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The number of zeros of $P_\nu(z)$ *in the sector*

$$\alpha \leq \arg z < \beta$$

is called $N_\nu(\alpha, \beta)$. *We say that the zeros of (the polynomials in) the sequence are equi-distributed if*

$$N_\nu(\alpha, \beta) = \frac{\beta - \alpha}{2\pi} n_\nu + o(n_\nu)$$

for all α *and* β *satisfying* $0 < \beta - \alpha < 2\pi$. Evidently this property of the sequence may depend on the choice of origin.

In his paper [4] CARLSON also showed that every power series representing an entire function of infinite order has a sequence of partial sums whose zeros are equi-distributed.

A theorem useful for the study of equi-distribution was published by ERDÖS and TURÁN in 1950 [6]:

Let $N(\alpha, \beta)$ *be the number of roots of*

$$a_0 + a_1 z + \cdots + a_n z^n = 0$$

in the sector

$$\alpha \leq \arg z < \beta.$$

Then

$$\left| N(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} \right| \leq c \sqrt{n \cdot \log P}$$

where

$$P = \frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}$$

and c *is a numerical constant.*

They showed that the theorem is true with $c = 16$ and that SZEGÖ's result easily follows from it.

From another point of view, the theorem of JENTZSCH tells us that every boundary point of the domain of uniform convergence for the partial sums of a power series is a limit-point of zeros of the partial sums. This, of course, is not true for general sequences of polynomials. OSTROWSKI [16, 1922] and SZEGÖ [21, 1922] have given supplementary conditions sufficient to ensure that the boundary points of the domain of uniform convergence for a sequence of polynomials are limit-points of their zeros.

1.2. Let us consider a sequence of polynomials converging uniformly in a neighborhood of the origin to a limit-function, not identically zero, and let us suppose that all the zeros of the polynomials belong to a given set E.

In connection with the results mentioned in the beginning of 1.1, it seems natural to ask: *For which sets* E *is it true that every sequence of this type converges uniformly at every point of the plane (to an entire function).* A related problem is treated in the dissertation of KOREVAAR [11, 1950] and a review of results in this field is given by OBRECHKOFF [15, 1942]. As we have remarked above, the existence of a zero-free sector with vertex at the origin is sufficient in the case of partial sums. For general sequences we have to make more farreaching assumptions. As a typical example we quote the following theorem of LINDWART and PÓLYA [13, 1914]:

If a sequence of polynomials converges uniformly in a domain containing the origin, to a limit-function, not identically zero, and if every polynomial has all its zeros in the half-plane

$$|\arg z| \leq \frac{\pi}{2},$$

then the sequence converges uniformly in every bounded domain, the limit being, therefore, an entire function. The order of this function does not exceed 2.

If we put $\zeta = e^z$ in a polynomial $P(\zeta) = \sum_{\mu=0}^{\nu} a_\mu \zeta^\mu$ we get an exponential polynomial $E(z) = P(e^z) = \sum_{\mu=0}^{\nu} a_\mu e^{\mu z}$. Only a few of the theorems quoted above have been generalized to the wider class of exponential polynomials $\sum_{\mu=0}^{\nu} a_\mu e^{\lambda_\mu z}$, with arbitrary real λ_μ. That a theorem corresponding to JENTZSCH's is true for the sections of Dirichlet series has been proved by KNOPP (the result is communicated in [9]). Examples showing the distribution of zeros of some exponential polynomials are given in a paper by TURÁN [14, 1948], where he proves that information on the distribution of zeros of the partial sums of the series for Riemann's ζ-function may increase our knowledge of the distribution of the zeros of the function.

1.3. In this paper we study various types of distributions of zeros for sequences of polynomials and exponential polynomials more general than partial sums of a series.

In the following section, we derive some formulas for zeros of analytic functions which will be used in the sequel. Conditions for equi-distribution of zeros of sequences of polynomials and the corresponding problem for exponential polynomials are studied in the third section. We also prove a theorem on conjugate harmonic functions which gives a simple proof of the theorem of ERDÖS and TURÁN with an improved value of the constant c. In section 5 we study the number of zeros of exponential polynomials in rectangular regions and give another generalization of the theorem of ERDÖS and TURÁN.

In the sixth section we consider sequences of analytic functions and characterize the distribution of the zeros in terms of certain functions of which

$$\overline{\lim} \frac{1}{n_\nu} \overset{+}{\log} |P_\nu(z)|$$

in the polynomial case is a typical representative. We generalize a recent result of ROSENBLOOM [17] by proving a theorem which yields the above-mentioned theorem of CARLSON, if applied to the partial sums of functions of positive order.

In section 7, we prove a theorem of the same type as that of LINDWART and PÓLYA but with weaker conditions on the zero-free region.

Some formulas concerning the distribution of zeros of analytic functions

2.1. In the sequel we will use various formulas related to the well-known theorem of Jensen. We have collected them in this section and give the proofs in such a way that the relations between them become apparent.

The theorem of Jensen reads as follows.

If $f(z)$ is analytic for $|z| \leq r$ and $|f(0)| = 1$ then

$$\sum_{\nu=1}^{n} \log \frac{r}{r_\nu} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r e^{i\theta})| \, d\theta \tag{2.1.1}$$

where $r_1, r_2, \ldots, r_n$ are the moduli of the zeros of $f(z)$ in the circle $|z| \leq r$, a zero of order p being counted p times.

A well-known method of proof is based upon Green's formula

$$\int_D (G \cdot \Delta H - H \cdot \Delta G) \, dx \, dy = \int_C \left(G \frac{\partial H}{\partial n} - H \frac{\partial G}{\partial n} \right) ds \tag{2.1.2}$$

which is valid if G and H are continuous and have continuous derivatives of the first two orders in D and on its boundary C $\left(\frac{\partial}{\partial n}\right.$ denotes differentiation along the outward normal$\left.\right)$.

Let $G(z, \zeta)$ be the Green's function for D, singular at ζ, and put $H = \log |f(z)|$, where we suppose that $f(z)$ is analytic in D. An application of (2.1.2) gives

$$\sum_{z_\nu \in D} G(z_\nu, \zeta) + \log |f(\zeta)| + \frac{1}{2\pi} \int_C \log |f(z)| \frac{\partial G}{\partial n} ds = 0. \tag{2.1.3}$$

We must exclude from D small circles around ζ and around the zeros z_ν of $f(z)$ when we apply (2.1.2); the integrals over the corresponding contours give the first two terms in (2.1.3). If D is the circle $|z| \leq r$ and if we put $\zeta = 0$, we obtain (2.1.1), as Green's function for the circle is

$$G_C(z, \zeta) = \log \left| \frac{r^2 - z\bar{\zeta}}{r(z - \zeta)} \right|.$$

We now give the corresponding formula for a sector. Suppose that ζ is real. The Green's function for the region $S(\lambda; R)$ defined by

$$|z| \leq R, \quad |\arg z| \leq \frac{\pi}{2\lambda}$$

and singular at ζ is

$$G_S(z, \zeta) = -\log \left| \frac{z^\lambda - \zeta^\lambda}{z^\lambda + \zeta^\lambda} \cdot \frac{R^{2\lambda} + z^\lambda \zeta^\lambda}{R^{2\lambda} - z^\lambda \zeta^\lambda} \right|$$

as is found by the conformal mapping of $S(\lambda; R)$ on the unit circle $|w| \leq 1$ by the transformation

$$w = \frac{Z^2 + 2Z - 1}{Z^2 - 2Z - 1}, \quad Z = \left(\frac{z}{R}\right)^{\lambda}.$$

Calculation of the normal derivatives in (2.1.3) gives

$$\log|f(\zeta)| + \sum_{z_\nu \in S} G_S(z_\nu, \zeta) = \frac{2\lambda}{\pi} R^{\lambda} \zeta^{\lambda} (R^{2\lambda} - \zeta^{2\lambda}) \int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \frac{\cos\lambda\theta \cdot \log|f(Re^{i\theta})|\, d\theta}{R^{4\lambda} - 2R^{2\lambda}\zeta^{2\lambda}\cos 2\lambda\theta + \zeta^{4\lambda}} +$$

$$(2.1.4) \qquad + \frac{\lambda}{\pi} \zeta^{\lambda} \int_0^R \left[\frac{\varrho^{\lambda-1}}{\zeta^{2\lambda} + \varrho^{2\lambda}} - \frac{R^{2\lambda}\varrho^{\lambda-1}}{R^{4\lambda} + \zeta^{2\lambda}\varrho^{2\lambda}}\right] \log\left|f\left(\varrho e^{\frac{\pi i}{2\lambda}}\right) f\left(\varrho e^{-\frac{\pi i}{2\lambda}}\right)\right| d\varrho.$$

Some special cases of this formula will be noted.

a) If $|f(z) - 1| = o(|z|^{\lambda})$ when $z \to 0$, we divide by ζ^{λ}. Letting ζ tend to 0, we find $(z_\nu = r_\nu e^{i\theta_\nu})$

$$\sum_{z_\nu \in S} \left(\frac{R^{\lambda}}{r_\nu^{\lambda}} - \frac{r_\nu^{\lambda}}{R^{\lambda}}\right) \cos\lambda\theta_\nu =$$

$$= \frac{\lambda}{\pi} \int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \cos\lambda\theta \cdot \log|f(Re^{i\theta})|\, d\theta + \frac{\lambda}{2\pi} \int_0^R \left(\frac{R^{\lambda}}{\varrho^{\lambda}} - \frac{\varrho^{\lambda}}{R^{\lambda}}\right) \log\left|f\left(\varrho e^{\frac{\pi i}{2\lambda}}\right) f\left(\varrho e^{-\frac{\pi i}{2\lambda}}\right)\right| \frac{d\varrho}{\varrho}$$

which is a formula used by Carlson in his investigations of sections of power series [4].

b) Let $f(z)$ be analytic in the sectors $S(\lambda; R)$ for all R and suppose that there is a $\mu < \lambda$ such that

$$\lim_{\varrho \to \infty} \frac{1}{\varrho^{\mu}} \log|f(\varrho e^{i\theta})| = 0$$

for $|\theta| \leq \frac{\pi}{2\lambda}$. Letting R tend to infinity we find that

$$(2.1.5) \quad \log|f(\zeta)| + \sum_{z_\nu \in S(\lambda;\, \infty)} \log\left|\frac{z_\nu^{\lambda} + \zeta^{\lambda}}{z_\nu^{\lambda} - \zeta^{\lambda}}\right| = \frac{\lambda}{\pi} \int_0^{\infty} \frac{\zeta^{\lambda}\varrho^{\lambda}}{\zeta^{2\lambda} + \varrho^{2\lambda}} \log\left|f\left(\varrho e^{\frac{\pi i}{2\lambda}}\right) f\left(\varrho e^{-\frac{\pi i}{2\lambda}}\right)\right| \frac{d\varrho}{\varrho}.$$

We now suppose that $f(z)$ is a polynomial $P(z)$. Equation (2.1.5) is multiplied by ζ^{-1} and integrated with respect to ζ from 0 to R. When R tends to infinity we obtain the following result:

If $P(z)$ is a polynomial of degree n with zeros $z_\nu = r_\nu e^{i\theta_\nu}$ then

$$\sum_{|\theta_\nu|<\varphi} (\varphi - |\theta_\nu|) = \frac{n\varphi^2}{2\pi} + \frac{1}{2\pi}\int_0^\infty \log \frac{|P(\varrho e^{i\varphi}) P(\varrho e^{-i\varphi})|}{|P(\varrho)|^2} \cdot \frac{d\varrho}{\varrho}. \tag{2.1.6}$$

A simpler proof is obtained by observing that, according to the factorization theorem for polynomials, formula (2.1.6) is equivalent to the corresponding formula for $P_1(z) = 1 - \frac{z}{z_1}$. Putting $r = \frac{\varrho}{|z_1|}$ we find that we must prove that $(0 \le |\theta_\nu| \le \pi)$

$$\int_0^\infty \log \frac{1 + 2r\cos\theta_1 + r^2}{1 + 2r\cos\theta_2 + r^2} \cdot \frac{dr}{r} = \theta_2^2 - \theta_1^2. \tag{2.1.7}$$

That is easily done, for example, by differentiating with respect to θ_1.

2.2. In order to study also exponential polynomials we give formulas for the strip B, defined by $|Im(z)| \le b$, and corresponding to (2.1.5) and (2.1.6). The function $w = e^{\frac{\pi z}{2b\lambda}}$ maps B on the sector $|\arg w| \le \frac{\pi}{2\lambda}$. From the formula (2.1.5) (in the w-plane) one obtains

$$\log|f(\zeta)| + \sum_{z_\nu \in B} \log \left| \frac{e^{\frac{\pi z_\nu}{2b}} + e^{\frac{\pi \zeta}{2b}}}{e^{\frac{\pi z_\nu}{2b}} - e^{\frac{\pi \zeta}{2b}}} \right| = \frac{1}{2b} \int_{-\infty}^{\infty} \frac{\log|f(x+ib) f(x-ib)|}{e^{\frac{\pi}{2b}(x-\zeta)} + e^{-\frac{\pi}{2b}(x-\zeta)}} dx \tag{2.2.1}$$

if, for example, $f(z)$ is of finite order; z_ν are the zeros of $f(z)$. (The same transformation applied to (2.1.4) gives a result of which a special case is used in 5.5.)

For an exponential polynomial

$$E(z) = a_0 e^{\lambda_0 z} + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z}$$

in which $\{\lambda_\nu\}_0^n$ are real numbers satisfying

$$\lambda_n > \lambda_{n-1} > \cdots > \lambda_1 > \lambda_0 = 0$$

and $a_0 a_n \ne 0$, we deduce, corresponding to (2.1.6), the formula

$$\sum_{|y_\nu|<b} (b - |y_\nu|) = \frac{\lambda_n b^2}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \log \frac{|E(x+ib) E(x-ib)|}{|E(x)|^2} dx, \tag{2.2.2}$$

where $z_\nu = x_\nu + i y_\nu$ are the zeros of $E(z)$.

To obtain a proof of this formula in the manner sketched in the text preceding (2.1.6), we apply (2.2.1) to

$$E_1(z) = \frac{E(z)}{a_0}$$

and to

$$E_2(z) = \frac{E(z)}{a_n} e^{-\lambda_n z}.$$

We put $b = \frac{\pi}{2}$ and observe that the general result is afterwards obtained by a simple transformation from the formula in this special case. The formula for $E_1(z)$ is integrated with respect to ζ from $-\infty$ to 0, and that for $E_2(z)$ from 0 to ∞.

Thus we obtain

$$\text{(2.2.3 a)} \quad \int_{-\infty}^{0} \log |E_1(\zeta)| \, d\zeta + \sum_{|Im(z_\nu)|<\frac{\pi}{2}} \int_{-\infty}^{0} \log \left| \frac{e^{z_\nu} + e^{\zeta}}{e^{z_\nu} - e^{\zeta}} \right| d\zeta =$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{arc\,tg} e^{-x} \cdot \log \left| E_1\left(x + i\frac{\pi}{2}\right) E_1\left(x - i\frac{\pi}{2}\right) \right| dx$$

and

$$\text{(2.2.3 b)} \quad \int_{0}^{\infty} \log |E_2(\zeta)| \, d\zeta + \sum_{|Im(z_\nu)|<\frac{\pi}{2}} \int_{0}^{\infty} \log \left| \frac{e^{z_\nu} + e^{\zeta}}{e^{z_\nu} - e^{\zeta}} \right| d\zeta =$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{arc\,tg}\, e^{x} \cdot \log \left| E_2\left(x + i\frac{\pi}{2}\right) E_2\left(x - i\frac{\pi}{2}\right) \right| dx.$$

If

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \operatorname{arc\,tg}\, e^{-x} \cdot \log |E_1(x)| \, dx$$

is subtracted from both members of (2.2.3 a) and

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \operatorname{arc\,tg}\, e^{x} \cdot \log |E_2(x)| \, dx$$

from those of (2.2.3 b), we find, after addition of these formulas, that

$$\text{(2.2.4)} \quad \sum_{|Im(z_\nu)|<\frac{\pi}{2}} \int_{-\infty}^{\infty} \log \left| \frac{e^{z_\nu} + e^{\zeta}}{e^{z_\nu} - e^{\zeta}} \right| d\zeta = \frac{1}{2} \int_{-\infty}^{\infty} \log \frac{\left| E\left(x + i\frac{\pi}{2}\right) E\left(x - i\frac{\pi}{2}\right) \right|}{|E(x)|^2} dx -$$
$$- \frac{2}{\pi} \int_{-\infty}^{0} \operatorname{arc\,tg}\, e^{x} \cdot \log \left| \frac{E_1(x)}{E_2(x)} \right| dx + \frac{2}{\pi} \int_{0}^{\infty} \operatorname{arc\,tg}\, e^{-x} \cdot \log \left| \frac{E_1(x)}{E_1(x)} \right| dx.$$

To obtain this formula we have used the obvious equalities

$$\text{arc tg}\, e^{-x} + \text{arc tg}\, e^{x} = \frac{\pi}{2}$$

and

$$\frac{\left|E_1\left(x+i\frac{\pi}{2}\right) E_1\left(x-i\frac{\pi}{2}\right)\right|}{|E_1(x)|^2} = \frac{\left|E_2\left(x+i\frac{\pi}{2}\right) E_2\left(x-i\frac{\pi}{2}\right)\right|}{|E_2(x)|^2} = \frac{\left|E\left(x+i\frac{\pi}{2}\right) E\left(x-i\frac{\pi}{2}\right)\right|}{|E(x)|^2}.$$

Putting $e^{\zeta - x_\nu} = r$ in the first member of (2.2.4), we find, according to (2.1.7), that

$$(2.2.5) \qquad \int_{-\infty}^{\infty} \log \left|\frac{e^{z_\nu}+e^{\zeta}}{e^{z_\nu}-e^{\zeta}}\right| d\zeta = \frac{1}{2}\int_0^{\infty} \log \left|\frac{e^{i y_\nu}+r}{e^{i y_\nu}-r}\right|^2 \frac{dr}{r} = \pi\left(\frac{\pi}{2} - |y_\nu|\right).$$

After a simple transformation the sum of the last two integrals in (2.2.4) may be written

$$(2.2.6) \qquad \frac{4}{\pi}\int_0^{\infty} \text{arc tg}\, e^{-x} \cdot \lambda_n x\, dx = \frac{2\lambda_n}{\pi}\int_0^{\infty} \frac{x^2\, dx}{e^x + e^{-x}} = \frac{\lambda_n \pi^2}{8}$$

where the second member is obtained through integration by parts. The value of this integral may be found by a simple contour integration. Insertion of (2.2.5) and (2.2.6) into (2.2.4) completes the proof of (2.2.2).

Equi-distribution of zeros of polynomials and exponential polynomials

3.1. We are going to study sequences of exponential polynomials $\{E_\nu(z)\}_1^\infty$ with

$$E_\nu(z) = \sum_{\mu=0}^{n_\nu} a_\mu^{(\nu)} e^{\lambda_\mu^{(\nu)} z}$$

where every $\lambda_\mu^{(\nu)}$ is real and non-negative.

A sequence of exponential polynomials is said to have its zeros equi-distributed in a strip

$$B > Im(z) > A$$

if the imaginary parts of the zeros of the polynomials form a sequence which is uniformly dense in the sense of Weyl (cf. ERDÖS-TURÁN [5] and references given there), i.e. if

$$\lim_{\nu\to\infty} \frac{N_\nu(\alpha,\beta)}{\lambda_\nu} = \frac{\beta-\alpha}{2\pi}$$

for all α and β satisfying $B > \beta > \alpha > A$. $N_\nu(\alpha,\beta)$ denotes the number of zeros of $E_\nu(z)$ in the strip $\beta > Im(z) \geq \alpha$, and $\lambda_\nu = \operatorname*{Max}_{\mu} \lambda_\mu^{(\nu)}$.

For sequences of polynomials a corresponding definition has already been given in the introduction. To avoid complications when $P_\nu(0)=0$ for some or all ν, we count a zero at the origin as $\frac{\theta}{2\pi}$ zeros in every sector of angle θ. If $\operatorname*{Min}_{\mu} \lambda_\mu^{(\nu)} = \zeta > 0$ in $E_\nu(z)$ we count this zero at $-\infty$ as $\frac{\lambda}{2\pi} b$ zeros in every strip of breadth b.

3.2. Some simple conditions for equi-distribution can be derived from the formulas of the preceding section.

Theorem. *A necessary and sufficient condition that the sequence $\{E_\nu(z)\}_1^\infty$ should have its zeros equi-distributed in the strip $B > Im(z) > A$ is that*

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \int_{-\infty}^{\infty} \log \frac{|E_\nu(x+i\alpha)\, E_\nu(x+i\beta)|}{\left|E_\nu\left(x+i\,\frac{\alpha+\beta}{2}\right)\right|^2}\, dx = 0 \tag{3.2.1}$$

for all α and β satisfying $B > \beta > \alpha > A$.

We introduce the notation

$$\varepsilon_\nu(y,\eta) = \frac{1}{2\pi\lambda_\nu} \int_{-\infty}^{\infty} \log \frac{|E_\nu(x+i[y+\eta])\, E_\nu(x+i[y-\eta])|}{|E_\nu(x+iy)|^2}\, dx.$$

That (3.2.1) is a necessary condition is seen from (2.2.2) in the following way. We suppose that the zeros are equi-distributed in $|Im(z)| < B$. Other cases can be handled after performing a translation. Let β be a positive number less than B. (2.2.2) may be rewritten

$$\frac{1}{\lambda_\nu} \int_0^\beta N_\nu(-\theta,\theta)\, d\theta = \frac{\beta^2}{2\pi} + \varepsilon_\nu(0,\beta) \tag{3.2.2}$$

because

$$\sum_{|y_\nu|<\beta} (\beta - |y_\nu|) = \int_0^\beta (\beta-\theta)\, dN_\nu(-\theta,\theta) = \int_0^\beta N_\nu(-\theta,\theta)\, d\theta.$$

From the assumption of equi-distribution we infer that

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} N_\nu(-\varphi,\varphi) = \frac{\varphi}{\pi}$$

for $0 < \varphi < B$. Since $N_\nu(-\theta,\theta) \le N_\nu(-\beta,\beta)$, $\frac{1}{\lambda_\nu} N_\nu(-\theta,\theta)$ is uniformly bounded for $0 < \theta < \beta$. Hence

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \int_0^\beta N_\nu(-\theta, \theta)\, d\theta = \int_0^\beta \lim_{\nu\to\infty} \frac{1}{\lambda_\nu} N_\nu(-\theta, \theta)\, d\theta = \frac{\beta^2}{2\pi}$$

and it follows from (3.2.2) that $\lim_{\nu\to\infty} \varepsilon_\nu(0, \beta) = 0$. The necessity of (3.2.1) for equi-distribution follows readily.

To prove that (3.2.1) is a sufficient condition we proceed as follows. After a translation, (2.2.2) may be written

$$\text{(3.2.3)} \qquad \frac{1}{\lambda_\nu} \sum_{|y_\nu - y| < b} (b - |y_\nu - y|) = \frac{b^2}{2\pi} + \varepsilon_\nu(y, b).$$

We now integrate both members of (3.2.3) with respect to y from $\alpha - b$ to $\beta + b$ (cf. 5.8). Every zero of $E_\nu(z)$ in $\alpha \le Im(z) < \beta$ (we suppose that $A < \alpha < < \beta < B$) gives a contribution b^2 in the sum of the left member; some other zeros may give a contribution and hence

$$\frac{b^2}{\lambda_\nu} N_\nu(\alpha, \beta) \le \frac{b^2}{2\pi}(\beta - \alpha + 2b) + \int_{\alpha - b}^{\beta + b} \varepsilon_\nu(y, b)\, dy.$$

A corresponding integration from $\alpha + b$ to $\beta - b$ gives (we suppose that $\alpha + b < \beta - b$)

$$\frac{b^2}{\lambda_\nu} N_\nu(\alpha, \beta) \ge \frac{b^2}{2\pi}(\beta - \alpha - 2b) + \int_{\alpha + b}^{\beta - b} \varepsilon_\nu(y, b)\, dy$$

and thus

$$\text{(3.2.4)} \qquad -\frac{b}{\pi} + \frac{1}{b^2} \int_{\alpha+b}^{\beta-b} \varepsilon_\nu(y, b)\, dy \le \frac{1}{\lambda_\nu} N_\nu(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} \le \frac{b}{\pi} + \frac{1}{b^2} \int_{\alpha-b}^{\beta+b} \varepsilon_\nu(y, b)\, dy.$$

Let ε be a positive number satisfying

$$\varepsilon < \frac{1}{\pi} \operatorname{Min} \{B - \beta,\ \alpha - A,\ \beta - \alpha\}.$$

If we put $b = \frac{\varepsilon\pi}{2}$, we have

$$\lim_{\nu\to\infty} \varepsilon_\nu(y, b) = 0$$

for every y in both intervals of integration. From (3.2.3) it may be seen that

$$0 \le \frac{b^2}{2\pi} + \varepsilon_\nu(y, b) \le \frac{\left(\frac{\beta - \alpha}{2} + 2b\right)^2}{2\pi} + \varepsilon_\nu\left(\frac{\alpha + \beta}{2}, \frac{\beta - \alpha}{2} + 2b\right)$$

for $\alpha - b < y < \beta + b$ and thus our assumption implies that $\varepsilon_\nu(y, b)$ is uniformly bounded with respect to y for fixed b. Hence

$$\lim_{\nu\to\infty} I_\nu = \lim_{\nu\to\infty} \int_{\alpha+b}^{\beta-b} \varepsilon_\nu(y, b)\, dy = 0, \quad \lim_{\nu\to\infty} J_\nu = \lim_{\nu\to\infty} \int_{\alpha-b}^{\beta+b} \varepsilon_\nu(y, b)\, dy = 0.$$

We now choose ν_0 such that $|I_\nu| < \frac{\varepsilon^3 \pi^2}{8}$ and $|J_\nu| < \frac{\varepsilon^3 \pi^2}{8}$ for $\nu > \nu_0$. Then follows from (3.2.4) that

$$\left| \frac{1}{\lambda_\nu} N_\nu(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} \right| \le \varepsilon$$

for $\nu > \nu_0$, and the equi-distribution follows. Thus it is proved that (3.2.1) is a sufficient condition.

3.3. Theorem 3.2 yields immediately a corresponding condition for equi-distribution of the zeros of a sequence of polynomials. We are going to study this polynomial case a little more thoroughly.

A simple transformation of (3.2.1) shows that the condition for equi-distribution of the zeros of $\{P_\nu(z)\}_1^\infty$ is

$$\lim_{\nu\to\infty} \varepsilon_\nu(\psi, \varphi) = 0$$

for all φ and ψ, where

$$\varepsilon_\nu(\psi, \varphi) = \frac{1}{2\pi n_\nu} \int_0^\infty \log \frac{|P_\nu(r e^{i(\psi+\varphi)}) P_\nu(r e^{i(\psi-\varphi)})|}{|P_\nu(r e^{i\psi})|^2} \cdot \frac{dr}{r} \tag{3.3.1}$$

and where n_ν is the degree of $P_\nu(z)$.

We introduce the notation

$$V_\nu(\theta) = \begin{cases} N_\nu(0, \theta) & \text{if } \theta > 0 \\ 0 & \text{if } \theta = 0 \\ -N_\nu(\theta, 0) & \text{if } \theta < 0 \end{cases}$$

where $N_\nu(\alpha, \beta)$ now is the number of zeros of $P_\nu(z)$ in the sector $\beta > \arg z \ge \alpha$. If $\theta > 2\pi$ we define $N_\nu(0, \theta)$ by

$$N_\nu(0, \theta) = N_\nu(0, \theta - 2\pi) + n_\nu$$

and in a similar way $V_\nu(\theta)$ is defined for every value of θ. With these notations, (2.1.6) gives

$$\frac{1}{n_\nu} \int_{-\varphi}^{\varphi} (\varphi - |\theta|)\, d V_\nu(\theta) = \frac{\varphi^2}{2\pi} + \varepsilon_\nu(0, \varphi). \tag{3.3.2}$$

Integration with respect to φ from 0 to π gives on the left side

$$(3.3.3)\quad \frac{1}{n_\nu}\int_0^\pi d\varphi \int_{-\varphi}^{\varphi} (\varphi - |\theta|)\, d V_\nu(\theta) = \frac{1}{2 n_\nu}\int_{-\pi}^{\pi} (\pi - |\theta|)^2\, d V_\nu(\theta) = \frac{1}{2 n_\nu}\int_{-\pi}^{\pi} \theta^2\, d V_\nu(\theta+\pi).$$

In the right member we change the order of integration and apply Jensen's formula (2.1.1) written in the following form

$$(3.3.4)\qquad \int_0^r \frac{R_\nu(u)}{u}\, du = \frac{1}{2\pi}\int_0^\pi \log \frac{|P_\nu(r e^{i\varphi})\, P_\nu(r e^{-i\varphi})|}{|P_\nu(0)|^2}\cdot d\varphi.$$

$R_\nu(u)$ is the number of zeros of $P_\nu(z)$ in $|z|\le u$. We obtain in this way from equation (3.3.2), applied to $P_\nu(z e^{i\psi})$, that

$$(3.3.5)\quad \frac{1}{2 n_\nu}\int_{-\pi}^{\pi} \theta^2\, d_\theta V_\nu(\theta+\psi+\pi) - \frac{\pi^2}{6} = \frac{1}{n_\nu}\int_0^\infty \left[\int_0^r \frac{R_\nu(u)}{u}\, du - \log\left|\frac{P_\nu(r e^{i\psi})}{P_\nu(0)}\right|\right]\frac{dr}{r}.$$

The common value of the members is called $\delta_\nu(\psi)$, that is

$$(3.3.6\text{ a})\qquad \delta_\nu(\psi) = \frac{1}{2 n_\nu}\int_{-\pi}^{\pi} \theta^2\, d_\theta V_\nu(\theta+\psi+\pi) - \frac{\pi^2}{6}$$

and

$$(3.3.6\text{ b})\quad \delta_\nu(\psi) = \int_0^\pi \varepsilon_\nu(\psi,\varphi)\, d\varphi = \frac{1}{n_\nu}\int_0^\infty \left[\int_0^r \frac{R_\nu(u)}{u}\, du - \log\left|\frac{P_\nu(r e^{i\psi})}{P_\nu(0)}\right|\right]\frac{dr}{r}.$$

Evidently $\delta_\nu(\psi)$ has the period 2π.

Formula (3.3.5) establishes a connection between the distributions of the moduli and of the arguments of the zeros of a polynomial.

3.4. We prove the following theorem and show how a well-known sufficient condition for equi-distribution is derived from it.

Theorem. *A necessary and sufficient condition that the zeros of* $\{P_\nu(z)\}_1^\infty$ *should be equi-distributed is that*

$$(3.4.1\text{ a})\qquad \lim_{\nu\to\infty} \operatorname*{Max}_{\psi} \delta_\nu(\psi) = 0.$$

An equivalent condition is that

$$(3.4.1\text{ b})\qquad \lim_{\nu\to\infty} \operatorname*{Min}_{\psi} \delta_\nu(\psi) = 0.$$

The function $\delta_\nu(\psi)$ is given by (3.3.6).

We first prove that the conditions (3.4.1 a) and (3.4.1 b) are equivalent by showing that each one implies the other.

From (3.3.6 a) it is easily seen that

$$\int_{-\pi}^{\pi} \delta_\nu(\psi)\, d\psi = 0.$$

That

$$|\delta_\nu(\psi+h) - \delta_\nu(\psi)| \leq 2\pi\, |h| \tag{3.4.2}$$

follows, for instance, from a study of a single term in the sum, giving $\delta_\nu(\psi)$, in (3.3.6 a).

Suppose that

$$\operatorname*{Max}_{\psi} \delta_\nu(\psi) = \varkappa_\nu$$

and that

$$\operatorname*{Min}_{\psi} \delta_\nu(\psi) = -\lambda_\nu < 0.$$

According to (3.4.2), the graph of the function to the right of the minimal point must be situated below a straight line with slope 2π through the minimal point. To the left of this point the graph is below a line with slope -2π.

Hence, if

$$\delta_\nu^+(\psi) = \operatorname{Max}\,\{\delta_\nu(\psi), 0\}$$

and

$$\delta_\nu^-(\psi) = \operatorname{Min}\,\{\delta_\nu(\psi), 0\},$$

we evidently have

$$\int_{-\pi}^{\pi} \delta_\nu^+(\psi)\, d\psi \leq 2\pi\, \varkappa_\nu$$

and

$$\int_{-\pi}^{\pi} \delta_\nu^-(\psi)\, d\psi \leq -\lambda_\nu \cdot \frac{\lambda_\nu}{2\pi}.$$

Now

$$\int_{-\pi}^{\pi} \delta_\nu^+(\psi)\, d\psi + \int_{-\pi}^{\pi} \delta_\nu^-(\psi)\, d\psi = \int_{-\pi}^{\pi} \delta_\nu(\psi)\, d\psi = 0,$$

and hence

$$\lambda_\nu \leq 2\pi \sqrt{\varkappa_\nu}.$$

Thus (3.4.1 a) implies (3.4.1 b) and a similar discussion shows that implication is also valid in the other direction.

By aid of this preliminary result, theorem 3.4 follows from theorem 3.2, as it is seen from (3.3.1) and (3.3.6 b) that

$$2\pi \cdot \varepsilon_\nu(\psi, \varphi) = 2\,\delta_\nu(\psi) - \delta_\nu(\psi+\varphi) - \delta_\nu(\psi-\varphi), \tag{3.4.3}$$

and from (3.3.6 b) that

$$\delta_\nu(\psi) = \int_{0}^{\pi} \varepsilon_\nu(\psi, \varphi)\, d\varphi. \tag{3.4.4}$$

Equation (3.4.3) yields the sufficiency of the conditions given in theorem 3.4, and equation (3.4.4) yields their necessity, because the sequence $\{\varepsilon_\nu(\psi, \varphi)\}$ is bounded according to (3.2.3) and because $\delta_\nu'(\psi) < 2\pi$.

3.5. Among the sufficient conditions for equi-distribution the following condition is perhaps the best known one.

Put

$$P_\nu(z) = \sum_{\mu=0}^{n_\nu} a_\mu^{(\nu)} z^\mu,$$

and

$$M_\nu = \operatorname*{Max}_{\varphi} |P_\nu(e^{i\varphi})|.$$

A sufficient condition that $\{P_\nu(z)\}_1^\infty$ should have its zeros equi-distributed is that simultaneously

$$\lim_{\nu\to\infty} \frac{1}{n_\nu} \log M_\nu = 0, \tag{3.5.1 a}$$

$$\lim_{\nu\to\infty} \frac{1}{n_\nu} \log |a_0^{(\nu)} a_{n_\nu}^{(\nu)}| = 0. \tag{3.5.1 b}$$

(We suppose for simplicity that $a_0^{(\nu)} \neq 0$ for all ν.)

This condition is equivalent to the condition for equi-distribution which follows from the previously quoted theorem of ERDÖS and TURÁN. We are going to study this theorem more closely in the following section.

We now show how this condition can be derived from theorem 3.4. If we put

$$P_\nu(z) = a_0^{(\nu)} \prod_{\mu=1}^{n_\nu} \left(1 - \frac{z}{z_\mu}\right)$$

with

$$z_\mu = \varrho_\mu e^{i\theta_\mu}$$

and form the polynomial

$$p_\nu(z) = \prod_{\mu=1}^{n_\nu} (1 - z \cdot e^{-i\theta_\mu}),$$

then it is true (SCHUR [19]) that

$$m_\nu = \operatorname*{Max}_{\varphi} |p_\nu(e^{i\varphi})| \leq \frac{1}{\sqrt{|a_0^{(\nu)} a_{n_\nu}^{(\nu)}|}} \operatorname*{Max}_{\varphi} |P_\nu(e^{i\varphi})| = \frac{M_\nu}{\sqrt{|a_0^{(\nu)} a_{n_\nu}^{(\nu)}|}}. \tag{3.5.2}$$

The proof of (3.5.2) is obtained by observing that

$$\varrho_\mu \left|1 - \frac{e^{i\theta}}{z_\mu}\right|^2 = \varrho_\mu + \frac{1}{\varrho_\mu} - 2\cos(\theta - \theta_\mu) \geq 2 - 2\cos(\theta - \theta_\mu) = \left|1 - \frac{e^{i\theta}}{e^{i\theta_\mu}}\right|^2.$$

Hence

$$\prod_{\mu=1}^{n_\nu} \varrho_\mu \left| 1 - \frac{e^{i\theta}}{z_\mu} \right|^2 \geq \prod_{\mu=1}^{n_\nu} \left| 1 - \frac{e^{i\theta}}{e^{i\theta_\mu}} \right|^2,$$

and the result easily follows.

To prove that the zeros of $\{P_\nu(z)\}_1^\infty$ are equi-distributed, we consider $\{p_\nu(z)\}_1^\infty$, as the zeros of $p_\nu(z)$ have the same arguments as those of $P_\nu(z)$.

All zeros of the polynomials $p_\nu(z)$ have modulus 1 and thus the function $\delta_\nu(\psi)$ defined by (3.3.6 b) is

$$\delta_\nu^*(\psi) = -\frac{1}{n_\nu}\int_0^1 \log|p_\nu(r e^{i\psi})|\frac{dr}{r} - \frac{1}{n_\nu}\int_1^\infty \log\left|\frac{p_\nu(r e^{i\psi})}{r^{n_\nu}}\right|\cdot\frac{dr}{r} =$$

$$= -\frac{1}{n_\nu}\int_0^1 \log|p_\nu(r e^{i\psi})\, q_\nu(r e^{-i\psi})|\frac{dr}{r},$$

where

$$q_\nu(z) = z^{n_\nu} p_\nu\left(\frac{1}{z}\right).$$

We define m_ν by

$$m_\nu = \operatorname*{Max}_\psi |q_\nu(e^{i\psi})| = \operatorname*{Max}_\psi |p_\nu(e^{i\psi})|.$$

According to a theorem related to Schwarz's lemma (see e. g. KOEBE [10, p. 59])

$$\log|p_\nu(r e^{i\psi})| \leq \frac{2r}{1+r}\log m_\nu$$

for $0 \leq r \leq 1$, and $q_\nu(z)$ satisfies the same inequality. Thus

$$-\delta_\nu^*(\psi) \leq \frac{1}{n_\nu}\cdot 4\cdot\log 2\cdot\log m_\nu$$

and by (3.5.2)

$$\operatorname*{Min}_\psi \delta_\nu^*(\psi) \geq -\frac{4\cdot\log 2}{n_\nu}\cdot\log\frac{M_\nu}{\sqrt{|a_0^{(\nu)} a_{n_\nu}^{(\nu)}|}}.$$

Hence the equi-distribution follows from theorem 3.4 if (3.5.1) is satisfied.

3.6. As an example, we consider a power series

$$f(z) = \sum_{\mu=0}^\infty a_\mu z^\mu$$

with radius of convergence 1 $(a_0 \neq 0)$.

Then, if

$$P_\nu(z) = \sum_{\mu=0}^\nu a_\mu z^\mu$$

there is a sub-sequence $\{P_{\nu_i}(z)\}_{i=1}^{\infty}$ for which

$$\lim_{\nu_i\to\infty} \operatorname{Max}_{\varphi} |P_{\nu_i}(re^{i\varphi})|^{\frac{1}{\nu_i}} = \begin{cases} 1 & \text{for } r \le 1 \\ r & \text{for } r \ge 1 \end{cases}$$

and

$$\lim_{\nu_i\to\infty} |a_{\nu_i}|^{\frac{1}{\nu_i}} = 1.$$

Hence the zeros of the polynomials in the sub-sequence are equi-distributed as is seen from (3.5.1) and this conclusion is the previously mentioned result of SZEGÖ.

That the above statement is true for the partial sums of a function of infinite order was first shown by CARLSON [3, 4]. Using the estimates given by CARLSON [4, p. 5—6], we can, by a transformation $z = K_\nu \zeta$ in the ν:th partial sum, obtain a sub-sequence fulfilling the requirements given in (3.5.1).

This method cannot successfully be applied to all sequences with equi-distributed zeros. We give an example of a sequence which cannot in this way be brought to satisfy the condition that

$$\lim_{\nu\to\infty} \frac{1}{n_\nu} \log \frac{M_\nu}{\sqrt{|a_0^{(\nu)} a_{n_\nu}^{(\nu)}|}} = 0. \tag{3.6.1}$$

Let

$$P_\nu(z) = \prod_{\mu=1}^{\nu} \left(1 - \frac{z}{z_\mu^{(\nu)}}\right) = \sum_{\mu=0}^{\nu} b_\mu^{(\nu)} z^\mu,$$

with

$$z_\mu^{(\nu)} = \left|\sec \frac{4\mu\pi}{2\nu+1}\right| e^{\frac{4\mu\pi i}{2\nu+1}}.$$

The zeros all satisfy $|Re(z_\mu^{(\nu)})| = 1$ and they are evidently equi-distributed.

We put $z = k_\nu \zeta$ in $P_\nu(z)$, and write $p_\nu(\zeta) = P_\nu(\zeta k_\nu)$ and $m_\nu = \operatorname{Max}_{\varphi} |p_\nu(e^{i\varphi})|$.

We now try to determine k_ν so that

$$\lim_{\nu\to\infty} \frac{1}{\nu} \log \frac{m_\nu}{\sqrt{|b_\nu^{(\nu)} k_\nu^\nu|}} = 0.$$

As

$$\lim_{\nu\to\infty} \frac{1}{\nu} \log |b_\nu^{(\nu)}| = \log \frac{1}{2}$$

a suitable value of k_ν must be > 2.

A simple calculation shows that

$$\lim_{\nu\to\infty} \frac{1}{\nu} \log |P_\nu(re^{i\varphi})| = \frac{1}{4} \log \frac{r^2(r^2 + 4r|\cos s| + 4)}{16},$$

if $r|\cos s| \ge 1$.

Hence

$$\varliminf_{\nu\to\infty} \frac{1}{\nu} \log \frac{m_\nu}{\sqrt{|b_\nu^{(\nu)} k_\nu^{(\nu)}|}} \geq \varliminf_{\nu\to\infty} \frac{1}{2} \log \frac{\frac{k_\nu (k_\nu + 2)}{4}}{\frac{1}{2} k_\nu} = \varliminf_{\nu\to\infty} \frac{1}{2} \log \frac{k_\nu + 2}{2} \geq \frac{1}{2} \log 2,$$

and (3.6.1) cannot be satisfied.

A theorem on conjugate harmonic functions

4.1. In the previous section, we referred to a theorem of ERDÖS and TURÁN [6]. In their proof no use was made of the results of the theory of analytic functions. Their theorem is a generalisation of a result of Schmidt and SCHUR [19] which gives an upper bound for the number of real roots of a polynomial. A simple function-theoretic proof of the theorem of Schmidt has been given by LITTLEWOOD and OFFORD [14].

We are going to prove a theorem for a pair of conjugate harmonic functions, which implies the result of ERDÖS and TURÁN and which gives a better value of the constant occurring in that theorem.

4.2. If the inequality of SCHUR (3.5.2) is compared with that of ERDÖS and TURÁN given in the introduction, we see that it is sufficient to prove that *if*

$$P(z) = \prod_{\nu=1}^{n} (1 - z \cdot e^{-i\theta_\nu}), \quad 0 \leq \theta_\nu < 2\pi$$

and if $N(\alpha, \beta)$ *is the number of* θ_ν *such that*

$$0 \leq \alpha \leq \theta_\nu < \beta < 2\pi$$

then

$$\left| N(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} n \right| \leq c \sqrt{n \cdot \operatorname{Max}_\theta \log |P(e^{i\theta})|}. \tag{4.2.1}$$

(A proof which does not require SCHUR's inequality follows from the investigations in the following section.)

To show the connection with the theorem on harmonic functions, we observe that the function ($z = \varrho e^{i\theta}$)

$$u_1(z) = \frac{1}{\pi} \log |1 - \varrho e^{i(\theta - \theta_1)}|$$

which is harmonic for $\varrho < 1$, has as a conjugate function

$$v_1(z) = \frac{1}{\pi} \arg (1 - \varrho e^{i(\theta - \theta_1)}).$$

It is easily seen that

$$\frac{\partial v_1(\varrho e^{i\theta})}{\partial \theta} < \frac{1}{2\pi}$$

for $\varrho<1$. On $\varrho=1$, $v_1(z)$ takes the boundary values

$$V_1(\theta)=C_1+\frac{\theta}{2\pi}-\mu_1(\theta)$$

where C_1 is an appropriate constant and $\mu_1(\theta)$ is defined by

$$\mu_1(\theta)=\begin{cases}0 \text{ if } \theta<\theta_1\\ 1 \text{ if } \theta>\theta_1\end{cases}.$$

Thus it is seen that a harmonic function $v(z)$ conjugate to

$$u(z)=\frac{1}{\pi}\log|P(\varrho e^{i\theta})| \tag{4.2.2}$$

has the boundary values

$$V(\theta)=C_0+\frac{n\theta}{2\pi}-N(0,\theta) \tag{4.2.3}$$

on $\varrho=1$. The derivative satisfies

$$\frac{\partial v}{\partial\theta}<\frac{n}{2\pi} \tag{4.2.4}$$

if $\varrho<1$.

In the sequel we do not restrict θ to the interval $(0, 2\pi)$. The value of $N(0,\theta)$ is given by the periodic continuation of $V(\theta)$ if θ does not belong to this interval.

If (4.2.2), (4.2.3) and (4.2.4) are compared with (4.2.1) it seems natural to state the following theorem.

4.3. Theorem. *Suppose that $f(z)=u+iv$, $f(0)=0$ is a function which is regular for $|z|<1$ and suppose that*

$$u<H,\ \frac{\partial v}{\partial\theta}<K \tag{4.3.1}$$

for $|z|<1$. Then

$$|v(z)|<C\sqrt{HK} \tag{4.3.2}$$

for $|z|<1$ where C is a constant e. g. 13.

Remark. It will be seen below that the value of C cannot be smaller than π. Since it follows from $\frac{\partial v}{\partial\theta}<K$ and

$$\int_0^{2\pi} v(\varrho e^{i\theta})\,d\theta=2\pi v(0)=0$$

that $|v(z)|<\pi K$ the theorem is obvious if $H>K$. We may suppose in the sequel that $H\leq K$.

As $\dfrac{\partial v}{\partial \theta} = \varrho \dfrac{\partial u}{\partial \varrho}$ the theorem holds true if (4.3.1) is replaced by

$$u < H, \ \frac{\partial u}{\partial \varrho} < K. \tag{4.3.1'}$$

Proof. To simplify the notations we suppose in the proof that $f(z)$ is regular for $|z| < 1 + \varepsilon$ and prove that (4.3.2) holds for $|z| = 1$. The theorem follows by a simple transformation.

For the proof we use a formula related to (2.1.5) but valid for every harmonic function regular in $|z| \leq 1$. We consider the domain consisting of the points z which satisfy the inequalities

$$|z| < 1, \ |\arg z| < \frac{\pi}{2\lambda}, \ |z-1| > \varepsilon.$$

Green's formula (2.1.2) is applied to the functions

$$G(z) = \log \left| \frac{1+z^\lambda}{1-z^\lambda} \right|, \ H(z) = u(z e^{i\varphi}).$$

G equals zero on the straight lines of the boundary and $\dfrac{\partial G}{\partial n} = 0$ on the arcs of the circle $|z| = 1$. If ε tends to zero, the integral $\int \dfrac{\partial G}{\partial n} \cdot H \cdot ds$ along the arc of the circle $|z-1| = \varepsilon$ tends to $\pi u(1, \varphi)$ and we find $(u(\varrho, \theta) = u(\varrho e^{i\theta}))$

$$2\lambda \int_0^1 \frac{u\left(\varrho, \varphi - \frac{\pi}{2\lambda}\right) + u\left(\varrho, \varphi + \frac{\pi}{2\lambda}\right)}{1+\varrho^{2\lambda}} \varrho^{\lambda-1} d\varrho +$$

$$+ \int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log \left| \cot \frac{\lambda\theta}{2} \right| \cdot u'_\varrho(1, \varphi + \theta) \, d\theta = \pi u(1, \varphi).$$

As

$$u'_\varrho(1, \varphi + \theta) = v'_\theta(1, \varphi + \theta)$$

an integration with respect to φ from $-\xi$ to ξ gives

$$\int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log \left| \cot \frac{\lambda\theta}{2} \right| \cdot [v(1, \theta + \xi) - v(1, \theta - \xi)] \, d\theta = \tag{4.3.3}$$

$$= \int_{-\xi}^{\xi} \left[\pi u(1, \varphi) - 2\lambda \int_0^1 \frac{u\left(\varrho, \varphi - \frac{\pi}{2\lambda}\right) + u\left(\varrho, \varphi + \frac{\pi}{2\lambda}\right)}{1+\varrho^{2\lambda}} \varrho^{\lambda-1} d\varrho \right] d\varphi.$$

Now $\dfrac{\partial v}{\partial \theta} < K$, hence

$$v(1,\,\theta+\xi) - v(1,\,\theta-\xi) \le v\left(1,\,\xi - \frac{\pi}{2\lambda}\right) + K\left(\frac{\pi}{2\lambda}+\theta\right) -$$

$$- \left[v\left(1,\,-\xi+\frac{\pi}{2\lambda}\right) - K\left(\frac{\pi}{2\lambda}-\theta\right)\right] = v\left(1,\,\xi-\frac{\pi}{2\lambda}\right) - v\left(1,\,-\xi+\frac{\pi}{2\lambda}\right) + K\frac{\pi}{\lambda}$$

$$\text{for } |\theta| \le \frac{\pi}{2\lambda}.$$

We put $\chi = \xi - \dfrac{\pi}{2\lambda}$ and observe that

$$\int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log\left|\cot\frac{\lambda\theta}{2}\right| d\theta = 2\int_0^{\frac{\pi}{2\lambda}} \log\cot\frac{\lambda\theta}{2}\, d\theta = \frac{4}{\lambda}\int_1^\infty \frac{\log t}{1+t^2}\,dt = \frac{4}{\lambda}k$$

where

$$k = \int_1^\infty \frac{\log t}{1+t^2}\,dt = \int_0^1 \frac{\operatorname{arc\,tg} t}{t}\,dt = \sum_{m=0}^\infty (-1)^m \frac{1}{(2m+1)^2} = 0.916\cdots$$

is known as Catalan's constant. In this way (4.3.3) is transformed into the inequality

$$v(1,\,\chi) - v(1,\,-\chi) \ge -K\frac{\pi}{\lambda} -$$

$$-\frac{\lambda}{4k}\int_{-\chi-\frac{\pi}{2\lambda}}^{\chi+\frac{\pi}{2\lambda}} \left[2\lambda\int_0^1 \frac{u\left(\varrho,\,\varphi-\frac{\pi}{2\lambda}\right) + u\left(\varrho,\,\varphi+\frac{\pi}{2\lambda}\right)}{1+\varrho^{2\lambda}}\,\varrho^{\lambda-1}\,d\varrho - \pi u(1,\,\varphi)\right] d\varphi.$$

We interchange the order of integration and use the inequality

$$\left|\int_\alpha^\beta u(\varrho,\,\theta)\,d\theta\right| \le \operatorname*{Max}_\theta u(\varrho,\,\theta)\cdot 2\pi < 2\pi H$$

which is evidently true because

$$\int_0^{2\pi} u(\varrho,\,\theta)\,d\theta = 2\pi u(0) = 0.$$

Thus

$$v(1, \chi) - v(1, -\chi) \geq -K\frac{\pi}{\lambda} - \frac{\lambda}{4k}\left[8\pi H \int_0^1 \frac{\lambda \varrho^{\lambda-1} d\varrho}{1+\varrho^{2\lambda}} + 2\pi^2 H\right] = -K\frac{\pi}{\lambda} - \frac{\lambda}{k}\pi^2 H.$$

We choose

$$\lambda = \sqrt{\frac{K k}{\pi H}}$$

and get

$$v(1, \chi) - v(1, -\chi) \geq -2\pi\sqrt{\frac{\pi}{k} H K}. \tag{4.3.4}$$

The same result holds evidently for $v(1, \chi_0 + \chi) - v(1, \chi_0 - \chi)$ for every χ_0.

As there certainly exists a θ_0, such that $v(1, \theta_0) = 0$, it follows that

$$|v(1, \theta)| \leq 2\pi\sqrt{\frac{\pi}{k} H K}.$$

The result of ERDÖS and TURÁN follows from (4.3.4) since this inequality is valid for $V(\theta)$ as given in (4.2.3). We have $H = \frac{1}{\pi}\log(\operatorname{Max}_\theta |P(e^{i\theta})|)$ and $K = \frac{n}{2\pi}$ and therefore

$$\left|\frac{N(\alpha, \beta)}{n} - \frac{\beta - \alpha}{2\pi}\right| \leq \sqrt{\frac{2\pi}{k} \cdot \frac{\log \operatorname{Max} |P(e^{i\theta})|}{n}}.$$

The simple example $P(z) = (1+z)^n$ shows that no number smaller than $\frac{1}{\sqrt{\log 2}}$ can be substituted for $\sqrt{\frac{2\pi}{k}}$. Hence the best possible value of the constant C in theorem 4.3 must be larger than $\pi\sqrt{\frac{2}{\log 2}}$.

On the distribution of the zeros of exponential polynomials

5.1. In this section we give some results concerning the zeros of exponential polynomials. In combination with a method used in a preceding section these results lead to a proof of a theorem which is a generalisation of that of ERDÖS and TURÁN.

We are going to study the number of zeros of

$$E(z) = a_0 + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z}$$

in rectangular regions $R(\delta, T)$ defined by

$$|Re(z)| \leq \delta, \ |Im(z)| \leq T.$$

We suppose that

$$a_0 a_n \neq 0$$

and that

$$\lambda_{\mu+1} - \lambda_\mu \geq \gamma > 0$$

for $\mu = 0, 1, \ldots, n-1$. We put $\lambda_0 = 0$. In the sequel we assume that $\gamma \leq 1$.

The function $D(z)$ is defined by

$$D(z) = e^{-\lambda_n z} E(z).$$

We put

$$e(x) = \sup_y |E(x+iy)|$$

and

$$d(x) = \sup_y |D(x+iy)|.$$

If

$$Q = |a_0| + |a_1| + \cdots + |a_n|$$

we evidently have

$$e(x) \leq Q \text{ for } x \leq 0$$

$$d(x) \leq Q \text{ for } x \geq 0.$$

We also put

$$S = \frac{Q}{\sqrt{|a_0 a_n|}}$$

and denote by $\{z_\nu\}$ the zeros of $E(z)$ in some order.

We are going to prove the following theorem.

Theorem. *Let $N(\delta, T)$ denote the number of zeros of $E(z)$ in the region $R(\delta, T)$ defined above. Then*

$$-3\left[\phi^{\frac{1}{2}} + 3\frac{\phi}{\delta}\right] \leq \frac{N(\delta, T)}{\lambda_n} - \frac{T}{\pi} \leq 6\left[e^{\frac{\delta}{2}} \phi^{\frac{1}{2}} + \phi\right]$$

where

$$\phi = \left(T + \frac{\pi}{\gamma}\right) \frac{\log S}{\lambda_n}.$$

For $\delta \to \infty$ this theorem yields the inequality

$$\frac{N(\infty, T)}{\lambda_n} \geq \frac{T}{\pi} - 3\sqrt{\left(T + \frac{\pi}{\gamma}\right)\frac{\log S}{\lambda_n}}$$

and if we apply this result to the polynomial case, where we are interested only in values of $T < 2\pi$ and $\gamma = 1$, we find

$$\frac{N(\infty, T)}{\lambda_n} \geq \frac{T}{\pi} - 3\sqrt{3\pi}\sqrt{\frac{\log S}{\lambda_n}}.$$

It is easily seen that this result implies the theorem of ERDÖS and TURÁN with $c = 3\sqrt{3\pi}$.

5.2. To simplify our equations we shall suppose from now on that we have multiplied $E(z)$ by a suitable constant so that $Q=1$.

Lemma. *If $T\geq 0$ and $x\leq 0$ we have*

$$\int_{-T}^{T}\log|E(x+iy)|\,dy\geq -2\pi\left(T+\frac{\pi}{\gamma}\right)\log\frac{2}{|a_0|}. \tag{5.2.1}$$

Let ζ be a number with $Re(\zeta)\leq 0$. We define

$$h(z)=E(\zeta+z).$$

Green's function for the left half-plane $Re(z)\leq 0$, singular at $-a$ $(a>0)$ is $\log\left|\frac{z-a}{z+a}\right|$.

A formula corresponding to (2.1.3) is clearly valid for the exponential polynomial $h(z)$ if the domain D is the left half-plane. We find

$$\sum_{\varkappa_\nu\in D}\log\left|\frac{\varkappa_\nu-a}{\varkappa_\nu+a}\right|+\log|h(-a)|-\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\log|h(iy)|}{a^2+y^2}\,dy=0$$

where $\{\varkappa_\nu\}$ are the zeros of $h(z)$. As the Green's function is not negative in D, it follows that

$$\int_{-\infty}^{\infty}\frac{\log|h(iy)|}{a^2+y^2}\,dy\geq\frac{\pi}{a}\log|h(-a)|. \tag{5.2.2}$$

Now $|h(iy)|\leq 1$ and thus for $u\geq 0$

$$\frac{1}{a^2+u^2}\int_{-u}^{u}\log|h(iy)|\,dy\geq\int_{-u}^{u}\frac{\log|h(iy)|}{a^2+y^2}\,dy\geq\int_{-\infty}^{\infty}\frac{\log|h(iy)|}{a^2+y^2}\,dy, \tag{5.2.3}$$

and if we use (5.2.2) we find that

$$\int_{-u}^{u}\log|h(iy)|\,dy\geq\pi\frac{a^2+u^2}{a}\log|h(-a)|.$$

Hence, if $a=u$, then

$$\int_{-u}^{u}\log|h(iy)|\,dy\geq 2\pi u\log|h(-u)|. \tag{5.2.4}$$

We need a lower bound for $|h(-u)|$.

According to a result of INGHAM [8]

$$\frac{1}{V}\int_{V_0-V}^{V_0+V} |E(x+iy)|\,dy \ge |a_0|$$

if $V=\frac{\pi}{\gamma}$ and V_0 is an arbitrary real number.

Thus, for fixed x, there is a y_0 in every interval of length $\frac{2\pi}{\gamma}$ such that

$$|E(x+iy_0)| \ge \frac{|a_0|}{2}.$$

Suppose now that η_0 is the real number with the smallest absolute value which satisfies

$$\left|E\left(x-T-\frac{\pi}{\gamma}+i\eta_0\right)\right| \ge \frac{|a_0|}{2}.$$

Then

$$|\eta_0| \le \frac{\pi}{\gamma}.$$

If we put $\zeta = x+i\eta_0$, it follows from (5.2.4) that

$$\int_{-\left(T+\frac{\pi}{\gamma}\right)}^{T+\frac{\pi}{\gamma}} \log|E(x+i(y+\eta_0))|\,dy \ge 2\pi\left(T+\frac{\pi}{\gamma}\right)\log\left|E\left(x-T-\frac{\pi}{\gamma}+i\eta_0\right)\right| \ge \tag{5.2.5}$$

$$\ge 2\pi\left(T+\frac{\pi}{\gamma}\right)\log\frac{|a_0|}{2},$$

and as $|E(x+iy)|\le 1$ when $x\le 0$ and $|\eta_0|\le\frac{\pi}{\gamma}$, we get

$$\int_{-T}^{T}\log|E(x+iy)|\,dy \ge \int_{-\left(T+\frac{\pi}{\gamma}\right)}^{T+\frac{\pi}{\gamma}} \log|E(x+i(y+\eta_0))|\,dx. \tag{5.2.6}$$

But (5.2.5) and (5.2.6) evidently imply (5.2.1) and the lemma is therefore proved.

In the same way it follows that

$$\int_{-T}^{T}\log|D(x+iy)|\,dy \ge 2\pi\left(T+\frac{\pi}{\gamma}\right)\log\frac{|a_n|}{2} \tag{5.2.7}$$

for $x\ge 0$ and as $|D(iy)|=|E(iy)|$ we also have

$$\int_{-T}^{T} \log |E(iy)|\, dy \geq \pi \left(T + \frac{\pi}{\gamma}\right) \log \frac{|a_0 a_n|}{4}. \tag{5.2.8}$$

5.3. We now turn to a study of the function

$$\varphi(\sigma) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left| \frac{1 + \sigma e^{iy}}{1 - \sigma e^{iy}} \right| dy. \tag{5.3.1}$$

In 4.2 we used the fact that for all σ

$$|\varphi(\sigma)| \leq \varphi(1) = 4k$$

where k is Catalan's constant.

For $|\sigma| \leq 1$

$$\varphi(\sigma) = 4 \sum_{m=0}^{\infty} (-1)^m \frac{\sigma^{2m+1}}{(2m+1)^2} \tag{5.3.2}$$

as is found by a series development of the integrand in (4.3.1). For other values of σ, we get $\varphi(\sigma)$ from the relations

$$\varphi(\sigma) = -\varphi(-\sigma) = \varphi\left(\frac{1}{\sigma}\right).$$

For $|\sigma| < 1$, it is true that

$$\varphi'(\sigma) = 4 \frac{\operatorname{arc\,tg} \sigma}{\sigma}.$$

Thus, for $0 \leq \sigma < 1$,

$$\pi < \varphi'(\sigma) \leq 4. \tag{5.3.3}$$

If $\varkappa \geq 1$ and $0 \leq \varkappa\sigma \leq 1$ we find

$$\varphi(\varkappa\sigma) - \varphi\left(\frac{1}{\varkappa}\sigma\right) = \sigma \int_{\frac{1}{\varkappa}}^{\varkappa} \varphi'(s\sigma)\, ds \geq \pi\sigma\left(\varkappa - \frac{1}{\varkappa}\right). \tag{5.3.4}$$

We also observe that for $0 \leq \sigma \leq 1$

$$\varphi(1) - \varphi(\sigma) = \int_0^1 \varphi'(s)\, ds \leq 4(1 - \sigma) \tag{5.3.5}$$

and

$$\varphi(\sigma) \leq 4\sigma. \tag{5.3.6}$$

5.4. The inequalities of 5.3 will now be used in the proofs of some lemmas on the distribution of zeros of exponential polynomials. They concern the zeros, $z_\nu = x_\nu + i y_\nu$, of the exponential polynomial $E(z)$ which is supposed to be normed so that $Q = 1$.

The following notations will be used for the regions with which we are going to deal. They are all bounded by the lines $|Im(z)| = T$ and by vertical lines. If δ is a non-negative number we define

$$\begin{aligned}
&W_1(\delta, T): Re(z) \le -\delta && |Im(z)| \le T,\\
&W_2(\delta, T): \delta \le Re(z) && |Im(z)| \le T,\\
&R_1(\delta, T): -\delta \le Re(z) \le 0 && |Im(z)| \le T,\\
&R_2(\delta, T): 0 \le Re(z) \le \delta && |Im(z)| \le T,\\
&W(\delta, T) = W_1(\delta, T) \cup W_2(\delta, T), &&\\
&R(\delta, T) = R_1(\delta, T) \cup R_2(\delta, T), &&\\
&W(T): |Im(z)| \le T. &&
\end{aligned}$$

5.5. Lemma. *If* $b > 0$, *then*

$$\sum_{z_\nu \in W_1(\delta, T)} e^{\frac{\pi x_\nu}{2b}} \le \frac{1}{\delta}\left(T + b + \frac{\pi}{\gamma}\right) \log \frac{2}{|a_0|}. \tag{5.5.1}$$

For the proof we put $h(z) = E(z + i\eta)$ and apply a formula we have mentioned in the beginning of 2.2, that is the extension of (2.1.4), with $R = 1$, to exponential polynomials. If z_ν is a zero of $E(z)$, the corresponding zero of $h(z)$ is $z_\nu - i\eta$. As $|h(z)| \le 1$ for $Re(z) \le 0$ we find

$$\begin{aligned}
&\log|E(\zeta + i\eta)| + \\
&+ \sum_{z_\nu - i\eta \in W_1(0, b)} \left[\log\left|\frac{e^{\frac{\pi(z_\nu - i\eta)}{2b}} + e^{\frac{\pi\zeta}{2b}}}{e^{\frac{\pi(z_\nu - i\eta)}{2b}} - e^{\frac{\pi\zeta}{2b}}}\right| - \log\left|\frac{1 + e^{\frac{\pi}{2b}(z_\nu - i\eta + \zeta)}}{1 - e^{\frac{\pi}{2b}(z_\nu - i\eta + \zeta)}}\right|\right] \le 0.
\end{aligned} \tag{5.5.2}$$

The left member of this inequali.y is now integrated with respect to η from $-T-b$ to $T+b$ (cf. the derivation of (3.2.1)). According to the definition of $\varphi(\sigma)$ we find that, as every term in the sum in (5.5.2) is positive,

$$\int_{-T-b}^{T+b} \log|E(\zeta + i\eta)|\, d\eta + \frac{2b}{\pi} \sum_{z_\nu \in W_1(0, T)} \left[\varphi\left(e^{\frac{\pi(x_\nu - \zeta)}{2b}}\right) - \varphi\left(e^{\frac{\pi(x_\nu + \zeta)}{2b}}\right)\right] \le 0.$$

We put $\zeta = -\delta \le 0$ and apply lemma 5.2. Then

$$\sum_{z_\nu \in W_1(0, T)} \left[\varphi\left(e^{\frac{\pi(x_\nu + \delta)}{2b}}\right) - \varphi\left(e^{\frac{\pi(x_\nu - \delta)}{2b}}\right)\right] \le \frac{\pi^2}{b}\left(T + b + \frac{\pi}{\gamma}\right) \log \frac{2}{|a_0|}. \tag{5.5.3}$$

The terms of the sum are evidently positive, and if we omit some of them, the value of the sum is certainly not increased. Hence

$$\sum_{z_\nu \in W_1(\delta,\, T)} \left[\varphi\left(e^{\frac{\pi(x_\nu+\delta)}{2b}}\right) - \varphi\left(e^{\frac{\pi(x_\nu-\delta)}{2b}}\right)\right] \leq \frac{\pi^2}{b}\left(T + b + \frac{\pi}{\gamma}\right) \log \frac{2}{|a_0|}. \tag{5.5.4}$$

But from (5.3.4) it follows, as $x_\nu + \delta \leq 0$, that

$$\varphi\left(e^{\frac{\pi(x_\nu+\delta)}{2b}}\right) - \varphi\left(e^{\frac{\pi(x_\nu-\delta)}{2b}}\right) \geq \pi\, e^{\frac{\pi x_\nu}{2b}}\left(e^{\frac{\pi\delta}{2b}} - e^{-\frac{\pi\delta}{2b}}\right) \geq \frac{\pi^2 \delta}{b}\, e^{\frac{\pi x_\nu}{2b}},$$

and if this inequality is applied to the terms of the summation in (5.5.4) the lemma follows.

Of course there is a similar result for $D(z)$ and $W_2(\delta, T)$. If the formula corresponding to (5.5.1) is added to (5.5.1), we obtain

$$\sum_{z_\nu \in W(\delta,\, T)} e^{-\frac{\pi |x_\nu|}{2b}} \leq \frac{1}{\delta}\left(T + b + \frac{\pi}{\gamma}\right) \log \frac{4}{|a_0\, a_n|}. \tag{5.5.5}$$

5.6. Lemma.

$$\sum_{z_\nu \in R_1(\delta,\, T)} |x_\nu| \leq 2\, e^{\delta}\left(T + \frac{\pi}{\gamma}\right) \log \frac{2}{|a_0|}. \tag{5.6.1}$$

From (5.5.3) it is seen that

$$\sum_{z_\nu \in R_1(\delta,\, T)} \left[\varphi\left(e^{-\frac{\pi(x_\nu+\delta)}{2b}}\right) - \varphi\left(e^{\frac{\pi(x_\nu-\delta)}{2b}}\right)\right] \leq \frac{\pi^2}{b}\left(T + b + \frac{\pi}{\gamma}\right) \log \frac{2}{|a_0|}. \tag{5.6.2}$$

Application of (5.3.4) gives, in this case, as $x_\nu + \delta \geq 0$

$$\varphi\left(e^{-\frac{\pi(x_\nu+\delta)}{2b}}\right) - \varphi\left(e^{\frac{\pi(x_\nu-\delta)}{2b}}\right) \geq \pi \cdot e^{-\frac{\pi\delta}{2b}}\left(e^{-\frac{\pi x_\nu}{2b}} - e^{\frac{\pi x_\nu}{2b}}\right) \geq \frac{\pi^2}{b}\, e^{-\frac{\pi\delta}{2b}}\, |x_\nu|. \tag{5.6.3}$$

We put $b = \frac{\pi}{2}$ and (5.6.1) follows from (5.6.2) and (5.6.3), as

$$T + b + \frac{\pi}{\gamma} < 2\left(T + \frac{\pi}{\gamma}\right)$$

for all T if $b = \frac{\pi}{2}$ and $\gamma \leq 1$.

As above, we get a corresponding formula for $D(z)$ and $R_2(\delta, T)$. Addition gives

$$\sum_{z_\nu \in R(\delta,\, T)} |x_\nu| \leq 2\, e^{\delta}\left(T + \frac{\pi}{\gamma}\right) \log \frac{4}{|a_0\, a_n|}. \tag{5.6.4}$$

5.7. In order to prove theorem 5.1 we note the following two consequences of our lemmas.

$$\sum_{z_\nu \in R(\delta,\, T)} \left[\varphi(1) - \varphi\left(e^{-\frac{\pi|x_\nu|}{2b}}\right)\right] \le \frac{4\pi}{b} e^{\delta} \left(T + \frac{\pi}{\gamma}\right) \log \frac{4}{|a_0 a_n|}. \tag{5.7.1}$$

By (5.3.5)

$$\varphi(1) - \varphi\left(e^{-\frac{\pi|x_\nu|}{2b}}\right) \le 4\left(1 - e^{-\frac{\pi|x_\nu|}{2b}}\right) \le \frac{2\pi|x_\nu|}{b}$$

and (5.7.1) follows by applying (5.6.4).

$$\sum_{z_\nu \in W(\delta,\, T)} \varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) \le \frac{4}{\delta}\left(T + b + \frac{\pi}{\gamma}\right) \log \frac{4}{|a_0 a_n|}. \tag{5.7.2}$$

By (5.3.6)

$$\varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) \le 4\, e^{-\frac{\pi|x_\nu|}{2b}}$$

and (5.7.2) follows from (5.5.5).

5.8. We now turn to the proof of theorem 5.1.

We rewrite (2.2.1) as we have previously done with similar formulas:

$$\log|E(i\eta)| + \sum_{z_\nu - i\eta \in W(b)} \log \left| \frac{e^{\frac{\pi z_\nu}{2b}} + e^{i\frac{\pi\eta}{2b}}}{e^{\frac{\pi z_\nu}{2b}} - e^{-i\frac{\pi\eta}{2b}}} \right| = \tag{5.8.1}$$

$$= \frac{1}{2b} \int_{-\infty}^{\infty} \frac{\log|E(x + i(\eta + b))\, E(x + i(\eta - b))|}{e^{\frac{\pi x}{2b}} + e^{-\frac{\pi x}{2b}}}\, dx.$$

We then integrate with respect to η from $-T-b$ to $T+b$. We have $|E(z)| \le 1$ for $Re(z) \le 0$ and $|D(z)| \le 1$ for $Re(z) \ge 0$. Hence, if we put $E(z) = e^{\lambda_n z} D(z)$ when $Re(z) > 0$,

$$\int_{-T-b}^{T+b} \log|E(i\eta)|\, d\eta + \frac{2b}{\pi} \sum_{z_\nu \in W(T)} \varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) \le \frac{1}{2b} \int_{-T-b}^{T+b} d\eta \int_0^{\infty} \frac{2\lambda_n x}{e^{\frac{\pi x}{2b}} + e^{-\frac{\pi x}{2b}}}\, dx.$$

We apply (5.2.8) to the first integral. In the integral on the right hand side, we put $x = \dfrac{2b}{\pi} \log t$. Thus it is seen that

$$\sum_{z_\nu \in W(T)} \varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) \le \frac{4\lambda_n k}{\pi}(T+b) + \frac{\pi^2}{2b}\left(T+b+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0 a_n|} \tag{5.8.2}$$

where as before

$$k = \int_1^\infty \frac{\log t}{1+t^2}\,dt.$$

If we integrate (5.8.1) from $-T+b$ to $T-b$, we can obtain a lower bound.

$$\frac{2b}{\pi}\sum_{z_\nu \in W(T)} \varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) \ge \frac{1}{2b}\int_{-T+b}^{T-b} d\eta \int_0^\infty \frac{2\lambda_n x}{e^{\frac{\pi x}{2b}} + e^{-\frac{\pi x}{2b}}}\,dx -$$

$$- \frac{1}{2b}\int_{-\infty}^{0} \frac{4\pi\left(T-b+\frac{\pi}{\gamma}\right)\log\frac{2}{|a_0|}}{e^{\frac{\pi x}{2b}} + e^{-\frac{\pi x}{2b}}}\,dx - \frac{1}{2b}\int_0^\infty \frac{4\pi\left(T-b+\frac{\pi}{\gamma}\right)\log\frac{2}{|a_n|}}{e^{\frac{\pi x}{2b}} + e^{-\frac{\pi x}{2b}}}\,dx.$$

We have used (5.2.1) and (5.2.7) to obtain the right member. Further simplifications yield

$$\begin{aligned}\sum_{z_\nu \in W(T)} \varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) &\ge \frac{4\lambda_n k}{\pi}(T-b) - \frac{\pi^2}{2b}\left(T-b+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0 a_n|} \ge \\ &\ge \frac{4\lambda_n k}{\pi}(T-b) - \frac{\pi^2}{2b}\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0 a_n|}.\end{aligned} \tag{5.8.3}$$

$N(\delta, T)$ was defined as the number of zeros of $E(z)$ in $R(\delta, T)$. We will use (5.8.2) and (5.8.3) to obtain bounds for $N(\delta, T)$. Evidently

$$4kN(\delta, T) = \sum_{z_\nu \in R(\delta, T)} \varphi(1) = \sum_{z_\nu \in R(\delta, T)} \varphi\left(e^{\frac{\pi x_\nu}{2b}}\right) + \sum_{z_\nu \in R(\delta, T)} \left[\varphi(1) - \varphi\left(e^{-\frac{\pi |x_\nu|}{2b}}\right)\right].$$

But then the first sum is less than the right member of (5.8.2) and the second sum is less than the right member of (5.7.1). Hence

$$4kN(\delta, T) \le \frac{4\lambda_n k}{\pi}(T+b) + \frac{\pi^2}{2b}\left(T+b+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0 a_n|} + \frac{4\pi}{b}e^\delta\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0 a_n|}$$

or

$$N(\delta, T) - \frac{T}{\pi}\lambda_n \le \frac{\lambda_n}{\pi}b + \frac{3\pi}{2kb}e^\delta\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0 a_n|} + \frac{\pi^2}{8k}\log\frac{4}{|a_0 a_n|}.$$

We choose

$$b=\pi\sqrt{\frac{3\,e^{\delta}\left(T+\dfrac{\pi}{\gamma}\right)}{2\,k\,\lambda_n}\log\frac{4}{|a_0\,a_n|}}$$

and find

$$\frac{N(\delta,T)}{\lambda_n}-\frac{T}{\pi}\le 3\,e^{\frac{\delta}{2}}\sqrt{\frac{T+\dfrac{\pi}{\gamma}}{\lambda_n}\log\frac{4}{|a_0\,a_n|}}+\frac{\pi^2}{8\,k\,\lambda_n}\log\frac{4}{|a_0\,a_n|}. \tag{5.8.4}$$

To find a lower bound for $N(\delta, T)$, we observe that

$$4\,k\,N(\delta,T)=\sum_{z_\nu\in R(\delta,T)}\varphi(1)\ge\sum_{z_\nu\in R(\delta,T)}\varphi\left(e^{\frac{\pi x_\nu}{2b}}\right)\ge\sum_{z_\nu\in W(T)}\varphi\left(e^{\frac{\pi x_\nu}{2b}}\right)-\sum_{z_\nu\in W(\delta,T)}\varphi\left(e^{\frac{\pi x_\nu}{2b}}\right).$$

But then by (5.8.3) and (5.7.2)

$$4\,k\,N(\delta,T)\ge\frac{4\,\lambda_n\,k}{\pi}(T-b)-\frac{\pi^2}{2\,b}\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0\,a_n|}-\frac{4}{\delta}\left(T+b+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0\,a_n|}.$$

In the last term we substitute $T+\dfrac{\pi}{\gamma}$ for b. This substitution is evidently legitimate if $b\le T$, and if $b>T$, the inequality only asserts that $N(\delta, T)$ is not less than a negative number. Thus

$$N(\delta,T)-\frac{T}{\pi}\lambda_n\ge-\frac{\lambda_n}{\pi}b-\frac{\pi^2}{8\,b\,k}\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0\,a_n|}-\frac{2}{\delta\,k}\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0\,a_n|}.$$

We choose

$$b=\pi\sqrt{\frac{\pi\left(T+\dfrac{\pi}{\gamma}\right)}{8\,k\,\lambda_n}\log\frac{4}{|a_0\,a_n|}}$$

and find

$$\frac{N(\delta,T)}{\lambda_n}-\frac{T}{\pi}\ge-1.5\sqrt{\frac{T+\dfrac{\pi}{\gamma}}{\lambda_n}\log\frac{4}{|a_0\,a_n|}}-\frac{2}{\delta\,k\,\lambda_n}\left(T+\frac{\pi}{\gamma}\right)\log\frac{4}{|a_0\,a_n|}. \tag{5.8.5}$$

To remove the restriction imposed by our normalization of $E(z)$, we substitute $\dfrac{Q^2}{|a_0\,a_n|}$ for $\dfrac{1}{|a_0\,a_n|}$. As

$$\frac{Q}{\sqrt{|a_0\,a_n|}}=\frac{|a_0|+|a_1|+\cdots+|a_n|}{\sqrt{|a_0\,a_n|}}\ge 2$$

it must be true that

$$\log\frac{4\,Q^2}{|a_0\,a_n|}\le\log\frac{Q^4}{|a_0\,a_n|^2}=4\log\frac{|a_0|+|a_1|+\cdots+|a_n|}{\sqrt{|a_0\,a_n|}}=4\log S.$$

We put

$$\phi = \frac{\left(T + \frac{\pi}{\gamma}\right) \log S}{\lambda_n}, \tag{5.8.6}$$

note that (5.8.4) and (5.8.5) may be written

$$6\, e^{\frac{\delta}{2}} \sqrt{\phi} + 2\,\phi \geq \frac{N(\delta, T)}{\lambda_n} - \frac{T}{\pi} \geq -3\sqrt{\phi} - \frac{9}{\delta}\phi$$

and theorem 5.1 is proved.

5.9. If the theorem just proved is applied to a sequence of exponential polynomials it gives a sufficient condition for equi-distribution.

Let $\{E_\nu(z)\}$ be a sequence of exponential polynomials of the type considered in 5.1 and suppose that

$$E_\nu(z) = \sum_{\mu=0}^{n_\nu} a_\mu^{(\nu)} e^{\lambda_\mu^{(\nu)} z}, \qquad \gamma_\nu = \operatorname*{Min}_{\mu} (\lambda_{\mu+1}^{(\nu)} - \lambda_\mu^{(\nu)}).$$

Put

$$\phi_\nu = \frac{T + \frac{\pi}{\gamma_\nu}}{\lambda_{n_\nu}^{(\nu)}} \log \frac{|a_0^{(\nu)}| + \cdots + |a_{n_\nu}^{(\nu)}|}{\sqrt{|a_0^{(\nu)} a_{n_\nu}^{(\nu)}|}}$$

and suppose that

$$\lim_{\nu \to \infty} \phi_\nu = 0.$$

If $N_\nu(\delta, T)$ is the number of zeros of $E_\nu(z)$ in $R(\delta, T)$, then

$$\frac{1}{\lambda_{n_\nu}^{(\nu)}} N_\nu(\delta, T) = \frac{T}{\pi} + 0\,(\sqrt{\phi_\nu}).$$

5.10. In this formulation for the exponential case our theorem gives immediately a bound for the difference between $\frac{\theta}{2\pi} n_\nu$ and the number of zeros of the partial sums of a power series, with radius of convergence 1, in a region

$$|\arg z - \theta_0| \leq \frac{\theta}{2}, \quad 1 - \varepsilon \leq R \leq 1 + \varepsilon.$$

On some other types of distributions of zeros

6.1. In a previous section we derived certain conditions for equi-distribution of zeros from formula (2.1.6).

If we want to characterize other types of distributions of zeros, we can proceed in a similar way. Let us consider a sequence of polynomials $\{P_\nu(z)\}_1^\infty$ and let us suppose, for example, that

$$N_\nu(-\varphi, \varphi) = o(n_\nu) \tag{6.1.1}$$

where n_ν is the degree of $P_\nu(z)$ and $N_\nu(-\varphi, \varphi)$ is the number of zeros of $P_\nu(z)$ in the sector $|\arg z| < \varphi$. If (2.1.6) is applied, we find

$$\frac{1}{n_\nu}\int_0^\infty \log \frac{|P_\nu(r e^{i\varphi}) P_\nu(r e^{-i\varphi})|}{|P_\nu(r)|^2} \cdot \frac{dr}{r} + \varphi^2 = \frac{2\pi}{n_\nu} \sum_{|\theta_\nu|<\varphi} (\varphi - |\theta_\nu|).$$

Thus, if the number of zeros of $P_\nu(z)$ in the sector is $o(n_\nu)$, then

$$\lim_{\nu\to\infty} \frac{1}{n_\nu}\int_0^\infty \log \frac{|P_\nu(r e^{i\varphi}) P_\nu(r e^{-i\varphi})|}{|P_\nu(r)|^2} \cdot \frac{dr}{r} = -\varphi^2. \tag{6.1.2}$$

In this section we are going to study the consequences of assumptions analogous to (6.1.1) from a different point of view. In that way we shall obtain results that are more suitable for applications than (6.1.2).

6.2. Let $\{f_\nu(z)\}_1^\infty$ be a sequence of analytic functions which converges uniformly in a certain domain (that is an open connected set of points). If, for a point ζ, there is a neighborhood in which the sequence converges uniformly, we define the domain of uniform convergence containing ζ as the largest domain D which satisfies the following conditions.

A. The sequence is uniformly convergent on every compact sub-set of D.
B. $\zeta \in D$.

If the set of points in which the functions $f_\nu(z)$ take the values a or b $(a \neq b)$ is considered, it is well-known (see for instance [9]) that the derived set contains every boundary point of every domain of uniform convergence. Of course there may be one value a, for which no boundary point is a limit-point of the a-points of the functions. We give the following example of a sequence of polynomials.

Example. Consider the sequence $\{P_\nu(z)\}_3^\infty$ where

$$P_\nu(z) = \left[1 + \frac{1}{\nu}\left(z + \frac{z^2}{2} + \cdots + \frac{z^{[\log \nu]}}{[\log \nu]}\right)\right]^\nu.$$

Evidently

$$\lim_{\nu\to\infty} P_\nu(z) = e^{-\log(1-z)} = \frac{1}{1-z}$$

uniformly on every compact sub-set of $|z| < 1$. For $|z| > 1$, $\lim\limits_{\nu\to\infty} P_\nu(z)$ does not exist. There are no zeros in $|z| \le e(1-\varepsilon)$ if $\varepsilon > 0$ since, in this case,

$$\left|z + \frac{z^2}{2} + \cdots + \frac{z^{[\log \nu]}}{[\log \nu]}\right| \le [\log \nu]\,\frac{(e(1-\varepsilon))^{\log \nu}}{[\log \nu]} = \nu^{1+\log(1-\varepsilon)} < \nu.$$

$\left(\text{We suppose that } \varepsilon < 1 - \frac{2}{e}\cdot\right)$

We do not intend to give additional requirements to ensure that every boundary point of a domain of uniform convergence is a limit-point of the zeros of the functions. Instead we shall try to characterize the limit-points by means other than the uniform convergence of our sequence.

From the original proof of JENTZSCH's theorem on the clustering of the zeros of the polynomial sections of a power series it is seen that the essential fact is not that the circle of convergence is the boundary of the domain of uniform convergence. It is the behaviour of the function

$$\mu(z) = \overline{\lim_{\nu\to\infty}} \, |P_\nu(z)|^{\frac{1}{n_\nu}}$$

which is important.

In JENTZSCH's case obviously $\mu(z) \le 1$, if z is inside the circle of convergence. That $\mu(z) > 1$ outside this circle follows from the relation

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|a_\nu|}\, \varrho = \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|P_\nu(\varrho e^{i\varphi})| - |P_{\nu-1}(\varrho e^{i\varphi})|} \le \mu(\varrho e^{i\varphi}),$$

where $P_\nu(z) = \sum_{\mu=0}^{\nu} a_\mu z^\mu$.

We shall now prove a theorem which is a generalisation of JENTZSCH's theorem. It is related to some recent results communicated without proof by ROSENBLOOM [17]. He derives his results by aid of potential theory and the theory of subharmonic functions. We prefer to give a rather simple proof of our theorem based on the classical theory of analytic functions. If our theorem is combined with well-known theorems concerning the modulus of analytic functions, sharper results can be found than those given by ROSENBLOOM [17, p. 137] for sequences of polynomials.

6.3. Theorem. *Let $\{f_\nu(z)\}_1^\infty$ be a sequence of analytic functions, regular in the closure $\bar{\Omega}$ of a bounded domain Ω and let z_0 be a point of Ω.*

Define λ_ν by

$$\lambda_\nu = \sup_{z\in\bar{\Omega}} \log |f_\nu(z)|.$$

We suppose that

$$\lim_{\nu\to\infty} \lambda_\nu = \infty$$

and that

$$|f_\nu(z_0)| \ge m > 0$$

for all ν.

Suppose that there is a domain, containing z_0, in which

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)| = 0 \tag{6.3.1}$$

and let E be the largest domain such that $z_0 \in E$ and such that (6.3.1) *is true for every $z \in E$.*

Let ζ be a boundary point of E that belongs to Ω. Then, to every neighborhood V of ζ there are a positive number $k(V)$ and a sub-sequence $\{f_{\nu_i}(z)\}_{i=1}^\infty$ so that the number of zeros of $f_{\nu_i}(z)$ in V is not less than $k(V)\cdot\lambda_{\nu_i}$.

This theorem is an immediate consequence of the following two lemmas. Let B, C and D represent any domains with the properties

$$z_0 \in B, \quad \bar{B} \subset C, \quad \bar{C} \subset D, \quad \bar{D} \subset \Omega$$

and let c_ν and d_ν denote the number of zeros of $f_\nu(z)$ in C and D.

Lemma I. *If*

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)| = 0$$

for $z \in D$, *then*

$$\lim_{\nu\to\infty} \frac{c_\nu}{\lambda_\nu} = 0.$$

Lemma II. *If*

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)| = 0$$

for $z \in B$, *and if*

$$\lim_{\nu\to\infty} \frac{d_\nu}{\lambda_\nu} = 0,$$

then

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)| = 0$$

uniformly for $z \in C$.

Proof of lemma I. Let Γ be a sufficiently regular curve in D such that $\bar{C}$ belongs to the interior of Γ. We apply Green's formula (2.1.3) to Γ and $\log |f_\nu(z)|$. As singular point in Green's function we take z_0. If the zeros of $f_\nu(z)$ are called $\{z_\mu^{(\nu)}\}$ we find that

$$\overline{\lim_{\nu\to\infty}} \frac{1}{\lambda_\nu} \sum_{z_\mu^{(\nu)} \in C} G_\Gamma(z_\mu^{(\nu)}) \le \int_\Gamma \left[-\frac{\partial G_\Gamma}{\partial n}\right] \overline{\lim_{\nu\to\infty}} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)|\, ds + \overline{\lim_{\nu\to\infty}} \frac{1}{\lambda_\nu} \log \frac{1}{m} = 0.$$

For $z \in C$ we have

$$G_\Gamma(z) \ge \delta > 0$$

where δ is independent of ν and hence our lemma is proved.

Proof of lemma II. We may evidently assume that $z_0 = 0$. The following simple inequality for polynomials will be used.

If $P(z)$ *is a polynomial of degree* n *and if* $P(0) = 1$, *then every interval* $[a, 2a]$ $(a > 0)$ *contains a real number* r *such that*

$$\operatorname*{Min}_\theta |P(re^{i\theta})| > e^{-9n}. \tag{6.3.2}$$

An inequality of this type follows easily from general theorems on the minimum modulus of analytic functions.[1] We are content with the result just given and sketch a simple proof.

We observe that

$$\operatorname*{Min}_{\theta} |P(\varrho e^{i\theta})| \geq \prod_{\nu=1}^{n} \left|1-\frac{\varrho}{\varrho_\nu}\right|$$

where $\{\varrho_\nu\}_1^n$ are the moduli of the zeros. We write

$$\prod_{\nu=1}^{n} \left|1-\frac{\varrho}{\varrho_\nu}\right| = \prod_{\varrho_\nu<4a} \left|1-\frac{\varrho}{\varrho_\nu}\right| \cdot \prod_{\varrho_\nu\geq 4a} \left|1-\frac{\varrho}{\varrho_\nu}\right| \cdot$$

The second product is certainly $\geq 2^{-n} > e^{-n}$ for every $\varrho \leq 2a$. Since

$$\sum_{\varrho_\nu<4a} \int_a^{2a} \log\left|1-\frac{\varrho}{\varrho_\nu}\right| d\varrho =$$

$$= \sum_\nu \varrho_\nu \int_{\frac{a}{\varrho_\nu}}^{\frac{2a}{\varrho_\nu}} \log|1-t|\,dt \geq \sum_\nu \varrho_\nu \int_0^2 \log|1-t|\,dt \geq -2\sum_\nu \varrho_\nu \geq -8an,$$

there is an r in the given interval such that the first product is not less than e^{-8n}.

To prove lemma II we consider the functions

$$g_\nu(z) = \frac{f_\nu(z)}{f_\nu(0)\,p_\nu(z)} \tag{6.3.3}$$

where

$$p_\nu(z) = \prod_{\mu=1}^{d_\nu} \left(1-\frac{z}{z_\mu^{(\nu)}}\right)$$

and $\{z_\mu^{(\nu)}\}$ are the zeros of $f_\nu(z)$ in D.

Every $g_\nu(z)$ is regular in D and has no zeros there. Let L be a contour in Ω such that the minimum distance from a point in $\bar{D}$ to L is $\eta > 0$. Then

---

[1] If $P(0)=1$, there is a constant $C(a,b)$ such that $(a>0)$

$$\operatorname*{Max}_{a\leq r\leq b} \operatorname*{Min}_{\theta} |P(re^{i\theta})| \geq e^{-n\,C(a,b)}.$$

A study of the equilibrium distribution of the unit mass on (a,b) shows that the best possible value is

$$C(a,b) = \log\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\cdot$$

(6.3.4) $$\operatorname*{Max}_{z\in L} |g_\nu(z)| \leq \frac{e^{\lambda_\nu}}{m} \prod_{\mu=1}^{d_\nu} \frac{|z_\mu^{(\nu)}|}{\eta} \leq \left(\frac{R}{\eta}\right)^{d_\nu} \frac{e^{\lambda_\nu}}{m},$$

if $\operatorname*{Max}_{z\in\Omega} |z| = R.$

The function

$$h_\nu(z) = e^{\frac{1}{\lambda_\nu}\log g_\nu(z)}$$

(we choose that branch of $\log g_\nu(z)$ which is $=0$ for $z=0$) is regular and single-valued in D and by (6.3.4)

$$|h_\nu(z)| = e^{\frac{1}{\lambda_\nu}\log|g_\nu(z)|} \leq \left(\frac{R}{\eta}\right)^{\frac{d_\nu}{\lambda_\nu}} \cdot \frac{e}{m^{\frac{1}{\lambda_\nu}}}$$

for $z\in D$. As

$$\lim_{\nu\to\infty} \frac{d_\nu}{\lambda_\nu} = 0$$

we see that

(6.3.5) $$|h_\nu(z)| \leq M$$

for $z\in D$ where M is independent of ν.

We now choose a so that the circle $|z| = 2a$ belongs to the domain B where

$$\lim_{\nu\to\infty} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)| = 0.$$

Then there is a number r satisfying $a \leq r \leq 2a$ such that

$$|p_\nu(re^{i\theta})| \geq e^{-9d_\nu}$$

according to (6.3.2). Since

$$\log|h_\nu(z)| = \frac{1}{\lambda_\nu}\log|g_\nu(z)| = \frac{1}{\lambda_\nu}\log|f_\nu(z)| - \frac{1}{\lambda_\nu}\log|f_\nu(0)| - \frac{1}{\lambda_\nu}\log|p_\nu(z)|,$$

we infer that

(6.3.6) $$\lim_{\nu\to\infty} \overset{+}{\log} |h_\nu(re^{i\theta}| = 0.$$

The sequence $\{h_\nu(z)\}_1^\infty$ is bounded in D and if $h(z)$ is a limit-function when $\nu\to\infty$, we have

$$h(0) = 1$$

and, according to (6.3.6)

$$|h(re^{i\theta}) \leq 1$$

for all θ.

Thus $h(z) \equiv 1$, and

$$\lim_{\nu \to \infty} \frac{1}{\lambda_\nu} \log g_\nu(z) = 0$$

uniformly in the closed sub-domain $\bar{C}$ of D.

Hence lemma II is proved as

$$0 \leq \frac{\overset{+}{\log} |f_\nu(z)|}{\lambda_\nu} \leq \frac{\overset{+}{\log} |f_\nu(0)|}{\lambda_\nu} + \frac{\overset{+}{\log} |p_\nu(z)|}{\lambda_\nu} + \frac{\overset{+}{\log} |g_\nu(z)|}{\lambda_\nu}$$

and as

$$\log |p_\nu(z)| = \sum_{\mu=1}^{d_\nu} \log \frac{|z - z_\mu^{(\nu)}|}{|z_\mu^{(\nu)}|} < d_\nu \log \frac{2R}{a} + \sum_{|z_\mu^{(\nu)}| < a} \log \frac{a}{|z_\mu^{(\nu)}|} \leq$$

$$\leq d_\nu \log \frac{2R}{a} + \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f_\nu(a e^{i\theta})}{f_\nu(0)} \right| d\theta.$$

Proof of theorem 6.3. Suppose that there is a boundary point ζ_0 of E with a neighborhood V_0 such that the number of zeros of $f_\nu(z)$ in V_0 is $o(\lambda_\nu)$. We consider a domain F such that

$$\zeta_0 \in F, \quad \bar{F} \subset E \cup V_0 \quad \text{and} \quad \bar{F} \subset \Omega.$$

According to lemma I and our assumptions the number of zeros of $f_\nu(z)$ in F is $o(\lambda_\nu)$ and hence it follows from lemma II that

$$\lim_{\nu \to \infty} \frac{1}{\lambda_\nu} \overset{+}{\log} |f_\nu(z)| = 0$$

in every closed sub-domain of F, that is ζ_0 is no boundary point of E. Thus our assumption that the number of zeros in V_0 is $o(\lambda_\nu)$ is false and theorem 6.3 is proved.

6.4. We give five illustrations of theorem 6.3 as applied to polynomials. In the first three λ_ν equals the degree of the ν:th polynomial but the last two show other possibilities.

α. (Szegö, cf. p. 42 below)

$$P_\nu(z) = \frac{1 - z^\nu}{1 - z} \cdot \frac{1 - \left(\frac{z}{2}\right)^\nu}{1 - \frac{z}{2}}.$$

$\{P_\nu(z)\}_1^\infty$ is uniformly convergent on every compact sub-set of $|z| < 1$. If Ω is $|z| < 3$, we have $\lambda_\nu = 2\nu \cdot \log \frac{3}{\sqrt{2}}$. E then is $|z| < 1$.

β. $$P_\nu(z) = 1 - z^\nu (2-z)^\nu$$

The domain of uniform convergence containing $z_0 = 0$ is the left part of the lemniscate $|z(2-z)| < 1$. If Ω is $|z| < 3$, then $\lambda_\nu = \nu \cdot \log 15$. E coincides with the domain of uniform convergence and the zeros are

$$z_\mu^{(\nu)} = 1 \pm e^{i\frac{\theta_\mu^{(\nu)} + 3\pi}{4}} \sqrt{2 \cdot \sin \frac{\theta_\mu^{(\nu)}}{2}}$$

where $\theta_\mu^{(\nu)} = \frac{\mu}{\nu} 2\pi$. $\mu = 1, 2, \ldots \nu$.

γ. $$P_\nu(z) = 1 - e^{i\nu^2\varphi} z^\nu (2 - z e^{i\nu\varphi})^\nu$$

with φ/π irrational.

$\{P_\nu(z)\}_1^\infty$ is uniformly bounded on every closed subset of $|z| < \sqrt{2} - 1$. The boundary of E is $|z| = \sqrt{2} - 1$ if we take the same Ω as in example (β). The zeros are $e^{-i\nu\varphi} z_\mu^{(\nu)}$ if $z_\mu^{(\nu)}$ are the zeros in (β).

δ. $$P_\nu(z) = (1 + z^{[\sqrt{\nu}]}) \left(1 + \left(\frac{z}{3}\right)^\nu\right).$$

If Ω is $|z| < 2$, then $\lambda_\nu = \log 2 \cdot [\sqrt{\nu}]$ and E is given by $|z| < 1$. If Ω is $|z| < 4$, then $\lambda_\nu = \log \frac{4}{3} \cdot \nu$ and E is given by $|z| < 3$.

ε. $$P_\nu(z) = \left(1 - \frac{z}{\sqrt{\nu}}\right)^\nu \left[\left(1 + \frac{z}{\sqrt{\nu}}\right)^\nu + \left(1 - \frac{z}{\sqrt{\nu}}\right)^\nu\right].$$

$\lim\limits_{\nu \to \infty} P_\nu(z) = e^{-z^2}$ if $Re(z) > 0$.

As Ω we take a circle

$$|z - a| < A$$

where a and A are real numbers satisfying $0 < a < A$. We put $z_0 = a$. Then $\lambda_\nu = 2(A - a)\sqrt{\nu}$ and E is bounded by the imaginary axis. A neighborhood of a point on this axis and belonging to Ω contains $C\sqrt{\nu} + o(\sqrt{\nu})$ zeros of $P_\nu(z)$ where C is independent of ν and positive.

6.5. In the proof of our next theorem we use the following theorem of Beurling [1, p. 96].

Let $F(z)$ be holomorphic for $|z| < R$; let $0 \le r_1 < r_2 < R$ and $M(r_1) > \mu > 0$, where $M(r) = \operatorname{Max}\limits_{|z|=r} |F(z)|$. The set in (r_1, r_2) where $\operatorname{Min}\limits_{|z|=r} |F(z)| \le \mu$ is denoted by $E(r_1, r_2)$. Then

$$\log \frac{M(r_2)}{\mu} > \frac{1}{2} e^{\frac{1}{2}\int\limits_E \frac{dr}{r}} \cdot \log \frac{M(r_1)}{\mu}. \tag{6.5.1}$$

This result will enable us to prove the following theorem.

Theorem. *Let* $\{P_\nu(z)\}_1^\infty$ *be a sequence of polynomials and suppose that* $\lim_{\nu\to\infty} n_\nu = \infty$, *if* n_ν *is the degree of* $P_\nu(z)$. *Put* $M_\nu(r) = \operatorname{Max}_{|z|=r} |P_\nu(z)|$. *Suppose that*

$$P_\nu(0) = 1 \tag{6.5.2}$$

for all ν, *and that there are two numbers* r_0 *and* ϱ, $r_0 > \varrho > 0$, *such that*

$$\lim_{\nu\to\infty} \frac{1}{n_\nu} \log M_\nu(\varrho) = 0 \tag{6.5.3}$$

$$\overline{\lim_{\nu\to\infty}} \frac{1}{n_\nu} \log M_\nu(r_0) = \eta > 0. \tag{6.5.4}$$

Then there exists a sub-sequence $\{P_{\nu_i}(z)\}_{i=1}^\infty$ *with the following property: for every domain* S *containing the origin and containing points arbitrarily far from the origin, there is a positive number* $k(S)$ *such that the number of zeros of* $P_{\nu_i}(z)$ *in* S *is not less than* $k(S)\cdot n_{\nu_i}$.

In the proof, we consider a sub-sequence which, for the sake of simplicity, we also denote $\{P_\nu(z)\}$ and for which

$$\lim_{\nu\to\infty} \frac{1}{n_\nu} \log M_\nu(r_0) = \eta. \tag{6.5.5}$$

We shall show that the assumption that the polynomials in this sequence have $o(n_\nu)$ zeros in S leads to a contradiction.

We choose a point $z_1 \in S$ such that

$$|z_1| = R = \varrho \cdot e^{\frac{64}{\eta_0}\sqrt{\frac{r_0}{\varrho}}}$$

where $\eta_0 = \operatorname{Min}(\eta, 32)$. This point can be joined to the origin by a polygonal line in S. We may suppose that this line has no other points in common with the circle $|z| = R$.

There exists a sub-domain S_1 of S such that S_1 contains the line connecting the origin and z_1 and such that $\bar{S}_1 \subset S$. According to lemma II and our assumptions

$$\lim_{\nu\to\infty} \frac{1}{n_\nu} \overset{+}{\log} |P_\nu(z)| = 0$$

uniformly for all $z \in S_1$. We determine ν_1 so that

$$\frac{1}{n_\nu} \log M_\nu(r_0) > \frac{3\eta_0}{4} \tag{6.5.6}$$

for $\nu > \nu_1$, and ν_2 so that

$$\frac{1}{n_\nu} \log |P_\nu(z)| \leq \frac{\eta_0}{4}$$

for $z \in \tilde{S}_1$ and $\nu > \nu_2$. Then

$$\operatorname{Min}_{\theta} |P_\nu(r e^{i\theta})| \le e^{\frac{1}{4}\eta_0 n_\nu}$$

for $0 \le r \le R$. We put $\nu_0 = \operatorname{Max}(\nu_1, \nu_2)$ and apply BEURLING's theorem to $P_\nu(z)$ for $\nu > \nu_0$. Since

$$R = \varrho\, e^{\frac{64}{\eta_0}\sqrt{\frac{r_0}{\varrho}}}$$

and $e^t > \frac{1}{2} t^2$ for $t > 0$, we infer that $R > 2 r_0$. If $\mu = e^{\frac{1}{4}\eta_0 n_\nu}$ we find by (6.5.6) that

$$\log\left(M_\nu(R)\, e^{-\frac{\eta_0 n_\nu}{4}}\right) \ge \frac{1}{2} e^{\frac{1}{2}\int_{r_0}^{R}\frac{dr}{r}} \log\left(M_\nu(r_0)\, e^{-\frac{\eta_0 n_\nu}{4}}\right) > \frac{\eta_0 n_\nu}{4}\left(\frac{R}{r_0}\right)^{\frac{1}{2}}.$$

Hence

$$\overline{\lim_{\nu\to\infty}} \frac{1}{n_\nu} \log M_\nu(R) \ge \frac{\eta_0}{4}\left(\frac{R}{r_0}\right)^{\frac{1}{2}}. \tag{6.5.7}$$

But for every polynomial

$$\left|\frac{P_\nu(z)}{z^{n_\nu}}\right| \le \frac{M_\nu(\varrho)}{\varrho^{n_\nu}}$$

for $|z| \ge \varrho$, and thus according to (6.5.3)

$$\overline{\lim_{\nu\to\infty}} \frac{1}{n_\nu} \log M_\nu(R) \le \log \frac{R}{\varrho}. \tag{6.5.8}$$

The inequalities (6.5.7) and (6.5.8) show the contradiction, since we have chosen R so that

$$\frac{\eta_0}{4}\left(\frac{R}{r_0}\right)^{\frac{1}{2}} = \frac{\eta_0}{4}\left(\frac{\varrho}{r_0}\right)^{\frac{1}{2}} e^{\frac{32}{\eta_0}\left(\frac{r_0}{\varrho}\right)^{\frac{1}{2}}} > \frac{128}{\eta_0}\left(\frac{r_0}{\varrho}\right)^{\frac{1}{2}} = 2\log\frac{R}{\varrho}.$$

Thus our assumption that the number of zeros of $P_\nu(z)$ in S is $o(n_\nu)$ is false and theorem 6.5 is proved.

An immediate consequence of this theorem is the following. If a sequence of polynomials satisfies (6.5.2) and (6.5.3) and if there exists a domain S of the type just considered in which the number of zeros of $P_\nu(z)$ is $o(n_\nu)$, then

$$\overline{\lim_{\nu\to\infty}} \frac{1}{n_\nu} \overset{+}{\log} |P_\nu(z)| = 0$$

for all z, and the number of zeros in a circle $|z| < R$ is $o(n_\nu)$ for every R, as follows from Jensen's theorem (lemma I in 6.3).

6.6. CARLSON's results for the partial sums of entire functions of order ϱ, $0 < \varrho < \infty$, may be derived from theorem 6.5. The following example shows in which way that is done.

Let $\{P_\nu(z)\}_1^\infty$ be the sequence of partial sums of

$$f(z) = \sum_{\mu=0}^{\infty} \left(\frac{z}{\mu}\right)^\mu$$

which is an entire function of order 1.

We put

$$Q_\nu(z) = z^\nu P_\nu\left(\frac{\nu}{z}\right) = \sum_{\mu=0}^{\nu} a_\mu^{(\nu)} z^\mu$$

where

$$a_\mu^{(\nu)} = \left(\frac{\nu}{\nu-\mu}\right)^{\nu-\mu}.$$

Then $a_\mu^{(\nu)} < e^\mu$ and we see that

$$|Q_\nu(z)| \leq \sum_{\mu=0}^{\nu} e^\mu \frac{(1-\varepsilon)^\mu}{e^\mu} \leq \frac{1}{\varepsilon} \tag{6.6.1}$$

for $|z| \leq \dfrac{1-\varepsilon}{e}$.

We have $a_\nu^{(\nu)} = 1$ for all ν and thus

$$\operatorname*{Max}_{|z|=2} |Q_\nu(z)| \geq 2^\nu. \tag{6.6.2}$$

The inequalities (6.6.1) and (6.6.2) show that theorem 6.5 is applicable. As domain S we take the union of a sector with vertex at the origin and a neighborhood of the origin. We find that $Q_\nu(z)$, and hence $P_\nu(z)$, have zeros in every sector with vertex at the origin and we get the estimate, given by CARLSON, for the number of zeros. From (6.6.1) and from Hurwitz's theorem as applied to $P_\nu(z)$, we conclude that if u is fixed, the number N_ν of zeros of $P_\nu(z)$ in the domain

$$u < |z| < \frac{\nu e}{1-\varepsilon}$$

satisfies

$$\lim_{\nu\to\infty} \frac{N_\nu}{\nu} = 1.$$

For estimates by general entire functions which show that theorem 6.5 is applicable, we refer to CARLSON [4, p. 5—7].

6.7. In a paper of 1922 SZEGÖ [21] studied sequences of polynomials which are uniformly convergent in every domain interior to a curve C. Let the sequence be $\{P_\nu(z)\}_1^\infty$, with

$$P_\nu(z) = \sum_{\mu=0}^{\nu} a_\mu^{(\nu)} z^\mu .$$

SZEGÖ proved that, *if*

$$w = \varphi(z) = z + c_0 + \frac{c_1}{z} + \cdots$$

maps the exterior of C *conformally on* $|w| > \gamma$, *then*

$$\overline{\lim} \, |a_\nu^{(\nu)}|^{\frac{1}{\nu}} \le \frac{1}{\gamma}. \tag{6.7.1}$$

If there is equality in (6.7.1), *then every point of* C *is a limit-point of zeros of the polynomials in the sequence.*

If C_1 is a curve in the interior of C and $\varphi_1(z)$ is the corresponding function, mapping the exterior of C_1 on $|w| > \gamma_1$, SZEGÖ considered $P_\nu(z)\,[\varphi_1(z)]^{-\nu}$. Since this function is regular outside C_1, it follows from the maximum-modulus theorem that

$$|a_\nu^{(\nu)}| \le \operatorname*{Max}_{z \in C_1} \left| \frac{P_\nu(z)}{(\varphi(z))^\nu} \right| = \frac{1}{\gamma_1^\nu} \operatorname*{Max}_{z \in C_1} |P_\nu(z)|. \tag{6.7.2}$$

The inequality (6.7.1) follows easily from this relation as we can find curves with γ_1 arbitrarily close to γ. It may also be seen from (6.7.2) that theorem 6.3 implies the second proposition of SZEGÖ. We want to prove that every point of C is a limit-point of the zeros of the polynomials of the sequence. For that purpose we consider a curve C_2 enclosing a point ζ of C but interior to the union of a neighborhood of ζ and the interior of C. The curve C_2 is taken so close to C that $\gamma_2 > \gamma$ (cf. SZEGÖ's paper). Then it follows from (6.7.2) that

$$\operatorname*{Max}_{z \in C_2} \lim_{\nu \to \infty} |P_\nu(z)|^{\frac{1}{\nu}} \ge \frac{\gamma_2}{\gamma} > 1$$

and 6.3 is applicable.

SZEGÖ points out that the second part of his theorem gives a sufficient but not a necessary condition that every point of C should be a limit-point. He gives the example (α) quoted on p. 37. He later showed [22] that *equality in* (6.7.1) *is the necessary and sufficient condition that all but* $o(\nu)$ *of the zeros should be arbitrarily near* C *for large* ν.

6.8. Similar problems for partial sums of power series are treated in a paper by CARLSON [2]. CARLSON also studies the number of zeros of arbitrary subsequences of the partial sums of a power series with finite radius of convergence. We conclude this section by proving certain corresponding results for the partial sums of entire functions.

Suppose that

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad a_0 = 1$$

is an entire function of order ϱ, that is

$$\varliminf_{\nu\to\infty} \frac{\log \frac{1}{|a_\nu|}}{\nu \log \nu} = \frac{1}{\varrho}.$$

We consider a sequence of partial sums

$$P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

where n runs through a sub-sequence $\{n_\nu\}$ of the positive integers, such that $\lim\limits_{\nu\to\infty} n_\nu = \infty$. To simplify our equations, we shall only write $\lim\limits_{n\to\infty}$ and presume that n goes through $\{n_\nu\}$. Then, if $M_n(r) = \max\limits_{|z|=r} |P_n(z)|$,

$$Q_n(\lambda) = \log M_n(n^\lambda) \geq 0 \tag{6.8.1}$$

will be a convex function of λ by the three-circles theorem.

Suppose that $a_{m_n} n^{\lambda m_n}$ is the maximal term in $P_n(n^\lambda)$. Then

$$M_n(n^\lambda) \leq (n+1) \cdot |a_{m_n}| \cdot n^{\lambda m_n}$$

and

$$q_n(\lambda) = \frac{Q_n(\lambda)}{n \log n} \leq \frac{\log(n+1)}{n \log n} + \frac{\log |a_{m_n}|}{n \log n} + \lambda \frac{m_n}{n}. \tag{6.8.2}$$

Now m_n obviously is a non-decreasing function of n and thus we have either $m_n < m$ or $\lim\limits_{n\to\infty} m_n = \infty$.

In the first case

$$\varlimsup_{n\to\infty} q_n(\lambda) \leq 0,$$

and in the second case

$$\varlimsup_{n\to\infty} q_n(\lambda) \leq \varlimsup_{n\to\infty} \left(\frac{\log |a_{m_n}|}{m_n \log m_n} + \lambda \frac{m_n}{n} \right) \leq \lambda - \frac{1}{\varrho}.$$

Hence there exists a convex function $q(\lambda)$ satisfying

$$0 \leq q(\lambda) = \varlimsup_{n\to\infty} q_n(\lambda) \leq \operatorname{Max}\left\{0, \lambda - \frac{1}{\varrho}\right\} \tag{6.8.3}$$

by virtue of (6.8.1).

From the convexity of $q(\lambda)$ we infer that the limit

$$\lim_{\lambda\to\frac{1}{\varrho}} \frac{q(\lambda)}{\lambda - \frac{1}{\varrho}} = q$$

exists and from (6.8.1) and (6.8.3) it is seen that $0 \le q \le 1$. If $N_n(n^\lambda)$ denotes the number of zeros of $P_n(z)$ in $|z| < n^\lambda$, we put

$$\overline{\lim_{n\to\infty}} \frac{N_n(n^\lambda)}{n} = p(\lambda) \le 1$$

and

$$p = p\left(\frac{1}{\varrho}\right).$$

We are going to show that for every sequence $\{n_\nu\}$ $p < q$. By Jensen's formula it is found that

$$0 \le \overline{\lim_{n\to\infty}} \frac{1}{n\log n} \int_0^{n^{\frac{1}{\varrho}}} \frac{N_n(r)}{r}\, dr = \overline{\lim_{n\to\infty}} \frac{1}{2\pi n \log n} \int_0^{2\pi} \log |P_n(n^\lambda e^{i\theta})|\, d\theta \le q\left(\frac{1}{\varrho}\right) = 0,$$

and

$$\overline{\lim_{n\to\infty}} \frac{1}{n\log n} \int_0^{n^{\frac{1}{\varrho}+\varepsilon}} \frac{N_n(r)}{r}\, dr \le q\left(\frac{1}{\varrho}+\varepsilon\right).$$

But

$$\int_{n^{\frac{1}{\varrho}}}^{n^{\frac{1}{\varrho}+\varepsilon}} \frac{N_n(r)}{r}\, dr \ge N_n\left(n^{\frac{1}{\varrho}}\right) \cdot \varepsilon \cdot \log n$$

and thus

$$\overline{\lim_{n\to\infty}} \frac{N_n\left(n^{\frac{1}{\varrho}}\right)}{n} \le \frac{q\left(\frac{1}{\varrho}+\varepsilon\right)}{\varepsilon}$$

for every $\varepsilon > 0$, and hence

$$0 \le p \le q \le 1.$$

A necessary and sufficient condition that $q < 1$ for a sequence of partial sums given by $\{n_\nu\}$ is that, for μ running through the integers given by

$$q' n_\nu < \mu \le n_\nu \tag{6.8.4}$$

where $q < q' < 1$, we have

$$\underline{\lim_{\mu\to\infty}} \frac{\log \frac{1}{|a_\mu|}}{\mu \log \mu} > \frac{1}{\varrho}.$$

The necessity is proved as follows.

For every $\mu \leq n$ we have

$$|a_\mu| n^{\lambda\mu} \leq M_n(n^\lambda)$$

that is

$$\frac{\log |a_\mu|}{n \log n} + \lambda \frac{\mu}{n} \leq q_n(\lambda)$$

and

$$\overline{\lim_{n\to\infty}} \left[\frac{\log |a_\mu|}{\mu \log \mu} \cdot \frac{\mu \log \mu}{n \log n} + \lambda \frac{\mu}{n}\right] \leq q(\lambda).$$

For μ given by (6.8.4) we evidently have

$$q' \cdot \overline{\lim_{\mu\to\infty}} \frac{\log |a_\mu|}{\mu \log \mu} + \lambda q' \leq q(\lambda).$$

Then there exists a $\delta(\varepsilon)$ such that

$$q' \overline{\lim_{\mu\to\infty}} \frac{\log |a_\mu|}{\mu \log \mu} + \lambda q' \leq \left(\lambda - \frac{1}{\varrho}\right)(q+\varepsilon)$$

if

$$\lambda \leq \frac{1}{\varrho} + \delta(\varepsilon)$$

that is

$$\overline{\lim_{\mu\to\infty}} \frac{\log |a_\mu|}{\mu \log \mu} < -\frac{1}{\varrho} - \delta(\varepsilon)\left(1 - \frac{q+\varepsilon}{q'}\right)$$

for μ given by (6.8.4).

But then we need only choose $\delta(\varepsilon)$ so that $q+\varepsilon < q'$ and we find

$$\overline{\lim_{\mu\to\infty}} \frac{\log |a_\mu|}{\mu \log \mu} < -\frac{1}{\varrho}$$

for the μ:s just mentioned.

The sufficiency of the condition follows easily from (6.8.2).

Everywhere convergent sequences of polynomials

7.1. We finally turn to the problem of restricting the position of the zeros of a sequence of polynomials in such a way that the uniform convergence of the sequence in some domain implies its convergence everywhere. We generalize the theorem of Lindwart and Pólya, quoted in the introduction, by weakening the conditions they imposed on the set occupied by the zeros. Generalizations of the theorem in other directions, for instance by weakening the convergence conditions, have been given by Szász [20] and Korevaar [11].

7.2. We formulate the following lemma which we will use in the proof of our theorem.

Lemma. *Let* $g(z)$ *be an analytic function regular in the sector* S_0 *defined by* $0 \leq |z| < r_0$, $|\arg z| \leq \psi_0$. *Suppose that*

$$|g(z)| \leq M$$

for $0 \leq |z| < \gamma$, $|\arg z| \leq \psi_0$, *and that* $(A > 0)$

$$Re(g(z)) \leq A$$

for $z \in S_0$. *Define* S_1 *by* $0 < |z| < r_1 < r_0$, $|\arg z| < \psi_1 < \psi_0$. *Then*

$$|g(z)| \leq C(S_0, S_1, \gamma)(A + M) \tag{7.2.1}$$

for $z \in S_1$, *where* $C(S_0, S_1, \gamma)$ *depends on the geometrical configuration but not on* A *or on* M.

The proof follows from a well-known theorem of Borel and Carathéodory (see for instance [23, p. 174]) by conformal mapping of the sector S_0 on the unit circle.

7.3. Theorem. *Suppose that a sequence of polynomials converges uniformly for* $|z| \leq 1$ *to a limit-function* $\not\equiv 0$ *and suppose that no polynomial has any zeros either in* $Re(z) \geq a$, *or in the sector of angle* ϑ *defined by*

$$-\frac{\pi}{2} \leq \varphi < \arg z < \varphi + \vartheta \leq \frac{\pi}{2}.$$

Then the sequence converges uniformly in every bounded domain, the limit-function being an entire function of order ≤ 2.

If $a < 1$ the theorem follows from that of LINDWART and PÓLYA, but other cases are not covered by that theorem. The first part of our proof coincides with the original proof of LINDWART and PÓLYA.

We call the sector and the half-plane mentioned in the theorem S och H respectively. The polynomials in the sequence are called $P_\nu(z)$ and we may suppose that $P_\nu(0) = 1$.

We choose a point z_0 and a positive number δ such that a circle with center z_0 and radius 2δ lies in the interior of both H and S. Let C denote the circle $|z - z_0| < \delta$. We consider the sequence of polynomials $\{Q_\nu(z)\}_1^\infty$, where

$$Q_\nu(z) = \frac{P_\nu(z_0 + z)}{P_\nu(z_0)}. \tag{7.3.1}$$

If we put

$$P_\nu(z) = \prod_{\mu=1}^{n_\nu} \left(1 - \frac{z}{z_\mu^{(\nu)}}\right),$$

we find

$$Q_\nu(z)=\prod_{\mu=1}^{n_\nu}\frac{1-\dfrac{z_0+z}{z_\mu^{(\nu)}}}{1-\dfrac{z_0}{z_\mu^{(\nu)}}}=\prod_{\mu=1}^{n_\nu}\left(1+\frac{z}{z_0-z_\mu^{(\nu)}}\right). \tag{7.3.2}$$

Hence

$$Q_\nu'(0)=\sum_{\mu=1}^{n_\nu}\frac{1}{z_0-z_\mu^{(\nu)}} \tag{7.3.3}$$

and

$$Re\,(Q_\nu'(0))=\sum_{\mu=1}^{n_\nu}\frac{Re\,(z_0-z_\mu^{(\nu)})}{|z_0-z_\mu^{(\nu)}|^2}\geq 2\,\delta\sum_{\mu=1}^{n_\nu}\frac{1}{|z_0-z_\mu^{(\nu)}|^2}. \tag{7.3.4}$$

To obtain a bound for $|Q_\nu(z)|$ we use the inequality

$$e^{Re\,(z)+\frac{1}{2}|z|^2}\geq|1+z| \tag{7.3.5}$$

which follows as $(z=x+iy)$

$$e^{x^2+y^2+2x}\geq 1+x^2+y^2+2x=(1+x)^2+y^2.$$

If we apply (7.3.5) to each factor in (7.3.2) we find, by use of (7.3.3) and of (7.3.4), that

$$\begin{aligned}\log|Q_\nu(z)|&\leq\sum_{\mu=1}^{n_\nu}\left[Re\left(\frac{z}{z_0-z_\mu^{(\nu)}}\right)+\frac{1}{2}\left|\frac{z}{z_0-z_\mu^{(\nu)}}\right|^2\right]\leq\\ &\leq|z|\left|\sum_{\mu=1}^{n_\nu}\frac{1}{z_0-z_\mu^{(\nu)}}\right|+\frac{|z|^2}{2}\sum_{\mu=1}^{n_\nu}\frac{1}{|z_0-z_\mu^{(\nu)}|^2}\leq|z|\cdot|Q_\nu'(0)|+\frac{|z|^2}{4\,\delta}|Q_\nu'(0)|.\end{aligned} \tag{7.3.6}$$

Now, as (7.3.1) shows,

$$|Q_\nu'(0)|=\left|\frac{1}{2\pi i}\int\limits_{|\zeta|=\delta}\frac{\log Q_\nu(\zeta)}{\zeta^2}\,d\zeta\right|\leq\frac{2}{\delta}\max_{z\in C}|\log P_\nu(z)|. \tag{7.3.7}$$

$\log P_\nu(z)$ denotes that branch of the function which tends to 0 when $z\to 0$, and is a regular function in the domain formed by S and H. But from (7.3.1), it is seen that

$$\log|P_\nu(z_0+z)|=\log|P_\nu(z_0)|+\log|Q_\nu(z)|$$

and application of (7.3.6) and (7.3.7) gives

$$\log|P_\nu(z_0+z)|\leq\left(1+\frac{2|z|}{\delta}+\frac{|z|^2}{2\,\delta^2}\right)\max_{z\in C}|\log P_\nu(z)|.$$

Hence

(7.3.8) $$\log |P_\nu(z)| \le F(|z|) \operatorname*{Max}_{z\in C} |\log P_\nu(z)|$$

where $F(|z|)$ is a quadratic function of $|z|$ whose coefficients depend only on z_0 and δ (and thus cn φ, ϑ and a) but not on ν. (If C is contained in the circle of uniform convergence, (7.3.8) gives an upper bound for $|P_\nu(z)|$ which is independent of ν and the theorem follows with the help of a well-known theorem of Vitali.)

To obtain the same result in our more general case we consider two bounded sectors S_0 and S_1 of the type studied in lemma 7.2 and such that $C \subset S_1 \subset$ $\subset S_0 \subset S$.

Then (7.3.8) tells us that

$$\operatorname*{Max}_{z\in S_0} \log |P_\nu(z)| \le K \cdot \operatorname*{Max}_{z\in C} |\log P_\nu(z)|$$

where K is independent of ν. Hence, from (7.2.1),

(7.3.9) $$\operatorname*{Max}_{z\in S_1} |\log P_\nu(z)| < K \cdot C(S_0, S_1, 1)\,[\operatorname*{Max}_{z\in C} |\log P_\nu(z)| + M],$$

f $|\log P_\nu(z)| \le M$ for $z \in S$ and $0 \le |z| \le 1$.

Since we want to prove that $\{|\log P_\nu(z)|\}_1^\infty$ is uniformly bounded for $z \in C$, it will be sufficient in the sequel to consider thcse $P_\nu(z)$ for which

$$\operatorname*{Max}_{z\in C} |\log P_\nu(z)| \ge M.$$

We then rewrite (7.3.9) as

(7.3.10) $$\operatorname*{Max}_{z\in S_1} |\log P_\nu(z)| \le L \cdot \operatorname*{Max}_{z\in C} |\log P_\nu(z)|.$$

Since C lies in the interior of S_1, it follows from the two-constant theorem that there exists a λ, $0 < \lambda < 1$, such that

(7.3.11] $$\operatorname*{Max}_{z\in C} |\log P_\nu(z)| \le [\operatorname*{Max}_{z\in U\cap S_1} |\log P_\nu(z)|]^\lambda [\operatorname*{Max}_{z\in S_1} |\log P_\nu(z)|]^{1-\lambda}$$

where U is $|z| \le 1$. If (7.3.10) is applied to (7.3.11), we find

$$\operatorname*{Max}_{z\in C} |\log P_\nu(z)| \le M^\lambda L^{1-\lambda} [\operatorname*{Max}_{z\in C} |\log P_\nu(z)|]^{1-\lambda}$$

that is

$$\operatorname*{Max}_{z\in C} |\log P_\nu(z)| \le M \cdot L^{\frac{1-\lambda}{\lambda}}.$$

We have thus found a bound for $\operatorname*{Max}_{z\in C} |\log P_\nu(z)|$ which is independent of ν.

From (7.3.8) we infer that $\{P_\nu(z)\}_1^\infty$ is bounded in every bounded domain and our theorem follows from the theorem of Vitali.

7.4. As the proof of theorem 7.3 indicates, it is not necessary that the zero-free domain, connecting the domain of convergence with the half-plane, be a sector. The assumption that there exists a zero-free half-plane, however, cannot be removed as the example below shows. Other similar examples may be found in a paper by SAXER [18] on sequences of rational functions.

Theorem 7.3, like the theorem of LINDWART and PÓLYA, may be easily extended to cover the case when there are a bounded number of zeros in the domains which we have previously required to be free from zeros.

Example. Suppose that $\omega(x) > 0$ is a function of x satisfying

$$\lim_{x\to\infty} \omega(x) = \infty,$$

$$\lim_{x\to\infty} \omega(x) \cdot x^{-\frac{1}{4}} = 0.$$

We consider the sequence of polynomials $\{P_\nu(z)\}_1^\infty$, where

$$P_\nu(z) = \left(1 - \frac{z\,\omega(\nu)}{\nu - i\sqrt{\nu}\,\omega(\nu)} + \frac{z^2}{\nu - i\sqrt{\nu}\,\omega(\nu)}\right)^\nu \cdot$$

$$\cdot \left[\left(1 - \frac{z\,\omega(\nu)}{\nu + [\omega(\nu)]^2} + \frac{z^2}{\nu + [\omega(\nu)]^2}\right)^\nu + \left(1 + \frac{z\,\omega(\nu)}{\nu + [\omega(\nu)]^2} + \frac{z^2}{\nu + [\omega(\nu)]^2}\right)^\nu\right].$$

The zeros of $P_\nu(z)$ are all purely imaginary with the exception of a zero of multiplicity ν at $z = \omega(\nu) + i\sqrt{\nu}$.

The sequence converges to e^{2z^2} if $Re(z) > 0$ but it is not convergent if $Re(z) < 0$. If we translate the origin to $z = 1$, we have an example of a sequence which is uniformly convergent in a circle around the origin and which does not converge to an entire function, although its zeros are situated in a domain that may, by suitable choice of the function $\omega(x)$, be made to approximate arbitrarily closely the domain in theorem 7.3.

These problems are related to one treated by KOREVAAR [11]. According to him, a set R is regular, if every entire function, all zeros of which belong to $\bar{R}$, is the uniform limit, in every bounded region, of a sequence of polynomials which have zeros only in R. He characterizes such regular sets. A set of the type considered in our example is regular in the sense of KOREVAAR, but that of the kind considered in theorem 7.3, is, of course, not regular.

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Tryckt den 2 oktober 1953

Uppsala 1953. Almqvist & Wiksells Boktryckeri AB