begin{aligned} b_0 &= A_0, \\ b_1 &= A_1 - a_1 A_0, \\ b_2 &= A_2 - a_1 A_1 - a_2 A_0, \\ &\cdots\cdots\cdots\cdots \\ b_{m-1} &= A_{m-1} - a_1 A_{m-2} - \cdots - a_{m-1} A_0. \end{aligned}\right\} \tag{14}$$

In virtue of (8) the product (13) is equal to the polynomial

$$b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1}.$$

This proves the first part of Theorem 2.

Suppose next that the numbers $b_0, b_1, \ldots, b_{m-1}$ are arbitrarily given and expand the rational function

$$\frac{b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1}}{1 - a_1 z - a_2 z^2 - \cdots - a_m z^m} \tag{15}$$

in a power series. If this series is given by (10), and if we multiply it by the polynomial (12), we find as above that the coefficients $A_0, A_1, A_2, \ldots, A_{m-1}$ are determined by the system (14) and further that the coefficients A_{m+n}, for all $n \geqq 0$, satisfy the recurrence (8).

The rational function (15) is the *generating function* of the recurrent sequence $\{A_n\}$. This function may be written in the form

$$\frac{\beta_0 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_{\mu-1} z^{\mu-1}}{1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_\mu z^\mu},$$

where the numerator and the denominator have no common divisor $z - \theta$, and where $\alpha_\mu \neq 0$. Then the order of the sequence $\{A_n\}$ is $= \mu$. In fact, suppose that it was of the order $\lambda < \mu$. Then, it would satisfy a homogeneous recurrence with the scale $[c_1, c_2, \ldots, c_\lambda]$ where $c_\lambda \neq 0$.

Hence we should have, in virtue of Theorem 2,

$$\frac{\beta_0 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_{\mu-1} z^{\mu-1}}{1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_\mu z^\mu} = \frac{e_0 + e_1 z + e_2 z^2 + \cdots + e_{\lambda-1} z^{\lambda-1}}{1 - c_1 z - c_2 z^2 - \cdots - c_\lambda z^\lambda},$$

where $e_0, e_1, \ldots, e_{\lambda-1}$ are constants. Thus

$$(\beta_0 + \beta_1 z + \cdots + \beta_{\mu-1} z^{\mu-1})(1 - c_1 z - \cdots - c_\lambda z^\lambda)$$
$$= (e_0 + e_1 z + \cdots + e_{\lambda-1} z^{\lambda-1})(1 - \alpha_1 z - \cdots - \alpha_\mu z^\mu).$$

Hence $1 - c_1 z - \cdots - c_\lambda z^\lambda$ would be divisible by $1 - \alpha_1 z - \cdots - \alpha_\mu z^\mu$. But this is impossible since $\lambda < \mu$.

There is of course an infinity of recurrent sequences of a given order m and satisfying the homogeneous recurrence with the given scale $[a_1, a_2, \ldots, a_m]$, $a_m \neq 0$.

4.—We add the following result:

Theorem 3. *Denote by $\theta_1, \theta_2, \theta_3$, etc. the distinct roots of the algebraic equation*

$$z^m - a_1 z^{m-1} - \cdots - a_{m-1} z - a_m = 0, \tag{16}$$

where $a_m \neq 0$. Then we obtain all the recurrent sequences $\{A_n\}$ satisfying the recurrence with the scale $[a_1, a_2, \ldots, a_m]$ by the following formula

$$A_n = \sum_{\theta_i} \left[d_{\nu_i, i} \binom{n + \nu_i - 1}{\nu_i - 1} + d_{\nu_i - 1, i} \binom{n + \nu_i - 2}{\nu_i - 2} + \cdots + d_{2, i} \binom{n+1}{1} + d_{1, i} \right] \theta_i^n, \tag{17}$$

where the sum is extended over all the distinct roots θ_i and where ν_i is the multiplicity of θ_i. The coefficients $d_{1,i}, d_{2,i}, \ldots, d_{\nu_i\, i}$ are arbitrary constants.

For the proof it suffices to observe that the function (15) may be written

$$\sum_{\theta_i} \left[\frac{d_{\nu_i, i}}{(1 - \theta_i z)^{\nu_i}} + \frac{d_{\nu_i - 1, i}}{(1 - \theta_i z)^{\nu_i - 1}} + \cdots + \frac{d_{1, i}}{1 - \theta_i z} \right],$$

and that we have the expansion

$$\frac{1}{(1-\theta_i z)^q} = \sum_{n=0}^{\infty} \binom{n+q-1}{q-1} \theta_i^n z^n.$$

The number of coefficients $d_{k,i}$ in formula (17) is equal to m. If the initial values $A_0, A_1, \ldots, A_{m-1}$ are given, and if the roots of equation (16) are known, we obtain from (17) a set of m linear equations for the determination of the coefficients $d_{k,i}$.

A corollary of Theorem 3 is the following proposition:

Let ν_1 denote the multiplicity of the root θ_1 of (16), and let A_n be given by (17). Then the difference

$$B_n = A_n - d_{\nu_1,1} \binom{n+\nu_1-1}{\nu_1-1} \theta_1^n$$

satisfies the recurrence

$$B_{m+n-1} = b_1 B_{m+n-2} + b_2 B_{m+n-3} + \cdots + b_{m-1} B_n,$$

where the coefficients $b_1, b_2, \ldots, b_{m-1}$ are determined by the identity

$$\frac{z^m - a_1 z^{m-1} - \cdots - a_m}{z-\theta_1} = z^{m-1} - b_1 z^{m-2} - \cdots - b_{m-1}.$$

We now turn to the inhomogeneous recurrences. Consider the recurrence of order m

$$A_{m+n} = a_1 A_{m+n-1} + a_2 A_{m+n-2} + \cdots + a_m A_n + b, \tag{19}$$

where a_m and b are $\neq 0$. Suppose first that

$$h = 1 - a_1 - a_2 - \cdots - a_m \neq 0,$$

and put $c = b/h$ and $\qquad B_n = A_n - c.$

Then it is easily seen that B_n satisfies the homogeneous recurrence

$$B_{m+n} = a_1 B_{m+n-1} + a_2 B_{m+n-2} + \cdots + a_m B_n. \tag{20}$$

Suppose next that $h=0$. Then the equation (16) has the root $z=1$. Denote by μ the multiplicity of this root. We may eliminate b between (19) and the formula

$$A_{m+n+1} = a_1 A_{m+n} + a_2 A_{m+n-1} + \cdots + a_m A_{n+1} + b.$$

Then

$$A_{m+n+1} = (a_1+1) A_{m+n} + (a_2-a_1) A_{m+n-1} + (a_3-a_2) A_{m+n-2} + \cdots + (a_m - a_{m-1}) A_{n+1} - a_m A_n.$$

This recurrence is homogeneous and its scale is

$$[a_1+1, a_2-a_1, a_3-a_2, \ldots, a_m-a_{m-1}, -a_m].$$

Plainly

$$z^{m+1} - (a_1+1) z^m - (a_2-a_1) z^{m-1} - \cdots - (a_m-a_{m-1}) z + a_m = (z-1)(z^m - a_1 z^{m-1} - a_2 z^{m-2} - \cdots - a_m).$$

Hence, in virtue of the corollary of Theorem 3, we have (for all $n \geqq 0$)

$$A_n = B_n + \binom{n+\mu}{\mu} d, \tag{21}$$

where B_n satifies (20), and where d is a certain constant (i. e. independent of n). To determine d we have the relations

$$A_{m+n} = B_{m+n} + \binom{m+n+\mu}{\mu} d$$

and

$$\sum_{i=1}^{m} a_i A_{m+n-i} = \sum_{i=1}^{m} a_i B_{m+n-i} + \sum_{i=1}^{m} a_i \binom{m+n+\mu-i}{\mu} d.$$

Since A_n and B_n satisfy (19) and (20) respectively, we get by subtraction

$$b = \left[\binom{m+n+\mu}{\mu} - \sum_{i=1}^{m} a_i \binom{m+n+\mu-i}{\mu}\right] d.$$

Since b and d are constants, we may take $n=0$. Hence

$$d = \frac{b}{\binom{m+\mu}{\mu} - \sum_{i=1}^{m} a_i \binom{m+\mu-i}{\mu}},$$

where μ denotes the multiplicity of the root $z=1$ in equation (16).

In this way the inhomogeneous case has been reduced to the homogeneous case.

5.—A general theory of recurrences is developed in N. E. Nörlund, *Vorlesungen über Differenzrechnung*, Berlin 1924 (Verl. Springer). Linear recurrences are treated in Kapitel 14, § 2. But the method employed is quite different from the simple one adopted in this note.

6.—We finish with a few examples:

(1) The sequence of the *Fibonacci numbers* F_n

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

satisfy the recurrence of the second order

$$F_{n+2} = F_{n+1} + F_n$$

and the initial conditions $F_1 = F_2 = 1$. One finds easily

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

(2) The general solution of the recurrence of the first order

$$A_{n+1} = a A_n + b$$

is easily found to be $$A_n = a^n A_0 + \frac{a^n - 1}{a - 1} b,$$

valid also for $a = 1$.

(3) The coefficients A_n in the expansion

$$\frac{1}{1 - 7z^2 - 6z^3} = \sum_{n=0}^{\infty} A_n z^n$$

satisfy the recurrence of the third order

$$A_{n+3} = 7A_{n+1} + 6A_n,$$

and the initial conditions $A_0 = 1$, $A_1 = 0$, $A_2 = 7$. Further we find

$$A_n = -\tfrac{1}{4}(-1)^n + \tfrac{4}{5}(-2)^n + \tfrac{9}{20} 3^n.$$

In fact the equation $$z^3 - 7z - 6 = 0$$

has the roots $z = -1, -2$, and 3.

(4) If we put in the recursive formula (3), for all n,

$$B_n = \log A_n,$$

we get the homogeneous linear recurrence

$$B_{n+2} = \tfrac{1}{2} B_{n+1} + \tfrac{1}{2} B_n.$$

Hence $$B_n = \alpha + \beta(-\tfrac{1}{2})^n,$$

where α and β are determined by the relations

$$B_0 = \alpha + \beta,\; B_1 = \alpha - \tfrac{1}{2}\beta.$$

Thus $$B_n = \tfrac{1}{3} B_0 + \tfrac{2}{3} B_1 + \tfrac{2}{3}(B_0 - B_1)(-\tfrac{1}{2})^n$$

and finally

$$A_n^3 = A_0^{2(-\frac{1}{2})^n + 1} A_1^{2 - 2(-\frac{1}{2})^n}.$$

Tryckt den 15 mars 1957

Uppsala 1957. Almqvist & Wiksells Boktryckeri AB