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The Serre spectral sequence of a noncommutative fibration for de Rham cohomology

by

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1. Introduction

This paper has three basic purposes:

(1) Developing a cohomology theory for modules with flat connections over noncommutative algebras, and showing that it has some properties in common with sheaf theory.

(2) Extending the Serre spectral sequence of a fibration in classical algebraic topology to the noncommutative domain.

(3) Examining the differential structure of quantum homogeneous spaces, and showing that many of them are 'fibrations' in a noncommutative sense.

In [9] methods of studying algebras by means of their differential calculi were introduced. We will apply Connes' differential methods to fibrations in algebraic topology.

In usual topology, sheaf cohomology and other methods allow cohomology with 'twisted' coefficients, i.e. coefficients which vary from point to point in the space. In the absence (so far at least) of a full sheaf cohomology construction in noncommutative geometry, we construct de Rham cohomology with twisted coefficients for algebras with differential structure. The allowed coefficients are modules with flat connection. Though there is a considerable similarity between de Rham cohomology with twisted coefficients in the noncommutative world and sheaf cohomology in the commutative world, it is quite possible that yet more general constructions, or constructions with additional properties, corresponding to sheaf theory exist in the noncommutative world. In the spirit of some developments in operator algebra (for example, see [10]), we show that bimodules can be used to replace algebra maps in constructing 'pullbacks' of the coefficient modules. In the special case of semi-free differential graded algebras this construction is shown to have an interesting interpretation in terms of *corings*.

In commutative algebraic topology, one of the most useful applications of twisted coefficients is to fibrations. For a locally trivial fibration, the Serre spectral sequence [14] starts with the cohomology of the base space with coefficients in the cohomology of the fibre (in general a twisted bundle), and converges to the cohomology of the total space. In producing a noncommutative analogue of this result, we not only have to find a proof which does not require local triviality, but also have to decide what a 'locally trivial' fibration should be in noncommutative differential geometry. Realistically we should define a fibration by the conditions which are required by the Serre spectral sequence. The seeming correspondence between sheaf theory and the cohomology we are considering leads us to suspect that yet more general 'Leray-type' spectral sequences exist.

We then discuss products in the Serre spectral sequence, which requires another condition to be imposed on the fibration. The product structure is not only important in its own right, but can frequently help in simplifying the calculation of spectral sequences.

Finally, we study fibrations given in terms of a coaction of a Hopf algebra on an algebra. As a nontrivial example of such a differential fibration we consider the quantum Hopf fibration $\iota: \mathcal{A}(S_q^2) \hookrightarrow \mathcal{A}(SL_q(2))$ with the 3-dimensional differential calculus on $\mathcal{A}(SL_q(2))$. As a further nontrivial class of examples of the fibrations discussed here, we look at the noncommutative homogeneous space construction with bicovariant differential calculi. This takes the classical construction of a group quotiented by a subgroup, and replaces it by two Hopf algebras with a surjective Hopf algebra map $\pi: X \to H$. We begin with such a π which is differentiable with respect to bicovariant differential structures on X and H [24]. Note that the bicovariant condition corresponds to the differentiability of the coproduct map, and it is reasonable to expect that this is the analogue of classical Lie groups. As in the classical case, some of the definitions can be given in terms of the Hopf–Lie algebras and their induced vector fields [1], [2]. The first stage is to identify the differential calculus for the homogeneous space $B = X^{\mathrm{co}\,H}$ (see Theorem 9.12) in a form suitable for calculation. Then it is shown that the inclusion map $B \to X$ is a fibration as defined earlier (see Theorem 10.5). For the development of noncommutative homogeneous spaces the reader should refer to [11] and [16]. Again the quantum Hopf fibration $\iota: \mathcal{A}(S_q^2) \hookrightarrow \mathcal{A}(SL_q(2))$ is an example of this situation, and we explicitly prove that it is a differentiable fibration for one of two standard 4-dimensional bicovariant calculi on $\mathcal{A}(SL_q(2))$.

All algebras are unital over a field k. The unadorned tensor product between vector spaces is over k. A Hopf algebra is always assumed to have a bijective antipode (this is not the most general situation algebraically, but the most natural from the point of view of noncommutative geometry).

2. Flat connections and cohomology with twisted coefficients

The classical Serre spectral sequence uses cohomology with a nontrivial coefficient bundle. In this section we discuss flat connections on modules in noncommutative geometry, and how this can be used to define de Rham cohomology with nontrivial coefficient modules.

By a *differential calculus* on a noncommutative algebra A we mean a differential graded algebra (d, Ω^*A) such that $\Omega^0 A = A$. The product in Ω^*A (for $*\geqslant 1$) is denoted by the wedge $\wedge$ (although Ω^*A is not graded anticommutative in general). The density condition says that $\Omega^{n+1}A \subset A \cdot d\Omega^n A$, but we will not require this till later.

The cohomology of (d, Ω^*A) is denoted by $H^*_{\mathrm{dR}}(A)$ and referred to as a *de Rham cohomology* of A. Recall that a *connection* in a left A-module E is a map $\nabla: E \to \Omega^1 A \otimes_A E$ satisfying the Leibniz rule, for all $a \in A$, $e \in E$, $\nabla(a \cdot e) = da \otimes e + a\nabla e$.

2.1. The construction of the cohomology

Definition 2.1. Given an algebra A with differential calculus (d, Ω^*A), we define the category ${}_A\mathcal{E}$ to consist of left A-modules E with connection $\nabla: E \to \Omega^1 A \otimes_A E$. A morphism $\phi: (E, \nabla) \to (F, \nabla)$ in the category is a left A-module map $\phi: E \to F$ which preserves the covariant derivative, i.e. $\nabla \circ \phi = (\mathrm{id} \otimes \phi) \circ \nabla: E \to \Omega^1 A \otimes_A F$.

Definition 2.2. Given $(E, \nabla) \in {}_A\mathcal{E}$, define

$$\begin{aligned} \nabla^{[n]}: \Omega^n A \otimes_A E &\longrightarrow \Omega^{n+1} A \otimes_A E, \\ \omega \otimes e &\longmapsto d\omega \otimes e + (-1)^n \omega \wedge \nabla e. \end{aligned}$$

Then the *curvature* is defined as $R = \nabla^{[1]}\nabla: E \to \Omega^2 A \otimes_A E$, and is a left A-module map.

The covariant derivative is called *flat* if the curvature is zero. We write ${}_A\mathcal{F}$ for the full subcategory of ${}_A\mathcal{E}$ consisting of left A-modules with flat connections.

PROPOSITION 2.3. *For all* $n \geqslant 0$, $\nabla^{[n+1]} \circ \nabla^{[n]} = \mathrm{id} \wedge R: \Omega^n A \otimes_A E \to \Omega^{n+2} A \otimes_A E$.

Proof. By explicit calculation,

$$\nabla^{[n+1]}(\nabla^{[n]}(\omega \otimes e)) = \nabla^{[n+1]}(d\omega \otimes e + (-1)^n \omega \wedge \nabla e).$$

Put $\nabla e = \xi_i \otimes e_i$ (summation implicit), and then

$$\begin{aligned} \nabla^{[n+1]}(\nabla^{[n]}(\omega \otimes e)) &= \nabla^{[n+1]}(d\omega \otimes e + (-1)^n \omega \wedge \xi_i \otimes e_i) \\ &= (-1)^{n+1} d\omega \wedge \nabla e + (-1)^n d\omega \wedge \xi_i \otimes e_i + \omega \wedge d\xi_i \otimes e_i - \omega \wedge \xi_i \wedge \nabla e_i \\ &= \omega \wedge (d\xi_i \otimes e_i - \xi_i \wedge \nabla e_i) \\ &= \omega \wedge R(e). \end{aligned}$$

□

Definition 2.4. Given $(E,\nabla)\in{}_A\mathcal{F}$, define $H^*(A;E,\nabla)$ to be the cohomology of the complex

$$E\xrightarrow{\nabla}\Omega^1A\otimes_A E\xrightarrow{\nabla^{[1]}}\Omega^2A\otimes_A E\xrightarrow{\nabla^{[2]}}\dots.$$

Note that $H^0(E,\nabla)=\Gamma E=\{e\in E:\nabla e=0\}$, the flat sections of E. We will often write $H^*(A;E)$, where there is no danger of confusing the covariant derivative.

PROPOSITION 2.5. *Given* $(E,\nabla)\in{}_A\mathcal{F}$, *the map*

$$\wedge:\Omega^nA\otimes(\Omega^rA\otimes_A E)\longrightarrow\Omega^{n+r}A\otimes_A E$$

defined by

$$\wedge(\xi\otimes(\omega\otimes e))=(\xi\wedge\omega)\otimes e$$

gives a graded left $H_{\mathrm{dR}}(A)$*-module structure on* $H^*(A;E,\nabla)$.

Proof. First calculate

$$\begin{aligned}\nabla^{[*]}(\xi\wedge(\omega\otimes e))&=\nabla^{[*]}((\xi\wedge\omega)\otimes e)\\&=d(\xi\wedge\omega)\otimes e+(-1)^{|\xi|+|\omega|}\xi\wedge\omega\wedge\nabla e\\&=d\xi\wedge(\omega\otimes e)+(-1)^{|\xi|}\xi\wedge\nabla^{[*]}(\omega\otimes e).\end{aligned}$$

This equation has the required immediate consequences:

If $d\xi=0$ and $\nabla^{[*]}(\omega\otimes e)=0$ then $\nabla^{[*]}(\xi\wedge(\omega\otimes e))=0$.

If $\nabla^{[*]}(\omega\otimes e)=0$ then $d\xi\wedge(\omega\otimes e)$ is in the image of $\nabla^{[*]}$.

If $d\xi=0$ then $\xi\wedge\nabla^{[*]}(\omega\otimes e)$ is in the image of $\nabla^{[*]}$. □

2.2. Mapping properties of the cohomology

In classical topology, maps on the cohomology can be induced by maps which change coefficients over the same topological space. Our analogue of this is the following theorem:

THEOREM 2.6. *The cohomology* H^* *in Definition* 2.4 *is a functor from* ${}_A\mathcal{F}$ *to graded left* $H^*_{\mathrm{dR}}(A)$*-modules, where the module structure is given in Proposition* 2.5.

Proof. Begin with a left A-module map $\phi:E\to F$ which preserves the covariant derivative, i.e. $\nabla\circ\phi=(\mathrm{id}\otimes\phi)\circ\nabla:E\to\Omega^1A\otimes_A F$. First show that the map $\mathrm{id}\otimes\phi:\Omega^*A\otimes_A E\to\Omega^*A\otimes_A F$ is a cochain map:

$$\begin{aligned}\nabla^{[*]}(\mathrm{id}\otimes\phi)(\omega\otimes e)&=\nabla^{[*]}(\omega\otimes\phi(e))\\&=d\omega\otimes\phi(e)+(-1)^{|\omega|}\omega\wedge\nabla\phi(e)\\&=d\omega\otimes\phi(e)+(-1)^{|\omega|}\omega\wedge(\mathrm{id}\otimes\phi)\nabla e\\&=(\mathrm{id}\otimes\phi)\nabla^{[*]}(\omega\otimes e).\end{aligned}$$

The functorial property is simply $(\mathrm{id}\otimes\phi)\circ(\mathrm{id}\otimes\psi)=\mathrm{id}\otimes(\phi\circ\psi)$, and the left module property is just $\xi\wedge(\omega\otimes\phi(e))=(\xi\wedge\omega)\otimes\phi(e)$. ☐

In classical topology, continuous functions between topological spaces also induce maps on the cohomology. One part of this is the pull-back construction for coefficients. Given the reversal of arrows which often occurs in considering algebras rather than spaces, this becomes a 'push-forward' construction in noncommutative geometry.

This would be an appropriate time to remind the reader that for algebras A and B with differentiable structure, an algebra map $\theta: A\to B$ is called differentiable if it extends to a map $\theta_*: \Omega^* A\to\Omega^* B$ of differential graded algebras.

LEMMA 2.7. *Given $(E,\nabla)\in{}_A\mathcal{E}$ and a differentiable algebra map $\theta: A\to B$, define*

$$\widehat{\nabla}: B\otimes_A E\longrightarrow\Omega^1 B\otimes_B B\otimes_A E=\Omega^1 B\otimes_A E,\quad \widehat{\nabla}(b\otimes e)=b\cdot(\theta_*\otimes\mathrm{id})(\nabla e)+db\otimes e.$$

Then $\theta_*(E,\nabla)=(B\otimes_A E,\widehat{\nabla})\in{}_B\mathcal{E}$, with right action of A on B given by $b\triangleleft a=b\theta(a)$.

Proof. To check that $\widehat{\nabla}$ is well defined, we must show that, for all $a\in A$, $b\in B$ and $e\in E$, $\widehat{\nabla}(b\theta(a)\otimes e)=\widehat{\nabla}(b\otimes ae)$:

$$\begin{aligned}\widehat{\nabla}(b\theta(a)\otimes e)&=b\theta(a)\cdot(\theta_*\otimes\mathrm{id})(\nabla e)+d(b\theta(a))\otimes e\\&=b\cdot(\theta_*\otimes\mathrm{id})(a\nabla e)+db\cdot\theta(a)\otimes e+b\cdot d\theta(a)\otimes e\\&=b\cdot(\theta_*\otimes\mathrm{id})(a\nabla e+da\otimes e)+db\otimes ae\\&=b\cdot(\theta_*\otimes\mathrm{id})\nabla(ae)+db\otimes ae\\&=\widehat{\nabla}(b\otimes ae).\end{aligned}$$

That $\widehat{\nabla}$ satisfies the Leibniz rule follows immediately from the definition (and the Leibniz rule for d). ☐

PROPOSITION 2.8. *If $\theta: A\to B$ is a differentiable algebra map and $(E,\nabla)\in{}_A\mathcal{F}$, then $\theta_*(E,\nabla)\in{}_B\mathcal{F}$.*

Proof. Following the notation of Lemma 2.7 and setting $\nabla e=\xi_i\otimes e_i$ (summation implied),

$$\begin{aligned}\widehat{\nabla}^{[1]}\widehat{\nabla}(b\otimes e)&=\widehat{\nabla}^{[1]}(b\cdot(\theta_*\otimes\mathrm{id})(\nabla e)+db\otimes e)\\&=\widehat{\nabla}^{[1]}(b\cdot\xi_i\otimes e_i+db\otimes e)\\&=d(b\cdot\xi_i)\otimes e_i+ddb\otimes e-b\cdot\xi_i\wedge\nabla e_i-db\wedge\nabla e\\&=db\wedge\xi_i\otimes e_i+b\cdot d\xi_i\otimes e_i-b\cdot\xi_i\wedge\nabla e_i-db\wedge\nabla e\\&=b\cdot(d\xi_i\otimes e_i-\xi_i\wedge\nabla e_i)\\&=b\cdot\nabla^{[1]}\nabla e\\&=0.\end{aligned}$$

☐

THEOREM 2.9. *For a differentiable algebra map* $\theta\colon A\to B$, *there is a functor* $\theta_*\colon {}_A\mathcal{E}\to {}_B\mathcal{E}$ *which is defined on objects as in Lemma* 2.7, *and where a morphism* $\phi\colon E\to F$ *is sent to the morphism* $\mathrm{id}\otimes\phi\colon B\otimes_A E\to B\otimes_A F$. *Further this functor restricts to a functor from* ${}_A\mathcal{F}$ *to* ${}_B\mathcal{F}$.

Proof. First, given a morphism $\phi\colon E\to F$ in ${}_A\mathcal{E}$, we need to show that $\mathrm{id}\otimes\phi\colon B\otimes_A E\to B\otimes_A F$ is a morphism in ${}_B\mathcal{E}$. Using the definition of $\widehat{\nabla}$ in Lemma 2.7,

$$\widehat{\nabla}(b\otimes\phi(e)) = b\cdot(\theta_*\otimes\mathrm{id})\nabla\phi(e)+db\otimes\phi(e),$$

and as ϕ is a morphism in ${}_A\mathcal{E}$,

$$\begin{aligned}\widehat{\nabla}(b\otimes\phi(e)) &= b\cdot(\theta_*\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\nabla e+db\otimes\phi(e)\\ &= (\mathrm{id}\otimes\phi)(b\cdot(\theta_*\otimes\mathrm{id})\nabla e+db\otimes e)\\ &= (\mathrm{id}\otimes\phi)\widehat{\nabla}(b\otimes e).\end{aligned}$$

The composition rule is just $(\mathrm{id}\otimes\phi)\circ(\mathrm{id}\otimes\psi)=\mathrm{id}\otimes(\phi\circ\psi)$. The restriction to flat connections is shown in Proposition 2.8. □

2.3. Generalised mapping properties

The mapping constructions can be generalised to bimodules rather than algebra maps, using the 'braiding' introduced by Madore [15].

Definition 2.10. A (B,A)-bimodule $M\in {}_B\mathcal{M}_A$ with additional structures

(a) a left B-connection $\check{\nabla}\colon M\to\Omega^1B\otimes_B M$;

(b) a (B,A)-bimodule map $\check{\sigma}\colon M\otimes_A\Omega^1A\to\Omega^1B\otimes_B M$

is called a *differentiable bimodule* if it satisfies the condition $\check{\nabla}(m\cdot a)=\check{\nabla}(m)\cdot a+\check{\sigma}(m\otimes da)$ for all $m\in M$ and $a\in A$.

Example 2.11. If $\theta\colon A\to B$ is a differentiable algebra map, take the bimodule $B\in {}_B\mathcal{M}_A$, with the usual left B-action, and right A-action given by $b\triangleleft a=b\theta(a)$. Also define $\check{\nabla}\colon B\to\Omega^1B\otimes_B B=\Omega^1B$ by $\check{\nabla}b=db$ and $\check{\sigma}\colon B\otimes_A\Omega^1A\to\Omega^1B\otimes_B B=\Omega^1B$ by $\check{\sigma}(b\otimes\xi)=b\cdot\theta_*(\xi)$. Now we check the condition

$$\check{\nabla}(b\triangleleft a)=\check{\nabla}(b\theta(a))=d(b\theta(a))=db\theta(a)+b\theta_*(da)=\check{\nabla}(b)\cdot a+\check{\sigma}(b\otimes da).$$

Hence B is a differentiable bimodule.

PROPOSITION 2.12. *Suppose that* $(M,\breve{\nabla},\breve{\sigma})$ *is a differentiable* (B,A)*-bimodule. Then the following defines a functor* $(M,\breve{\nabla},\breve{\sigma})_*:{}_A\mathcal{E}\to{}_B\mathcal{E}$:

On objects $(E,\nabla)\in{}_A\mathcal{E}$, *define* $(M,\breve{\nabla},\breve{\sigma})_*(E,\nabla)=(M\otimes_A E,\widehat{\nabla})$, *where*

$$\widehat{\nabla}(m\otimes e)=\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)$$

On morphisms $\phi:E\to F$, *define* $(M,\breve{\nabla},\breve{\sigma})_*\phi=\mathrm{id}\otimes\phi: M\otimes_A E\to M\otimes_A F$.

Proof. First we need to check that $\widehat{\nabla}$ is a well-defined function on $M\otimes_A E$:

$$\begin{aligned}\widehat{\nabla}(m\otimes ae)&=\breve{\nabla}m\otimes ae+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla(ae))\\&=(\breve{\nabla}m)\cdot a\otimes ae+(\breve{\sigma}\otimes\mathrm{id})(m\otimes a\nabla e)+(\breve{\sigma}\otimes\mathrm{id})(m\otimes da\otimes e).\end{aligned}$$

By using the differentiable bimodule condition this becomes

$$\widehat{\nabla}(m\otimes ae)=\breve{\nabla}(m\cdot a)\otimes ae+(\breve{\sigma}\otimes\mathrm{id})(m\cdot a\otimes\nabla e)=\widehat{\nabla}(m\cdot a\otimes e).$$

To check that $\widehat{\nabla}$ is a left-B-covariant derivative, as $\breve{\sigma}$ is a left B-module map,

$$\begin{aligned}\widehat{\nabla}(b\cdot m\otimes e)&=\breve{\nabla}(b\cdot m)\otimes e+(\breve{\sigma}\otimes\mathrm{id})(b\cdot m\otimes\nabla e)\\&=b\cdot\breve{\nabla}(m)\otimes e+db\otimes m\otimes e+b\cdot(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\&=b\cdot\widehat{\nabla}(m\otimes e)+db\otimes m\otimes e.\end{aligned}$$

Next we check the morphism condition:

$$\begin{aligned}\widehat{\nabla}(m\otimes\phi(e))&=\breve{\nabla}m\otimes\phi(e)+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla\phi(e))\\&=\breve{\nabla}m\otimes\phi(e)+(\breve{\sigma}\otimes\phi)(m\otimes\nabla e)\\&=(\mathrm{id}\otimes\mathrm{id}\otimes\phi)\widehat{\nabla}(m\otimes e).\end{aligned}$$ □

Definition 2.13. The differentiable (B,A)-bimodule $(M,\breve{\nabla},\breve{\sigma})$ is said to be *flat* if $\breve{\sigma}$ induces a (B,A)-bimodule map $\breve{\sigma}: M\otimes_A\Omega^2A\to\Omega^2B\otimes_B M$ so that the following conditions are satisfied:

(a) as a left B-connection on M, $\breve{\nabla}$ is flat;

(b) $(\mathrm{id}\wedge\breve{\sigma})(\breve{\sigma}\otimes\mathrm{id})=\breve{\sigma}(\mathrm{id}\otimes\wedge): M\otimes_A\Omega^1A\otimes_A\Omega^1A\to\Omega^2B\otimes_B M$.

For the rest of this subsection we assume the density condition for Ω^1A.

LEMMA 2.14. *If the differentiable* (B,A)*-bimodule* $(M,\breve{\nabla},\breve{\sigma})$ *is flat, then the following map vanishes*:

$$[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})]\breve{\sigma}-(\mathrm{id}\wedge\breve{\sigma})(\breve{\nabla}\otimes\mathrm{id})-\breve{\sigma}(\mathrm{id}\otimes d): M\otimes_A\Omega^1A\to\Omega^2B\otimes_B M.$$

Proof. First note that the displayed formula is well defined, as for all $m\in M$, $\eta\in\Omega^1B$ and $b\in B$,

$$[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})](\eta b\otimes m)=[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})](\eta\otimes bm).$$

Since Ω^1A satisfies the density condition, to prove the vanishing of the displayed formula, we now only have to apply it to elements of the form $m\otimes da$, and use the differentiable bimodule condition on $\breve{\sigma}$:

$$\begin{aligned}[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})]\breve{\sigma}(m\otimes da)&=[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})](\breve{\nabla}(m\cdot a)-(\breve{\nabla}m)\cdot a)\\&=[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})]\,\breve{\nabla}(m\cdot a)\\&\qquad-[(d\otimes\mathrm{id})-(\mathrm{id}\wedge\breve{\nabla})]\,\breve{\nabla}(m)\cdot a\\&\qquad+(\mathrm{id}\wedge\breve{\sigma})(\breve{\nabla}m\otimes da)\end{aligned}$$

and

$$[(\mathrm{id}\wedge\breve{\sigma})(\breve{\nabla}\otimes\mathrm{id})+\breve{\sigma}(\mathrm{id}\otimes d)](m\otimes da)=(\mathrm{id}\wedge\breve{\sigma})(\breve{\nabla}m\otimes da).$$

This means that the displayed formula applied to $m\otimes da$ gives $R(m\cdot a)-R(m)\cdot a$, where R is the curvature of the left B-connection on M, and this vanishes by Definition 2.13. □

PROPOSITION 2.15. *If the differentiable (B,A)-bimodule $(M,\breve{\nabla},\breve{\sigma})$ is flat, then the functor $(M,\breve{\nabla},\breve{\sigma})_*\colon {}_A\mathcal{E}\to{}_B\mathcal{E}$ restricts to a functor from ${}_A\mathcal{F}$ to ${}_B\mathcal{F}$.*

Proof. We need to show that the following expression vanishes, where E is a left A-module with flat connection ∇, and $e\in E$:

$$\begin{aligned}\widehat{\nabla}^{[1]}\widehat{\nabla}(m\otimes e)&=\widehat{\nabla}^{[1]}(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e))\\&=(d\otimes\mathrm{id}\otimes\mathrm{id})(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e))\\&\qquad-(\mathrm{id}\wedge\widehat{\nabla})(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e))\\&=(d\otimes\mathrm{id}\otimes\mathrm{id})(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e))\\&\qquad-(\mathrm{id}\wedge\breve{\nabla}\otimes\mathrm{id})(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e))\\&\qquad-(\mathrm{id}\wedge\breve{\sigma}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla)(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e))\\&=(d\otimes\mathrm{id}\otimes\mathrm{id})(\breve{\nabla}m\otimes e)+(d\otimes\mathrm{id}\otimes\mathrm{id})(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\&\qquad-(\mathrm{id}\wedge\breve{\nabla}\otimes\mathrm{id})(\breve{\nabla}m\otimes e)-(\mathrm{id}\wedge\breve{\nabla}\otimes\mathrm{id})(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\&\qquad-(\mathrm{id}\wedge\breve{\sigma}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla)(\breve{\nabla}m\otimes e+(\breve{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)).\end{aligned}$$

As the left-B-covariant derivative $\check{\nabla}$ on M is flat, the first and third terms cancel, giving

$$\begin{aligned}\widehat{\nabla}^{[1]}\widehat{\nabla}(m\otimes e) &= (d\otimes\mathrm{id}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)-(\mathrm{id}\wedge\check{\nabla}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla)(\check{\nabla}m\otimes e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla)(\check{\sigma}\otimes\mathrm{id}\otimes\mathrm{id})(m\otimes\nabla e)\\ &= (d\otimes\mathrm{id}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)-(\mathrm{id}\wedge\check{\nabla}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\check{\nabla}\otimes\mathrm{id}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla)(m\otimes\nabla e).\end{aligned}$$

Using property (b) of Definition 2.13 this becomes

$$\begin{aligned}\widehat{\nabla}^{[1]}\widehat{\nabla}(m\otimes e) &= (d\otimes\mathrm{id}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)-(\mathrm{id}\wedge\check{\nabla}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\check{\nabla}\otimes\mathrm{id}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\check{\sigma}\otimes\mathrm{id})(\mathrm{id}\otimes\wedge\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\nabla)(m\otimes\nabla e)\\ &= (d\otimes\mathrm{id}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)-(\mathrm{id}\wedge\check{\nabla}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\check{\nabla}\otimes\mathrm{id}\otimes\mathrm{id})(m\otimes\nabla e)-(\check{\sigma}\otimes\mathrm{id})(m\otimes(\mathrm{id}\wedge\nabla)\nabla e)\\ &= (d\otimes\mathrm{id}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)-(\mathrm{id}\wedge\check{\nabla}\otimes\mathrm{id})(\check{\sigma}\otimes\mathrm{id})(m\otimes\nabla e)\\ &\quad -(\mathrm{id}\wedge\check{\sigma}\otimes\mathrm{id})(\check{\nabla}\otimes\mathrm{id}\otimes\mathrm{id})(m\otimes\nabla e)-(\check{\sigma}\otimes\mathrm{id})(\mathrm{id}\otimes d\otimes\mathrm{id})(m\otimes\nabla e),\end{aligned}$$

where we have used the flatness of ∇ on E in the last equality. Now Lemma 2.14 completes the proof. □

2.4. The bicategory of differentiable bimodules

A possible way of understanding differentiable bimodules and induced functors between categories of connections is to construct a suitable *bicategory*. Recall that a bicategory [3] consists of three layers of structures: 0-cells, 1-cells defined for any pair of 0-cells, and 2-cells defined for each pair of 1-cells. There are two types of composition: the horizontal composition of 1-cells which is unital and associative up to isomorphisms and the vertical composition of 2-cells which is strictly associative and unital. The following gathers all the data that constitute a bicategory relevant to differential bimodules.

Definition 2.16. The bicategory DiffBim of *differentiable bimodules* contains the following data:

(a) 0-cells are differential graded algebras (Ω^*A, d); we write A for the zero-degree subalgebra of Ω^*A.

(b) A 1-cell $\Omega^*A \to \Omega^*B$ is given by a differentiable bimodule (M, ∇_M, σ_M), i.e. M is a (B, A)-bimodule, $\nabla_M : M \to \Omega^1 B \otimes_B M$ is a left B-connection and $\sigma_M : M \otimes_A \Omega^1 A \to \Omega^1 B \otimes_B M$ is a generalised flip satisfying the conditions of Definition 2.10.

(c) A 2-cell

$$\begin{tikzcd}
\Omega^*A \arrow[rr, "{(M,\nabla_M,\sigma_M)}"] & {} \arrow[d, Rightarrow, "\phi"] & \Omega^*B \\
\Omega^*A \arrow[rr, "{(N,\nabla_N,\sigma_N)}"'] & {} & \Omega^*B
\end{tikzcd}$$

is given as a (B, A)-bimodule map $\phi : M \to N$ that commutes with covariant derivatives and generalised flip operators, i.e. such that $\nabla_N \circ \phi = (\mathrm{id} \otimes \phi) \circ \nabla_M$ and $\sigma_N \circ (\phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \phi) \circ \sigma_M$.

The horizontal composition

$$\Omega^*A \xrightarrow{(M,\nabla_M,\sigma_M)} \Omega^*B \xrightarrow{(N,\nabla_N,\sigma_N)} \Omega^*C$$

is defined as a differentiable (C, A)-bimodule $(N \otimes_B M, \nabla_{N \otimes_B M}, \sigma_{N \otimes_B M})$, where

$$\nabla_{N \otimes_B M} = \nabla_N \otimes \mathrm{id} + (\sigma_N \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nabla_M) \quad \text{and} \quad \sigma_{N \otimes_B M} = (\sigma_N \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_M).$$

The vertical composition is the usual composition of mappings. The category of 1-cells $\Omega^*A \to \Omega^*B$ with morphisms provided by 2-cells is denoted by $\mathsf{DiffBim}(\Omega^*A, \Omega^*B)$.

It is left to the reader to check that the data collected in Definition 2.16 indeed constitute a bicategory. Essentially this requires similar computations to those in the proof of Proposition 2.12. The bicategory DiffBim contains all (left) connections in the following way:

LEMMA 2.17. *View k as a trivial differential graded algebra with the differential given by the zero map. Then*

$$\mathsf{DiffBim}(k, \Omega^*A) \equiv {}_A\mathcal{E}.$$

Proof. Since $\Omega^1 k = 0$, every generalised flip σ must be a zero map, and thus an object in the category $\mathsf{DiffBim}(k, \Omega^*A)$ is a left A-module M with a left A-connection $\nabla_M : M \to \Omega^1 A \otimes_A M$. As to the morphisms $\phi : M \to N$ in $\mathsf{DiffBim}(k, \Omega^*A)$, the commutativity with flips is trivially satisfied (as flips are zero maps), and hence only the condition $\nabla_N \circ \phi = (\mathrm{id} \otimes \phi) \circ \nabla_M$ remains. This is equivalent to saying that ϕ is a morphism in ${}_A\mathcal{E}$. ☐

In view of Lemma 2.17, the functor $(M, \nabla_M, \sigma_M)_* : {}_A\mathcal{E} \to {}_B\mathcal{E}$ constructed in Proposition 2.12 has a very simple and natural bicategorical explanation. Given a connection $(E,\nabla_E)\in {}_A\mathcal{E}\equiv \mathsf{DiffBim}(k,\Omega^*A)$ and a differentiable bimodule $(M,\nabla_M,\sigma_M)\in \mathsf{DiffBim}(\Omega^*A,\Omega^*B)$ one can construct a differentiable bimodule in $\mathsf{DiffBim}(k,\Omega^*B)\equiv {}_B\mathcal{E}$ as the horizontal composition of 1-cells

$$k \xrightarrow{(E,\nabla_E)} \Omega^*A \xrightarrow{(M,\nabla_M,\sigma_M)} \Omega^*B.$$

By the functoriality of the horizontal composition, this results in a functor ${}_A\mathcal{E}\to {}_B\mathcal{E}$ described in Proposition 2.12.

In a similar way one constructs a bicategory $\mathsf{FlatDiffBim}$ of *flat differentiable bimodules* with differential graded algebras (Ω^*A,d) such that Ω^1A satisfies the density condition as 0-cells, the 1-cells are given as flat differentiable bimodules $(M,\nabla_M,\sigma_M^1,\sigma_M^2)$, where σ_M^1 and σ_M^2 are flip operators of order one and two (cf. Definition 2.13), and the 2-cells are (B,A)-bimodule maps commuting with ∇_M, σ_M^1 and σ_M^2. The horizontal composition is given by

$$\begin{aligned}\nabla_{N\otimes_B M} &= \nabla_N\otimes \mathrm{id}+(\sigma_N^1\otimes \mathrm{id})\circ(\mathrm{id}\otimes\nabla_M),\\ \sigma^i_{N\otimes_B M} &= (\sigma_N^i\otimes\mathrm{id})\circ(\mathrm{id}\otimes\sigma_M^i), \quad i=1,2,\end{aligned}$$

and the vertical composition is the usual composition of mappings. One easily shows that ${}_A\mathcal{F}\equiv \mathsf{FlatDiffBim}(k,\Omega^*A)$ and then identifies the functor in Proposition 2.15 as the horizontal composition of 1-cells in $\mathsf{FlatDiffBim}$.

2.5. The case of semi-free differential graded algebras

Recall that Ω^*A is said to be *semi-free* if and only if Ω^*A is isomorphic to the tensor algebra of the A-bimodule Ω^1A. As observed in [19] there is a bijective correspondence between semi-free differential graded algebras over A and A-corings with a group-like element (cf. [7, §29.8]). The constructions in §§2.1–2.3 have very natural interpretations in terms of such corings and comodules. For more information on corings and comodules we refer to [7].

Starting with an A-coring $\mathfrak{C}$ and a group-like element $g\in\mathfrak{C}$, we define $\Omega^1A=\ker\varepsilon_{\mathfrak{C}}$, where $\varepsilon_{\mathfrak{C}}:\mathfrak{C}\to A$ is the counit of $\mathfrak{C}$. The differential is then defined by $d(a)=ga-ag$, for all $a\in A$, and, for all $c^1\otimes\ldots\otimes c^n\in(\ker\varepsilon_{\mathfrak{C}})^{\otimes_A n}$,

$$\begin{aligned}d(c^1\otimes\ldots\otimes c^n) &= g\otimes c^1\otimes\ldots\otimes c^n+(-1)^{n+1}c^1\otimes\ldots\otimes c^n\otimes g\\ &\quad+\sum_{i=1}^{n}(-1)^i c^1\otimes\ldots\otimes c^{i-1}\otimes\Delta_{\mathfrak{C}}(c^i)\otimes c^{i+1}\otimes\ldots\otimes c^n,\end{aligned}$$

where $\Delta_{\mathfrak{C}}: \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ is the coproduct in $\mathfrak{C}$. The *density condition* for $\Omega^1 A$, i.e. the requirement that any 1-form is a linear combination of $a\,da'$, is equivalent to the requirement that the map $A \otimes A \to \mathfrak{C}$, $a \otimes a' \mapsto aga'$, be surjective (note the similarity with the definition of a *space cover* in [12]).

Let E be a left A-module. As explained in [7, §29.11], connections

$$\nabla: E \longrightarrow \Omega^1 A \otimes_A E = \ker \varepsilon_{\mathfrak{C}} \otimes_A E$$

are in bijective correspondence with left A-module sections of $\varepsilon_{\mathfrak{C}} \otimes \mathrm{id}: \mathfrak{C} \otimes_A E \to E$, i.e. left A-linear maps $\varrho^E: E \to \mathfrak{C} \otimes_A E$ such that $(\varepsilon_{\mathfrak{C}} \otimes \mathrm{id}) \circ \varrho^E = \mathrm{id}$. Furthermore, flat connections are in bijective correspondence with left $\mathfrak{C}$-coactions in E. This correspondence, explicitly given by

$$\varrho^E(e) = g \otimes e - \nabla(e), \quad \text{for all } e \in E,$$

establishes an isomorphism between the categories of flat connections on A and left $\mathfrak{C}$-comodules.

Let $\mathfrak{C}$ be an A-coring with a group-like element $g_{\mathfrak{C}}$, and $\mathfrak{D}$ be a B-coring with a group-like element $g_{\mathfrak{D}}$. Recall that a morphism of corings consists of an algebra map $\theta_0: A \to B$ and an A-bimodule map $\theta_1: \mathfrak{C} \to \mathfrak{D}$ that respects the coproducts and counits (cf. [7, §24.1] for more details). Any morphism of corings (θ_0, θ_1) such that $\theta_1(g_{\mathfrak{C}}) = g_{\mathfrak{D}}$ is a differentiable algebra map. Incidentally, such a morphism of corings is termed a *morphism of space covers* in [12]. Let $\nabla: E \to \Omega^1 A \otimes_A E$ be a connection, and $\varrho^E: E \to \mathfrak{C} \otimes_A E$ be the corresponding section of $\varepsilon_{\mathfrak{C}} \otimes \mathrm{id}$. Then the section $\varrho^{B \otimes_A E}: E \to \mathfrak{D} \otimes_B E$ of $\varepsilon_{\mathfrak{D}} \otimes \mathrm{id}$ corresponding to the induced connection in $B \otimes_A E$ comes out as

$$\varrho^{B \otimes_A E}(b \otimes e) = b\theta_1(e_{[-1]}) \otimes e_{[0]},$$

where $\varrho^E(e) = e_{[-1]} \otimes e_{[0]}$ (summation implicitly understood). In view of the isomorphism ${}_A\mathcal{F} \cong {}^{\mathfrak{C}}\mathcal{M}$, the corresponding functor between the categories of flat connections described in Theorem 2.9 can be identified with the induction functor between categories of left comodules (cf. [7, §24.6]).

For differential graded algebras corresponding to an A-coring $\mathfrak{C}$ with a group-like element $g_{\mathfrak{C}}$ and a B-coring $\mathfrak{D}$ with a group-like element $g_{\mathfrak{D}}$, differentiable bimodules (M, ∇, σ) are in bijective correspondence with pairs (M, Φ), where M is a (B, A)-bimodule and $\Phi: M \otimes_A \mathfrak{C} \to \mathfrak{D} \otimes_B M$ is a (B, A)-bimodule map rendering commutative the following diagram:

$$\begin{tikzcd}
M \otimes_A \mathfrak{C} \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{id} \otimes \varepsilon_{\mathfrak{C}}"'] & & \mathfrak{D} \otimes_B M \arrow[dl, "\varepsilon_{\mathfrak{D}} \otimes \mathrm{id}"] \\
& M &
\end{tikzcd} \tag{2.1}$$

Furthermore, differentiable flat bimodules (M, ∇, σ) are in bijective correspondence with pairs (M, Φ) such that in addition to (2.1) also the diagram

$$\begin{tikzcd}
M\otimes_A \mathfrak{C} \arrow[rr, "\Phi"] \arrow[d, "\mathrm{id}\otimes\Delta_{\mathfrak{C}}"'] & & \mathfrak{D}\otimes_B M \arrow[d, "\Delta_{\mathfrak{D}}\otimes\mathrm{id}"] \\
M\otimes_A \mathfrak{C}\otimes_A \mathfrak{C} \arrow[dr, "\Phi\otimes\mathrm{id}"'] & & \mathfrak{D}\otimes_B \mathfrak{D}\otimes_B M \\
& \mathfrak{D}\otimes_B M\otimes_A \mathfrak{C} \arrow[ur, "\mathfrak{D}\otimes\Phi"'] &
\end{tikzcd} \tag{2.2}$$

is commutative. The correspondence is given by $\sigma = \Phi|_{M\otimes_A \ker\varepsilon_{\mathfrak{C}}}$ and

$$\Phi(m\otimes c) = g_{\mathfrak{D}}\otimes m\varepsilon_{\mathfrak{C}}(c) - \nabla(m)\varepsilon_{\mathfrak{C}}(c) + \sigma(m\otimes(c - g_{\mathfrak{C}}\varepsilon_{\mathfrak{C}}(c)))$$

for all $m\in M$ and $c\in\mathfrak{C}$. An interesting point to note here is that the map Φ is well defined, i.e. factors through the coequaliser defining $M\otimes_A\mathfrak{C}$, thanks to the last condition in Definition 2.10 (the compatibility between the connection and σ).

A pair (M, Φ) satisfying conditions (2.1) and (2.2) constitutes a 1-cell in the *left bicategory of corings* LEM(Bim) defined in [5] as the bicategory of comonads in the bicategory Bim of rings and bimodules following the general procedure in [21] and [13]. In view of the discussion in §2.4 and the present section, LEM(Bim) can be understood as a subbicategory of DiffBim.

3. The long exact sequence

Consider a short exact sequence $0\to E\to F\to G\to 0$ in ${}_A\mathcal{F}$, and suppose that the modules $\Omega^* A$ are flat (i.e. tensoring with them preserves exactness). We assume these conditions for the remainder of the section. From this we form the following diagram, where the rows are exact, and the columns form cochain complexes (i.e. the vertical maps compose to give zero):

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "\phi"] \arrow[d, "\nabla"] & F \arrow[r, "\psi"] \arrow[d, "\nabla"] & G \arrow[r] \arrow[d, "\nabla"] & 0 \\
0 \arrow[r] & \Omega^1 A\otimes_A E \arrow[r, "\mathrm{id}\otimes\phi"] \arrow[d, "\nabla^{[1]}"] & \Omega^1 A\otimes_A F \arrow[r, "\mathrm{id}\otimes\psi"] \arrow[d, "\nabla^{[1]}"] & \Omega^1 A\otimes_A G \arrow[r] \arrow[d, "\nabla^{[1]}"] & 0 \\
0 \arrow[r] & \Omega^2 A\otimes_A E \arrow[r, "\mathrm{id}\otimes\phi"] \arrow[d, "\nabla^{[2]}"] & \Omega^2 A\otimes_A F \arrow[r, "\mathrm{id}\otimes\psi"] \arrow[d, "\nabla^{[2]}"] & \Omega^2 A\otimes_A G \arrow[r] \arrow[d, "\nabla^{[2]}"] & 0 \\
& & & &
\end{tikzcd}$$

What follows is standard homological algebra, but not all readers may be familiar with it. Note that for (e.g.) $\psi: F\to G$ we write $\psi^{-1}(g)$ for $g\in G$ to mean a choice of $f\in F$ for which $\psi(f)=g$. It will turn out that the maps eventually defined by using such potentially multivalued maps will turn out to be unique, and we have no wish to introduce the complication of topologised cochain complexes, and so have no need to worry about the continuity of the resulting operations. It is merely notation used to try to clarify the definitions and proofs. Again take $\Gamma E=\{e\in E:\nabla e=0\}$.

PROPOSITION 3.1. *The sequence* $0\to\Gamma E\to\Gamma F\to\Gamma G$ *is exact.*

Proof. It is immediate that $\phi:\Gamma E\to\Gamma F$ is one-to-one, and that the composition $\Gamma E\to\Gamma F\to\Gamma G$ is zero. To show that $\Gamma E\to\Gamma F\to\Gamma G$ is exact, take $f\in\Gamma F$ with $\psi(f)=0$. As $E\to F\to G$ is exact, there is an $e\in E$ with $\phi(e)=f$. By following the top left commutative square in the diagram and using the fact that $\mathrm{id}\otimes\phi:\Omega^1A\otimes_A E\to\Omega^1A\otimes_A F$ is one-to-one, we see that $\nabla e=0$. □

PROPOSITION 3.2. *The (multivalued) map*

$$(\mathrm{id}\otimes\phi)^{-1}\nabla\psi^{-1}:\Gamma G\longrightarrow\Omega^1A\otimes_A E$$

quotients to a well-defined connecting map $\Gamma G\to H^1(A;E)$.

Proof. Begin with $g\in\Gamma G$, and take an $f\in F$ with $\psi(f)=g$. By using the top right commutative square in the diagram, $\nabla f\in\ker(\mathrm{id}\otimes\psi:\Omega^1A\otimes_A F\to\Omega^1A\otimes_A G)$. Then by the exactness of the rows, there is an $x\in\Omega^1A\otimes_A E$ with $(\mathrm{id}\otimes\phi)(x)=\nabla f$. By exactness of the second row, to show that $e'\to x\in\ker\nabla^{[1]}$ we only have to show that $\nabla^{[1]}(\mathrm{id}\otimes\phi)(x)=0$, i.e. that $\nabla^{[1]}\nabla f=0$, which is true. Then $[x]\in H^1(A:E)$, but now we ask if it is unique.

Suppose that we have $f'\in F$ with $\psi(f')=g$, and $x'\in\Omega^1A\otimes_A E$ with $(\mathrm{id}\otimes\phi)(x')=\nabla f'$. Then $f'-f=\phi(e)$ for some $e\in E$, and $(\mathrm{id}\otimes\phi)(x'-x)=\nabla(f'-f)=\nabla\phi(e)=(\mathrm{id}\otimes\phi)(\nabla e)$. As $\mathrm{id}\otimes\phi$ is one-to-one we deduce that $x'-x=\nabla e$. □

Remark 3.3. As this is not a text on homological algebra, we will now merely quote the result of continuing with the methods outlined: Given the conditions at the beginning of this section, there is a long exact sequence

$$\begin{aligned}H^0(A,E)\longrightarrow H^0(A,F)\longrightarrow H^0(A,G)\longrightarrow H^1(A,E)&\\ \longrightarrow H^1(A,F)\longrightarrow H^1(A,G)\longrightarrow H^2(A,E)\longrightarrow\dots\ .&\end{aligned}$$

4. Noncommutative fibre bundles

We consider a possible meaning for a differentiable algebra map $\iota\colon B\to X$ to be a 'fibration' with 'base algebra' B and 'total algebra' X. From here we will require that the differential calculi satisfy the density condition.

Definition 4.1. Define the cochain complexes

$$\Xi^0_m X=\iota_*\Omega^m B\cdot X \quad\text{and}\quad \Xi^n_m X=\frac{\iota_*\Omega^m B\wedge\Omega^n X}{\iota_*\Omega^{m+1}B\wedge\Omega^{n-1}X}\quad (n>0).$$

with differential $d\colon \Xi^n_m X\to\Xi^{n+1}_m X$ defined by $d[\omega]_m=[d\omega]_m$, where $\omega\in\iota_*\Omega^m B\wedge\Omega^n X$ and $[\cdot]_m$ is the corresponding quotient map.

The maps $\Theta_m\colon\Omega^m B\otimes_B\Xi^n_0 X\to\Xi^n_m X$ defined by $\Theta_m(\omega\otimes[\xi]_0)=[\iota_*\omega\wedge\xi]_m$ are cochain maps if $\Omega^m B\otimes_B\Xi^*_0 X$ is given the differential $(-1)^m\mathrm{id}\otimes d$.

Remark 4.2. To see that the differential in Definition 4.1 is well defined, note that for all $m,n\geqslant 0$, d maps $\iota_*\Omega^m B\wedge\Omega^n X$ into $\iota_*\Omega^m B\wedge\Omega^{n+1}X$. This is because $d\Omega^m B\subset\Omega^{m+1}B\subset\Omega^m B\wedge\Omega^1 B$ (note the use of the density condition here).

There is a left B-module structure for $\Xi^n_m X$ given by $b\cdot\xi=\iota(b)\xi$. As $d(\iota(b)\cdot\theta)=\iota_*(db)\wedge\theta+\iota(b)\cdot d\theta$, we see that $d\colon\Xi^n_m X\to\Xi^{n+1}_m X$ is a left B-module map, so the cohomology $H^n(\Xi^*_m X)$ inherits a left B-module structure.

In this degree of generality, this construction might be merely curious, but consider an example:

Example 4.3. Let $X=B\otimes F$, where F is an algebra with differential structure, and give X the tensor product differential structure. By definition, $\iota(b)=b\otimes 1$ and

$$\Omega^n X=(\Omega^0 B\otimes\Omega^n F)\oplus\ldots\oplus(\Omega^n B\otimes\Omega^0 F),$$

so there is an isomorphism of cochain complexes $B\otimes\Omega^n F\to\Xi^n_0 X$ given by $b\otimes\xi\mapsto\iota(b)\xi$. It follows that $H^n(\Xi^*_0 X)$ is just $B\otimes H^n_{\mathrm{dR}}(F)$, the fibre cohomology module. Also this module has a flat left B-connection $\nabla\colon B\otimes H^*_{\mathrm{dR}}(F)\to\Omega^1 B\otimes_B B\otimes H^*_{\mathrm{dR}}(F)$ given by $\nabla(b\otimes x)=db\otimes 1\otimes x$. The de Rham cohomology of B with coefficients in this module with flat connection is $H^*_{\mathrm{dR}}(B)\otimes H^*_{\mathrm{dR}}(F)$, which by the Künneth theorem is just the cohomology of $X=B\otimes F$.

In topology fibrations can be built from open covers of the base space, and a trivial fibration over each open set. Our example has just dealt with what would be a noncommutative trivial fibration, so we might ask what a more general noncommutative fibration would look like. By analogy we might consider $\Xi^n_0 X$ to be the 'vertical' or 'fibre' forms, and its cohomology to be the cohomology of the 'fibre' of the map. In the topological case, this cohomology can form a nontrivial bundle over the base space. We have seen that for noncommutative de Rham cohomology it is reasonable to have coefficient bundles with flat connection, and this is the route that we will take for our version of a fibration.

PROPOSITION 4.4. *Suppose that* $\Theta_1\colon\Omega^1B\otimes_B\Xi_0^*X\to\Xi_1^*X$ (*as defined in Definition* 4.1) *is invertible. Then there is a left-B-covariant derivative*

$$\nabla\colon H^n(\Xi_0^*X)\longrightarrow\Omega^1B\otimes_B H^n(\Xi_0^*X)$$

defined by $[\omega]\mapsto(\mathrm{id}\otimes[\,\cdot\,])\Theta_1^{-1}[d\omega]_1$.

Proof. If $[\omega]_0\in Z^n=\ker(d\colon\Xi_0^nX\to\Xi_0^{n+1}X)$, then $d\omega\in\iota_*\Omega^1B\wedge\Omega^nX$. Thus $[d\omega]_1\in\Xi_1^nX$ is a cocycle, so $(\mathrm{id}\otimes d)\Theta_1^{-1}[d\omega]_1=0\in\Omega^1B\otimes_B\Xi_0^{n+1}X$, i.e. $\Theta_1^{-1}[d\omega]_1\in\Omega^1B\otimes_B Z^n$.

Now suppose that $[\omega']_0=[\omega]_0\in Z^n$. Then $\omega'-\omega\in\iota_*\Omega^1B\wedge\Omega^{n-1}X$, so we get

$$\Theta_1^{-1}[\omega'-\omega]_1\in\Omega^1B\otimes_B\Xi_0^{n-1}X.$$

As Θ^{-1} is a cochain map, $-(\mathrm{id}\otimes d)\Theta_1^{-1}[\omega'-\omega]_1=\Theta_1^{-1}[d\omega'-d\omega]_1=\Theta_1^{-1}[d\omega']_1-\Theta_1^{-1}[d\omega]_1$. Thus

$$\Theta_1^{-1}[d\omega']_1-\Theta_1^{-1}[d\omega]_1\in\Omega^1B\otimes_B d\Xi_0^{n-1}X,$$

so we get a well-defined map $Z^n\to\Omega^1B\otimes_B H^n(\Xi_0^*X)$.

To finish showing that ∇ is well defined, we show that $d\Xi_0^nX$ maps to zero, which we see as $\nabla[d\xi]=(\mathrm{id}\otimes[\,\cdot\,])\Theta_1^{-1}[d^2\xi]_1=0$.

Finally we need to show the left connection condition:

$$\nabla[\iota(b)\cdot\omega]=(\mathrm{id}\otimes[\,\cdot\,])\Theta_1^{-1}[\iota(db)\wedge\omega+\iota(b)\cdot d\omega]=\iota(b)\wedge[\omega]+b\cdot\nabla[\omega].\qquad\square$$

PROPOSITION 4.5. *Suppose that* $\Theta_m\colon\Omega^mB\otimes_B\Xi_0^*X\to\Xi_m^*X$ (*as defined in Definition* 4.1) *is invertible for* $m=1,2$. *Then the curvature of the connection on* $H^n(\Xi_0^*X)$ *described in Proposition* 4.4 *is zero.*

Proof. Take $[\omega]_0\in Z^n=\ker(d\colon\Xi_0^nX\to\Xi_0^{n+1}X)$, and write

$$\Theta_1^{-1}[d\omega]_1=\sum_i\xi_i\otimes[\eta_i]_0\in\Omega^1B\otimes_B Z^n.$$

Likewise write $\Theta_1^{-1}[d\eta_i]_1=\sum_j\chi_{ij}\otimes[\mu_{ij}]_0\in\Omega^1B\otimes_B Z^n$. Now write the composition $\nabla^{[1]}\nabla$ as

$$[\omega]\longmapsto\sum_i\xi_i\otimes[\eta_i]\longmapsto\sum_i\Big(d\xi_i\otimes[\eta_i]-\sum_j\xi_i\wedge\chi_{ij}\otimes[\mu_{ij}]\Big).$$

Inserting the definition of Θ^{-1}, we get $[d\omega]_1=\sum_i[\iota_*\xi_i\wedge\eta_i]_1$ and $[d\eta_i]_1=\sum_j[\iota_*\chi_{ij}\wedge\mu_{ij}]_1$. This means that

$$d\omega-\sum_i\iota_*\xi_i\wedge\eta_i\in\iota_*\Omega^2B\wedge\Omega^{n-1}X.$$

so we write

$$d\omega-\sum_i\iota_*\xi_i\wedge\eta_i=\sum_k\iota_*\tau_k\wedge\lambda_k.$$

where $\tau_k\in\Omega^2B$ and $\lambda_k\in\Omega^{n-1}X$. Applying d to this, we get

$$\sum_k(\iota_*d\tau_k\wedge\lambda_k+\iota_*\tau_k\wedge d\lambda_k)=\sum_i(\iota_*\xi_i\wedge d\eta_i-\iota_*d\xi_i\wedge\eta_i).$$

Then we obtain $\sum_k[\iota_*\tau_k\wedge d\lambda_k]_2=\sum_{i,j}[\iota_*(\xi_i\wedge\chi_{ij})\wedge\mu_{ij}]_2-\sum_i[\iota_*(d\xi_i)\wedge\eta_i]_2$. Thus the two elements $\sum_k\tau_k\otimes[d\lambda_k]_0$ and $\sum_{i,j}\xi_i\wedge\chi_{ij}\otimes[\mu_{ij}]_0-\sum_i d\xi_i\otimes[\eta_i]_0$ of $\Omega^2B\otimes_B\Xi_0^{n-1}$ map to the same thing under Θ_2, so by our assumption they must be equal. Now as $[d\lambda_k]_0$ is a coboundary, the curvature must vanish. ☐

5. Spectral sequences

The reader should refer to [14] for the details of the homological algebra used to construct the spectral sequence. We will merely quote the results.

Remark 5.1. Start with a differential graded module C^n (for $n\geqslant0$) and $d\colon C^n\to C^{n+1}$ with $d^2=0$. Suppose that C has a filtration $F^mC\subset C=\bigoplus_{n\geqslant0}C^n$ for $m\geqslant0$ so that

(1) $dF^mC\subset F^mC$ for all $m\geqslant0$ (i.e. the filtration is preserved by d);
(2) $F^{m+1}C\subset F^mC$ for all $m\geqslant0$ (i.e. the filtration is decreasing);
(3) $F^0C=C$ and $F^mC^n=F^mC\cap C^n=\{0\}$ for all $m>n$ (a boundedness condition).

Then there is a spectral sequence $(E_r^{*,*},d_r)$ for $r\geqslant1$ with d_r of bidegree $(r,1-r)$ and

$$E_1^{p,q}=H^{p+q}(F^pC/F^{p+1}C)=\frac{\ker(d\colon F^pC^{p+q}/F^{p+1}C^{p+q}\to F^pC^{p+q+1}/F^{p+1}C^{p+q+1})}{\operatorname{im}(d\colon F^pC^{p+q-1}/F^{p+1}C^{p+q-1}\to F^pC^{p+q}/F^{p+1}C^{p+q})}.$$

In more detail, we define

$$\begin{aligned}Z_r^{p,q}&=F^pC^{p+q}\cap d^{-1}(F^{p+r}C^{p+q+1}),\\ B_r^{p,q}&=F^pC^{p+q}\cap d(F^{p-r}C^{p+q-1}),\\ E_r^{p,q}&=Z_r^{p,q}/(Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}).\end{aligned}$$

The differential

$$d_r\colon E_r^{p,q}\longrightarrow E_r^{p+r,q-r+1}$$

is the map induced on quotienting $d\colon Z_r^{p,q}\to Z_r^{p+r,q-r+1}$.

The spectral sequence converges to $H^*(C,d)$ in the sense that

$$E_\infty^{p,q}\cong\frac{F^pH^{p+q}(C,d)}{F^{p+1}H^{p+q}(C,d)},$$

where $F^pH^*(C,d)$ is the image of the map $H^*(F^pC,d)\to H^*(C,d)$ induced by the inclusion $F^pC\to C$.

Now take the case of a differentiable algebra map $\iota\colon B\to X$. We can give the following example of a spectral sequence:

Remark 5.2. Define the filtration $F^m\Omega^{n+m}X=\iota_*\Omega^mB\wedge\Omega^nX$ of Ω^*X. This obeys conditions (1) and (2) of Remark 5.1 as

$$\iota_*\Omega^{m+1}B\wedge\Omega^nX\subset\iota_*\Omega^mB\wedge\iota_*\Omega^1B\wedge\Omega^nX\subset\iota_*\Omega^mB\wedge\Omega^{n+1}X.$$

We have boundedness as $\iota_*\Omega^0B\wedge\Omega^nX=\Omega^nX$, and by convention, $\Omega^nX=0$ for $n<0$. Note that

$$\frac{F^p\Omega^{p+q}X}{F^{p+1}\Omega^{p+q}X}=\Xi_p^qX,$$

and we obtain a spectral sequence with $E_1^{p,q}\cong H^q(\Xi_p^*X)$ which converges to $H_{\mathrm{dR}}^*(X)$ in the sense described in Remark 5.1. The differential $d_1\colon H^q(\Xi_p^*X)\to H^q(\Xi_{p+1}^*X)$ is the map given by applying d to cocycles in Ξ_p^*X, taking care over the domains!

Definition 5.3. The differentiable algebra map $\iota\colon B\to X$ is called a *differential fibration* if $\Theta_m\colon\Omega^mB\otimes_B\Xi_0^*X\to\Xi_m^*X$ (as given in Definition 4.1) is invertible for all $m\geqslant 0$.

THEOREM 5.4. *Suppose that $\iota\colon B\to X$ is a differential fibration. Then there is a spectral sequence converging to $H_{\mathrm{dR}}^*(X)$ with*

$$E_2^{p,q}\cong H^p(B;H^q(\Xi_0^*X),\nabla).$$

Proof. We note that $\Theta_{m*}\colon\Omega^pB\otimes_BH^q(\Xi_0^*X)\to H^q(\Xi_p^*X)$ is an isomorphism, and that it commutes with the differential in the spectral sequence if we use the flat connection cochain complex on $\Omega^pB\otimes_BH^q(\Xi_0^*X)$. □

6. The multiplicative structure

Even if one is not a priori interested in a multiplicative structure on the cohomology theories, in algebraic topology a knowledge of the multiplicative structure can help to find the differentials in the spectral sequence. In this section we suppose that the differentiable algebra map $\iota: B \to X$ is a differential fibration, that each $\Omega^m B$ is flat as a right B-module, and that the following condition holds:

Definition 6.1. The map $\iota: B \to X$ will be said to satisfy the *differential braiding condition* if $\Omega^n X \wedge \iota_* \Omega^m B \subset \iota_* \Omega^m B \wedge \Omega^n X$ for all $n, m \geqslant 0$.

Remark 6.2. Note that the condition in Definition 6.1 means that the wedge multiplication preserves the filtration in the construction of the spectral sequence, as

$$(\iota_* \Omega^i B \wedge \Omega^j X) \wedge (\iota_* \Omega^k B \wedge \Omega^l X) \subset \iota_* \Omega^i B \wedge \iota_* \Omega^k B \wedge \Omega^j X \wedge \Omega^l X \subset \iota_* \Omega^{i+k} B \wedge \Omega^{j+l} X,$$

so there is a multiplicative structure on the spectral sequence. However, we have gone to considerable trouble to show that the E_2-page of the spectral sequence can be expressed in terms of a cohomology bundle with connection, so we shall look at what this multiplicative structure means in these terms.

PROPOSITION 6.3. *Define a map* $\hat{\sigma}: \Xi_0^n X \otimes_B \Omega^m B \to \Omega^m B \otimes_B \Xi_0^n X$ *by* $\hat{\sigma}([\xi]_0 \otimes \omega) = \omega_i' \otimes [\xi_i']_0$ *(summation implicit), where* $[\iota_* \omega_i' \wedge \xi_i']_m = (-1)^{nm} [\xi \wedge \iota_* \omega]_m$. *For the cochain structure on* Ξ_0^*,

$$\hat{\sigma}((\ker d) \otimes_B \Omega^m B) \subset \Omega^m B \otimes_B (\ker d) \quad \textit{and} \quad \hat{\sigma}((\operatorname{im} d) \otimes_B \Omega^m B) \subset \Omega^m B \otimes_B (\operatorname{im} d),$$

so there is a well-defined map $\sigma: H^n(\Xi_0^* X) \otimes_B \Omega^m B \to \Omega^m B \otimes_B H^n(\Xi_0^*)$.

Proof. First suppose that $[\xi]_0 \in \ker d \subset \Xi_0^n X$. We write $\hat{\sigma}([\xi]_0 \otimes \omega) = \omega_i' \otimes [\xi_i']_0$, where

$$\iota_* \omega_i' \wedge \xi_i' \cong (-1)^{nm} \xi \wedge \iota_* \omega \quad \bmod \ \iota_* \Omega^{m+1} B \otimes_B \Omega^{n-1} X.$$

On applying d,

$$\iota_* d\omega_i' \wedge \xi_i' + (-1)^m \iota_* \omega_i' \wedge d\xi_i' \cong (-1)^{nm} d\xi \wedge \iota_* \omega + (-1)^{nm+n} \xi \wedge \iota_* d\omega \\ \bmod \ \iota_* \Omega^{m+1} B \otimes_B \Omega^n X.$$

As $d\xi \in \iota_* \Omega^1 B \otimes_B \Omega^n X$, this shows that $[\iota_* \omega_i' \wedge d\xi_i']_m = 0$, and therefore the fibration condition gives $\omega_i' \otimes_B [d\xi_i']_0 = 0$. The result follows by flatness.

Now take $[\eta]_0 \in \Xi_0^{n-1} X$, and then find

$$\iota_* \omega_i' \wedge \eta_i' \cong \eta \wedge \iota_* \omega \quad \bmod \ \iota_* \Omega^{m+1} B \otimes_B \Omega^{n-2} X.$$

Applying d gives

$$\iota_* d\omega_i' \wedge \eta_i' + (-1)^m \iota_* \omega_i' \wedge d\eta_i' \cong d\eta \wedge \iota_* \omega - (-1)^n \eta \wedge \iota_* d\omega \quad \bmod \ \iota_* \Omega^{m+1} B \otimes_B \Omega^{n-1} X,$$

which reduces to

$$(-1)^m \iota_* \omega_i' \wedge d\eta_i' \cong d\eta \wedge \iota_* \omega \quad \bmod \ \iota_* \Omega^{m+1} B \otimes_B \Omega^{n-1} X. \qquad \square$$

PROPOSITION 6.4. *If the differential braiding condition holds, then there exists a well-defined map* $\wedge: \Xi_0^r X \otimes \Xi_0^s X \to \Xi_0^{r+s} X$ *defined by* $[\xi]_0 \wedge [\eta]_0 = [\xi \wedge \eta]_0$, *and this gives a well-defined map* $\wedge: H^r(\Xi_0^* X) \otimes H^s(\Xi_0^* X) \to H^{r+s}(\Xi_0^* X)$.

Proof. To show that the map $[\xi]_0 \otimes [\eta]_0 \mapsto [\xi \wedge \eta]_0$ is well defined we need to show that both $\iota_* \Omega^1 B \wedge \Omega^{r-1} X \wedge \Omega^s X$ and $\Omega^r X \wedge \iota_* \Omega^1 B \wedge \Omega^{s-1} X$ are contained in $\iota_* \Omega^1 B \wedge \Omega^{r+s-1} X$. The first inclusion is automatic, and the second follows from the differential braiding condition. The rest is left to the reader. □

PROPOSITION 6.5. *For all* $x \in H^n(\Xi_0^* X)$ *and* $\omega \in \Omega^m B$,

$$(\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\nabla x \otimes \omega) + \sigma(x \otimes d\omega) = [d \otimes \mathrm{id} + (-1)^m (\mathrm{id} \wedge \nabla)] \sigma(x \otimes \omega).$$

Proof. For all $\omega \in \Omega^m B$ and $\xi \in \Omega^n X$, we have defined $\hat{\sigma}(\xi \otimes \omega) = \omega_i \otimes \xi_i$, where

$$(-1)^{nm} \xi \wedge \iota_* \omega = \iota_* \omega_i \wedge \xi_i + \iota_* \phi_i \wedge \eta_i$$

for some $\phi_i \in \Omega^{m+1} B$ and $\eta_i \in \Omega^{n-1} X$. Taking d of this gives

$$\begin{aligned}(-1)^{nm} d\xi \wedge \iota_* \omega + (-1)^{nm+n} \xi \wedge \iota_* d\omega = \iota_* d\omega_i \wedge \xi_i + (-1)^m \iota_* \omega_i \wedge d\xi_i \\ + \iota_* d\phi_i \wedge \eta_i + (-1)^{m+1} \iota_* \phi_i \wedge d\eta_i.\end{aligned} \tag{6.1}$$

Now we suppose that $[\xi]_0 \in \ker(d: \Xi_0^n X \to \Xi_0^{n+1} X)$, and then we also have $[d\xi_i]_m = 0$. This means that all the terms of (6.1) are in $\iota_* \Omega^{m+1} B \wedge \Omega^n X$, and using the quotient map $[\cdot]_{m+1}$ we obtain

$$\begin{aligned}(-1)^{nm} [d\xi \wedge \iota_* \omega]_{m+1} + (-1)^{nm+n} [\xi \wedge \iota_* d\omega]_{m+1} \\ = [\iota_* d\omega_i \wedge \xi_i]_{m+1} + (-1)^m [\iota_* \omega_i \wedge d\xi_i]_{m+1} + (-1)^{m+1} [\iota_* \phi_i \wedge d\eta_i]_{m+1}.\end{aligned} \tag{6.2}$$

Now write $\nabla \xi = \psi_i \otimes [\zeta_i]_0 \in \Omega^1 B \otimes \Xi_0^n$ and $\nabla \xi_i = \upsilon_{ik} \otimes [\zeta_{ik}]_0$. Substituting this in (6.2) gives

$$\begin{aligned}(-1)^{nm} [\iota_* \psi_i \wedge \zeta_i \wedge \iota_* \omega]_{m+1} = [\iota_* d\omega_i \wedge \xi_i]_{m+1} + (-1)^m [\iota_* \omega_i \wedge \iota_* \psi_{ik} \wedge \zeta_{ik}]_{m+1} \\ + (-1)^{nm+n} [\xi \wedge \iota_* d\omega]_{m+1} + (-1)^{m+1} [\iota_* \phi_i \wedge d\eta_i]_{m+1}.\end{aligned} \tag{6.3}$$

On passing to the cohomology the last term in (6.3) vanishes, giving the result. □

PROPOSITION 6.6. *For* $x, y \in H^*(\Xi_0^* X)$, $\nabla(x \wedge y) = \nabla x \wedge y + (\sigma \wedge \mathrm{id})(x \otimes \nabla y)$.

Proof. Suppose that x and y are given by $[\xi]_0 \in \Xi_0^r X$ and $[\eta]_0 \in \Xi_0^s X$, respectively. Set $\nabla x = \omega_i \otimes [\xi_i]$ and $\nabla y = \phi_i \otimes [\eta_i]$ for $[\xi_i]_0 \in \Xi_0^r X$ and $[\eta_i]_0 \in \Xi_0^s X$. Then

$$d(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^r \xi \wedge d\eta = \iota_* \omega_i \wedge \xi_i \wedge \eta + (-1)^r \xi \wedge \iota_* \phi_i \wedge \eta_i. \qquad \square$$

Proposition 6.7. *For all* $x\in H^*(\Xi_0^*X)$ *and* $\omega,\phi\in\Omega^*B$,

$$(\mathrm{id}\wedge\sigma)(\sigma(x\otimes\omega)\otimes\phi) = \sigma(x\otimes(\omega\wedge\phi)).$$

Proof. Set $x=[\xi]_0$. We will use explicit summations in this proof. We obtain

$$(\mathrm{id}\wedge\sigma)(\sigma([\xi]_0\otimes\omega)\otimes\phi)=\sum_i(\mathrm{id}\wedge\sigma)(\omega_i'\otimes[\xi_i']_0\otimes\phi)=\sum_{i,j}\omega_i'\wedge\phi_{ij}'\otimes[\xi_{ij}'']_0,$$

where

$$\sum_i[\iota_*\omega_i'\wedge\xi_i']_{|\omega|}=(-1)^{|\xi|\,|\omega|}[\xi\wedge\iota_*\omega]_{|\omega|}\quad\text{and}\quad\sum_j[\iota_*\phi_{ij}'\wedge\xi_{ij}'']_{|\phi|}=(-1)^{|\xi|\,|\phi|}[\xi_i'\wedge\iota_*\phi]_{|\phi|}.$$

From this we obtain

$$\begin{aligned}\sum_{i,j}[\iota_*(\omega_i'\wedge\phi_{ij}')\wedge\xi_{ij}'']_{|\omega|+|\phi|}&=\sum_i(-1)^{|\xi|\,|\phi|}[\iota_*(\omega_i')\wedge\xi_i'\wedge\iota_*\phi]_{|\omega|+|\phi|}\\&=(-1)^{|\xi|(|\phi|+|\omega|)}[\xi\wedge\iota_*\omega\wedge\iota_*\phi]_{|\omega|+|\phi|}.\end{aligned}$$ □

The reader will recall that in the construction of the spectral sequence the vector spaces $\Omega^nB\otimes_BH^m(\Xi_0^*X)$ appear. This is not such a simple thing as a tensor product differential complex, as the derivative involves a connection which does not map $H^*(\Xi_0^*X)$ to itself. It is therefore not surprising that the product structure has to be rather more complicated than the graded tensor product. In fact, we have already given all the ingredients required for the product, it only remains to state them in a more coherent manner:

Definition 6.8. Take $(E^m,\nabla)\in{}_A\mathcal{F}$ for all $m\geqslant0$, and suppose that each E^m is an A-bimodule. Give $e\in E^m$ the grade $|e|=m$. A *product structure* on this family consists of

(1) A-bimodule maps $\sigma\colon E^m\otimes_A\Omega^nA\to\Omega^nA\otimes_AE^m$;

(2) a product $\wedge\colon E^m\otimes_AE^{m'}\to E^{m+m'}$ which satisfy the following conditions, for all $e,f\in E^*$ and $\xi,\eta\in\Omega^*A$:

(a) the product $(\xi\otimes e)\wedge(\eta\otimes f)=(-1)^{|e|\,|\eta|}\xi\wedge\sigma(e\otimes\eta)\wedge f$ on $\Omega^*A\otimes E^*$ is associative;

(b) $(\mathrm{id}\wedge\sigma)(\nabla e\otimes\xi)+\sigma(e\otimes d\xi)=[d\otimes\mathrm{id}+(-1)^{|\xi|}(\mathrm{id}\wedge\nabla)]\sigma(e\otimes\xi)$;

(c) $\nabla(e\wedge f)=\nabla e\wedge f+(\sigma\wedge\mathrm{id})(e\otimes\nabla f)$;

(d) $(\mathrm{id}\wedge\sigma)(\sigma(e\otimes\xi)\otimes\eta)=\sigma(e\otimes(\xi\wedge\eta))$.

PROPOSITION 6.9. *In Definition 6.8 the derivative $\nabla^{[*]}$ is a graded derivation over the given product structure on $\Omega^*A\otimes E^*$, i.e.*

$$\nabla^{[*]}((\xi\otimes e)\wedge(\eta\otimes f))=\nabla^{[*]}(\xi\otimes e)\wedge(\eta\otimes f)+(-1)^{|\xi|+|e|}(\xi\otimes e)\wedge\nabla^{[*]}(\eta\otimes f).$$

Thus there is an induced product structure on the cohomology,

$$\wedge\colon H^n(A,E^m,\nabla)\otimes H^{n'}(A,E^{m'},\nabla)\longrightarrow H^{n+n'}(A,E^{m+m'},\nabla).$$

Proof. Begin with

$$\begin{aligned}(-1)^{|e|\,|\eta|}\nabla^{[*]}((\xi\otimes e)\wedge(\eta\otimes f))&=(d\otimes\mathrm{id}+(-1)^{|\xi|+|\eta|}(\mathrm{id}\wedge\nabla))(\xi\wedge\sigma(e\otimes\eta)\wedge f)\\&=d\xi\wedge\sigma(e\otimes\eta)\wedge f+(-1)^{|\xi|}\xi\wedge(d\otimes\mathrm{id})\sigma(e\otimes\eta)\wedge f\\&\quad+(-1)^{|\xi|+|\eta|}\xi\wedge(\mathrm{id}\wedge\nabla)\sigma(e\otimes\eta)\wedge f\\&\quad+(-1)^{|\xi|+|\eta|}\xi\wedge(\mathrm{id}\wedge\sigma\wedge\mathrm{id})(\sigma(e\otimes\eta)\otimes\nabla f).\end{aligned}$$

Using property (b) of Definition 6.8, this becomes

$$\begin{aligned}(-1)^{|e|\,|\eta|}\nabla^{[*]}((\xi\otimes e)\wedge(\eta\otimes f))&=d\xi\wedge\sigma(e\otimes\eta)\wedge f+(-1)^{|\xi|}\xi\wedge(\mathrm{id}\wedge\sigma)(\nabla e\otimes\eta)\wedge f\\&\quad+(-1)^{|\xi|}\xi\wedge\sigma(e\otimes d\eta)\wedge f\\&\quad+(-1)^{|\xi|+|\eta|}\xi\wedge(\mathrm{id}\wedge\sigma\wedge\mathrm{id})(\sigma(e\otimes\eta)\otimes\nabla f).\end{aligned}\tag{6.4}$$

Next we calculate

$$(-1)^{|e|\,|\eta|}\nabla^{[*]}(\xi\otimes e)\wedge(\eta\otimes f)=(-1)^{|e|\,|\eta|}(d\xi\otimes e+(-1)^{|\xi|}\xi\wedge\nabla e)\wedge(\eta\otimes f),$$

which is the same as the first two terms of (6.4). Next

$$\begin{aligned}&(-1)^{|e|\,|\eta|+|\xi|+|e|}(\xi\otimes e)\wedge\nabla^{[*]}(\eta\otimes f)\\&\qquad=(-1)^{|e|\,|\eta|+|\xi|+|e|}(\xi\otimes e)\wedge(d\eta\otimes f+(-1)^{|\eta|}\eta\wedge\nabla f)\\&\qquad=(-1)^{|\xi|}\xi\wedge\sigma(e\otimes d\eta)\wedge f+(-1)^{|\xi|+|\eta|}\xi\wedge(\sigma\wedge\mathrm{id})(e\otimes\eta\wedge\nabla f),\end{aligned}$$

so to prove the result we only need to verify

$$(\mathrm{id}\wedge\sigma\wedge\mathrm{id})(\sigma(e\otimes\eta)\otimes\nabla f)=(\sigma\wedge\mathrm{id})(e\otimes\eta\wedge\nabla f),$$

which is given by property (d) of Definition 6.8. □

7. Coactions of Hopf algebras

In classical topology, fibrations arise whenever there is a continuous (compact) group action on a (compact) Hausdorff space (e.g. a free action gives rise to a principal fibration). A base of the fibration is then identified with the quotient of the total space by this action. In noncommutative geometry this corresponds to a coaction of a Hopf algebra on an algebra. This is the case that we consider in this section and, indeed in all the remaining sections.

7.1. Differential calculi on Hopf algebras

For more details on this subject, the reader should see [24]. Suppose that a Hopf algebra H with coproduct Δ_H, counit ε_H and the invertible antipode S has a differential calculus Ω^*H. We write the coproduct in H as $\Delta_H(h)=h_{(1)}\otimes h_{(2)}$, $\Delta^2_H(h)=h_{(1)}\otimes h_{(2)}\otimes h_{(3)}$, etc., and the left H-coaction on Ω^*H as $\xi\mapsto\xi_{[-1]}\otimes\xi_{[0]}$ (summation understood). If there is no danger of confusion we will simply write Δ and ε for Δ_H and ε_H (this convention applies to all other Hopf algebras as well). In this section we shall not assume that the coproduct is differentiable (this would give a bicovariant calculus), but only that the left H-coaction $\lambda\colon\Omega^*H\to H\otimes\Omega^*H$ is defined. L^nH denotes the space of left-invariant n-forms on H, that is,

$$L^nH={}^{\mathrm{co}\,H}(\Omega^nH)=\{\xi\in\Omega^nH:\xi_{[-1]}\otimes\xi_{[0]}=1_H\otimes\xi\}.$$

The Hopf–Lie algebra $\mathfrak{h}$ of H is defined to be

$$\mathfrak{h}=\{\alpha\colon\Omega^1H\to k:\alpha(\eta h)=\alpha(\eta)\varepsilon(h)\text{ for all }\eta\in\Omega^1H\text{ and }h\in H\}.$$

Note that defining $\mathfrak{h}$ as a vector space only requires a 'classical point', that is, an algebra map $\varepsilon\colon H\to k$.

LEMMA 7.1. *If, for a left-invariant* $\eta\in\Omega^1H$, $\alpha(\eta)=0$ *for all* $\alpha\in\mathfrak{h}$, *then* $\eta=0$.

Proof. For any k-linear map $T\colon L^1H\to k$, define $\alpha_T\colon\Omega^1H\to k$ by

$$\alpha_T(\xi)=T(\xi_{[0]}S^{-1}(\xi_{[-1]})).$$

Then for $h\in H$,

$$\alpha_T(\xi h)=T(\xi_{[0]}h_{(2)}S^{-1}(h_{(1)})S^{-1}(\xi_{[-1]}))=\alpha_T(\xi)\varepsilon(h),$$

so $\alpha_T\in\mathfrak{h}$. For a left-invariant $\eta\in\Omega^1H$, choose T so that $T(\eta)\neq0$, and then $\alpha_T(\eta)\neq0$. $\square$

7.2. Differentiable right coactions

Suppose that the algebra X has a differentiable right coaction ϱ (written on elements as $\varrho(x)=x_{[0]}\otimes x_{[1]}\in X\otimes H$, summation understood) by the Hopf algebra H which makes it into a comodule algebra. This means that $\varrho: X\to X\otimes H$ is a coaction and a differentiable algebra map, so we obtain a map of differential graded algebras (under the $\wedge$-multiplication)

$$\varrho_*:\Omega^nX\longrightarrow\Omega^n(X\otimes H)=\bigoplus_{0\leqslant r\leqslant n}\Omega^rX\otimes\Omega^{n-r}H.$$

Write $\Pi_{m,n-m}$ for the corresponding projection from $\Omega^n(X\otimes H)$ to $\Omega^mX\otimes\Omega^{n-m}H$. Note that the maps $\Pi_{n,0}\varrho_*$ define the right coactions of H on Ω^nX. These are also denoted by ϱ. The subalgebra $B\subset X$ is defined to be the coinvariants for the right H-coaction, i.e. $B=X^{\mathrm{co}\,H}:=\{b\in X:\varrho(b)=b\otimes 1_H\}$. We now define the calculus on B by $\Omega^1B=B\cdot dB\subset\Omega^1X$ and $\Omega^nB=\bigwedge^n\Omega^1B\subset\Omega^nX$. It is immediate that $\Omega^nB\subset(\Omega^nX)^{\mathrm{co}\,H}$, the H-invariant n-forms on X. However, we can be rather more restrictive:

Definition 7.2. Define

$$\mathcal{H}^nX=\bigcap_{n>m\geqslant 0}\ker(\Pi_{m,n-m}\varrho_*:\Omega^nX\to\Omega^mX\otimes\Omega^{n-m}H).$$

The elements of $\mathcal{H}^nX$ are called *horizontal* n-forms.

Remark 7.3. It is immediate that $\Omega^nB\subset\mathcal{H}^nX$. and we might conjecture that in 'nice' cases we should have $\Omega^nB=(\Omega^nX)^{\mathrm{co}\,H}\cap\mathcal{H}^nX$. The reader should note that in the case of a bicovariant calculus on H, the differential algebra Ω^*H is itself a graded Hopf algebra (see [4]), and then the conjecture is that Ω^*B is the invariant part of Ω^*X under the right Ω^*H-coaction.

Remark 7.4. As in the classical case, it is possible to define horizontal 1-forms with reference to the Hopf–Lie algebra. Remember from [1] that the vector fields on X are the right X-module maps from Ω^1X to X. Every $\alpha\in\mathfrak{h}$ gives a vector field $\widehat{\alpha}$ on X defined by $\widehat{\alpha}(\xi)=(\mathrm{id}\otimes\alpha)\Pi_{0,1}\varrho_*(\xi)$ for every $\xi\in\Omega^1X$.

PROPOSITION 7.5. $\mathcal{H}^1X=\bigcap_{\alpha\in\mathfrak{h}}\ker(\widehat{\alpha}:\Omega^1X\to X)$.

Proof. First the reader should recall the definition of the cotensor product $U\Box_HV$ of a right H-comodule U and a left H-comodule V [17]. This is the subset of $U\otimes V$ consisting of all $u\otimes v$ (summation implicit) where $u_{[0]}\otimes u_{[1]}\otimes v=u\otimes v_{[-1]}\otimes v_{[0]}\in U\otimes H\otimes V$. Note that we can restrict the codomain to get $\Pi_{0,1}\varrho_*:\Omega^1X\to X\Box_H\Omega^1H$. Now there is a one-to-one correspondence between $X\Box_H\Omega^1H$ and $X\otimes L^1H$ given by $x\Box_H\xi\mapsto x_{[0]}\otimes\xi S^{-1}(x_{[1]})$ and $y\otimes\eta\mapsto y_{[0]}\Box_H\eta y_{[1]}$. This combines with Lemma 7.1 to prove the result. $\square$

7.3. When the algebra coacted on is a Hopf algebra

A special case of interest, corresponding to homogeneous spaces, is when the algebra X is itself a Hopf algebra. Suppose that the Hopf algebra X has a differentiable right coaction ϱ of the Hopf algebra H which makes it into a comodule algebra. We shall also assume that ϱ commutes with the coproduct Δ_X of X, i.e.

$$(\mathrm{id}\otimes\varrho)\Delta_X=(\Delta_X\otimes\mathrm{id})\varrho\colon X\longrightarrow X\otimes X\otimes H.$$

This is the case if and only if the map $\pi\colon X\to H$ defined by $\pi(x)=(\varepsilon_X\otimes\mathrm{id})\varrho(x)$ is a bialgebra map, and then $\varrho(x)=(\mathrm{id}\otimes\pi)\Delta_X(x)$. Let $B:=X^{\mathrm{co}\,H}$.

LEMMA 7.6. $\Delta_X B\subset X\otimes B$.

Proof. By the definition of coinvariants, for all $b\in B$, $\varrho(b)=b\otimes 1_H$, so

$$(\Delta_X\otimes\mathrm{id})\varrho(b)=b_{(1)}\otimes b_{(2)}\otimes 1_H=b_{(1)}\otimes\varrho(b_{(2)}). \qquad \square$$

Lemma 7.6 means that the Hopf algebra X left coacts on B. Thus B can be viewed as a noncommutative generalisation of a homogeneous space of X [18].

8. The noncommutative Hopf fibration with a nonbicovariant calculus

In this section we give an explicit example of a noncommutative differentiable fibration. It is well known that the underlying algebra inclusion is a *quantum principal bundle* [6]. Our aim, however, is to show that it is a differentiable fibration in the sense of Definition 5.3.

8.1. Example: The quantum Hopf fibration

This is an example of the type of coaction discussed in §7.3. Consider the complex Hopf algebra $X=\mathcal{A}(SL_q(2))$ generated by $\{\alpha,\beta,\gamma,\delta\}$ with the relations

$$\begin{gathered}\alpha\beta=q\beta\alpha,\quad \alpha\gamma=q\gamma\alpha,\quad \beta\gamma=\gamma\beta,\quad \beta\delta=q\delta\beta,\quad \gamma\delta=q\delta\gamma,\\ \alpha\delta=\delta\alpha+(q-q^{-1})\beta\gamma\quad\text{and}\quad \alpha\delta-q\beta\gamma=1,\end{gathered}\tag{8.1}$$

where q is a complex number which is not a root of unity. On this level of algebraic generality, there is no need to make further restrictions on q, although geometrically most interesting is the case $0<q<1$, whereby X can be made into a $*$-algebra and extended to a C^*-algebra of functions on the quantum group $SU_q(2)$ (cf. [23]). The coproduct is given by

$$\begin{gathered}\Delta\alpha=\alpha\otimes\alpha+\beta\otimes\gamma,\quad \Delta\beta=\alpha\otimes\beta+\beta\otimes\delta,\\ \Delta\gamma=\gamma\otimes\alpha+\delta\otimes\gamma,\quad \Delta\delta=\delta\otimes\delta+\gamma\otimes\beta,\end{gathered}\tag{8.2}$$

and counit and antipode by

$$\varepsilon(\alpha)=\varepsilon(\delta)=1,\quad \varepsilon(\beta)=\varepsilon(\gamma)=0,$$
$$S(\alpha)=\delta,\quad S(\delta)=\alpha,\quad S(\beta)=-q^{-1}\beta,\quad S(\gamma)=-q\gamma.$$

We will take H to be the group algebra of $\mathbf{Z}$, which we take as generated by z and z^{-1} with $\Delta z^{\pm 1}=z^{\pm 1}\otimes z^{\pm 1}$, $S(z^{\pm 1})=z^{\mp 1}$ and $\varepsilon(z^{\pm 1})=1$. The Hopf algebra map $\pi\colon X\to H$ is given by

$$\pi(\alpha)=z,\quad \pi(\delta)=z^{-1}\quad\text{and}\quad \pi(\beta)=\pi(\gamma)=0.$$

The right H-coaction ϱ on X is then given by

$$\varrho(\alpha)=\alpha\otimes z,\quad \varrho(\beta)=\beta\otimes z^{-1},\quad \varrho(\gamma)=\gamma\otimes z\quad\text{and}\quad \varrho(\delta)=\delta\otimes z^{-1}.$$

The invariant part of X, $B=X^{\mathrm{co}\,H}=\mathcal{A}(S_q^2)$, is generated as an algebra by $\{\alpha\beta,\alpha\delta,\gamma\delta\}$ and is known as (an algebra of functions on) the standard quantum 2-sphere [18].

8.2. The 3D nonbicovariant calculus on $\mathcal{A}(SL_q(2))$

This left-covariant differential calculus on $X=\mathcal{A}(SL_q(2))$ was introduced by Woronowicz in [23] and is generated by three left-invariant 1-forms $\{\omega^0,\omega^1,\omega^2\}$. The differentials of the generators are given by

$$\begin{aligned} d\alpha&=\alpha\omega^1-q\beta\omega^2, & d\beta&=\alpha\omega^0-q^2\beta\omega^1,\\ d\gamma&=\gamma\omega^1-q\delta\omega^2, & d\delta&=\gamma\omega^0-q^2\delta\omega^1. \end{aligned}\tag{8.3}$$

We have the commutation relations

$$\begin{aligned} \omega^0\alpha&=q^{-1}\alpha\omega^0, & \omega^0\beta&=q\beta\omega^0,\\ \omega^1\alpha&=q^{-2}\alpha\omega^1, & \omega^1\beta&=q^2\beta\omega^1,\\ \omega^2\alpha&=q^{-1}\alpha\omega^2, & \omega^2\beta&=q\beta\omega^2, \end{aligned}\tag{8.4}$$

and similarly for replacing α with γ and β with δ. For the higher forms we have exterior derivatives

$$d\omega^0=q^2(q^2+1)\omega^0\wedge\omega^1,\quad d\omega^1=q\omega^0\wedge\omega^2,\quad d\omega^2=q^2(q^2+1)\omega^1\wedge\omega^2,\tag{8.5}$$

and wedge multiplication

$$\begin{gathered} \omega^0\wedge\omega^0=\omega^1\wedge\omega^1=\omega^2\wedge\omega^2=0,\\ \omega^2\wedge\omega^0=-q^2\omega^0\wedge\omega^2,\quad \omega^1\wedge\omega^0=-q^4\omega^0\wedge\omega^1,\quad \omega^2\wedge\omega^1=-q^4\omega^1\wedge\omega^2. \end{gathered}\tag{8.6}$$

8.3. The differentiable coaction

We need the map π given in §8.1 to extend to a map π_* of differential graded algebras. Such an extension of π exists, provided there is a suitable differential structure on H, which can be constructed as follows. From (8.3) we obtain $dz{=}z\pi_*(\omega^1)$, $0{=}z\pi_*(\omega^0)$, $0{=}-qz^{-1}\pi_*(\omega^2)$ and $d(z^{-1}){=}-q^2z^{-1}\pi_*(\omega^1)$. This can be summarised by

$$\pi_*(\omega^0)=\pi_*(\omega^2)=0, \quad \pi_*(\omega^1)=z^{-1}\cdot dz \quad \text{and} \quad z\cdot dz=q^2dz\cdot z. \tag{8.7}$$

(To see this, note that from $z\cdot z^{-1}{=}1$ we use the derivation property for d to get $d(z^{-1}){=}-z^{-1}\cdot dz\cdot z^{-1}$.) It is easily checked that the map π_* defined in this fashion satisfies all the relations and that the constructed differential calculus on H is bicovariant. However, the cost of differentiability of π_* is that the commutative algebra H is given a noncommutative differential structure!

To find ϱ_* we look at (8.3), and use $\varrho_*(d\alpha){=}d(\varrho(\alpha))$, etc., to give

$$\varrho_*(\omega^0)=\omega^0\otimes z^{-2}, \quad \varrho_*(\omega^1)=1\otimes z^{-1}\cdot dz+\omega^1\otimes 1 \quad \text{and} \quad \varrho_*(\omega^2)=\omega^2\otimes z^2. \tag{8.8}$$

To check that this gives a well-defined map on Ω^1X, one needs to check that it is consistent with the relations in (8.4)—this is left to the reader. Then to define ϱ_* on the higher forms by using the wedge product, we only have to check the relations in (8.5) and (8.6), which is easily done by a straightforward calculation.

To find the horizontal 1-forms we apply $\Pi_{0,1}$ to (8.8) to get

$$\Pi_{0,1}\,\varrho_*(\omega^1)=1\otimes z^{-1}\cdot dz \quad \text{and} \quad \Pi_{0,1}\,\varrho_*(\omega^0)=\Pi_{0,1}\,\varrho_*(\omega^2)=0.$$

It follows that the horizontal 1-forms are precisely those of the form $a\omega^0+b\omega^2$ for $a,b\in X$. We can also calculate the right H-coaction by applying $\Pi_{1,0}$ to (8.8) to get

$$\Pi_{1,0}\,\varrho_*(\omega^1)=\omega^1\otimes 1, \quad \Pi_{1,0}\,\varrho_*(\omega^0)=\omega^0\otimes z^{-2} \quad \text{and} \quad \Pi_{1,0}\,\varrho_*(\omega^2)=\omega^2\otimes z^2.$$

Then the invariant horizontal 1-forms are precisely those of the form $a\omega^0+b\omega^2$, where $\varrho(a){=}a\otimes z^2$ and $\varrho(b){=}b\otimes z^{-2}$.

8.4. The corresponding calculus on $B{=}\mathcal{A}(S_q^2)$

We can calculate

$$\begin{aligned} d(\alpha\beta)&=\alpha^2\omega^0-q^2\beta^2\omega^2,\\ qd(\beta\gamma)&=\alpha\gamma\omega^0-q^2\beta\delta\omega^2,\\ d(\gamma\delta)&=\gamma^2\omega^0-q^2\delta^2\omega^2. \end{aligned} \tag{8.9}$$

From this we get

$$\delta d(\alpha\beta)-q^{-1}\beta d(\beta\gamma)=\alpha\omega^0,$$
$$q\delta d(\beta\gamma)-q^{-1}\beta d(\gamma\delta)=\gamma\omega^0.$$

By left multiplying these last equations by α and γ, we see that $\alpha^2\omega^0$, $\alpha\gamma\omega^0$ and $\gamma^2\omega^0$ are all in $B\cdot dB$. From (8.9) we deduce that $\beta^2\omega^2$, $\beta\delta\omega^2$ and $\delta^2\omega^2$ are also all in $B\cdot dB$.

Given a monomial a in the generators $\{\alpha,\beta,\gamma,\delta\}$ with $\varrho(a)=a\otimes z^2$, we can reorder it as either $a=x\alpha^2$, $a=x\alpha\gamma$ or $a=x\gamma^2$, where $x\in B$. Thus we have $a\omega^0\in B\cdot dB$. Likewise for a monomial b with $\varrho(b)=b\otimes z^{-2}$ we have $b\omega^2\in B\cdot dB$. From this and the discussion in §8.3 we conclude that Ω^1B is precisely the horizontal invariant 1-forms on X.

Now we shall consider the 2-forms. Since ϱ_* is a graded algebra map, we immediately obtain

$$\varrho_*(\omega^0\wedge\omega^2)=\omega^0\wedge\omega^2\otimes 1 \quad \text{and} \quad \varrho_*(\omega^l\wedge\omega^1)=\omega^l\otimes z^{2l-3}\cdot dz+\omega^l\wedge\omega^1\otimes z^{2l-2},\ l=0,2.$$

Hence the horizontal 2-forms are multiples of $\omega^0\wedge\omega^2$. Then the invariant horizontal 2-forms are $B\cdot\omega^0\wedge\omega^2$. To see that Ω^2B is all of this, we use the relation

$$\alpha^2\delta^2-(q+q^{-1})\alpha\gamma\beta\delta+q^2\gamma^2\beta^2=1.$$

By using $\alpha^2\omega^0\wedge\delta^2\omega^2=q^2\alpha^2\delta^2\omega^0\wedge\omega^2$ and similar calculations, we see that $\omega^0\wedge\omega^2$ is contained in $\Omega^1B\wedge\Omega^1B$.

All 3-forms are multiples of $\omega^0\wedge\omega^1\wedge\omega^2$, but none of these (except zero) are horizontal, so we conclude that $\Omega^3B=0$.

8.5. An easy example of a spectral sequence

We will use the notation $\langle\dots\rangle$ to denote the right X-module generated by the listed elements. Then as right X-modules, $B\otimes_B X\cong X$, $\Omega^1B\otimes_B X\cong\langle\omega^0,\omega^2\rangle$ and $\Omega^2B\otimes_B X\cong\langle\omega^0\wedge\omega^2\rangle$. We can calculate the $\Xi^n_m X$ as shown in the following table:

$\Xi^n_m X$	$n=0$	$n=1$	$n>1$
$m=0$	X	$\langle\omega^1\rangle$	0
$m=1$	$\langle\omega^0,\omega^2\rangle$	$\langle\omega^0\wedge\omega^1,\omega^2\wedge\omega^1\rangle$	0
$m=2$	$\langle\omega^0\wedge\omega^2\rangle$	$\langle\omega^0\wedge\omega^1\wedge\omega^2\rangle$	0
$m>2$	0	0	0

It follows that

$$\Theta_m: \Omega^m B \otimes_B \Xi_0^n X = \Omega^m B \otimes_B X \otimes_X \Xi_0^n X \longrightarrow \Xi_m^n X$$

(as defined in Definition 4.1) is an isomorphism, and that the quantum Hopf fibration $\iota: \mathcal{A}(S_q^2) \hookrightarrow \mathcal{A}(SL_q(2))$ is a differential fibration for this differential structure.

Now we shall calculate the E_2-page of the spectral sequence in this case. The first thing to do is to look at $H^*(\Xi_0^* X)$. Recall that we consider only the generic case, where q is not a root of unity. Note that the coaction ϱ makes X into a $\mathbf{Z}$-graded algebra with the grading $\deg \alpha = \deg \gamma = 1$, $\deg \beta = \deg \delta = -1$ and $\deg 1 = 0$.

Lemma 8.1. *For any homogeneous $x \in X$, the differential*

$$d: \Xi_0^0 X = X \longrightarrow \Xi_0^1 X = \Omega^1 X / \langle \omega^0, \omega^2 \rangle$$

gives

$$dx = [\deg x; q^{-2}] x \omega^1,$$

where $[n; q^{-2}] = (q^{-2n} - 1)/(q^{-2} - 1)$ is a q^{-2}-integer.

Proof. This is most easily proved by checking the formula on the generators of X, and then showing that if the formula holds for homogeneous $a, b \in X$ then it also holds for $x = ab$. This uses the Leibniz rule and (8.4). □

Proposition 8.2. *As left B-modules, $H^0(\Xi_0^* X) = B$, $H^1(\Xi_0^* X) = B \cdot \omega^1$ and, for $n > 1$, $H^n(\Xi_0^* X) = 0$.*

Proof. This comes from Lemma 8.1 and $\Xi_0^n X = 0$ for $n > 1$. □

Remark 8.3. We now have to find the left B-connection ∇ described in Proposition 4.4. As each $H^n(\Xi_0^* X)$ is a finitely generated B-module, it is enough to find ∇ on the generators. Choose generators 1_B and ω^1 in $H^0(\Xi_0^* X)$ and $H^1(\Xi_0^* X)$, respectively; an explicit calculation then implies that $\nabla 1_B = 0$ and $\nabla \omega^1 = 0$. Now we can calculate the ∇-cohomology of the $H^n(\Xi_0^* X)$-module, which is given by the cochain complex

$$H^n(\Xi_0^* X) \longrightarrow \Omega^1 B \otimes_B H^n(\Xi_0^* X) \longrightarrow \Omega^2 B \otimes_B H^n(\Xi_0^* X) \longrightarrow \dots .$$

Using the generators, we identify this with the usual de Rham complex

$$B \longrightarrow \Omega^1 B \longrightarrow \Omega^2 B \longrightarrow \dots ,$$

and so we get $E_2^{p,r} \cong H_{\mathrm{dR}}^p(B)$ for $r = 0, 1$, and $E_2^{p,r} \cong 0$ for other values of r. This gives the E_2-page of the Serre spectral sequence (we display only potentially nonzero terms):

$$\begin{array}{c|cccccc}
r & & & & & & \\
1 & H^0_{\mathrm{dR}}(B) & H^1_{\mathrm{dR}}(B) & H^2_{\mathrm{dR}}(B) & H^3_{\mathrm{dR}}(B) & & \\
0 & H^0_{\mathrm{dR}}(B) & H^1_{\mathrm{dR}}(B) & H^2_{\mathrm{dR}}(B) & H^3_{\mathrm{dR}}(B) & & \\
\hline
 & 0 & 1 & 2 & 3 & & p
\end{array}$$

The only possibly nonzero differentials on this page are $d_2\colon(0,1)\to(2,0)$ and $d_2\colon(1,1)\to(3,0)$. All further pages have all differentials zero, just from considering the indices. From this we see that $H^3_{\mathrm{dR}}(B)\cong H^4_{\mathrm{dR}}(X)$, but $H^4_{\mathrm{dR}}(X)=0$ as $\Omega^4X=0$, and so $H^3_{\mathrm{dR}}(B)=0$. Using this, we get $H^2_{\mathrm{dR}}(B)\cong H^3_{\mathrm{dR}}(X)$. Also we obtain $H^0_{\mathrm{dR}}(B)\cong H^0_{\mathrm{dR}}(X)$ and the more complicated cases

$$\begin{aligned}
&H^1_{\mathrm{dR}}(X)\cong H^1_{\mathrm{dR}}(B)\oplus\ker(d_2\colon H^0_{\mathrm{dR}}(B)\to H^2_{\mathrm{dR}}(B)),\\
&H^2_{\mathrm{dR}}(X)\cong H^1_{\mathrm{dR}}(B)\oplus\operatorname{coker}(d_2\colon H^0_{\mathrm{dR}}(B)\to H^2_{\mathrm{dR}}(B)).
\end{aligned}$$

To get any further, we would have to use additional information about either B or X. However, this is one of the primary reasons why the Serre spectral sequence is useful, it turns information about one space into information about the other space.

9. A construction for bicovariant calculi

In this section we consider Hopf algebras X and H with bicovariant differential calculi. We assume that there exists a differentiable surjective Hopf algebra map $\pi\colon X\to H$. The right H-coaction on X is given by $\varrho=(\mathrm{id}\otimes\pi)\Delta\colon X\to X\otimes H$ (cf. §7.3). Since the calculus on X is bicovariant, the coproduct Δ in X is a differentiable map, and hence also the coaction ϱ is differentiable (as a composition of differentiable maps).

9.1. Left-invariant forms and coactions

We first study the covariance properties of the spaces of horizontal n-forms (see Definition 7.2).

PROPOSITION 9.1. *$\mathcal{H}^nX$ is preserved by the right H-coaction, i.e.*

$$\varrho(\mathcal{H}^nX)\subset\mathcal{H}^nX\otimes H.$$

Proof. Start with any $\eta\in\mathcal{H}^nX$. To check that $\varrho(\eta)\in\mathcal{H}^nX\otimes H$ we need to show that $(\Pi_{m,n-m}\varrho_*\otimes\mathrm{id})\varrho(\eta)=0$ for all $n>m\geqslant 0$. Inserting the definition of the right coaction, we need to show that $(\Pi_{m,n-m}\varrho_*\otimes\mathrm{id})\Pi_{n,0}\varrho_*(\eta)=0$, for all $n>m\geqslant 0$. This, using more projections to forms, is the same as $\Pi_{m,n-m,0}(\varrho_*\otimes\mathrm{id})\varrho_*(\eta)=0$ (here we have extended the projection Π to three factors in the obvious manner). By the coaction property this is $\Pi_{m,n-m,0}(\mathrm{id}\otimes\Delta_*)\varrho_*(\eta)=0$. However, as $\eta\in\mathcal{H}^nX$ we know that $\varrho_*(\eta)\in\Omega^nX\otimes H$, giving $(\mathrm{id}\otimes\Delta_*)\varrho_*(\eta)\in\Omega^nX\otimes H\otimes H$, and applying the projection gives zero. ☐

PROPOSITION 9.2. *The space of horizontal n-forms $\mathcal{H}^nX$ is preserved by the left X-coaction, i.e.* $\Pi_{0,n}\Delta_*(\mathcal{H}^nX)\subset X\otimes\mathcal{H}^nX$.

Proof. For $\eta\in\mathcal{H}^nX$ and all $0\leqslant m<n$ we need to show that

$$(\mathrm{id}\otimes\mathrm{id}\otimes\pi_*)(\mathrm{id}\otimes\Pi_{m,n-m}\Delta_*)\Pi_{0,n}\Delta_*\eta$$

vanishes. By coassociativity, this is the same as

$$\Pi_{0,m,n-m}(\Delta_*\otimes\mathrm{id})(\mathrm{id}\otimes\pi_*)\Delta_*\eta.$$

Now $(\mathrm{id}\otimes\pi_*)\Delta_*\eta\in\Omega^nX\otimes H$, so applying the projection gives zero, as $n-m>0$. ☐

9.2. The 1-forms on the base B

To identify the differential forms on the base B, we require that $\pi: X\to H$ satisfies an additional condition, and this is best phrased in terms of the space

$$\mathcal{K}=\ker(\pi_*: L^1X\to L^1H). \tag{9.1}$$

Note that $\mathcal{K}$ is simply the space of horizontal left-invariant 1-forms on X.

Definition 9.3. We say that $\pi: X\to H$ satisfies *Condition K* if $\mathcal{K}\subset dB\cdot X$.

Note that checking that π satisfies Condition K is easier than it might seem, as often the left-invariant 1-forms on X form a finite-dimensional space (see the explicit example in §11.3).

PROPOSITION 9.4. *If $\pi: X\to H$ satisfies Condition K, then $\mathcal{H}^1X=dB\cdot X$.*

Proof. Take any $\eta\in\mathcal{H}^1X$. By Proposition 9.2 on the left X-coaction,

$$\eta_{[-1]}\otimes\eta_{[0][-1]}\otimes\eta_{[0][0]}\in X\otimes X\otimes\mathcal{H}^1X.$$

Remember, for any 1-form ξ, that $\xi_{[0]}\cdot S^{-1}(\xi_{[-1]})$ is left-invariant. It follows that

$$\eta_{[-1]}\otimes\eta_{[0][0]}\cdot S^{-1}(\eta_{[0][-1]})\in X\otimes dB\cdot X,$$

so

$$\eta=\eta_{[0][0]}\cdot S^{-1}(\eta_{[0][-1]})\eta_{[-1]}\in dB\cdot X.$$

The other inclusion is immediate. □

Definition 9.5. ([22]) A *normalised left integral* for H is a map $\int : H\to k$ with $(\mathrm{id}\otimes\int)\Delta_H=1_H\cdot\int : H\to H$ and $\int 1_H=1$.

A Hopf algebra H has a normalised left integral if and only if it is cosemisimple. One easily checks that if $\int$ is a normalised left integral for H, then, for any right H-comodule Y with coaction $\varrho : Y\to Y\otimes H$, the map $(\mathrm{id}\otimes\int)\varrho : Y\to Y$ is a projection onto the coinvariant subspace $Y^{\mathrm{co}\,H}:=\{y\in Y : \varrho(y)=y\otimes 1\}$.

PROPOSITION 9.6. *Suppose that H has a normalised left integral, and that $\pi : X\to H$ satisfies Condition K. Then $\Omega^1B=dB\cdot B=(\mathcal{H}^1X)^{\mathrm{co}\,H}$.*

Proof. Apply the left integral to the result of Proposition 9.4. □

9.3. Bicovariant calculi on Hopf algebras using left-invariant 1-forms

In the case where the coproduct is differentiable, there is a construction of the calculus on a Hopf algebra X in terms of the left-invariant 1-forms L^1X which is due to Woronowicz [24].

There is an isomorphism of X-modules and -comodules (the module/comodule structures indicated by the dots)

$$\begin{aligned} {}^{\bullet}\Omega^1X^{\bullet}_{\bullet} &\longrightarrow (L^1X)^{\bullet}\otimes{}^{\bullet}X^{\bullet}_{\bullet}, \\ \xi &\longmapsto \xi_{[0]}\,S^{-1}(\xi_{[-1](2)})\otimes\xi_{[-1](1)}, \end{aligned} \tag{9.2}$$

with the inverse given by the product map. It is also a left X-module map, but with the left action on $L^1X\otimes X$ given by $x\triangleright(\eta\otimes y)=x_{(2)}\triangleright\eta\otimes x_{(1)}y$, and $x\triangleright\eta=x_{(2)}\eta S^{-1}(x_{(1)})$ for $\eta\in L^1X$ and $x,y\in X$.

The relation between the left X-action on L^1X and the right X-coaction $\varrho_X : L^1X\to L^1X\otimes X$ is summarised in the equation $\varrho_X(x\triangleright\eta)=x_{(2)}\triangleright\eta_{[0]}\otimes x_{(3)}\eta_{[1]}S^{-1}(x_{(1)})$. This fits the left action/right coaction version of a Yetter–Drinfeld module (cf. [8, §5]). By the standard results on Yetter–Drinfeld modules there is a braiding $\sigma : L^1X\otimes L^1X\to L^1X\otimes L^1X$ defined by $\sigma(\xi\otimes\eta)=\eta_{[0]}\otimes S(\eta_{[1]})\triangleright\xi$, with inverse $\sigma^{-1}(\xi\otimes\eta)=\xi_{[1]}\triangleright\eta\otimes\xi_{[0]}$.

We define the wedge product on L^1X as a quotient

$$L^1X\wedge L^1X=\frac{L^1X\otimes L^1X}{\ker(\sigma-\mathrm{id}\otimes\mathrm{id} : L^1X\otimes L^1X\to L^1X\otimes L^1X)},$$

and extend this to higher wedge products. We choose to do this extension by quotienting pairwise in each adjacent factor in $(L^1X)^{\otimes n}$, rather than using Woronowicz's antisymmetriser, and in general this might give a different choice of higher differential calculi. There is (as a matter of definition of the higher forms) an isomorphism

$$\Omega^n X \longrightarrow (L^1X)^{\wedge n} \otimes X, \tag{9.3}$$

with wedge product defined by $(\xi\otimes x)\wedge(\eta\otimes y)=\xi\wedge(x_{(2)}\triangleright\eta)\otimes x_{(1)}y$.

9.4. Identifying the higher-dimensional calculus on the base algebra

PROPOSITION 9.7. *Using the isomorphism* (9.2), $\mathcal{H}^1X$ *corresponds to* $\mathcal{K}\otimes X$.

Proof. Begin with a horizontal 1-form $\xi\in\mathcal{H}^1X$, and apply the isomorphism (9.2) to get $\xi_{[0]}S^{-1}(\xi_{[-1](2)})\otimes\xi_{[-1](1)}$. We need to show that $\pi_*(\xi_{[0]})\pi(S^{-1}(\xi_{[-1](2)}))\otimes\xi_{[-1](1)}=0$. As $\xi\in\mathcal{H}^1X$ we know that $\xi_{[-1]}\otimes\pi_*(\xi_{[0]})=0$, and the required result follows from this.

For the other direction, take $\eta\otimes x\in\mathcal{K}\otimes X$. Then applying the left X-coaction to ηx gives $x_{(1)}\otimes\eta x_{(2)}$, and applying $\mathrm{id}\otimes\pi_*$ to this gives $x_{(1)}\otimes\pi_*(\eta)\pi(x_{(2)})=0$. □

PROPOSITION 9.8. *The usual right* H*-coaction on* L^1X *restricts to one on* $\mathcal{K}$. *Also the usual left* X*-action on* L^1X *restricts to one on* $\mathcal{K}$.

Proof. For the coaction, for $\eta\in\mathcal{K}$ we need to show that $\Pi_{1,0}(\mathrm{id}\otimes\pi_*)\Delta_*\eta\in\mathcal{K}\otimes H$. To do this we need to show the vanishing of $\Pi_{1,0}(\pi_*\otimes\pi_*)\Delta_*\eta=\Pi_{1,0}\Delta_*\pi_*(\eta)=0$ (using the fact that π is a coalgebra map).

For the action, we have

$$\pi_*(x\triangleright\eta)=\pi_*(x_{(2)}\eta S^{-1}(x_{(1)}))=\pi(x_{(2)})\,0\,\pi(S^{-1}(x_{(1)}))=0. \qquad \square$$

COROLLARY 9.9. *If* $\pi\colon X\to H$ *satisfies Condition K, and* H *has a normalised left integral, then, using the isomorphism* (9.2), Ω^1B *corresponds to* $(\mathcal{K}\otimes X)^{\mathrm{co}\,H}$.

LEMMA 9.10. *If* H *has a normalised left integral* $\int$, *then for all* $a,c\in H$,

$$c_{(2)}\int(aS(c_{(1)}))=a_{(1)}\int(a_{(2)}S(c)).$$

Proof. For all $a,b\in H$ the left integral property gives

$$a_{(1)}\otimes a_{(2)}b_{(1)}\int(a_{(3)}b_{(2)})=a_{(1)}\otimes 1_H\int(a_{(2)}b).$$

Applying S^{-1} to the last factor gives

$$a_{(1)}\otimes S^{-1}(b_{(1)})S^{-1}(a_{(2)})\int(a_{(3)}b_{(2)})=a_{(1)}\otimes 1_H\int(a_{(2)}b).$$

Now multiply the second factor on the right by the first to get

$$S^{-1}(b_{(1)})\int(ab_{(2)})=a_{(1)}\int(a_{(2)}b).$$

Finally, putting $b=S(c)$ gives the result. □

LEMMA 9.11. *If H has a normalised left integral, then there is a projection $p\colon L^1X\to L^1X$ with image $\mathcal{K}$ which preserves the right H-coaction (i.e. p is right H-colinear, i.e. $\varrho p=(p\otimes \mathrm{id})\varrho$).*

Proof. Take any linear projection $p_0\colon L^1X\to L^1X$ with image $\mathcal{K}$, and define (using square brackets for the H-coaction)

$$p(\xi)=p_0(\xi_{[0]})_{[0]}\int(p_0(\xi_{[0]})_{[1]}S(\xi_{[1]})).$$

First we show that p is a projection to $\mathcal{K}$. Since the image of p_0 is $\mathcal{K}$, and $\mathcal{K}$ is coacted on by H, it is obvious from the formula that the image of p is contained in $\mathcal{K}$. Now suppose that $\xi\in\mathcal{K}$, and then

$$p(\xi)=\xi_{[0][0]}\int(\xi_{[0][1]}S(\xi_{[1]}))=\xi_{[0]}\int(\xi_{[1](1)}S(\xi_{[1](2)}))=\xi_{[0]}\,\varepsilon(\xi_{[1]})=\xi.$$

Finally we need to show the H-colinearity of p:

$$\begin{aligned}p(\xi)_{[0]}\otimes p(\xi)_{[1]}&=p_0(\xi_{[0]})_{[0][0]}\otimes p_0(\xi_{[0]})_{[0][1]}\int(p_0(\xi_{[0]})_{[1]}S(\xi_{[1]}))\\&=p_0(\xi_{[0]})_{[0]}\otimes p_0(\xi_{[0]})_{[1]}\int(p_0(\xi_{[0]})_{[2]}S(\xi_{[1]})),\end{aligned}$$

$$\begin{aligned}p(\xi_{[0]})\otimes\xi_{[1]}&=p_0(\xi_{[0][0]})_{[0]}\otimes\xi_{[1]}\int(p_0(\xi_{[0][0]})_{[1]}S(\xi_{[0][1]}))\\&=p_0(\xi_{[0]})_{[0]}\otimes\xi_{[2]}\int(p_0(\xi_{[0]})_{[1]}S(\xi_{[1]})).\end{aligned}$$

Now Lemma 9.10 gives the equality of these expressions. □

THEOREM 9.12. *If* $\pi: X \to H$ *satisfies Condition K, and* H *has a normalised left integral, then, using the isomorphism* (9.3), $\Omega^n B = (\mathcal{K}^{\wedge n} \otimes X)^{\mathrm{co}\,H}$ *(with the tensor coaction).*

Proof. We shall prove this by induction on n, starting at $\Omega^1 B = (\mathcal{K} \otimes X)^{\mathrm{co}\,H}$, which has been done in Corollary 9.9. Now assume the statement for n.

To show that $\Omega^{n+1} B \subset (\mathcal{K}^{\wedge(n+1)} \otimes X)^{\mathrm{co}\,H}$, we use $\Omega^{n+1} B \subset \Omega^n B \wedge \Omega^1 B$, the formula for the wedge product given in §9.3 and Proposition 9.8.

To show that $(\mathcal{K}^{\wedge(n+1)} \otimes X)^{\mathrm{co}\,H} \subset \Omega^{n+1} B$, take $(\xi \wedge \eta) \otimes x \in (\mathcal{K}^{\wedge(n+1)} \otimes X)^{\mathrm{co}\,H}$ (summation indices omitted for clarity), with $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^{\wedge n}$. Then

$$(\xi \wedge \eta) \otimes x = (p(\xi_{[0]}) \otimes S(\xi_{[1](1)})) \wedge (\xi_{[1](2)} \triangleright (\eta \otimes x)), \tag{9.4}$$

where we use square brackets for the right X-coaction on $L^1 X$ and $p: L^1 X \to L^1 X$ is the projection given in Lemma 9.11. In (9.4) the first factor in the wedge product is in $\mathcal{K} \otimes X$, and the second is in $\mathcal{K}^{\wedge n} \otimes X$. We use the colinearity of p to rewrite (9.4) as

$$(\xi \wedge \eta) \otimes x = (p(\xi_{[0][0]}) \otimes S(\xi_{[0][1]})) \wedge (\xi_{[1]} \triangleright (\eta \otimes x)), \tag{9.5}$$

and now it is evident that the first factor is in $(\mathcal{K} \otimes X)^{\mathrm{co}\,H} = \Omega^1 B$. It is not obvious that the second factor is H-invariant. However, we can integrate both sides of the equation over H, and this averages the second factor to be H-invariant, without changing the left-hand side. □

10. Homogeneous spaces as fibrations

In this section we still consider Hopf algebras X and H with bicovariant differential calculi. We assume that there exists a differentiable surjective Hopf algebra map $\pi: X \to H$. The differentiable right H-coaction on X is given by $\varrho = (\mathrm{id} \otimes \pi)\Delta: X \to X \otimes H$.

10.1. Checking the definition of fibration

LEMMA 10.1. *Suppose that* H *has a normalised left integral. For any right* H*-comodule and right* X*-module* V *such that the right action* $\triangleleft: V \otimes X \to V$ *is an* H*-comodule map (with the tensor product coaction), the action* $\triangleleft: V^{\mathrm{co}\,H} \otimes_B X \to V$ *is a bijective correspondence.*

Proof. Call the coaction $\varrho: V \to V \otimes H$. Take a linear map $\psi: H \to X$ so that $\pi \circ \psi = \mathrm{id}: H \to H$. Now define an inverse map $\tau: V \to V^{\mathrm{co}\,H} \otimes_B X$ for $\triangleleft: V^{\mathrm{co}\,H} \otimes_B X \to V$ by

$$\left(\left(\mathrm{id} \otimes \textstyle\int\right) \varrho\right)(a_{[0]} \triangleleft S(\psi(a_{[1]})_{(1)})) \otimes_B \psi(a_{[1]})_{(2)}.$$

The purpose of the operation $(\mathrm{id}\otimes\int)\varrho$ is to ensure that the result lies in $V^{\mathrm{co}\,H}\otimes_B X$ rather than just $V\otimes_B X$. The reader may verify that these two maps are inverse by direct calculation. The reader familiar with the Hopf–Galois theory may recognise this result as a consequence of [20, Theorem I]. □

COROLLARY 10.2. *If* $\pi: X\to H$ *satisfies Condition K, then, under the isomorphism* (9.3), $\Omega^m B\cdot X$ *corresponds to* $\mathcal{K}^{\wedge m}\otimes X$.

Proof. Put $V=\mathcal{K}^{\wedge m}\otimes X$ in Lemma 10.1, and use Theorem 9.12. □

COROLLARY 10.3. *For integer* $n\geqslant 1$, *we have that* $\Omega^m B\wedge\Omega^n X$ *corresponds to* $(\mathcal{K}^{\wedge m}\wedge(L^1X)^{\wedge n})\otimes X$, *and then*

$$\Xi^0_m X=\Omega^m B\cdot X=\mathcal{K}^{\wedge m}\otimes X,$$

$$\Xi^n_m X=\frac{\Omega^m B\wedge\Omega^n X}{\Omega^{m+1}B\wedge\Omega^{n-1}X}\cong\frac{\mathcal{K}^{\wedge m}\wedge(L^1X)^{\wedge n}}{\mathcal{K}^{\wedge(m+1)}\wedge(L^1X)^{\wedge(n-1)}}\otimes X,\quad n\geqslant 1.$$

Proof. Note that $\Omega^m B\wedge\Omega^n X=\Omega^m B\cdot X\wedge\Omega^n X$ and use Corollary 10.2. □

LEMMA 10.4. *For integers* $m\geqslant 0$ *and* $n\geqslant 1$, *there is an isomorphism*

$$\frac{\mathcal{K}^{\wedge m}\wedge(L^1X)^{\wedge n}}{\mathcal{K}^{\wedge(m+1)}\wedge(L^1X)^{\wedge(n-1)}}\cong\mathcal{K}^{\wedge m}\otimes\frac{(L^1X)^{\wedge n}}{\mathcal{K}\wedge(L^1X)^{\wedge(n-1)}}$$

induced by the map

$$[\varkappa_1\otimes\ldots\otimes\varkappa_m\otimes\xi_1\otimes\ldots\otimes\xi_n]\longmapsto(\varkappa_1\wedge\ldots\wedge\varkappa_m)\otimes[\xi_1\otimes\ldots\otimes\xi_n],$$

for $\varkappa_i\in\mathcal{K}$ *and* $\xi_i\in L^1X$, *and where* $[\cdot]$ *denotes equivalence class.*

Proof. First we must show that the map given in Lemma 10.4 is well defined, and to do this we must use the braiding σ in the definition of wedge product. The space on the left-hand side is defined to be the quotient of $\mathcal{K}^{\otimes m}\otimes(L^1X)^{\otimes n}$ by a subspace spanned by elements of the form $\varkappa_1\otimes\ldots\otimes\varkappa_m\otimes\xi_1\otimes\ldots\otimes\xi_n$ where at least one of the following statements is true:

(a) $\xi_1\in\mathcal{K}$;

(b) for some $1\leqslant i\leqslant m-1$, $\sigma(\varkappa_i\otimes\varkappa_{i+1})=\varkappa_i\otimes\varkappa_{i+1}$;

(c) $\sigma(\varkappa_m\otimes\xi_1)=\varkappa_m\otimes\xi_1$;

(d) for some $1\leqslant i\leqslant n-1$, $\sigma(\xi_i\otimes\xi_{i+1})=\xi_i\otimes\xi_{i+1}$.

We need to show that all these elements are mapped to zero. In cases (a) and (d) we have $[\xi_1\otimes\ldots\otimes\xi_n]=0$. In case (b) we have $\varkappa_1\wedge\ldots\wedge\varkappa_m=0$. In case (c),

$$\sigma(\varkappa_m\otimes\xi_1)=\xi_{1[0]}\otimes S(\xi_{1[1]})\triangleright\varkappa_m\in L^1X\otimes\mathcal{K},$$

since the left X-action restricts to $\mathcal{K}$. It then follows that we are in case (a).

The inverse map is

$$[\varkappa_1\otimes\ldots\otimes\varkappa_m]\otimes[\xi_1\otimes\ldots\otimes\xi_n]\longmapsto[\varkappa_1\otimes\ldots\otimes\varkappa_m\otimes\xi_1\otimes\ldots\otimes\xi_n],$$

and showing that this is well defined is rather easier than for the forward map. □

THEOREM 10.5. *Suppose that X and H are Hopf algebras with bicovariant differential structure, and that $\pi: X\to H$ is a surjective differentiable Hopf algebra map. Additionally suppose that*

(1) *$\pi: X\to H$ satisfies Condition K (see Definition* 9.3);

(2) *H has a normalised left integral.*

Then the inclusion $B=X^{\mathrm{co}\,H}\to X$ is a differentiable fibration (see Definition 5.3), *where $\varrho=(\mathrm{id}\otimes\pi)\Delta_X: X\to X\otimes H$. Here B has the differential structure given by Theorem* 9.12.

Proof. From Definition 5.3 we need to show that the map $\Theta_m:\Omega^m B\otimes_B\Xi_0^*X\to\Xi_m^*X$ defined by $\xi\otimes[\eta]_0\mapsto[\xi\wedge\eta]_m$ is invertible for all $m\geqslant 0$.

We begin by using the fact that $\Omega^m B\otimes_B\Xi_0^*X\cong\Omega^m B\otimes_B X\otimes_X\Xi_0^*X$. Since we have $\Omega^m B=(\mathcal{K}^{\wedge m}\otimes X)^{\mathrm{co}\,H}$, Lemma 10.1 gives $\Omega^m B\otimes_B\Xi_0^*X\cong(\mathcal{K}^{\wedge m}\otimes X)\otimes_X\Xi_0^*X$. From §9.3 the right X-action on $\mathcal{K}^{\wedge m}\otimes X$ is just multiplication on the second factor, so

$$\Omega^m B\otimes_B\Xi_0^*X\cong\mathcal{K}^{\wedge m}\otimes X\otimes_X\Xi_0^*X\cong\mathcal{K}^{\wedge m}\otimes\Xi_0^*X,$$

and the result follows from Corollary 10.3 and Lemma 10.4. □

10.2. Identifying the fibre of the fibration

Assume the conditions of Theorem 10.5.

LEMMA 10.6. *$L^1X\wedge\mathcal{K}\subset\mathcal{K}\wedge L^1X$.*

Proof. From Corollary 10.2, $X\cdot dB\subset X\cdot dB\cdot X=X\cdot(\mathcal{K}\otimes X)=(\mathcal{K}\otimes X)=dB\cdot X$. Applying d to this gives $dX\wedge dB\subset dB\wedge dX$. From this we conclude that $dX\wedge dB\cdot X\subset dB\wedge dX\cdot X$, so from Condition K, $dX\wedge\mathcal{K}\subset dB\wedge dX\cdot X$. Multiplying again by X,

$$X\cdot dX\wedge\mathcal{K}\subset X\cdot dB\wedge dX\cdot X\subset dB\wedge X\cdot dX\cdot X,$$

so $\Omega^1X\wedge\mathcal{K}\subset dB\wedge\Omega^1X$. From Proposition 9.7, $dB\subset\mathcal{K}\cdot X$, so $\Omega^1X\wedge\mathcal{K}\subset\mathcal{K}\wedge\Omega^1X$. Then $L^1X\wedge\mathcal{K}\subset\mathcal{K}\wedge\Omega^1X$. Note that $L^1X\wedge\mathcal{K}$ is left-invariant with respect to the left X-coaction, and consider $\xi\wedge\eta\in{}^{\mathrm{co}X}(\mathcal{K}\wedge\Omega^1X)$. Then, by invariance of $\xi\wedge\eta$ and ξ,

$$\begin{aligned}\xi\wedge\eta&=S((\xi\wedge\eta)_{[-1]})(\xi\wedge\eta)_{[0]}\\&=S(\eta_{[-1]})(\xi\wedge\eta_{[0]})\\&=S(\eta_{[-1]})_{(3)}\xi\wedge S^{-1}(S(\eta_{[-1]})_{(2)})S(\eta_{[-1]})_{(1)}\eta_{[0]}\\&=S(\eta_{[-1](1)})\xi S^{-1}(S(\eta_{[-1](2)}))\wedge S(\eta_{[-1](3)})\eta_{[0]}\\&=S(\eta_{[-1](1)})\triangleright\xi\wedge S(\eta_{[-1](2)})\eta_{[0]}\\&=S(\eta_{[-1]})\triangleright\xi\wedge S(\eta_{[0][-1]})\eta_{[0][0]}.\end{aligned}$$

As the left X-action restricts to $\mathcal{K}$, this is in $\mathcal{K}\otimes L^1X$. □

PROPOSITION 10.7. *The map* $\pi_*^{\otimes n}\colon(L^1X)^{\otimes n}\to(L^1H)^{\otimes n}$ *induces an invertible map*

$$\tilde{\pi}\colon\frac{(L^1X)^{\wedge n}}{\mathcal{K}\wedge(L^1X)^{\wedge(n-1)}}\longrightarrow(L^1H)^{\wedge n}.$$

Proof. The domain of $\tilde{\pi}$ is the quotient of $(L^1X)^{\otimes n}$ by a subspace spanned by elements of the form $\xi_1\otimes\ldots\otimes\xi_n$, where at least one of the following statements is true:

(a) $\xi_1\in\mathcal{K}$.

(b) For some $1\leqslant i\leqslant n-1$, $\sigma(\xi_i\otimes\xi_{i+1})=\xi_i\otimes\xi_{i+1}$.

The codomain of $\tilde{\pi}$ is the quotient of $(L^1H)^{\otimes n}$ by a subspace spanned by elements of the form $\eta_1\otimes\ldots\otimes\eta_n$, where the following statement is true:

(c) For some $1\leqslant i\leqslant n-1$, $\sigma(\eta_i\otimes\eta_{i+1})=\eta_i\otimes\eta_{i+1}$.

Case (a) maps to zero under $\pi_*^{\otimes n}$ by definition of $\mathcal{K}$. Case (b) maps to case (c) as π is a Hopf algebra map. Thus $\tilde{\pi}$ is well defined.

Given the hypothesis, it is automatic that $\tilde{\pi}$ is onto.

To show that $\tilde{\pi}$ is one-to-one, it is sufficient to show that the subspace quotienting $(L^1X)^{\otimes n}$ in the domain contains all elements of the form $\xi_1\otimes\ldots\otimes\xi_n$, where the following statement is true:

(d) For some $1\leqslant i\leqslant n$, $\xi_i\in\mathcal{K}$.

This follows from repeated application of Lemma 10.6. □

11. Example: The noncommutative Hopf fibration with a bicovariant calculus

In this section we return to the algebras X, H and B discussed in §8.1, but now we consider a (minimal) bicovariant differential calculus on $\mathcal{A}(SL_q(2))$. In view of the results of §§ 9 and 10 our task will be to construct a suitable calculus on H so that the map $\pi\colon X\to H$ is differentiable and then to check that π satisfies Condition K in Definition 9.3.

11.1. A 4D bicovariant calculus on $\mathcal{A}(SL_q(2))$

This differential calculus on $X=\mathcal{A}(SL_q(2))$ was introduced by Woronowicz in [24] and is generated by four left-invariant 1-forms $\{\omega^1,\omega^2,\omega^+,\omega^-\}$. The differentials of the generators are given by

$$\begin{aligned}
d\alpha &= \frac{q-q^{-1}-q^{-2}}{q+1}\alpha\omega^1 - q^{-2}\beta\omega^+ + \frac{q^{-1}}{q+1}\alpha\omega^2,\\
d\beta &= \frac{q}{q+1}\beta\omega^1 - q^{-2}\alpha\omega^- - \frac{q^{-2}}{q+1}\beta\omega^2,\\
d\gamma &= \frac{q-q^{-1}-q^{-2}}{q+1}\gamma\omega^1 - q^{-2}\delta\omega^+ + \frac{q^{-1}}{q+1}\gamma\omega^2,\\
d\delta &= \frac{q}{q+1}\delta\omega^1 - q^{-2}\gamma\omega^- - \frac{q^{-2}}{q+1}\delta\omega^2.
\end{aligned} \tag{11.1}$$

We have the commutation relations

$$\begin{aligned}
&\omega^2\alpha = q\alpha\omega^2 - (q-q^{-1})\beta\omega^+ + q(q-q^{-1})^2\alpha\omega^1, && \omega^2\beta = q^{-1}\beta\omega^2 - (q-q^{-1})\alpha\omega^-,\\
&\omega^-\alpha = \alpha\omega^- - (q^2-1)\beta\omega^1, && \omega^-\beta = \beta\omega^-,\\
&\omega^+\alpha = \alpha\omega^+, && \omega^+\beta = \beta\omega^+ - (q^2-1)\alpha\omega^1,\\
&\omega^1\alpha = q^{-1}\alpha\omega^1, && \omega^1\beta = q\beta\omega^1,
\end{aligned} \tag{11.2}$$

and these relations with the replacements $\alpha\mapsto\gamma$ and $\beta\mapsto\delta$.

11.2. The differentiability of $\pi\colon X\to H$

We use the Hopf algebra map π from §8.1, and assume that the map $\pi\colon X\to H$ is differentiable, i.e. that it extends to a map π_* of differential graded algebras. Applying π_* to the expression for $d\beta$ in (11.1) gives $z\pi_*(\omega^-)=0$, and since z is invertible we deduce that $\pi_*(\omega^-)=0$. Likewise the expression for $d\gamma$ in (11.1) gives $\pi_*(\omega^+)=0$. Now the sixth equation in (11.2) gives $(q^2-1)z\pi_*(\omega^1)=0$, so if $q\neq\pm1$ we get $\pi_*(\omega^1)=0$. Then the equations for $d\alpha$ and $d\delta$ in (11.1) give

$$dz = \frac{q^{-1}}{q+1}z\pi_*(\omega^2) \quad\text{and}\quad -z^{-1}\cdot dz\cdot z^{-1} = -\frac{q^{-2}}{q+1}z^{-1}\pi_*(\omega^2).$$

From this we get

$$\pi_*(\omega^2) = q(q+1)z^{-1}\cdot dz \quad\text{and}\quad dz\cdot z^{-1} = q^{-1}z^{-1}\cdot dz.$$

Just as in the case of the 3D calculus, we must have a noncommutative calculus for the commutative algebra H. Note that $\mathcal{K}$, the left-invariant forms which are in the kernel of π_*, has basis $\{\omega^1,\omega^+,\omega^-\}$.

11.3. Verifying Condition K

PROPOSITION 11.1. *All of ω^-, ω^+ and ω^1 are in $dB\cdot X$.*

Proof. Begin by calculating

$$\begin{aligned}
d\beta_{(2)}\cdot S^{-1}(\beta_{(1)}) &= d\beta\cdot\delta - q\,d\delta\cdot\beta \\
&= \frac{q}{q+1}\beta\omega^1\delta - q^{-2}\alpha\omega^-\delta - \frac{q^{-2}}{q+1}\beta\omega^2\delta \\
&\quad - \frac{q^2}{q+1}\delta\omega^1\beta + q^{-1}\gamma\omega^-\beta + \frac{q^{-1}}{q+1}\delta\omega^2\beta \\
&= -q^{-2}\alpha\delta\omega^- - \frac{q^{-2}}{q+1}\beta(q^{-1}\delta\omega^2 - (q-q^{-1})\gamma\omega^-) \\
&\quad + q^{-1}\gamma\beta\omega^- + \frac{q^{-1}}{q+1}\delta(q^{-1}\beta\omega^2 - (q-q^{-1})\alpha\omega^-) \\
&= -q^{-2}(\alpha\delta - q\gamma\beta)\omega^- - \frac{q^{-2}(q^2-1)}{q+1}(\delta\alpha - q^{-1}\beta\gamma)\omega^- \\
&= -q^{-1}\omega^-,
\end{aligned}$$

and also

$$\begin{aligned}
d\gamma_{(2)}\cdot S^{-1}(\gamma_{(1)}) &= d\gamma\cdot\alpha - q^{-1}d\alpha\cdot\gamma \\
&= \frac{q-q^{-1}-q^{-2}}{q+1}\gamma\omega^1\alpha - q^{-2}\delta\omega^+\alpha + \frac{q^{-1}}{q+1}\gamma\omega^2\alpha \\
&\quad - q^{-1}\frac{q-q^{-1}-q^{-2}}{q+1}\alpha\omega^1\gamma + q^{-3}\beta\omega^+\gamma - \frac{q^{-2}}{q+1}\alpha\omega^2\gamma \\
&= -q^{-2}\delta\alpha\omega^+ + \frac{q^{-1}}{q+1}\gamma(q\alpha\omega^2 - (q-q^{-1})\beta\omega^+ + q(q-q^{-1})^2\alpha\omega^1) \\
&\quad + q^{-3}\beta\gamma\omega^+ - \frac{q^{-2}}{q+1}\alpha(q\gamma\omega^2 - (q-q^{-1})\delta\omega^+ + q(q-q^{-1})^2\gamma\omega^1) \\
&= -q^{-2}\delta\alpha\omega^+ - \frac{q^{-1}}{q+1}(q-q^{-1})\gamma\beta\omega^+ \\
&\quad + q^{-3}\beta\gamma\omega^+ + \frac{q^{-2}}{q+1}(q-q^{-1})\alpha\delta\omega^+ \\
&= -q^{-2}\omega^+ + \frac{q^{-3}(q^2-1)}{q+1}\omega^+ \\
&= -q^{-3}\omega^+.
\end{aligned}$$

Now we have

$$\begin{aligned}
d(\alpha\beta)_{(2)}\cdot S^{-1}((\alpha\beta)_{(1)}) &= d\alpha_{(2)}\cdot S^{-1}(\alpha_{(1)})\,\varepsilon(\beta)-q^{-1}\alpha_{(2)}\omega^{-}S^{-1}(\alpha_{(1)})\\
&= \gamma\omega^{-}\beta-q^{-1}\alpha\omega^{-}\delta\\
&= -q^{-1}(\alpha\delta-q\gamma\beta)\,\omega^{-}\\
&= -q^{-1}\omega^{-},\\
d(\gamma\beta)_{(2)}\cdot S^{-1}((\gamma\beta)_{(1)}) &= d\gamma_{(2)}\cdot S^{-1}(\gamma_{(1)})\,\varepsilon(\beta)-q^{-1}\gamma_{(2)}\omega^{-}S^{-1}(\gamma_{(1)})\\
&= q^{-2}\alpha\omega^{-}\gamma-q^{-1}\gamma\omega^{-}\alpha\\
&= q^{-2}\alpha(\gamma\omega^{-}-(q^{2}-1)\delta\omega^{1})-q^{-1}\gamma(\alpha\omega^{-}-(q^{2}-1)\beta\omega^{1})\\
&= q^{-1}(q^{2}-1)\gamma\beta\omega^{1}-q^{-2}(q^{2}-1)\alpha\delta\omega^{1}\\
&= (q^{-2}-1)(\alpha\delta-q\beta\gamma)\omega^{1}\\
&= (q^{-2}-1)\,\omega^{1},\\
d(\delta\gamma)_{(2)}\cdot S^{-1}((\delta\gamma)_{(1)}) &= d\delta_{(2)}\cdot S^{-1}(\delta_{(1)})\,\varepsilon(\gamma)-q^{-3}\delta_{(2)}\omega^{+}S^{-1}(\delta_{(1)})\\
&= q^{-4}\beta\omega^{+}\gamma-q^{-3}\delta\omega^{+}\alpha\\
&= -q^{-3}(\delta\alpha-q^{-1}\beta\gamma)\,\omega^{+}\\
&= -q^{-3}\omega^{+}.
\end{aligned}$$

This proves the claim since, for all $b\in B$, $\Delta b\in X\otimes B$. □

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