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The reality of the eigenvalues of certain integral equations

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With 4 figures in the text

§ 1. Introduction

In this paper we shall study the reality of the eigenvalues in some integral equations of the Fredholm type

$$\varphi(x) = \lambda \int_0^1 K(x, y)\, \varphi(y)\, dy.$$

The kernel $K(x, y)$ is assumed to be 0 above a certain curve in the square $0 \le {x \atop y} \le 1$ where it is defined. Below the curve we suppose that $K(x, y) = P(x)\, Q(y)$. Let the curve have the equation $y = f(x)$ and make the following assumptions:

(α) $f(x)$ is non-decreasing,

(β) $\lim\limits_{t=+0} f(x-t) > x$ except possibly for $x = 0$ and $x = 1$,

(γ) $P(x)\, Q(x)$ is integrable in $0 \le x \le 1$.

We shall study two types of kernels:

Kernel A: The curve does not pass through $(0, 0)$ nor through $(1, 1)$ (fig. 1).
Kernel B: The curve goes through $(0, 0)$ or $(1, 1)$ or both points (fig. 2).

In [1] I have obtained explicit expressions for the corresponding denominators of Fredholm. In equation A they are polynomials in λ of degree depending only on the curve $y = f(x)$. I shall give an account of the formulas in question.

Let $f^2(x)$ mean $f(f(x))$, generally $f^n(x)$ the nth iterated function. We also introduce the in an appropriate way defined inverse $f^{-1}(x)$ which we give the value 0 for $0 \le x \le f(0)$. In the integral equation A, restricted to the square $0 \le {x \atop y} \le \alpha$, the denominator of Fredholm becomes:

$$D(\alpha, \lambda) = 1 - \lambda F_1(\alpha) + \lambda^2 F_2(\alpha) - \cdots + (-\lambda)^n F_n(\alpha), \tag{1}$$

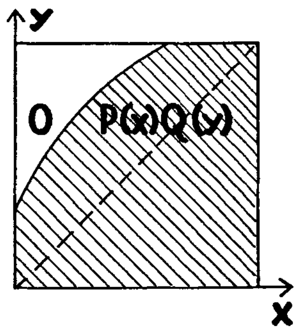

Fig. 1.

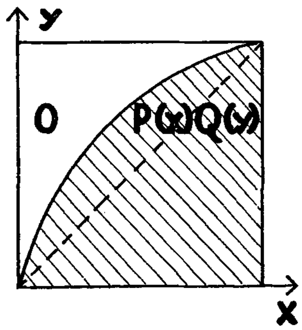

Fig. 2.

where n is determined by the inequality

$$f^{n-1}(0) < \alpha \leq f^{n}(0).$$

The coefficients $F_1(\alpha)$, $F_2(\alpha)$, ... are obtained from:

$$\begin{cases} F_1(\alpha) = \int\limits_0^{\alpha} P(y)\,Q(y)\,dy; \\ F_2(\alpha) = \int\limits_0^{\alpha} F_1(f^{-1}(y))\,P(y)\,Q(y)\,dy; \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ F_\nu(\alpha) = \int\limits_0^{\alpha} F_{\nu-1}(f^{-1}(y))\,P(y)\,Q(y)\,dy; \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{cases} \tag{2}$$

The denominator of Fredholm in the equation B regarding the square $0 \leq {x \atop y} \leq 1$ is an integral function

$$D(\lambda) = 1 + \sum_{\nu=1}^{\infty} (-\lambda)^{\nu} F_\nu(1), \tag{3}$$

with coefficients determined by (2).

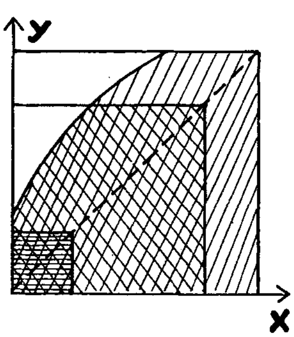

Fig. 3.

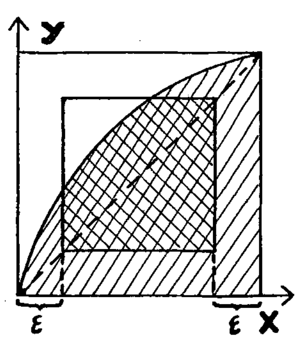

Fig. 4.

We shall show that the eigenvalues of A and B are real, if $P(x)Q(x) \geq 0$ (almost everywhere). For this purpose we shall examine the succession of kernels which we get with a fixed curve $y = f(x)$ and a variable square $0 \leq \frac{x}{y} \leq \alpha$, $0 < \alpha \leq 1$ (fig. 3). When $\alpha \leq f(0)$ the kernel is $P(x)Q(y)$ in the whole square. It has a single eigenvalue which is real. For increasing α the number of eigenvalues is increasing.

§ 2. Proof of the reality of the eigenvalues of equation A when $P(x)\,Q(x) \geq 0$.

In order to obtain a relation between denominators of Fredholm corresponding to different squares we note that (2) gives:

$$F_\nu(\alpha) = F_\nu(\beta) + \int_\beta^\alpha F_{\nu-1}(f^{-1}(y))\,P(y)\,Q(y)\,dy.$$

Introduce this into (1):

$$D(\alpha, \lambda) = D(\beta, \lambda) - \lambda \int_\beta^\alpha D(f^{-1}(y), \lambda)\,P(y)\,Q(y)\,dy. \tag{4}$$

In the following we shall generally denote by $\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, \ldots, \lambda_n^{(\alpha)}, \ldots$ the eigenvalues corresponding to the square $0 \leq \frac{x}{y} \leq \alpha$ arranged so that their moduli form a non-decreasing sequence. We first suppose that $P(x)\,Q(x) > 0$.

Theorem 1. *When $P(x)\,Q(x) > 0$, the eigenvalues of A are real, positive and simple. Further, if $f^{-1}(\alpha) \leq \beta < \alpha$, we have*

$$0 < \lambda_1^{(\alpha)} < \lambda_1^{(\beta)} < \cdots < \lambda_{n-1}^{(\alpha)} < \lambda_{n-1}^{(\beta)} < \lambda_n^{(\alpha)} < \lambda_n^{(\beta)}. \tag{5}$$

We have assumed that $f^{n-1}(0) < \alpha \leq f^n(0)$. Note that $\lambda_n^{(\beta)}$ is missing if $f^{-1}(\alpha) \leq \beta \leq f^{n-1}(0)$. (5) indicates that every eigenvalue is decreasing for increasing α. If we let α decreasing tend to $f^{n-1}(0)$, $\lambda_n^{(\alpha)}$ tends to $+\infty$. Hence theorem 1 involves that the new eigenvalues, gradually appearing as α increases, are entering from $+\infty$. The theorem is proved by induction.

1. *Theorem 1 is valid when $0 < \alpha \leq f(0)$.*

In this case $K(x, y) = P(x)\,Q(y)$ in the whole square and the single eigenvalue is

$$\lambda_1^{(\alpha)} = \frac{1}{\int_0^\alpha P(y)\,Q(y)\,dy}.$$

Since $P(x)\,Q(x) > 0$ $\lambda_1^{(\alpha)}$ is positive and decreasing for increasing α, which proves theorem 1.

It remains to show that if theorem 1 is valid when $\alpha \le f^{n-1}(0)$, it is valid when $\alpha \le f^n(0)$, too. We illustrate the method by first proving:

2. *Theorem 1 is valid when* $f(0) < \alpha \le f^2(0)$.

We treat the cases $\beta \le f(0)$ and $\beta > f(0)$ separately.

2 a. $f^{-1}(\alpha) \le \beta \le f(0)$.

To the square $0 \le {x \atop y} \le \beta$ belongs a single eigenvalue $\lambda_1^{(\beta)}$. We shall prove that $0 < \lambda_1^{(\alpha)} < \lambda_1^{(\beta)} < \lambda_2^{(\alpha)}$ by examining the sign of $D(\alpha, \lambda)$, when $\lambda = 0$, $\lambda = \lambda_1^{(\beta)}$ and $\lambda = +\infty$. Putting $\lambda_1^{(\beta)}$ for λ into (4) we get:

$$D(\alpha, \lambda_1^{(\beta)}) = -\lambda_1^{(\beta)} \int_\beta^\alpha D(f^{-1}(y), \lambda_1^{(\beta)}) P(y) Q(y) \, dy. \tag{6}$$

When $\beta \le y \le \alpha$ we have $f^{-1}(\beta) \le f^{-1}(y) \le f^{-1}(\alpha) \le \beta$. By **1** the zero $\lambda_1^{(f^{-1}(y))}$ of $D(f^{-1}(y), \lambda)$ is $\ge \lambda_1^{(\beta)}$. Since $D(\alpha, 0) > 0$ it follows that $D(f^{-1}(y), \lambda_1^{(\beta)})$ is ≥ 0 in the right member of (6). To show that this expression cannot be zero identically we note that there exists an interval $\beta \le y \le \beta + \Delta\beta$, where $f^{-1}(y)$ is $< \beta$ on account of the conditions (α) and (β) on $f(y)$. In this interval **1** involves that $D(f^{-1}(y), \lambda_1^{(\beta)}) > 0$. By $P(x)Q(x) > 0$ we conclude from (6) that $D(\alpha, \lambda_1^{(\beta)}) < 0$. Thus

$$D(\alpha, 0) > 0,$$

$$D(\alpha, \lambda_1^{(\beta)}) < 0,$$

$$D(\alpha, +\infty) > 0.$$

We see that the zeros of $D(\alpha, \lambda)$ are real, positive and simple and

$$0 < \lambda_1^{(\alpha)} < \lambda_1^{(\beta)} < \lambda_2^{(\alpha)}.$$

2 b. $f(0) < \beta < \alpha$.

By **2 a** both $D(\alpha, \lambda)$ and $D(\beta, \lambda)$ have two positive, simple zeros. We have to prove that

$$0 < \lambda_1^{(\alpha)} < \lambda_1^{(\beta)} < \lambda_2^{(\alpha)} < \lambda_2^{(\beta)}. \tag{7}$$

Put $\lambda = \lambda_\nu^{(\beta)}$, $\nu = 1, 2$, into (4):

$$D(\alpha, \lambda_\nu^{(\beta)}) = -\lambda_\nu^{(\beta)} \int_\beta^\alpha D(f^{-1}(y), \lambda_\nu^{(\beta)}) \, P(y) Q(y) \, dy. \tag{8}$$

Since when $\beta \le y \le \alpha$ we have

$$f^{-1}(\beta) \le f^{-1}(y) \le f^{-1}(\alpha) \le f(0) < \beta,$$

2 a shows that the single zero $\lambda_1^{(f^{-1}(y))}$ of $D(f^{-1}(y), \lambda)$ satisfies

$$\lambda_1^{(\beta)} < \lambda_1^{(f^{-1}(y))} < \lambda_2^{(\beta)}.$$

Hence

$$D(f^{-1}(y), \lambda_1^{(\beta)}) > 0 \quad \text{and} \quad D(f^{-1}(y), \lambda_2^{(\beta)}) < 0.$$

From this and from $P(x)Q(x) > 0$ we infer by (8)

$$D(\alpha, \lambda_1^{(\beta)}) < 0, \quad D(\alpha, \lambda_2^{(\beta)}) > 0.$$

Since $D(\alpha, 0) > 0$ this proves (7).

3. *Assuming that theorem 1 is valid when* $0 < \alpha < f^{n-1}(0)$, *it is valid also when* $f^{n-1}(0) < \alpha \le f^n(0)$.

3 a. $f^{-1}(\alpha) \le \beta \le f^{n-1}(0)$.

By the assumption $D(\beta, \lambda)$ has $n-1$ positive, simple zeros $\lambda_1^{(\beta)}, \lambda_2^{(\beta)}, \ldots, \lambda_{n-1}^{(\beta)}$. We have to prove that the n zeros of $D(\alpha, \lambda)$ are positive and simple satisfying (5), where $\lambda_n^{(\beta)}$ is missing.

As in **2** we use (4) to determine the sign of $D(\alpha, \lambda_\nu^{(\beta)})$ where $\nu = 1, 2, \ldots, n-1$. The result is once more formula (8).

When $\beta \le y \le \alpha$ we have further

$$f^{n-3}(0) < f^{-1}(\beta) \le f^{-1}(y) \le f^{-1}(\alpha) \le \beta \le f^{n-1}(0).$$

$D(f^{-1}(y), \lambda)$ has $n-2$ or $n-1$ zeros, which by the assumption are located according to

$$0 < \lambda_1^{(\beta)} \le \lambda_1^{(f^{-1}(y))} < \cdots < \lambda_{n-2}^{(\beta)} \le \lambda_{n-2}^{(f^{-1}(y))} < \lambda_{n-1}^{(\beta)} \le \lambda_{n-1}^{(f^{-1}(y))}. \tag{9}$$

The signs of equality are applicable to the case $f^{-1}(y) = \beta$ only. Since $\lambda_\nu^{(\beta)}$ is situated between the $(\nu-1)$st and νth zeros of $D(f^{-1}(y), \lambda)$, we infer that

$$(-1)^{\nu-1} D(f^{-1}(y), \lambda_\nu^{(\beta)}) \ge 0. \tag{10}$$

On account of the conditions (α) and (β) on $f(x)$ there exists an interval $\beta \le y \le \beta + \Delta\beta$ where $f^{-1}(y) < \beta$. For y in that interval the sign of equality cannot appear in (9) and (10). Since $P(x)Q(x) > 0$ we infer by (8)

$$(-1)^\nu D(\alpha, \lambda_\nu^{(\beta)}) > 0, \qquad \nu = 1, 2, \ldots, n-1.$$

As $D(\alpha, 0) > 0$ this proves (5).

3 b. $f^{n-1}(0) < \beta < \alpha$.

In consequence of **3 a** $D(\alpha, \lambda)$ and $D(\beta, \lambda)$ both have n positive, simple zeros. It is required to prove the inequality (5). We again use (8) for examining the sign of $D(\alpha, \lambda_\nu^{(\beta)})$, $\nu = 1, 2, \ldots, n$.

For y in the interval $\beta \leq y \leq \alpha$ we get

$$f^{n-2}(0) < f^{-1}(\beta) \leq f^{-1}(y) \leq f^{-1}(\alpha) \leq f^{n-1}(0) < \beta.$$

The $n-1$ zeros of $D(f^{-1}(y), \lambda)$ satisfy by **3 a**

$$0 < \lambda_1^{(\beta)} < \lambda_1^{(f^{-1}(y))} < \cdots < \lambda_{n-1}^{(f^{-1}(y))} < \lambda_n^{(\beta)}.$$

We infer that $(-1)^{\nu-1} D(f^{-1}(y), \lambda_\nu^{(\beta)}) > 0$ in the whole interval $\beta \leq y \leq \alpha$. By (8) we find $(-1)^\nu D(\alpha, \lambda_\nu^{(\beta)}) > 0$ for all ν, which proves (5).

The proof of theorem 1 is now completed. Our next object will be to relax the restriction $P(x)Q(x) > 0$.

Theorem 2. *When $P(x)Q(x) \geq 0$ the eigenvalues of A are real, positive.*

Define $[P(x)Q(x)]_\varepsilon$ as $P(x)Q(x)$ if $P(x)Q(x) \geq \varepsilon$, as ε if $P(x)Q(x) < \varepsilon$. $(\varepsilon > 0)$.

To $f(x)$ and $[P(x)Q(x)]_\varepsilon$ there corresponds an integral equation of type A, for which theorem 1 is applicable. Its denominator of Fredholm $D(\varepsilon, \alpha, \lambda)$ is:

$$D(\varepsilon, \alpha, \lambda) = 1 - \lambda F_1(\varepsilon, \alpha) + \lambda^2 F_2(\varepsilon, \alpha) - \cdots + (-\lambda)^n F_n(\varepsilon, \alpha), \tag{11}$$

with

$$\begin{cases} F_1(\varepsilon, \alpha) = \int_0^\alpha [P(y)Q(y)]_\varepsilon \, dy; \\ F_2(\varepsilon, \alpha) = \int_0^\alpha F_1(\varepsilon, f^{-1}(y)) [P(y)Q(y)]_\varepsilon \, dy; \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ F_\nu(\varepsilon, \alpha) = \int_0^\alpha F_{\nu-1}(\varepsilon, f^{-1}(y)) [P(y)Q(y)]_\varepsilon \, dy; \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \end{cases} \tag{12}$$

For every fixed y the integrands of (12) are positive functions of ε, non-increasing for decreasing ε, converging to the integrands of (2) when ε tends to 0. Hence, when ε tends to 0, every $F_\nu(\varepsilon, \alpha)$ converges to $F_\nu(\alpha)$ and we infer that the polynomial $D(\varepsilon, \alpha, \lambda)$ converges to $D(\alpha, \lambda)$. The zeros of $D(\alpha, \lambda)$ are the limits of the zeros of $D(\varepsilon, \alpha, \lambda)$ when ε tends to 0. As limits of real, positive numbers they are real, positive.

Theorem 3. *When $P(x)Q(x) \geq 0$ the eigenvalues of A are non-increasing for increasing α.*

Take out an arbitrary eigenvalue $\lambda_\nu^{(\alpha)}$ of the integral equation. It is the limit when ε tends to 0 of an eigenvalue $\lambda_\nu^{(\alpha)}(\varepsilon)$ of the integral equation used in the proof of theorem 2. By theorem 1 $\lambda_\nu^{(\alpha)}(\varepsilon)$ is a decreasing function of α. Hence $\lambda_\nu^{(\alpha)} = \lim_{\varepsilon=0} \lambda_\nu^{(\alpha)}(\varepsilon)$ is non-increasing.

Theorem 4. *Let the "existence-square" of the kernel of A be $\alpha \leq {x \atop y} \leq 1$. If $P(x)Q(x) \geq 0$ its eigenvalues are non-increasing for decreasing α.*

By the change of variables $\xi = 1-y$, $\eta = 1-x$ the integral equation A is transformed into another of the same type with the same eigenvalues. The square $\alpha \leq {x \atop y} \leq 1$ is transformed into the square $0 \leq {\xi \atop \eta} \leq 1-\alpha$. Thus theorem 4 is an immediate consequence of theorem 3.

To conclude this study of equation A we shall show that the assumption $P(x)Q(x) \geq 0$ is essential in order that the eigenvalues shall be real. We shall give a simple example of an equation of the type A where $P(x)Q(x)$ changes its sign and the eigenvalues are complex.

Define the kernel in $0 \leq {x \atop y} \leq 1$ by $f(x) = x + \frac{2}{3}$ and $Q(x) = 1$. The corresponding denominator of Fredholm is by (1) and (2):

$$D(\lambda) = 1 - \lambda \int_0^1 P(y)\,dy + \lambda^2 \int_{\frac{2}{3}}^1 P(x)\,dx \int_0^{x-\frac{2}{3}} P(y)\,dy.$$

Choose $P(x) > 0$ in the intervals $0 \leq x \leq \frac{1}{3}$ and $\frac{2}{3} \leq x \leq 1$, and $P(x) < 0$ in the interval $\frac{1}{3} < x < \frac{2}{3}$, so that $\int_0^1 P(y)\,dy = 0$. We get: $D(\lambda) = 1 + \lambda^2 C$ with $C > 0$, hence the eigenvalues are non-real.

§ 3. Proof of the reality of the eigenvalues of equation *B*.

When the curve goes through one or both of the points (0, 0) and (1, 1), the denominator of Fredholm of the kernel defined in $0 \leq {x \atop y} \leq 1$ is an integral function. We shall examine the reality of its zeros when $P(x)Q(x) \geq 0$.

Restricting the "existence-square" of the kernel of B to $\varepsilon \leq {x \atop y} \leq 1-\varepsilon$, $\frac{1}{2} > \varepsilon > 0$, we get an equation of the type A (fig. 4, p. 80). We shall show that the denominator of Fredholm $D(\varepsilon, \lambda)$ of the new equation converges to $D(\lambda)$ when ε tends to 0. Since by theorem 2 the zeros of $D(\varepsilon, \lambda)$ are real, positive, we infer that the eigenvalues of B are likewise real, positive.

Let $[P(x)Q(x)]_\varepsilon$ denote $P(x)Q(x)$ in the interval $\varepsilon \leq x \leq 1-\varepsilon$ and 0 in the intervals $0 \leq x < \varepsilon$ and $1-\varepsilon < x \leq 1$. $D(\varepsilon, \lambda)$ is simply obtained by putting $[P(x)Q(x)]_\varepsilon$ for $P(x)Q(x)$ into (2) and (3). We get formulas of the form (11) and (12) where $D(\varepsilon, \lambda) = D(\varepsilon, 1-\varepsilon, \lambda)$ and with n determined by

$$f^{n-1}(\varepsilon) < 1-\varepsilon \leq f^n(\varepsilon), \qquad n = n(\varepsilon).$$

Because the integrands of (12) are positive functions of ε, non-decreasing for decreasing ε, and converging to the integrands of (2), every coefficient $F_\nu(\varepsilon, 1)$ converges non-decreasing to $F_\nu(1)$ when ε decreases to 0.

When $|\lambda| \le R$ the moduli of the terms of the series $D(\varepsilon, \lambda)$ and $D(\lambda)$ are smaller than the corresponding terms of the convergent series

$$\sum_{\nu=0}^{\infty} F_\nu(1) R^\nu = D(-R).$$

Since every term of $D(\varepsilon, \lambda)$ converges to a term in $D(\lambda)$, we conclude that the convergence of $D(\varepsilon, \lambda)$ to $D(\lambda)$ is uniform in every circle $|\lambda| \le R$.

Theorem 5. *The eigenvalues of B are real, positive if $P(x) Q(x) \ge 0$.*

Since $D(\varepsilon, \lambda)$ tends to $D(\lambda)$ uniformly in every circle $|\lambda| \le R$ we can apply a theorem of Hurwitz [2]. By it the zeros of $D(\lambda)$ are exactly the limits of the zeros of $D(\varepsilon, \lambda)$ when ε tends to 0. As limits of real, positive numbers the eigenvalues of B are real, positive and theorem 5 is proved.

Denote the zeros of $D(\varepsilon, \lambda)$ by $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_{n(\varepsilon)}(\varepsilon)$ and the zeros of $D(\lambda)$ by $\lambda_1, \lambda_2, \ldots$ arranged so that their moduli form a non-decreasing sequence. We have $\lim_{\varepsilon=0} \lambda_\nu(\varepsilon) = \lambda_\nu$ for all ν. Combining theorems 3 and 4 we infer that every $\lambda_\nu(\varepsilon)$ is non-increasing for decreasing ε.

Theorem 6. *Putting $\int_0^1 P(y) Q(y)\, dy = M$ we have*

$$\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} = M.$$

From (11) we get:

$$\sum_{\nu=1}^{n(\varepsilon)} \frac{1}{\lambda_\nu(\varepsilon)} = F_1(\varepsilon, 1) = \int_\varepsilon^{1-\varepsilon} P(y) Q(y)\, dy.$$

Hence we can to every $\eta > 0$ find a number $\varepsilon_0(\eta) > 0$ such that

$$M - \eta < \sum_{\nu=1}^{n(\varepsilon_0)} \frac{1}{\lambda_\nu(\varepsilon_0)} \le M.$$

$\frac{1}{\lambda_\nu(\varepsilon)}$ is non-decreasing for decreasing ε. Hence:

$$M - \eta < \sum_{\nu=1}^{n(\varepsilon_0)} \frac{1}{\lambda_\nu(\varepsilon_0)} \le \sum_{\nu=1}^{n(\varepsilon_0)} \frac{1}{\lambda_\nu} \le M.$$

Because η is arbitrary, we get $\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} = M$.

Theorem 7. *$D(\lambda)$ is of genus* 0: $D(\lambda)=\prod_{\nu=1}^{\infty}\left(1-\frac{\lambda}{\lambda_\nu}\right)$.

Put $\frac{1}{\lambda_\nu(\varepsilon)}=0$ if $\nu>n(\varepsilon)$. The convergence of the infinite product $\prod_{\nu=1}^{\infty}\left(1-\frac{\lambda}{\lambda_\nu(\varepsilon)}\right)$ is uniform in ε since $\frac{1}{\lambda_\nu(\varepsilon)}$ is non-decreasing for decreasing ε and $\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu}=M$. Hence

$$D(\lambda)=\lim_{\varepsilon=0}D(\varepsilon,\lambda)=\lim_{\varepsilon=0}\prod_{\nu=1}^{\infty}\left(1-\frac{\lambda}{\lambda_\nu(\varepsilon)}\right)=\prod_{\nu=1}^{\infty}\lim_{\varepsilon=0}\left(1-\frac{\lambda}{\lambda_\nu(\varepsilon)}\right)=\prod_{\nu=1}^{\infty}\left(1-\frac{\lambda}{\lambda_\nu}\right).$$

Example. The function

$$D(\lambda)=1+\sum_{\nu=1}^{\infty}(-\lambda)^\nu\frac{1}{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^2}\right)\cdots\left(1+\frac{1}{a}+\cdots+\frac{1}{a^{\nu-1}}\right)},$$

$0<a<1$, is of genus 0 and has its zeros real, positive. The fact is that it is the denominator of Fredholm of the integral equation of type B defined by

$$f(x)=x^a,\ 0<a<1,\ P(x)Q(x)=1.$$

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[1] U. Hellsten, Determination of the denominator of Fredholm in some types of integral equations. Acta Mathematica 79, 1947, p. 105.
[2] E. C. Titchmarsh, The theory of functions, second edition, Oxford, 1939, p. 119.

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