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On the average distance property and certain energy integrals

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Abstract. One of our main results is the following: Let X be a compact connected subset of the Euclidean space $\mathbf{R}^n$ and $r(X, d_2)$ the rendezvous number of X, where d_2 denotes the Euclidean distance in $\mathbf{R}^n$. (The rendezvous number $r(X, d_2)$ is the unique positive real number with the property that for each positive integer n and for all (not necessarily distinct) $x_1, x_2, \dots, x_n$ in X, there exists some x in X such that $(1/n)\sum_{i=1}^n d_2(x_i, x) = r(X, d_2)$.) Then there exists some regular Borel probability measure μ_0 on X such that the value of $\int_X d_2(x, y)\, d\mu_0(y)$ is independent of the choice x in X, if and only if $r(X, d_2) = \sup_\mu \int_X \int_X d_2(x, y)\, d\mu(x)\, d\mu(y)$, where the supremum is taken over all regular Borel probability measures μ on X.

1. Introduction

In 1964 O. Gross published the following remarkable result.

Theorem. (Gross) *Let (X, d) be a compact connected metric space. Then there is a unique positive real number $r(X, d)$ with the property that for each positive integer n and for all (not necessarily distinct) $x_1, x_2, \dots, x_n$ in X, there exists an x in X such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(x_i, x) = r(X, d).$$

For a proof of this theorem see [6]. An excellent survey on this topic is given in [4]. See also for example [9], [10], [11], [15], and [16]–[19].

Remark 1.

(a) In the situation of Gross's theorem we say that (X, d) has the average distance property with rendezvous number $r(X, d)$.

(b) $\frac{1}{2}D(X, d) \le r(X, d) < D(X, d)$, where $D(X, d)$ is the diameter of X. For a proof see Theorem 2 in [6]. The positive real number $m(X, d) = r(X, d)/D(X, d)$ is often called the dispersion or magic number of X.

(c) Graham Elton first generalized Gross's theorem in the following sense (for a proof see [4]).

Let (X,d) be a compact connected metric space. Then $r(X,d)$ is the unique positive real number with the following property: for each regular Borel probability measure μ on X, there exists an x in X such that

$$\int_X d(x,y)\,d\mu(y) = r(X,d).$$

Moreover there are regular Borel probability measures μ and ν on X such that

$$\int_X d(x,y)\,d\nu(y) \le r(X,d) \le \int_X d(x,y)\,d\mu(y)$$

for all x in X.

(d) $$r(X,d) = \inf_\mu \max_{x\in X} \int_X d(x,y)\,d\mu(y) = \sup_\mu \min_{x\in X} \int_X d(x,y)\,d\mu(y),$$

where the infimum and supremum is taken over all regular Borel probability measures μ on X. For a proof see [4].

It turns out, that for a given compact connected metric space (X,d) the explicit calculation of the rendezvous number $r(X,d)$ is often rather difficult. For example $r(X,d)$ is still unknown for such a nice space X as a general ellipse in $\mathbf{R}^2$ and d the Euclidean distance.

As pointed out in [4] the key trick for calculating $r(X,d)$ in some cases is to find some regular probability measure μ_0 on X such that the value of $\int_X d(x,y)\,d\mu_0(y)$ is independent of the choice of x.

Then by the definition of $r(X,d)$ and Elton's generalization of Gross's theorem we get $r(X,d)=\int_X d(x,y)\,d\mu_0(y)$ for an arbitrary choice of x (compare with Theorem 1 in [10]).

S. Morris and P. Nickolas used this result to calculate $r(S^n,d)$, where S^n is the sphere of radius $\frac{1}{2}$ in the Euclidean space $\mathbf{R}^{n+1}$ equipped with the Euclidean distance d, $n\ge 1$. They show the following theorem.

Theorem. (Morris, Nickolas) *Let S^n and d be defined as above and let λ be the normalisation of the usual n-dimensional Lebesgue measure on S^n, then we have*

$$r(S^n,d) = \int_{S^n} d(x,y)\,d\lambda(y) = \frac{2^{n-1}\left[\Gamma\left(\frac{1}{2}(n+1)\right)\right]^2}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(2n+1)\right)}$$

for any x in S^n, where Γ is the gamma function.

For a proof see [10].

In [4] the existence of such a μ_0 is used to calculate $r(X_\varphi, d)$, where X_φ is any arc of a circle with radius $\frac{1}{2}$ subtending an angle φ at the centre and d denotes the Euclidean distance in $\mathbf{R}^2$ (see Example 5 in [4]).

Unfortunately such a probability measure μ_0 does not exist in many cases (for example all convex compact subsets in the Euclidean space $\mathbf{R}^n$, which are not line segments), because of the following result due to David Wilson.

Theorem. (Wilson) *Let X be a compact connected subset of a rotund normed space. (This means that $\|x+y\|<\|x\|+\|y\|$ unless x and y are linearly dependent. Observe that the Euclidean space $\mathbf{R}^n$, for any n, is a rotund normed space.)*

Suppose that, for some regular Borel probability measure μ_0 on X the value of $\int_X \|x-y\|\, d\mu_0(y)$ is independent of the choice of x, and that X is not a line segment. Then no three points of X are colinear.

For a proof see Proposition 2 in [4].

Remark 2. If we consider normed spaces which are not rotund, the situation can be rather different:

For example let $X=\left\{x\in\mathbf{R}^2 : \frac{1}{2}(\|x-(1,1)\|_1+\|x+(1,1)\|_1)=2\right\}$, where $\|\cdot\|_1$ denotes the usual 1-norm in $\mathbf{R}^2$. Since $(1,1)$ and $(-1,-1)$ are elements of X, the regular Borel probability measure μ_0 on X, defined as $\mu_0=\frac{1}{2}(\delta_{(1,1)}+\delta_{(-1,-1)})$ (δ_x denotes the point measure concentrated on x) has the property, that $\int_X \|x-y\|_1\, d\mu_0(y)=2$ for all x in X. But we get easily that $X=\{x=(x_1,x_2)\in\mathbf{R}^2 : \max(|x_1|,|x_2|)\le 1\}$ and hence X is convex.

In [11] P. Nickolas and D. Yost first noted the connection between Gross's theorem and certain energy integrals studied in earlier papers on distance geometry, especially by R. Alexander and K. B. Stolarsky (see also G. Björck in [2]). A survey on this topic is given in [1]. See also [13] and [14].

In [11] Nickolas and Yost give a new proof of a result essentially due to Stolarsky (see Theorem 2 in [13]), which also follows from the earlier work of Björck (see [2]). They also find an elegant proof in the case $n=1$ by using Gross's theorem.

Theorem. (Stolarsky, Björck) *Let S^n be the sphere of radius $\frac{1}{2}$ in the Euclidean space $\mathbf{R}^{n+1}$ equipped with Euclidean distance. Then we have*

$$\sup_\mu \int_{S^n}\int_{S^n} d(x,y)\, d\mu(x)\, d\mu(y) = r(S^n, d),$$

where the supremum is taken over all regular Borel probability measures μ on S^n.

For a proof see Proposition 4 in [11].

We will generalize this result in Proposition 3 of this paper (see also the above mentioned theorem of Morris and Nickolas).

In this paper we develop a close connection between the relation of the smallest upper bound of energy integrals of the form $\int_X \int_X d(x,y)\,d\mu(x)\,d\mu(y)$, μ a regular Borel probability measure on the compact connected metric space (X,d), and the rendezvous number $r(X,d)$ on one hand, and the existence of some regular Borel probability measure μ_0 on X such that the value of $\int_X d(x,y)\,d\mu_0(y)$ is independent of the choice of x, on the other hand.

The last part of the paper will use a uniform distribution method for recursive approximation of such a μ_0 (if it exists), in the case of compact connected subsets of the Euclidean space $\mathbf{R}^n$.

2. Basic definitions and notation

Let (X,d) be a compact metric space. By $C(X)$ we denote the Banach space of all real valued continuous functions on X equipped with the usual sup-norm. By $M(X)$ we denote the space of all regular Borel probability measures on X.

It is well known that $M(X)$ equipped with the w^*-topology (a net μ_α tends to μ if and only if $\int_X f(x)\,d\mu_\alpha(x)$ tends to $\int_X f(x)\,d\mu(x)$ for all f in $C(X)$) becomes a compact convex space, such that the w^*-topology can be metrized (for example by the so called Prohorov metric). For x in X let δ_x, $\delta_x \in M(X)$, be the point measure concentrated on x.

It follows that the set $\{\delta_x : x \in X\}$ is the set of extreme points of $M(X)$ and hence the Krein–Milman theorem applies, that $M(X)$ is the w^*-closure of the convex hull of $\{\delta_x : x \in X\}$. For basic properties of $M(X)$ see for example [3] and [12]. With d_2 we always denote the Euclidean distance in some $\mathbf{R}^n$.

Now we follow essentially the notation of Björck in [2].

For any $(\mu,\nu) \in M(X) \times M(X)$ we define

$$I(\mu,\nu) = \int_X \int_X d(x,y)\,d\mu(x)\,d\nu(y), \quad I(\mu) = I(\mu,\mu).$$

For $\mu \in M(X)$ let d_μ be defined as

$$d_\mu : X \to \mathbf{R}, \quad d_\mu(x) = \int_X d(x,y)\,d\mu(y).$$

Of course

$$I(\mu,\nu) = \int_X d_\mu(x)\,d\nu(x) = \int_X d_\nu(x)\,d\mu(x).$$

The positive real number $M(X,d)$ is defined as

$$M(X,d) = \sup_{\mu \in M(X)} I(\mu).$$

It is easily shown that

$$\tfrac{1}{2}D(X,d) \leq M(X,d) \leq D(X,d),$$

where $D(X,d)$ denotes the diameter of X.

Furthermore μ_1 in $M(X)$ is called maximal if $M(X,d)=I(\mu_1)$ and μ_0 in $M(X)$ is called d-invariant if the function d_{μ_0} is constant on X.

3. The results

The first proposition is an easy consequence of w^*-compactness of $M(X)$, mentioned in Theorem 1 in [2] for the case $d=d_2^\lambda$, $\lambda>0$.

Proposition 1. *Let (X,d) be a compact metric space. Then there exists some maximal measure μ_1 in $M(X)$.*

The next result is due to Björck.

Theorem. (Björck) *Let X be a compact subset of the Euclidean space $\mathbf{R}^n$. Then we have:*

(1) *The maximal measure μ_1 in $M(X)$ is unique.*

(2) *Let X_{μ_1} be the support of the unique maximal measure μ_1. It follows that* $(d_2)_{\mu_1}(x)=M(X,d_2)$ *for all x in X_{μ_1} and* $(d_2)_{\mu_1}(x)\leq M(X,d_2)$ *for all x in X.*

Moreover X_{μ_1} is a subset of the boundary of X for $n\geq 2$.

For a proof see Theorems 1, 2, 3 in [2].

The first part of the following theorem follows easily from the definition of $r(X,d)$ and $M(X,d)$ and is mentioned on p. 267 in [2] for $d=d_2^\alpha$, $\alpha>0$.

Theorem 1. *Let (X,d) be a compact connected metric space. Then we have:*

(1) $r(X,d)\leq M(X,d)$.

(2) *If $r(X,d)=M(X,d)$, then there is some d-invariant measure μ_0 in $M(X)$.*

The question arises if the existence of some d-invariant measure μ_0 implies $r(X,d)=M(X,d)$. In general the answer is no, because of the following result.

Proposition 2. *There is some compact connected metric space (X,d) with some d-invariant measure μ_0, such that $r(X,d)<M(X,d)$.*

Now it turns out that the so called quasihypermetric property implies the reversed implication in Theorem 1, part (2). Let us recall the definition and some examples for quasihypermetric spaces.

A metric space (X,d) is called quasihypermetric if

$$\sum_{i,j=1}^{n} c_i c_j d(x_i,x_j) \leq 0,$$

for all $n \in \mathbf{N}$, $x_1, \dots, x_n$ in X and all $c_1, \dots, c_n \in \mathbf{R}$ with $c_1 + \dots + c_n = 0$.

Furthermore remember that a real linear normed space E is called embeddable in L_p (for some fixed $1 \leq p < \infty$) if there exists some measure space (Ω, Σ, μ) and a linear isometry T from E into $L_p(\Omega, \Sigma, \mu)$.

In [8] P. Lévy proved that $(E, \|\cdot\|)$ is embeddable in L_p, $1 \leq p \leq 2$, if and only if, for all $n \in \mathbf{N}$, $x_1, \dots, x_n \in E$ and all $c_1, \dots, c_n \in \mathbf{R}$ with $c_1 + \dots + c_n = 0$ we have

$$\sum_{i,j=1}^{n} c_i c_j \|x_i - x_j\|^p \leq 0.$$

In [5] it is shown that $\mathbf{R}^n$ equipped with the usual p-norm ($1 \leq p \leq \infty$) is embeddable in L_1 if and only if $1 \leq p \leq 2$.

Furthermore it is known that each two dimensional real normed space is L^1 embeddable (for example see [7]).

From all this we have the following examples for quasihypermetric spaces:

(1) The Euclidean space $\mathbf{R}^n$, $n \geq 1$.

(2) The n-dimensional space $\mathbf{R}^n$ equipped with the usual p-norm, $1 \leq p \leq 2$, $n \geq 1$.

(3) All two dimensional real normed spaces.

The n-dimensional space $\mathbf{R}^n$ equipped with the usual p-norm, $2 < p \leq \infty$, $n \geq 3$ is not quasihypermetric.

Back to the question raised after Theorem 1 we have the following theorem.

Theorem 2. *Let (X,d) be a compact connected quasihypermetric space with some d-invariant measure μ_0 in $M(X)$. It follows that $r(X,d) = M(X,d)$ and μ_0 is maximal on X.*

We obtain the following corollary.

Corollary 1. *Let (X,d) be a compact connected quasihypermetric space.*

Then there exists some d-invariant measure μ_0 in $M(X)$ (which is then maximal too) if and only if $r(X,d) = M(X,d)$.

Since many papers on Gross's theorem deal with the Euclidean case (for example see [11] and [15]) the following result is of special interest.

Proposition 3. *Let X be a compact connected subset of the Euclidean space $\mathbf{R}^n$. Then there exists some d_2-invariant measure μ_0 in $M(X)$ if and only if $r(X,d)=M(X,d)$.*

Moreover if such a μ_0 exists, it is unique and $\mu_0=\mu_1$, where μ_1 is the unique maximal measure on X due to the theorem of Björck mentioned above.

Compare this result to the theorem of Stolarsky (resp. Björck) mentioned in the introduction of this paper. (Let $(X,d)=(S^{n-1},d_2)$ and $\mu_0=\mu_1=\lambda$, the normalized Lebesgue measure on S^{n-1}.) It is worth noticing that the proofs of this theorem given in [13] and [2] used the quasihypermetric property of the Euclidean space $\mathbf{R}^n$ in a more or less hidden form (see Theorem 1 in [13] and Lemma 1 in [2] and compare with formula (*) resp. (**) in the proof of Theorem 2).

Remark 3. If a compact connected space (X,d) has a d-invariant measure μ_0 in $M(X)$, it is not unique in general.

For example, let $X=S^1$ and $d(e^{i\alpha},e^{i\beta})=\min(|\alpha-\beta|,|2\pi-(\alpha-\beta)|)$ for $0\le\alpha,\beta<2\pi$. It follows easily that for all $0\le\alpha<\pi$ the measures $\mu_\alpha=\frac{1}{2}(\delta_{e^{i\alpha}}+\delta_{e^{i(\alpha+\pi)}})$ are d-invariant on (X,d).

We now focus our attention on compact subsets (X,d_2) of the Euclidean space $\mathbf{R}^n$. Stolarsky showed (see Theorem 2 in [14]) that the term

$$r(S^{n-1},d_2)-\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)$$

is essentially the L_2-cap discrepancy of the finite point set $\{x_1,\dots,x_N\}$ on $X=S^{n-1}$, the Euclidean sphere for all $n\ge 2$.

Therefore a sequence $(X_N)_{N\ge1}$ on S^{n-1} is uniformly distributed with respect to $\mu_0=\mu_1=\lambda$ (the normalized Lebesgue measure on S^{n-1}) if and only if

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)\to r(S^{n-1},d_2),\quad \text{as } N\to\infty.$$

We generalize this as an easy consequence of Björck's theorem.

Proposition 4. *Let (X,d_2) be a compact subset of the Euclidean space $\mathbf{R}^n$ and μ_1 its unique maximal measure, due to the theorem of Björck.*

Then a sequence $(x_N)_{N\ge1}$ in X is uniformly distributed with respect to μ_1 if and only if

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)\to M(X,d_2),\quad \textit{as } N\to\infty.$$

From this we obtain the following result.

Proposition 5. *Let (X, d_2) be a compact connected subset of the Euclidean space $\mathbf{R}^n$ such that $r(X, d_2) = M(X, d_2)$.*

Furthermore let μ_0 be its unique d_2-invariant measure, $\mu_0 = \mu_1$, μ_1 the unique maximal measure on X (see Proposition 3).

Now choose some arbitrary point x_1 in X. For $N \geq 1$ choose some point x_{N+1} in X, which exists by Gross's theorem, such that

$$\frac{1}{N}\sum_{i=1}^{N} d(x_i, x_{N+1}) \geq r(X, d_2).$$

Then the so obtained sequence $(x_N)_{N\geq 1}$ in X is uniformly distributed with respect to μ_0.

As a consequence of Proposition 5 we obtain the following recursive method for approximating both $r(X, d_2)$ and μ_0 if (X, d_2) fulfills the conditions of Proposition 5.

Corollary 2. *Let (X, d_2) be as in Proposition 5. Now choose some arbitrary point x_1 in X. For $N \geq 1$ choose some point x_{N+1} in X such that*

$$\frac{1}{N}\sum_{i=1}^{N} d_2(x_i, x_{N+1}) = \max_{x\in X} \frac{1}{N}\sum_{i=1}^{N} d(x_i, x).$$

Then we have:

(1) *The so obtained sequence $(x_N)_{N\geq 1}$ in X is uniformly distributed with respect to μ_0.*

(2)

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i, x_j) \to r(X, d_2), \quad \text{as } N \to \infty.$$

We illustrate Corollary 2 by a very simple example.

Example 1. Let $X = [0, 1]$. Of course $\mu_0 \in M(X)$, $\mu_0 = \frac{1}{2}(\delta_0 + \delta_1)$ is the unique d_2-invariant measure on (X, d_2) and hence

$$r(X, d_2) = \tfrac{1}{2}(d_2(0,0) + d_2(0,1)) = \tfrac{1}{2}.$$

Now we construct a sequence $(x_N)_{N\geq 1}$ in X as given in Corollary 2.

Let $x_1 = 0$. Hence $x_2 = 1$. Now x_3 can be chosen arbitrarily, let us say $x_3 = 0$. Hence $x_4 = 1$ and so on. Therefore (for example)

$$x_N = \begin{cases} 0, & \text{if } N \text{ is odd,} \\ 1, & \text{if } N \text{ is even.} \end{cases}$$

It follows that

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)=\begin{cases}\dfrac{N^2-1}{N^2}\cdot\dfrac{1}{2}, & \text{if } N \text{ is odd,}\\ \dfrac{1}{2}, & \text{if } N \text{ is even}\end{cases}$$

and

$$\mu_N=\begin{cases}\dfrac{N+1}{N}\cdot\dfrac{1}{2}\delta_0+\dfrac{N-1}{N}\cdot\dfrac{1}{2}\delta_1, & \text{if } N \text{ is odd,}\\ \dfrac{1}{2}\delta_0+\dfrac{1}{2}\delta_1, & \text{if } N \text{ is even}\end{cases}$$

where

$$\mu_N=\frac{1}{N}\sum_{i=1}^{N}\delta x_i.$$

We see that

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)\to\frac{1}{2}=r(X,d_2)$$

and

$$\mu_N\to\tfrac{1}{2}(\delta_0+\delta_1)=\mu_0 \text{ (with respect to the } w^*\text{-topology),}$$

as $N\to\infty$.

As an application of Corollary 2 we give the following concluding examples.

Example 2. Let $(X,d)=(S^1,d_2)$. Notice that the well-known van der Corput sequence (which is uniformly distributed on the torus S^1 with respect to the normalized Lebesgue measure $\lambda=\mu_0$ on S_1) follows the construction method given in Corollary 2.

Example 3. Let $(X,d)=(T,d_2)$, T the Reuleaux triangle, which consists of the vertices of an equilateral triangle in $\mathbf{R}^2$ together with three arcs of circles, each circle having centre at one of the vertices and endpoints at the other two vertices. Without loss of generality take T such that the diameter of T is equal to 1.

This space (T,d_2) is of special interest, because it is conjectured in [4] and [11] that $m(T,d_2)=g(\mathbf{R}^2)$, where $g(\mathbf{R}^2)$ is the supremum of the numbers $m(X,d_2)$ as X ranges over all compact connected subsets of the Euclidean space $\mathbf{R}^2$. (Remember that $m(X,d_2)=r(X,d_2)/D(X,d_2)$, where $D(X,d_2)$ is the diameter of X.)

The calculations given in [11] indicate that $r(T,d_2)=m(T,d_2)$ lies between 0.6675276 and 0.6675284. We did some calculations for $M(T,d_2)$ and conjecture that $r(T,d_2)=M(T,d_2)$. Following this trace computer calculations (applying

Corollary 2) lead to $x_1, \dots, x_N$ on T such that $(1/N)\sum_{i=1}^{N}\delta_{x_i}$ is almost d_2-invariant on T.

We are working on a paper in which exhaustive computer calculations will hopefully light up the question, if indeed such a d_2-invariant measure μ_0 exists on T and what it looks like.

4. The proofs

Let us first collect some simple well-known properties of functions related to a metric d.

Lemma 1. *Let (X,d) be a compact metric space. Then we have:*

(1) For each μ in $M(X)$ it follows that $|d_\mu(x)-d_\mu(x')|\le d(x,x')$ for all x, x' in X.

(2) Let $(f_n)_{n\ge1}$ be a sequence in $C(X)$ and f in $C(X)$ such that $f_n(x)\to f(x)$, as $n\to\infty$ for all x in X and $|f_n(x)-f_n(x')|\le cd(x,x')$, $|f(x)-f(x')|\le cd(x,x')$ for some $c>0$ and all x, x' in X and all $n\ge1$. Then $\|f_n-f\|\to0$, as $n\to\infty$, in $C(X)$.

(3) Let $(\mu_n)_{n\ge1}$ be a sequence in $M(X)$ and $\mu\in M(X)$. If $\mu_n\to\mu$, as $n\to\infty$, with respect to the w^*-topology of $M(X)$, it follows that $I(\mu_n)\to I(\mu)$, as $n\to\infty$.

Proof. (1) follows by triangle inequality and (2) is obtained by routine calculations using an ε-net for X by compactness. Now let $\mu_n\to\mu$, as $n\to\infty$, with respect to the w^*-topology on $M(X)$. Hence $d_{\mu_n}(x)\to d_\mu(x)$, as $n\to\infty$, for all x in X. Applying (1) and (2) we get $\|d_{\mu_n}-d_\mu\|\to0$, as $n\to\infty$, in $C(X)$. Now

$$\begin{aligned}|I(\mu_n)-I(\mu)| &\le |I(\mu_n)-I(\mu,\mu_n)|+|I(\mu,\mu_n)-I(\mu)|\\ &\le \|d_{\mu_n}-d_\mu\|+|I(\mu,\mu_n)-I(\mu)|.\end{aligned}$$

Again $\mu_n\to\mu$ implies $I(\mu,\mu_n)\to I(\mu,\mu)=I(\mu)$, as $n\to\infty$, and so we are done. □

Proof of Proposition 1. The assertion follows by compactness of $M(X)$ and Lemma 1, part (3).

Proof of Theorem 1. (1) By Remark 1, part (c) take some μ in $M(X)$ such that $d_\mu(x)\ge r(X,d)$ for all x in X. Hence $M(X,d)\ge I(\mu)\ge r(X,d)$.

(2) Again choose some μ in $M(X)$ such that $d_\mu(x)\ge r(X,d)$ for all x in X.

Let us assume that there exists some x_0 in X such that $d_\mu(x_0)>r(X,d)$. Define $\varepsilon>0$ such that $d_\mu(x_0)=r(X,d)+\varepsilon$ and let $\lambda=(r(X,d)+\varepsilon)/(r(X,d)+2\varepsilon)$. Of course we have $0<\lambda<1$. Now let $\nu\in M(X)$, $\nu=\lambda\mu+(1-\lambda)\delta_{x_0}$. Since $d_\mu(x)\ge r(X,d)$ for all x in X we get $M(X,d)\geq I(\mu)\geq r(X,d)$ and since $r(X,d)=M(X,d)$ this implies $r(X,d)=I(\mu)$. Now it follows that

$$I(\nu)=\lambda^2 I(\mu)+2\lambda(1-\lambda)d_\mu(x_0)=r(X,d)+\frac{\varepsilon^2}{r(X,d)+2\varepsilon}>r(X,d)=M(X,d),$$

which is a contradiction to the definition of $M(X,d)$. Therefore $d_\mu(x)=r(X,d)$ for all x in X and $\mu_0=\mu$ is a d-invariant measure on X. $\square$

Proof of Proposition 2. Consider $\mathbf{R}^3$ equipped with the usual ∞-norm. Define

$$x_0=(1,1,1),\quad x_1=(-1,1,1),\quad x_2=(1,-1,1),\quad x_3=(1,1,-1)$$

and

$$f\colon \mathbf{R}^3\to\mathbf{R}\quad\text{as } f(x)=\frac{1}{8}\sum_{i=0}^{3}(\|x-x_i\|_\infty+\|x+x_i\|_\infty).$$

It is easy to see that $A\subseteq\mathbf{R}^3$, $A=\{x\in\mathbf{R}^3:f(x)\leq\frac{7}{4}\}$ is a compact convex centrally symmetric subset of $\mathbf{R}^3$ with 0 in the interior of A.

Let $X=\{x\in\mathbf{R}^3:f(x)=\frac{7}{4}\}$ and d be the metric induced by the ∞-norm. Since $\{x\in\mathbf{R}^3:f(x)<\frac{7}{4}\}$ is an open convex subset of A, (X,d) is a compact connected metric space. Now $f(x_i)=f(-x_i)=\frac{7}{4}$ for $0\leq i\leq 3$ and hence x_i and $-x_i$ are elements of X for $0\leq i\leq 3$. Define μ_0 in $M(X)$ as

$$\mu_0=\frac{1}{8}\sum_{i=0}^{3}(\delta_{x_i}+\delta_{-x_i}).$$

By the definition of X it follows that μ_0 is a d-invariant measure on X and $r(X,d)=\frac{7}{4}$. Since $f(\frac{3}{2},0,0)=\frac{7}{4}$, we obtain that $x_4=(\frac{3}{2},0,0)$ is an element of X. Let ν belong to $M(X)$, $\nu=\frac{1}{5}(\delta_{x_0}+\delta_{-x_1}+\delta_{x_2}+\delta_{x_3}+\delta_{-x_4})$. It follows that

$$M(X,d)\geq I(\nu)=\tfrac{44}{25}>\tfrac{7}{4}=r(X,d).\quad\square$$

Proof of Theorem 2. Let $n\in\mathbf{N}$ and $x_1,\dots,x_n$ be elements in X. Choose arbitrary $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\in[0,1]$ such that

$$\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\beta_i=1.$$

Now let $c_i=\alpha_i-\beta_i$ for $1\leq i\leq n$. Since $c_i\in\mathbf{R}$ and $\sum_{i=1}^{n}c_i=0$ the quasihypermetric property of (X,d) implies the formula

$$(*)\qquad I\Big(\sum_{i=1}^{n}\alpha_i\delta_{x_i}\Big)+I\Big(\sum_{i=1}^{n}\beta_i\delta_{x_i}\Big)\leq 2I\Big(\sum_{i=1}^{n}\alpha_i\delta_{x_i},\sum_{i=1}^{n}\beta_i\delta_{x_i}\Big).$$

Of course the function $(\mu,\nu)\mapsto I(\mu,\nu)$ on $M(X)\times M(X)$ is w^*-continuous in each variable. Together with the fact that $M(X)$ is a compact metric space, the Krein–Milman theorem (see Chapter 2 of this paper) and applying Lemma 1, part (3) yield the formula

$$(**) \qquad I(\mu)+I(\nu)\leq 2I(\mu,\nu)$$

for all μ,ν in $M(X)$.

By assumption there exists some μ_0 in $M(X)$ such that $d_{\mu_0}(x)=r(X,d)$ for all x in X.

Let $\nu=\mu_0$ in formula $(**)$. Hence

$$I(\mu)\leq r(X,d) \quad \text{for all } \mu \text{ in } M(X).$$

Therefore $M(X,d)\leq r(X,d)$ and applying Theorem 1, part (1) we get

$$M(X,d)=r(X,d).$$

Since $M(X,d)=r(X,d)=I(\mu_0)$ we obtain that μ_0 is maximal on X. □

Proof of Corollary 1. Apply Theorem 1, part (2) and Theorem 2.

Proof of Proposition 3. Apply Corollary 1 and Björck's theorem, part (1), mentioned above.

Proof of Proposition 4. Let $(x_N)_{N\geq 1}$ be a sequence in X which is uniformly distributed with respect to μ_1. Hence $(1/N)\sum_{i=1}^{N}\delta_{x_i}\to\mu_1$, as $N\to\infty$, with respect to the w^*-topology on $M(X)$. Now Lemma 1, part (3) applies

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)=I\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\right)\to I(\mu_1)=M(X,d_2), \quad \text{as } N\to\infty.$$

On the other hand consider some sequence $(x_N)_{N\geq 1}$ in X such that

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)\to M(X,d_2), \quad \text{as } N\to\infty.$$

Since $M(X)$ is a compact metric space (see Chapter 2 of this paper) the set of all accumulation points of $((1/N)\sum_{i=1}^{N}\delta_{x_i})_{N\geq 1}$ is not empty and each accumulation point μ in $M(X)$ can be obtained by some subsequence $((1/N_k)\sum_{i=1}^{N_k}\delta_{x_i})_{k\geq 1}$ tending to μ in the w^*-topology on $M(X)$. Hence we get

$$\frac{1}{N_k^2}\sum_{i,j=1}^{N_k} d_2(x_i,x_j)\to M(X,d_2)$$

and

$$\frac{1}{N_k^2}\sum_{i,j=1}^{N_k} d_2(x_i,x_j)=I\left(\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i}\right)\to I(\mu),$$

by Lemma 1, part (3), as $k\to\infty$. Therefore we have $I(\mu)=M(X,d_2)$. Applying Björck's theorem, part (1) we get $\mu=\mu_1$. □

Proof of Proposition 5. By definition of $(x_N)_{N\geq 1}$ we get

$$r(X,d_2)=M(X,d_2)\geq I\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\right)=\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)\geq \frac{N-1}{N}r(X,d_2).$$

Hence

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d_2(x_i,x_j)\to r(X,d_2)=M(X,d_2),\quad \text{as } N\to\infty.$$

Applying Proposition 4 we obtain that $(x_N)_{N\geq 1}$ is uniformly distributed with respect to $\mu_1=\mu_0$. □

Proof of Corollary 2. (1) By Remark 1, part (d) we have

$$\max_{x\in X}\frac{1}{N}\sum_{i=1}^{N} d_2(x_i,x)\geq r(X,d_2)$$

for all $N\geq 1$. Hence by definition of $(x_N)_{N\geq 1}$ and Proposition 5 we are done.

(2) Applying part (1), $r(X,d_2)=M(X,d_2)$, $\mu_0=\mu_1$ and Proposition 4, the assertion follows. □

References

1. Alexander, R. and Stolarsky, K. B., Extremal problems of distance geometry related to energy integrals, *Trans. Amer. Math. Soc.* **193** (1974), 1–31.
2. Björck, G., Distributions of positive mass, which maximize a certain generalized energy integral, *Ark. Mat.* **3** (1958), 255–269.
3. Choquet, G., *Lectures on Analysis, Vol. II: Representation Theory*, Benjamin, New York–Amsterdam, 1969.
4. Cleary, J., Morris, S. A. and Yost, D., Numerical geometry – numbers for shapes, *Amer. Math. Monthly* **93** (1986), 260–275.
5. Dor, L. E., Potentials and isometric embedding in L_1, *Israel J. Math.* **24** (1976), 260–268.
6. Gross, O., The rendezvous value of a metric space, in *Advances in Game Theory* (Dresher, M., Shapley, L. S. and Tucker, A. W., eds.), Ann. of Math. Stud. **52**, pp. 49–53, Princeton Univ. Press, Princeton, N. J., 1964.

7. Herz, C., A class of negative definite functions, *Proc. Amer. Math. Soc.* **14** (1963), 670–676.
8. Lévy, P., *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris, 1937.
9. Lin, P.-K., The average distance property of Banach spaces, *Arch. Math.* (*Basel*) **68** (1997), 496–502.
10. Morris, S. A. and Nickolas, P., On the average distance property of compact connected metric spaces, *Arch. Math.* (*Basel*) **40** (1983), 459–463.
11. Nickolas, P. and Yost, D., The average distance property for subsets of Euclidean space, *Arch. Math.* **50** (1988), 380–384.
12. Parthasarathy, K. R., *Probability Measures on Metric Spaces*, Academic Press, New York–London, 1967.
13. Stolarsky, K. B., Sums of distances between points on a sphere, *Proc. Amer. Math. Soc.* **35** (1972), 547–549.
14. Stolarsky, K. B., Sums of distances between points on a sphere, II, *Proc. Amer. Math. Soc.* **41** (1973), 575–582.
15. Strantzen, J., An average distance result in Euclidean n-space, *Bull. Austral. Math. Soc.* **26** (1982), 321–330.
16. Wolf, R., On the average distance property of spheres in Banach spaces, *Arch. Math.* **62** (1994), 378–384.
17. Wolf, R., On the average distance property in finite dimensional real Banach spaces, *Bull. Austral. Math. Soc.* **51** (1994), 87–101.
18. Wolf, R., Averaging distances in certain Banach spaces, *Bull. Austral. Math. Soc.* **55** (1997), 147–160.
19. Yost, D., Average distances in compact connected spaces, *Bull. Austral. Math. Soc.* **26** (1982), 331–342.

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