Ark. Mat., **33** (1995), 249–279

The propagation of polarization for systems of transversal type

Nils Dencker

1. Introduction

In this paper we study the propagation and distribution of polarization sets for solutions to systems having characteristics of transversal involutive self-intersection. Thus, we assume that the characteristic set is micro-locally a union of two non-radial hypersurfaces, which have transversal involutive intersection at the double characteristics. We also assume that the principal symbol vanishes of first order on the two-dimensional kernel at the intersection. These types of systems we call systems of transversal type. The propagation of singularities for the corresponding scalar wave operator was considered in [11].

We shall consider the propagation of $H_{(s)}$ polarization sets of the solutions. This polarization set indicates those components of the distribution, which are not in $H_{(s)}$. Outside the intersection of the characteristics, the polarizations for solutions propagate along Hamilton orbits, which are unique liftings of the bicharacteristics. The limits of polarizations from outside the double characteristic set, are called real polarizations, the others are called complex polarizations. It follows from the conditions that there are only two linearly independent real polarizations over the double characteristic set. The real polarizations are foliated by limits of Hamilton orbits, which we call limit Hamilton orbits. The results on the propagation of polarization depend on whether the polarization is contained in (limit) Hamilton orbits or not. When it is, we can define an invariant, called the trace of the orbits (see Definition 5.3). If this trace satisfies a second order transport equation along the bicharacteristics, we obtain propagation of polarization according to Theorem 6.2. When the polarization is either complex, or real and transversal to limit Hamilton orbits, we prove propagation of polarization in Theorems 6.3 and 6.4, respectively.

When we have a polarization condition, i.e., one component of the solution is in $H_{(s)}$, then the singularities of the solutions must either be contained in limit

Hamilton orbits with vanishing trace or in other types of orbits, which we call complex and coherent Hamilton orbits, see Theorems 7.1 and 7.2.

The plan of the paper is as follows. In Section 2, we define the systems of transversal type and the real and complex polarizations. The systems are reduced to normal forms in Section 3. We compute the transport equations for the limit Hamilton orbits in Section 4. Also, some other invariants are computed. In Section 5, we show that the real polarizations are foliated by limit Hamilton orbits, prove the invariance of the trace, and define the complex and coherent Hamilton orbits. We prove the propagation results in Section 6, and analyze the distribution of polarization in Section 7. Finally, we prove some technical lemmas in Appendix A. The energy estimates we are going to use are derived in Appendix B, and we estimate the regularity of an important coupling term in Appendix C.

The systems of transversal type have some similarities with the systems of conical refraction type and the systems of uniaxial type, which both occur in double refraction. The corresponding propagation of polarization for these systems was studied in [3] and [5]. Because of this similarity, we have been able to utilize some of the results of [3] in the present paper.

The results are only proved for distributions with values in $\mathbf{C}^N$, but since they are microlocal and invariant under multiplication with elliptic $N\times N$ systems of pseudo-differential operators, they easily carry over to sections of vector bundles.

2. Definitions

Let $P\in\Psi^m_{\mathrm{phg}}$ be an $N\times N$ system of classical pseudo-differential operators on a C^∞ manifold X. Let $p=\sigma(P)$ be the principal symbol, $\det p=|p|$ the determinant of p and $\Sigma=(\det p)^{-1}(0)$ the characteristics of P. Let

$$\Sigma_2=\{(x,\xi)\in\Sigma : d(\det p)=0 \text{ at } (x,\xi)\}, \tag{2.1}$$

and $\Sigma_1=\Sigma\setminus\Sigma_2$. The following definition makes it clear which type of systems we are going to study.

Definition 2.1. The system P is of transversal type at $w_0\in\Sigma_2$ if

$$\Sigma_2 \text{ is a non-radial involutive manifold of codimension 2,} \tag{2.2}$$

$$\begin{gathered}\det p=e\cdot q, \text{ where } e\neq 0 \text{ and } q \text{ is real valued}\\ \text{with Hessian having rank 2 and positivity 1,}\end{gathered} \tag{2.3}$$

$$\dim\operatorname{Ker} p=2 \text{ on } \Sigma_2, \tag{2.4}$$

microlocally near w_0.

The definition of transversal type is clearly invariant under symplectic changes of coordinates and multiplication of P with elliptic systems of pseudo-differential operators. It is clear that the adjoint P^* is of transversal type at w_0 if P is. Observe that conditions (2.2) and (2.3) imply that

$$\Sigma = S_1 \cup S_2, \tag{2.5}$$

where S_1 and S_2 are C^∞ surfaces, which intersect transversally and involutively at Σ_2. Condition (2.4) means that p vanishes of first order on its kernel over Σ_2 according to the following lemma.

Lemma 2.2. *Assume that p satisfies* (2.2) *and* (2.3), *then* (2.4) *is equivalent to*

$$\begin{aligned} &\forall\, w \in \Sigma_2,\ \exists\, \varrho \in T_w(T^*X) \text{ such that } \pi_C(\partial_\varrho p)(w) \\ &\text{is a bijection between } \operatorname{Ker} p(w) \text{ and } \operatorname{Coker} p(w). \end{aligned} \tag{2.6}$$

Here π_C is the quotient mapping $\mathbf{C}^n \to \mathbf{C}^n / \operatorname{Im} p(w) = \operatorname{Coker} p(w)$. We also find that (2.6) *holds if and only if $\partial_\varrho^2 \det p(w) \neq 0$, implying that $\varrho \in N_w\Sigma_2 = T_wX/T_w\Sigma_2$, which is the normal bundle of Σ_2.*

The lemma follows from the proof of [3, Lemma 2.2]. Observe that condition (2.6) is invariant under multiplication of P by elliptic $N \times N$ systems of pseudo-differential operators and symplectic changes of coordinates. This follows from the fact that

$$d(apb) = (da)pb + a(dp)b + ap\, db, \tag{2.7}$$

which also gives the invariance of the following definition.

Definition 2.3. We say that $\varrho \in T_w(T^*X)$, $w \in \Sigma_2$, is non-degenerate with respect to P if condition (2.6) holds. The variable $t \in C^\infty(T^*X)$ is non-degenerate with respect to P at $w \in \Sigma_2$ if the corresponding Hamilton field H_t is.

In the following we shall use the notation

$$\mathcal{N}_P = \operatorname{Ker} \sigma(P) \subseteq (T^*X \setminus 0) \times \mathbf{C}^N.$$

Now, if $z_1 \perp \operatorname{Im} p$ and $z_2 \in \mathcal{N}_P$ then ${}^t z_1 dp z_2$ is zero on $T\Sigma_2$, thus it defines an element in $N^*\Sigma_2$. In fact, because of (2.4), we find that $\mathcal{N}_P$ and $\operatorname{Coker} p = \mathcal{N}_{P^*}$ are 2 dimensional C^∞ vector bundles over Σ_2. By extending z_1 and z_2 to C^1 sections over Σ_2, we find

$$0 = d({}^t z_1 p z_2) = {}^t z_1 dp z_2 \quad \text{on } T\Sigma_2.$$

We could also obtain this from Proposition 3.1 by using (2.7).

Definition 2.4. Let P be of transversal type and $(w;z)\in\mathcal{N}_P$, $w\in\Sigma_2$. Then we say that $(w;z)$ is a real polarization for P if

$$\pi_C dp(w)z\in N^*_w\Sigma_2 \tag{2.8}$$

has kernel of dimension 1. If the kernel has dimension 0, then we say that $(w;z)$ is a complex polarization for P. We denote the real polarizations by $\mathcal{N}_{\mathbf{R}}$ and the complex polarizations by $\mathcal{N}_{\mathbf{C}}$.

It follows from (2.7) that this definition is invariant under multiplication of P by elliptic $N\times N$ systems of pseudo-differential operators and symplectic changes of coordinates. Observe that it follows from Proposition 3.1 together with (2.7) that the kernel of $\pi_C dp(w)z$ is contained in the radical of $\operatorname{Hess}\det p(w)$, so the dimension must be ≤ 1, which gives $\mathcal{N}_{\mathbf{R}}\cup\mathcal{N}_{\mathbf{C}}=\mathcal{N}_P$ over Σ_2. Next, we define the $H_{(s)}$ polarization set, where $H_{(s)}$ is the usual Sobolev space.

Definition 2.5. For $u\in\mathcal{D}'(X,\mathbf{C}^N)$ we define the polarization sets

$$\mathrm{WF}^s_{\mathrm{pol}}(u)=\bigcap\mathcal{N}_B\subseteq(T^*X\setminus0)\times\mathbf{C}^N, \tag{2.9}$$

where $\mathcal{N}_B=\operatorname{Ker}\sigma(B)$, and the intersection is taken over those $1\times N$ systems $B\in\Psi^0_{\mathrm{phg}}$ such that $Bu\in H_{(s)}$.

If $P\in\Psi^m_{\mathrm{phg}}$ is of transversal type and $Pu\in H_{(s)}$ near $w\in\Sigma_1$, then we find that $\mathrm{WF}^{s+m}_{\mathrm{pol}}(u)\subseteq\mathcal{N}_P$ and also that $\mathrm{WF}^{s+m-1}_{\mathrm{pol}}(u)$ is a union of Hamilton orbits in $\mathcal{N}_P$ near w. In fact, P is essentially a scalar operator then, so this follows from [7, Section 26.1]. The Hamilton orbits are unique line bundles in $\mathcal{N}_P$ over bicharacteristics of Σ_1 by [2, Definition 4.1].

We shall consider the limits of $\mathcal{N}_P|_{\Sigma_1}$ when we approach Σ_2. Let

$$\mathcal{N}^j_P=\mathcal{N}_P|_{S_j\setminus\Sigma_2},\quad j=1,2, \tag{2.10}$$

which are the kernels over the simple characteristics.

Definition 2.6. For $j=1$, 2, we define the limit polarizations

$$\partial\mathcal{N}^j_P=\left\{(w,z)\in\Sigma_2\times\mathbf{C}^N : z=\lim_{k\to\infty}z_k\right\}, \tag{2.11}$$

where $z_k\in\operatorname{Ker}p(w_k)$ and $S_j\setminus\Sigma_2\ni w_k\to w$.

It is clear that $\partial\mathcal{N}^j_P$ is conical in ξ and linear in the fiber. It follows from Proposition 3.1 and Remark 3.2 that $\partial\mathcal{N}^j_P$ is a C^∞ line bundle over Σ_2, $j=1,2$, and that $\mathcal{N}_{\mathbf{R}}=\partial\mathcal{N}^1_P\cup\partial\mathcal{N}^2_P$.

3. The normal form

We shall now obtain normal forms for systems of pseudo-differential operators of transversal type. Since the kernel of p has dimension two on Σ_2, we shall use the following base for $\mathbf{GL}(2,\mathbf{C})$: let $\mathbf{I}=\mathrm{Id}_2$,

$$\mathbf{J}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

Proposition 3.1. *Let $P\in\Psi^1_{\mathrm{phg}}$ be an $N\times N$ system of transversal type at $w_0\in\Sigma_2$, and let $t\in C^\infty(T^*X)$ be homogeneous of order 0 and non-degenerate with respect to P at $w_0\in\Sigma_2$. Then by completing t to a symplectic coordinate system $(t,x;\tau,\xi)$ of T^*X and multiplying with elliptic $N\times N$ systems of pseudo-differential operators of order 0, we may assume that $w_0=(0;(0,0,\dots,1))\in T^*\mathbf{R}^n$,*

$$\Sigma_2=\{\tau=\xi_1=0\}, \tag{3.2}$$

$$P\sim\begin{pmatrix} Q & 0 \\ 0 & E \end{pmatrix} \mod C^\infty, \tag{3.3}$$

where $E\in\Psi^1_{\mathrm{phg}}$ is an elliptic $(N-2)\times(N-2)$ system of pseudo-differential operators and

$$Q=D_t\mathbf{I}+\alpha(t,x,D_x)D_{x_1}\mathbf{J}+Q_0(t,x,D_x), \tag{3.4}$$

microlocally near $w_0\in\Sigma_2$. Here $0<\alpha(t,x,\xi)$ is homogeneous of degree 0 in ξ, and $Q_0\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$. If $\sigma(P)$ is real valued, then the elliptic systems can be chosen with real principal symbols.

Proof. Since the result is local and we have invariant conditions, we may assume that $X=\mathbf{R}^n$. Since t is non-degenerate with respect to P, the Hamilton field H_t is not tangent to Σ_2. Thus we may complete t to a homogeneous symplectic coordinate system $(t,x;\tau,\xi)$, microlocally near $w_0\in\Sigma_2$, so that $\tau=\xi_1=0$ on Σ_2, which gives (3.2).

By choosing suitable homogeneous bases for $\operatorname{Ker} p$ and the orthogonal complement of $\operatorname{Im} p$ in $\mathbf{C}^N$ on Σ_2, and extending to bases of $\mathbf{C}^N$ in a neighborhood, we obtain P on the form

$$P=\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\in S^1. \tag{3.5}$$

Here P_{22} is an elliptic $(N-2)\times(N-2)$ system and the principal symbols of P_{11}, P_{12} and P_{21} vanish on Σ_2. By constructing a parametrix for P_{22} and multiplying

P from left and right with suitable homogeneous elliptic systems of order 0, we obtain P on the form (3.3) near $w_0 \in \Sigma_2$ (see the proof of [3, Proposition 2.5]). If p is real valued, this can be done using elliptic systems having real principal symbols, making $\sigma(Q)$ and $\sigma(E)$ real.

Now consider the 2×2 system $Q=P_{11}$ and let q be the principal symbol of Q, then $q(w_0)=0$. Since t is non-degenerate with respect to P, we find that $\det \partial_\tau q(w_0) = \partial_\tau^2 \det q(w_0) \neq 0$. Thus we can use the matrix version of Malgrange's preparation theorem in [6, Theorem 5.3] and homogeneity, to obtain

$$q = e(\tau \mathbf{I} + k(t,x,\xi)), \tag{3.6}$$

where $k \in C^\infty(\mathbf{R}, S^1)$ satisfies $k=0$ on Σ_2, and e is a 2×2 homogeneous system satisfying $|e|\neq 0$. If q is real valued, then we can make e and k real valued. We find that $\det p = |e|(\tau^2 + \tau \operatorname{Tr} k + |k|)$, thus condition (2.5) implies that $\operatorname{Tr} k$ and $\det k$ are real valued. By multiplication with an elliptic system of pseudo-differential operators of order 0, we obtain that $e \equiv \mathbf{I}$. Since $k(t,x,\xi) \equiv 0$ when $\xi_1 = 0$, we may complete $(t, \tau + \operatorname{Tr} k/2)$ to a homogeneous, symplectic coordinate system so that (3.2) is conserved and $\operatorname{Tr} k \equiv 0$ which implies $\det k \le 0$. We also find that

$$k(t,x,\xi) = A(t,x,\xi)\xi_1 \tag{3.7}$$

where $A \in C^\infty(\mathbf{R}, S^0)$ and $\det A$ is real valued. Since $\operatorname{Tr} A \equiv 0$ we obtain

$$A \equiv \alpha_1 \mathbf{J} + \alpha_2 \mathbf{K} + \alpha_3 \mathbf{L} \tag{3.8}$$

with $\alpha_j \in C^\infty(\mathbf{R}, S^0)$.

Next, we are going to obtain a normal form for A. If $B \in \mathbf{SL}(2,\mathbf{C})$, then the conjugation $A \mapsto B^{-1}AB$ preserves the determinant and the trace, thus $\det A = -\alpha_1^2 - \alpha_2^2 + \alpha_3^2$ is preserved. This gives a Lie group homomorphism $\mathbf{SL}(2,\mathbf{C}) \to \mathbf{SO}_{\mathbf{C}}(2,1)$ which is easily seen to be surjective. In fact, it is a double cover of $\mathbf{SO}_{\mathbf{C}}(2,1) \cong \mathbf{SO}(3,\mathbf{C})$, which gives a representation of the complex spinor group $\operatorname{Spin}(3,\mathbf{C}) \cong \mathbf{SL}(2,\mathbf{C})$. When $B \in \mathbf{SL}(2,\mathbf{R})$ is real valued, we get the Lie group homomorphism $\mathbf{SL}(2,\mathbf{R}) \to \mathbf{SO}_{\mathbf{R}}(2,1)$ which also is surjective (it gives a representation of the real spinor group $\operatorname{Spin}(3,\mathbf{R})$). Condition (2.3) implies that $\det A < 0$, thus by conjugating with a homogeneous elliptic system of pseudo-differential operators of order 0, we obtain that $A \equiv \alpha \mathbf{J}$, where $\alpha(t,x,\xi) \in C^\infty(\mathbf{R}, S^0)$ satisfies $\alpha > 0$ in a conical neighborhood of $w_0 \in \Sigma_2$. When A is real valued, i.e., α_j are real valued $\forall j$, this can be done using elliptic systems with real principal symbols.

If $q_0 \in S^0$ is the term homogeneous of degree 0 in the expansion of Q, then the matrix version of Malgrange's division theorem in [6, Theorem 5.9] and homogeneity give

$$q_0 = B_{-1} q + Q_0, \tag{3.9}$$

where $Q_0(t,x,\xi)\in C^\infty(\mathbf{R},S^0)$ and $B_{-1}\in S^{-1}$. By multiplying Q with an operator with symbol $\mathbf{I}-B_{-1}$, we may assume $B_{-1}\equiv 0$. Using this repeatedly, we obtain (3.4) by induction over lower order terms. □

The normal form (3.4) immediately gives the following

Remark 3.2. If we consider the 2×2 system $P=Q$ in (3.4), and put $S_j=\{\tau=(-1)^j\alpha\xi_1\}$ then

$$\partial\mathcal{N}_P^j=\Sigma_2\times\{z\cdot{}^t(\cos(\pi j/2),\sin(\pi j/2)):z\in\mathbf{C}\}, \tag{3.10}$$

and it is easy to see that $\partial\mathcal{N}_P^1\cup\partial\mathcal{N}_P^2=\mathcal{N}_\mathbf{R}$. We also find that $\mathcal{N}_P^j$ extends to a C^∞ line bundle over S_j, $j=1,2$. We shall later use the fact that, if $w\in\partial\mathcal{N}_{P*}^j$ and $v\in\partial\mathcal{N}_P^k$ at $w_0\in\Sigma_2$ and $j\neq k$, then (2.7) and (3.4) imply that $w^*dp(w_0)v\equiv 0$. Observe that by using for example [11, Proposition 2.1] we may choose homogeneous, symplectic coordinates such that $\Sigma=\{\tau^2\equiv\xi_1^2\}$, which gives $\det A\equiv -1$ in (3.7) and $\alpha\equiv 1$ in (3.4). But then we may have to change the t variable.

4. The transport equations

In this section, we shall compute some higher order invariants of the systems of transversal type. Let P be a 2×2 system of transversal type near $w_0\in\Sigma_2$, on the form in (3.4). Thus,

$$P=D_t\mathbf{I}+\zeta(t,x,D_x)\mathbf{J}+p_0(t,x,D_x), \tag{4.1}$$

where $\zeta(t,x,\xi)=\alpha(t,x,\xi)\xi_1$, α and $p_0\in C^\infty(\mathbf{R},S^0_{\mathrm{phg}})$, and $\alpha(t,x,\xi)>0$. Recall that $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{L}$ are defined by (3.1), thus $\det\sigma(P)\equiv\tau^2-\zeta^2$.

Now we shall study how the symbol behaves, when applied on a C^∞ section of $\mathbf{C}^2$ over Σ_2. In order to keep P a system of pseudo-differential operators in the x variables and $\sigma(P)$ symmetric, we shall conjugate P with systems independent of τ, with values in $\mathbf{SO}(2,\mathbf{C})$. In what follows, we shall suppress the t dependence and write S^m instead of $C^\infty(\mathbf{R},S^m)$ for example.

Thus, let $A\in\Psi^0_{\mathrm{phg}}$ be a 2×2 system with homogeneous principal symbol

$$a=\cos(\theta)\mathbf{I}+\sin(\theta)\mathbf{L},\quad \theta(t,x,\xi)\in S^0. \tag{4.2}$$

Then A is elliptic, with microlocal parametrix A^{-1}. We find $\sigma(A^{-1})=\sigma({}^tA)=\cos(\theta)\mathbf{I}-\sin(\theta)\mathbf{L}$. We have

$$\sigma(APA^{-1})=\tau\mathbf{I}+\cos(2\theta)\zeta\mathbf{J}+\sin(2\theta)\zeta\mathbf{K}. \tag{4.3}$$

Let R_0 be the set of symbols $r\in S^0$, such that $r|_{\xi_1=0}\in S^{-1}$.

Lemma 4.1. *With P and A defined by (4.1)–(4.2), we get the symbol expansion*

(4.4)
$$APA^{-1}=\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{pmatrix}\sim\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{pmatrix}+\begin{pmatrix} D_{12}\theta & -D_{11}\theta \\ D_{22}\theta & -D_{12}\theta\end{pmatrix}+\begin{pmatrix} f_{11}(\theta) & f_{12}(\theta) \\ f_{21}(\theta) & f_{22}(\theta)\end{pmatrix}$$

modulo R_0. Here $p_{ij}=\sigma(P_{ij})$, $D_{jk}\theta=\partial_\tau p_{jk}D_t\theta+\partial_{\xi_1}p_{jk}D_{x_1}\theta\sim(1/i)H_{p_{jk}}\theta \mod R_0$, and $\{f_{jk}\}\in S^0$ are C^∞ functions of θ, depending linearly on p_0.

Proof. We only have to compute the term of order zero in the expansion of APA^{-1}, which is equal to

$$ap_0a^{-1}+a(D_ta^{-1}+\partial_{\xi_1}pD_{x_1}a^{-1}) \mod R_0, \tag{4.5}$$

since $AA^{-1}\sim\mathbf{I}$ modulo C^∞. Since $d(a^{-1})=-a^{-1}d\theta\,\mathbf{L}$, we find

$$a\partial_{\xi_1}pD_{x_1}a^{-1}=-a\partial_{\xi_1}pa^{-1}D_{x_1}\theta\,\mathbf{L}\sim-\partial_{\xi_1}(apa^{-1})D_{x_1}\theta\,\mathbf{L} \mod R_0, \tag{4.6}$$

which proves the lemma. □

Now recall that $\Sigma=S_1\cup S_2$, where we can put

$$S_j=\{\tau=(-1)^j\alpha\xi_1\}. \tag{4.7}$$

It is clear from the normal form (4.1) that the fiber of

$$\partial\mathcal{N}_P^j=\mathcal{N}_P|_{S_j\setminus\Sigma_2} \tag{4.8}$$

is spanned by ${}^t(0,1)$ when $j=1$, and ${}^t(1,0)$ when $j=2$.

Assume now that ${}^t(0,1)\in\operatorname{Ker}\sigma(APA^{-1})$ over S_1, which means that $\theta\equiv k\pi$ on S_1, where $k\in\mathbf{Z}$, say $k=0$. Since θ is independent of τ, we find $\theta\equiv0$. If P_{ij} is defined by (4.4), we find that $p_{12}\equiv0$. Then $P_{12}\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$ has principal symbol equal to

$$\sigma(P_{12})\sim f_{12}(0)=(p_0)_{12} \mod R_0. \tag{4.9}$$

By multiplying P with the transposed cofactor matrix, we obtain the wave system

$$Q={}^tP^{\mathrm{co}}P=\begin{pmatrix} P_{22}P_{11}-P_{12}P_{21} & [P_{22},P_{12}] \\ [P_{11},P_{21}] & P_{11}P_{22}-P_{21}P_{12}\end{pmatrix}=\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22}\end{pmatrix}, \tag{4.10}$$

which has principal symbol $\sigma(Q)=(\tau^2-\zeta^2)\mathbf{I}$. It is easy to see that Q satisfies the microlocal Levi condition: $\sigma_{\mathrm{sub}}(Q)=0$ on Σ_2. (Here $\sigma_{\mathrm{sub}}(Q)$ is the subprincipal symbol of Q.) We are going to compute the coupling term $q_{12}=[P_{22},P_{12}]$. Since we are going to compute the symbols modulo $R_0\supset S^{-1}$, we shall use the same notation for the operator as for its principal symbol.

Proposition 4.2. *Let P_{ij} be defined by (4.4). Then we find the commutator*

$$[P_{12}, P_{22}] \sim BP_{11} + K \mod I, \tag{4.11}$$

where I is the left module over Ψ^0_{phg} generated by P_{12}, P_{22}. Here B, $K \in \Psi^0_{\mathrm{phg}}$ have symbols

$$B \sim D_{22}\theta + \beta(\theta), \tag{4.12}$$

$$K \sim D_{11}D_{22}\theta - D_{12}D_{12}\theta + \gamma_1(\theta)D_{x_1}\theta + \gamma_0(\theta) \mod R_0 \tag{4.13}$$

where β and γ_j are C^∞ functions of θ, $j=0,1$, and $\beta(n\pi/2)=0$ for $n \in \mathbf{Z}$. When $p_0 \in R_0$ and $\partial_t \zeta = \{\tau, \zeta\}$ vanishes of second order at Σ_2 (i.e., $\partial_t \alpha \in R_0$), we obtain β and $\gamma_j \in R_0$. When θ is real valued, we find that $i\beta(\theta)$ is real valued.

This result follows immediately from [3, Proposition 3.2] by substituting ζ for ζ_1 and 0 for ζ_2, then $\mathrm{Ad}\, p = (0, 0, \partial_t \zeta)$, which implies $\beta(n\pi/2)=0$ for $n \in \mathbf{Z}$.

Next, we shall compute the coupling term $[P_{12}, P_{22}]$ when $2\theta/\pi \notin \mathbf{Z}$, i.e., dp_{12} and dp_{22} span the (complexified) conormal bundle of Σ_2. Thus ${}^t(-\sin(\theta), \cos(\theta))$ is a complex polarization vector for P i.e., ${}^t(0,1)$ is a complex polarization vector for the conjugated system in (4.4). It is then clear that $\{p_{12}, p_{22}\} = c_1 p_{12} + c_2 p_{22}$ for some homogeneous c_j.

Proposition 4.3. *Let P_{ij} be defined by (4.4), assume $w_0 \in \Sigma_2$ and $2\theta(w_0)/\pi \notin \mathbf{Z}$. Then the commutator*

$$[P_{12}, P_{22}] \sim K_0 \in \Psi^0_{\mathrm{phg}} \mod I, \tag{4.14}$$

microlocally near w_0, where I is the left Ψ^0_{phg} module generated by P_{12} and P_{22}. We have

$$\sigma(K_0) \sim D_t^2\theta - \alpha^2 D_{x_1}^2 \theta - 2\cot(2\theta)((D_t\theta)^2 - \alpha^2(D_{x_1}\theta)^2) + 2\cot(2\theta)D_{22}\theta f_{12}(\theta) \tag{4.15}$$

modulo R_0 and C^∞ functions of θ which are affine in $D\theta = (D_t\theta, D_{x_1}\theta)$.

Proof. We are going to use the formula (4.11). First, we note that when $2\theta/\pi \notin \mathbf{Z}$, then

$$p_{11} = p_{22} - 2\cot(2\theta)p_{12}, \tag{4.16}$$

thus we find from (4.4)

$$P_{11} \sim P_{22} + 2D_{12}\theta + f_{11} - f_{22} - 2\cot(2\theta)(P_{12} + D_{11}\theta - f_{12}) \mod R_0. \tag{4.17}$$

This implies that the symbol of $BP_{11} \mod I$ is equal to

$$2D_{22}\theta D_{12}\theta - 2\cot(2\theta)(D_{22}\theta+\beta(\theta))(D_{11}\theta - f_{12}) \tag{4.18}$$

modulo R_0 and C^∞ terms which are affine functions of $D\theta$. Since $\beta(\theta)=0$ when $\theta=n\pi/2$, we find that this is equal to

$$2D_{22}\theta D_{12}\theta - 2\cot(2\theta)(D_{22}\theta D_{11}\theta - D_{22}\theta f_{12}) \tag{4.19}$$

modulo C^∞ terms which are affine functions of $D\theta$. Now $2D_{22}\theta D_{12}\theta \sim 2D_t\theta D_{12}\theta + 2\cot(2\theta)\alpha^2\sin^2(2\theta)(D_{x_1}\theta)^2 \mod R_0$, so (4.19) is equal to

$$-2\cot(2\theta)((D_t\theta)^2 - \alpha^2(D_{x_1}\theta)^2 - f_{12}D_{22}\theta) + 2D_t\theta D_{12}\theta. \tag{4.20}$$

It is easy to compute from (4.13) that

$$K \sim D_t^2\theta - \alpha^2 D_{x_1}^2\theta - 2\alpha(\sin(2\theta)D_{11}\theta + \cos(2\theta)D_{12}\theta)D_{x_1}\theta \tag{4.21}$$

modulo R_0 and C^∞ terms which are affine in $D\theta$. Since $\sin(2\theta)D_{11}\theta + \cos(2\theta)D_{12}\theta = \sin(2\theta)D_t\theta$, we get the result from (4.20) and (4.21). □

Next, we shall consider the case when $\theta=0$ but $B\neq 0$ at $w_0\in\Sigma_2$, where B is given by (4.12). Let

$$P_0 = P_{11} + F, \tag{4.22}$$

where $F\in\Psi^0_{\mathrm{phg}}$ has principal symbol equal to K/B. By the formula (4.11) we find that $[P_{12}, P_{22}] \sim BP_0 \mod I$ and R_0. We shall now compute the higher order commutators. Since dp_{11} and dp_{22} span the normal bundle of Σ_2, we can write

$$[P_{j2}, P_0] \sim E_j \in \Psi^0_{\mathrm{phg}} \quad \mod M, \quad j=1,2, \tag{4.23}$$

where M is the left Ψ^0_{phg} module generated by P_0, P_{12} and P_{22}.

The following result follows from the proof of [3, Proposition 3.4] by substituting ζ for ζ_1 and 0 for ζ_2.

Proposition 4.4. *Let P_{ij} be given by* (4.4), *P_0 by* (4.22) *and E_j by* (4.23). *Then we find the symbols*

$$\begin{cases} E_1 \sim D_{11}G + D_{12}F + r_1(\theta, D_{x_1}\theta, G, F), \\ E_2 \sim D_{12}G + D_{22}F + r_2(\theta, D_{x_1}\theta, G, F) \end{cases} \tag{4.24}$$

modulo R_0. Here $G=2D_t\theta$ and $r_j\in\Psi^0_{\mathrm{phg}}$ is a C^∞ function of ξ, θ, G and F, an affine function of $D_{x_1}\theta$ and homogeneous in ξ, $j=1,2$.

Now, if $\theta=0$ and $B\neq 0$ at $w_0\in\Sigma_2$, then $D_{22}\theta\neq 0$, so $\theta\not\equiv 0$ near w_0. In that case both K_0 and E_j are defined, and we shall study the relation between their principal symbols.

Proposition 4.5. *Assume $\theta=0$ and $B\neq 0$ at $w_0\in\Sigma_2$, where B is given by* (4.12). *Let K_0 be defined by* (4.14) *and E_j by* (4.23). *When $2\theta/\pi\notin\mathbf{Z}$ we find*

$$E_j\sim B^{-1}D_{j2}K_0+C_jK_0 \quad \operatorname{mod} R_0,\ j=1,2, \tag{4.25}$$

for some $C_j\in C^\infty$. This implies that $E_j\not\sim 0\Longrightarrow K_0\not\sim 0 \operatorname{mod} S^{-1}$ on Σ_2.

Proof. Observe that, (4.14) implies that $K_0\in M$ when $2\theta/\pi\notin\mathbf{Z}$. From (4.11), (4.14) and (4.22) we find, when $2\theta/\pi\notin\mathbf{Z}$,

$$[P_{12},P_{22}]\sim BP_0\sim K_0 \quad \operatorname{mod} I. \tag{4.26}$$

If we commute elements in I with P_{j2} we get elements in M. Thus we find, when $2\theta/\pi\notin\mathbf{Z}$,

$$[P_{j2},B]P_0+B[P_{j2},P_0]\sim[P_{j2},K_0] \quad \operatorname{mod} M. \tag{4.27}$$

By (4.23), this gives (4.25). Since $d\theta\neq 0$ we obtain the last statement. □

Later, we shall use the wave operator to estimate the coupling term. For that purpose we need the following proposition, which follows from the proof of [3, Proposition 3.5].

Proposition 4.6. *Let P_0 be defined by* (4.22), *q_{22} be defined by* (4.10) *and $R_1=\{A\in\Psi^1_{\mathrm{phg}}:\sigma(A)=0$ on $\Sigma_2\}$. Then we find that the symbol of $[q_{22},P_0]$, modulo $\Psi^0_{\mathrm{phg}}q_{22}$, R_1P_0, $\Psi^0_{\mathrm{phg}}D_tP_0$, R_1P_{j2}, is equal to*

$$D_{11}E_2-D_{12}E_1+g_1E_1+g_2E_2 \quad \operatorname{mod} R_0, \tag{4.28}$$

where E_j is given by (4.23). *We find $g_j\in S^0$ is a C^∞ function of ξ, θ and F, an affine function of $D\theta$ and homogeneous in ξ, $j=1,2$.*

Now we want to change F so that (4.28) is in R_0 and the relation (4.11) is preserved, i.e.,

$$[P_{12},P_{22}]=BP_0+B_1P_{12}+B_2P_{22}+R \tag{4.29}$$

holds with $R\in R_0$ (which means that $BF=K$ on Σ_2). This is guaranteed by the following proposition, which follows from the proof of [3, Proposition 3.6]. This also gives an additional transport equation for E_1 and E_2.

Proposition 4.7. *Let P_{ij} be given by* (4.4), *P_0 by* (4.22) *with $F\in\Psi^0_{\mathrm{phg}}$ and R by* (4.29). *Assume that $R\in R_0$ when $t=0$, then $R\in R_0$ for $0\le t\le T$ if and only if*

$$D_{22}E_1-D_{12}E_2+f_1E_1+f_2E_2=0 \quad \text{on } \Sigma_2 \quad \text{for } 0\le t\le T, \tag{4.30}$$

modulo S^{-1}. Here E_j is given by (4.23) *and $f_j\in S^0$ is a C^∞ function of ξ, θ, an affine function of $D\theta$ and homogeneous in ξ, $j=1,2$.*

If we let $U={}^t(G,F,\theta,E_1,E_2)$, then by using (4.24), (4.30), the fact that $D_t\theta=G/2$, and letting (4.28) be zero on Σ_2, we get a first order quasilinear system in the variables (t,x_1) on Σ_2:

$$D_tU+A(t,x,\xi'',U,D_{x_1})U+A_0(t,x,\xi'',U)=0 \quad \text{on } \Sigma_2. \tag{4.31}$$

Here

$$A=\begin{pmatrix} -\alpha_1D_{x_1} & \alpha_2D_{x_1} & d_{13} & 0 & 0\\ \alpha_2D_{x_1} & \alpha_1D_{x_1} & d_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & d_{43} & \alpha_1D_{x_1} & -\alpha_2D_{x_1}\\ 0 & 0 & d_{53} & -\alpha_2D_{x_1} & -\alpha_1D_{x_1} \end{pmatrix} \tag{4.32}$$

with $\alpha_1=\alpha\cos(2\theta)$, $\alpha_2=\alpha\sin(2\theta)$, and d_{j3} is a first order differential operator in x_1, $j\neq3$, $x=(x_1,x'')$. This is a system of first order differential operators in x_1 depending C^∞ on t, x_1, the function U and the parameters (x'',ξ''), and A_0 is a C^∞ function of t, x, ξ'' and U. Since (4.31) is an equation on Σ_2, we find that $\xi_1=0$ in A and A_0.

Proposition 4.8. *Let A be given by* (4.32) *and A_0 be a C^∞ function of t, x, ξ'' and U with values in $\mathbf{C}^5$. Then the Cauchy problem*

$$\begin{cases} D_tU+A(t,x,\xi'',U,D_{x_1})U+A_0(t,x,\xi'',U)=0,\\ U(0,x,\xi'')=U_0(x,\xi'')\in C^\infty, \end{cases} \tag{4.33}$$

has a unique C^∞ solution U in a neighborhood Ω of $(0,x_0,\xi''_0)$. The solution U depends continuously on U_0, A and A_0 in the C^∞ topology. The neighborhood Ω only depends on the C^∞ norms of U_0, A and A_0.

Proof. We find that $\sigma(A)(\xi_1)$ is symmetrizable by putting $\eta_1=\alpha\sin(2\theta)\xi_1$ and $\eta_2=\alpha\cos(2\theta)\xi_1$ in the proof of Proposition 3.8 in [3]. As in that proof, we get the result from the proof of Theorem 5.6 in [10, Chapter 4]. $\square$

Remark 4.9. We find that the initial values $U|_{t=0}=U_0$ are determined uniquely by the values of $D_t^j\theta|_{t=0}$ for $j\le 3$. This follows from the fact that the mapping

$$(\theta, D_t\theta, D_t^2\theta, D_t^3\theta)|_{t=0} \mapsto (\theta, G, F, E_2)|_{t=0} \tag{4.34}$$

is a diffeomorphism when $B\neq 0$, and $E_1|_{t=0}$ is determined by (4.24). Since A and A_0 are homogeneous of degree 0 in ξ'', we may choose Ω conical and U homogeneous in ξ'', if U_0 is homogeneous in ξ''.

5. Invariants of the system

Let $P\in\Psi^m_{\mathrm{phg}}$ be an $N\times N$ system of transversal type. Then, we find that P is of real principal type on Σ_1, $\dim\mathcal{N}_P^j=1$, and $\mathcal{N}_P^j$ is foliated by Hamilton orbits, which are unique line bundles over bicharacteristics of Σ_1 (see Definitions 3.1 and 4.1 in [2]). We find that the polarization set $\mathrm{WF}_{\mathrm{pol}}^{s+m-1}(u)$ is a union of Hamilton orbits when $Pu\in H_{(s)}$, according to the proof of Theorem 4.2 in [2]. We shall analyze what happens when approaching Σ_2. In the following, a C^∞ curve on a C^∞ manifold M is an injective immersion of a compact interval $I\subset\mathbf{R}$ into M. We say that a sequence of C^∞ curves converges to a C^∞ curve, if there exist parametrizations of the curves, on a fixed interval, that converge in C^∞. A sequence of Hamilton orbits converges, if it does as a sequence of curves in $T^*X\times\mathbf{P}_{\mathbf{C}}^{N-1}$. Now, S_1 and S_2 are transversal at Σ_2, so their Hamilton fields are non-parallel on Σ_2. Since Σ_2 is involutive of codimension 2, the Hamilton fields of S_j are tangent to Σ_2 and generate the two-dimensional foliation of Σ_2. By using Proposition 3.1, we obtain the following proposition from Remark 3.2.

Proposition 5.1. *We find that $\partial\mathcal{N}_P^j$ is foliated by limit Hamilton orbits, which are limits of Hamilton orbits in $\mathcal{N}_P^j$, and are unique line bundles over bicharacteristics in S_j at Σ_2 for $j=1,2$.*

Over Σ_2, the singularities may be carried by limits of Hamilton orbits. We shall now consider the limit Hamilton orbit case. As before, we assume that $P\in\Psi^m_{\mathrm{phg}}$ is of transversal type at $w_0\in\Sigma_2$. Let $\mathcal{V}\subseteq\partial\mathcal{N}_P^j$ be a C^∞ line bundle over a leaf L of the foliation of Σ_2, it is no restriction to assume that $j=1$. Since $\partial\mathcal{N}_P^1$ is one dimensional and foliated by limit Hamilton orbits, we find that $\mathcal{V}$ is a union of limit Hamilton orbits. We shall define an invariant of $\mathcal{V}$. Choose $V\in\Psi^0_{\mathrm{phg}}$ and $W\in\Psi^{1-m}_{\mathrm{phg}}$ so that $\sigma(V)$ span $\mathcal{V}$ over L and $\sigma(W)$ span $\partial\mathcal{N}^2_{P^*}$ over Σ_2. (The adjoint P^* is also of transversal type.) By Remark 3.2 we find that $\sigma(W)^*dp\sigma(V)\equiv 0$ on L. Put

$$K=W^*PV\in\Psi^1_{\mathrm{phg}}, \tag{5.1}$$

and $\varkappa=\sigma_{\mathrm{sub}}(K)|_L$. Clearly, $\sigma(K)=0$ and $d\sigma(K)=0$ on L. Thus, the subprincipal symbol $\sigma_{\mathrm{sub}}(K)$ is well-defined on L.

Proposition 5.2. *Let K be given by* (5.1) *with the conditions above. Then we find that $\varkappa \sim \sigma_{\mathrm{sub}}(K)|_L$, modulo non-vanishing factors, is invariant under conjugation of P by elliptic, scalar Fourier integral operators, corresponding to homogeneous canonical transformations on T^*X, multiplication of P by elliptic $N \times N$ systems of pseudo-differential operators, and is independent of the choices of V and W.*

Proof. Since $\varkappa$ is the value of a subprincipal symbol for an operator whose principal symbol vanishes of second order, it is invariant under elliptic, scalar Fourier integral operators, corresponding to homogeneous canonical transformations on T^*X. Clearly, V and W are unique, modulo elliptic operators and terms in Ψ^0_{phg} and $\Psi^{1-m}_{\mathrm{phg}}$ having principal symbols vanishing on L and Σ_2, respectively. Let

$$R_L = \{A \in \Psi^1_{\mathrm{phg}} : \sigma(A) = d\sigma(a) = \sigma_{\mathrm{sub}}(A) = 0 \text{ on } L\}, \tag{5.2}$$

then terms in R_L do not change the value of $\varkappa$. If $A \in \Psi^{\mu}_{\mathrm{phg}}$ has principal symbol vanishing on Σ_2, $B \in \Psi^{\nu}_{\mathrm{phg}}$ has principal symbol vanishing on L, and $\mu+\nu=1$, then the calculus gives AB and $BA \in R_L$ because Σ_2 is involutive. Since $\sigma(PV)=0$ on L and $\sigma(W^*P)=0$ on Σ_2, this gives invariance of K modulo R_L when $\sigma(V)$ and $\sigma(W)$ are fixed on L and Σ_2 respectively. Now, replacing V by VA, and W by WB where A and B are elliptic, replaces K by B^*KA, which gives a non-vanishing factor in $\varkappa$. Multiplication of P by elliptic systems only has the effect of changing lower order terms in V and W, which proves the invariance. $\square$

Definition 5.3. We call $\varkappa \sim \sigma_{\mathrm{sub}}(K)|_L$ modulo non-vanishing factors, the trace of the C^∞ line bundle $\mathcal{V} \subseteq \mathcal{N}_{\mathbf{R}}$ over L, where L is a leaf of the foliation of Σ_2.

In particular, we find that the condition that the trace vanishes identically on L is invariant. Next, we shall consider the complex polarization case. Let L be a leaf of the foliation of Σ_2 and $\mathcal{V} \subset \mathcal{N}_{\mathbf{C}}$ a line bundle over L. Choose $V \in \Psi^0_{\mathrm{phg}}$ and W_1, $W_2 \in \Psi^{1-m}_{\mathrm{phg}}$ so that $\sigma(V)$ span $\mathcal{V}$ over L and $\sigma(W_1)$, $\sigma(W_2)$ span $\mathcal{N}_{P^*}$ over Σ_2. Put

$$P_j = W_j^* P V \in \Psi^1_{\mathrm{phg}}, \tag{5.3}$$

then dp_1 and dp_2 span $N_L^*\Sigma_2$ over $\mathbf{C}$ by Definition 2.4, since $\mathcal{V} \subset \mathcal{N}_{\mathbf{C}}$ and $\mathcal{N}_{P^*} \perp \operatorname{Im} p$. Since Σ_2 is involutive and $p_j=0$ on Σ_2, we can find a_1 and $a_2 \in S^0$ so that $\{p_1, p_2\} - a_1p_1 - a_2p_2$ vanishes of second order on L. Since dp_1 and dp_2 are linearly independent in $N_L^*\Sigma_2$, we find that a_1 and a_2 are uniquely determined on L. Let $A_j \in \Psi^0_{\mathrm{phg}}$ have principal symbol a_j, $j=1,2$, and put

$$k \sim \sigma_{\mathrm{sub}}([P_1, P_2] - A_1P_1 - A_2P_2)|_L, \tag{5.4}$$

which is then well-defined.

Proposition 5.4. *We find that k in (5.4), modulo non-vanishing factors, is invariant under conjugation of P by elliptic, scalar Fourier integral operators, corresponding to homogeneous canonical transformations on T^*X, multiplication of P by elliptic $N\times N$ systems of pseudo-differential operators, and is independent of the choices of V, W_j and A_j, $j=1,2$.*

Proof. Since k is the value of a subprincipal symbol for an operator whose principal symbol vanishes of second order, it is invariant under elliptic, scalar Fourier integral operators, corresponding to homogeneous canonical transformations on T^*X. Clearly, V and W_1, W_2 are unique, modulo elliptic operators and terms in Ψ^0_{phg} and $\Psi^{1-m}_{\mathrm{phg}}$ having principal symbols vanishing on L and Σ_2, respectively. Let R_L be defined by (5.2), then terms in R_L do not contribute to k. As before, if $A\in\Psi^{\mu}_{\mathrm{phg}}$ and $B\in\Psi^{\nu}_{\mathrm{phg}}$ have principal symbols vanishing on L and Σ_2 respectively, $\mu+\nu=1$, then the calculus gives AB and $BA\in R_L$. Since $\sigma(PV)=0$ on L and $\sigma(W_j^*P)=0$ on Σ_2, this gives invariance of k when $\sigma(V)$ and $\sigma(W_j)$ are fixed on L and Σ_2 respectively. Similarly, different choices of A_j also give terms in R_L.

Now, replacing V by VA, where $A\in\Psi^0_{\mathrm{phg}}$ is elliptic, replaces P_j by P_jA. Since

$$[P_1A,P_2A]=[P_1,A]P_2A+A[P_1,P_2]A+[A,P_2]P_1A \tag{5.5}$$

this gives a non-vanishing factor in k. Replacing W_j by $W_1B_j^1+W_2B_j^2$, replaces P_j by $B_j^{1*}P_1+B_j^{2*}P_2$. Since

$$[B_1^{j*}P_j,B_2^{k*}P_k]=B_1^{j*}[P_j,B_2^{k*}]P_k+B_1^{j*}B_2^{k*}[P_j,P_k]+[B_1^{j*},B_2^{k*}P_k]P_j \tag{5.6}$$

this also gives a non-vanishing factor in k. Multiplication of P by elliptic systems only has the effect of changing lower order terms in V and W_j, which proves the invariance. □

Definition 5.5. We call $k\sim\sigma(K)|_L$ modulo non-vanishing factors, the curvature of the C^∞ line bundle $\mathcal{V}\subseteq\mathcal{N}_{\mathbf{C}}$ over L, where L is a leaf of the foliation of Σ_2.

In particular, the condition that the curvature of the line bundle vanishes identically on L is invariant. We are going to show later that those line bundles may carry polarization, thus we make the following definition.

Definition 5.6. Let P be an $N\times N$ system of transversal type. A complex Hamilton orbit is a C^∞ line bundle $\mathcal{V}\subset\mathcal{N}_{\mathbf{C}}$ over a leaf L of the foliation of Σ_2, for which the curvature identically vanishes.

Example 5.7. If we consider the system in (4.4), then it follows that ${}^t(0,1)$ spans a complex Hamilton orbit over a leaf of Σ_2, if and only if $2\theta/\pi\notin\mathbf{Z}$ and $\sigma(K_0)|_{\Sigma_2}\equiv 0$ in (4.15).

Polarization may also be carried by line bundles, which are real at a point $w_0\in\Sigma_2$, but not tangent to any limit Hamilton orbit. This implies that $d\theta\neq 0$, thus the polarization is complex at points arbitrarily close to w_0. At these points, the curvature k is well-defined, according to Definition 5.5.

Definition 5.8. Let P be of transversal type at $w_0\in\Sigma_2$. A coherent Hamilton orbit through w_0 is a C^∞ line bundle $\mathcal{V}\subset\mathcal{N}_P$ over a leaf of the foliation L of Σ_2, such that $\mathcal{V}|_{w_0}\subset\mathcal{N}_{\mathbf{R}}$ but $\mathcal{V}$ is not tangent to any limit Hamilton orbit at w_0, and the curvature of $\mathcal{V}$ vanishes identically in the open set where $\mathcal{V}\subset\mathcal{N}_{\mathbf{C}}$.

Example 5.9. We find that ${}^t(0,1)$ spans a coherent Hamilton orbit through $w_0\in\Sigma_2$ over a leaf of Σ_2 for the system in (4.4), if and only if $2\theta(w_0)/\pi\in\mathbf{Z}$, $H_{p_{22}}\theta(w_0)=iB(w_0)\neq 0$ in (4.12), and $\sigma(K_0)|_{\Sigma_2}\equiv 0$ in (4.15) when $2\theta/\pi\notin\mathbf{Z}$. This implies that $\sigma(E_j)\equiv 0$ on Σ_2 by Proposition 4.5, where E_j is defined by (4.23).

6. The propagation of polarization

We shall now prove the results on the propagation of polarization. First we consider the case when there is no polarization condition. We say that $u\in H_{(s)}$ at $w\in T^*\mathbf{R}^n\setminus 0$, i.e., $w\notin \mathrm{WF}_{(s)}(u)$, if $u=u_1+u_2$ where $u_1\in H_{(s)}$ and $w\notin\mathrm{WF}(u_2)$. Let

$$s_u^*(w)=\sup\{s\in\mathbf{R}: u\in H_{(s)} \text{ at } w\},\quad w\in T^*X\setminus 0, \tag{6.1}$$

be the regularity function.

Since Σ_2 is involutive, it has a natural foliation given by the Hamilton fields of the normal vectors. The Hessian of q in (2.3) gives an invariant Lorentz structure on the leaves of this foliation, by the natural identification of the tangent space of the leaves with the normal bundle via the symplectic form (see [9, Lemma 2.5]). Since the Hessian is hyperbolic of rank 2, we get four different wave cones in the tangent space of the leaves. Let U be a conical neighborhood of $w_0\in\Sigma_2$. We define a local propagation cone $C^j_{w_0}\subset\overline{U}$, $0\leq j\leq 3$, so that $C^j_{w_0}$ is the closure of the set of $w\in\Sigma_2$ lying in the same leaf as w_0, such that there is a C^1 curve in U joining w_0 and w, having tangent which is everywhere in the interior of one of the wave cones.

Theorem 6.1. *Let $P\in\Psi^m_{\mathrm{phg}}(X)$ be an $N\times N$ system of transversal type at $w_0\in\Sigma_2$, and assume that $u\in\mathcal{D}'(X,\mathbf{C}^N)$ satisfies $s^*_{Pu}>s-m+1$ at w_0. If $s^*_u>s$ in one of the propagation cones $C^j_{w_0}\setminus w_0$ microlocally near w_0, then $s^*_u>s$ at w_0.*

Proof. By multiplication and conjugation with elliptic, scalar pseudo-differential operators, we may assume that $m=1$. By Proposition 3.1 we may assume that the coordinates (t,x) are chosen so that $\Sigma_2=\{\tau=\xi_1=0\}$, P is on the form (3.3), and $t=0$ at w_0. Also we may assume that $s^*_{Pu}>s$ in U, and $s^*_u>s$ in the propagation cone $\Omega=(C^j_{w_0}\setminus w_0)\cap U$ contained in $t<0$, for some conical neighborhood U of w_0. Clearly, we may assume that $N=2$ and $P=Q$ is on the form (3.4). In fact, since $E\in\Psi^1_{\mathrm{phg}}$ in (3.3) is elliptic, we obtain $u_j\in H_{(s+1)}$ when $j>2$.

By conjugating with a scalar pseudo-differential operator, it is clear that it suffices to prove that $u\in H_{(0)}$ at w_0 if $Pu\in H_{(\varepsilon)}$ in U and $u\in H_{(\varepsilon)}$ in Ω, for some $\varepsilon>0$ which is fixed in what follows. By Proposition A.1, we may assume that for any N and $\delta>0$ we have

$$|\hat{u}(\tau,\xi)|\le C_{\delta,N}\langle(\tau,\xi)\rangle^{-N}\quad\text{when } |\tau|\ge c_\delta(\langle\xi\rangle^\delta+\langle\xi_1\rangle). \tag{6.2}$$

which implies that $\xi\neq0$ in $\mathrm{WF}(u)$. We are going to use the Sobolev spaces $H_{(r,s)}$ and $H'_{(r,s)}$ with norms defined by (A.1) and (A.6). Let $Q={}^tP^{\mathrm{co}}P$, then $Qu\in H_{(\varepsilon,-1)}$ in U. Since we may assume $\delta\le1$, Lemma A.3 implies that Pu and Qu satisfy (6.2). If we choose $\delta\le\varepsilon$ in (6.2), we find $Qu\in H'_{(0,-1)}$ in $\pi_0(U\cap\Sigma_2)$ by (A.11), where $\pi_0(t,x;\tau,\xi)=(t,x,\xi)$. Similarly, since u and $Pu\in H_{(\varepsilon)}$ in Ω, we obtain that $u\in H'_{(0)}=H'_{(0,0)}$ and $D_tu\in H'_{(0,-1)}$ in $\pi_0\Omega$. Then

$$(u,D_tu)|_{t=r}\in H_{(0,0)}\times H_{(0,-1)}\quad\text{at } i_r(\Omega\cap\{t=r\}) \tag{6.3}$$

for almost all $r<0$, close to 0, where $i_r(r,x;0,\xi)=(x,\xi)$. Proposition B.1 in Appendix B with $Q={}^tP^{\mathrm{co}}P$ gives $u\in H'_{(0)}$ at π_0w_0, and Lemma A.3 gives $u\in H_{(0)}$ at w_0. $\square$

Over $\Sigma_1=\Sigma\setminus\Sigma_2$, the polarization set $\mathrm{WF}^s_{\mathrm{pol}}(u)$ is a union of Hamilton orbits when $Pu\in H_{(s-m+1)}$, since P is of real principal type there (see Definitions 3.1 and 4.1 in [2]). Over Σ_2, the singularities may be carried by limits of Hamilton orbits. We are going to consider the limit Hamilton orbit case. As before, we assume that $P\in\Psi^m_{\mathrm{phg}}$ is of transversal type at $w_0\in\Sigma_2$. Let $\mathcal{V}\subseteq\mathcal{N}_P$ be a C^∞ line bundle over S_j, for example $j=1$ (see Remark 3.2). Since S_1 is a union of (limits of) bicharacteristics, $\mathcal{V}$ is a union of (limits of) Hamilton orbits over S_1, and we have $\partial\mathcal{N}^1_P=\mathcal{V}|_{\Sigma_2}$. Let $\varkappa$ be the trace of $\mathcal{V}$ over Σ_2 as defined by Definition 5.3, and let π_1 be the projection $T^*\mathbf{R}^n\times\mathbf{C}^N\to T^*\mathbf{R}^n$ given by $\pi_1(w,z)=w$.

Theorem 6.2. *Let $P \in \Psi^m_{\mathrm{phg}}$ be an $N \times N$ system of transversal type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{\mathrm{phg}}$ be a $1 \times N$ system such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0, and $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ is a hypersurface near w_0. Let $\varkappa$ be the trace of $\mathcal{N}_A \cap \mathcal{N}_P$ over Σ_2 and $0 \neq D$ the Hamilton field of M_A. Assume that*

$$D^2\varkappa + c_1 D\varkappa + c_0\varkappa \equiv 0 \tag{6.4}$$

near w_0, for some $c_j \in C^\infty(\Sigma_2)$. If $u \in \mathcal{D}'(X, \mathbf{C}^N)$ satisfies $\min(s^*_{Pu} + m - 1, s^*_u + 1) > s$ at w_0 and $s^*_{Au} > s$ in one of the propagation cones $C^j_{w_0} \setminus w_0$ microlocally near w_0, then we find $s^*_{Au} > s$ at w_0.

In this case $M_A = S_j$ for some j, the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to one on S_j, $\mathcal{N}_A \cap \mathcal{N}_P|_{\Sigma_2} = \partial \mathcal{N}^j_P$, and $\mathcal{N}_A \cap \mathcal{N}_P$ is a union of (limit) Hamilton orbits. Observe that the trace is defined up to a non-vanishing factor, thus (6.4) is well-defined. The conclusion implies that $\mathrm{WF}^s_{\mathrm{pol}}(u) \subseteq \mathcal{N}_A \cap \mathcal{N}_P$ at w_0.

Proof. As in the proof of Theorem 6.1, we may assume that $m=1$, $N=2$ and $P=Q$ is on the form in Proposition 3.1. It suffices to prove that $Au \in H_{(0)}$ at w_0, if $Pu \in H_{(\varepsilon)}$, $u \in H_{(\varepsilon-1)}$ in U and $Au \in H_{(\varepsilon)}$ in $\Omega = (C^j_{w_0} \setminus w_0) \cap U$ for some conical neighborhood U of $w_0 \in \Sigma_2$ and some $\varepsilon > 0$, which is fixed in what follows. Clearly, we may assume the coordinates chosen so that $t=0$ at w_0 and $t<0$ in Ω.

By using the matrix version of the Malgrange preparation theorem in [6, Theorem 5.9] as in the proof of Proposition 3.1, we may assume that $A \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ is independent of τ. Clearly, it is no restriction to assume

$$M_A = S_1 = \{\tau = -\alpha\xi_1\}.$$

Since $\mathcal{N}_A \cap \mathcal{N}_P = \mathcal{N}_P$ over $S_1 \setminus \Sigma_2$, we find $\sigma(A) = {}^t(e, 0)$ on S_1, $0 \neq e \in S^0$. Since $\sigma(A)$ is independent of τ and $u \in H_{(\varepsilon-1)}$, we obtain that $u_1 \in H_{(\varepsilon)}$. We can assume that u satisfies (6.2) with $0 < \delta \leq 1$, which by Lemma A.3 also holds for Pu, Au and Qu, where $Q = {}^tP^{\mathrm{co}}P$. By choosing $\delta \leq \varepsilon$ in (6.2) we obtain $Qu \in H'_{(0,-1)}$ and $Pu \in H'_{(0)}$ (which implies $D_t Pu \in H'_{(0,-1)}$) in $\pi_0(U \cap \Sigma_2)$. Also, we find $u_1 \in H'_{(0)}$, $D_t u_1 \in H'_{(0,-1)}$ in $\pi_0\Omega$ and $u \in H'_{(-1)}$, $D_t u \in H'_{(-1,-1)}$ in $\pi_0(U \cap \Sigma_2)$ by Lemma A.3.

By Proposition 3.1, we find P_{12} and $P_{21} \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ and by the invariance given by Proposition 5.2 we find that $\sigma(P_{12})|_{\Sigma_2}$ is proportional to the trace $\varkappa$ of $\mathcal{N}_A \cap \mathcal{N}_P$ over Σ_2. Now (4.10) gives

$$q_{11}u_1 \sim -P_0 u_2 \mod H'_{(0,-1)} \quad \text{in } \pi_0(U \cap \Sigma_2). \tag{6.5}$$

Here $P_0 = [P_{22}, P_{12}] \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ has principal symbol equal to $D(c\varkappa)$ on Σ_2, where $c \neq 0$ and $D = -iH_{p_{22}}$ is proportional to the Hamilton field of S_1. Proposition C.1 implies that $P_0 u_2 \in H_{(\varrho,-1)}$ in Ω for any $\varrho < \varepsilon$, which implies $D_t P_0 u_2 \in$

$H_{(\varrho,-2)}$ in Ω. By taking $\delta<\frac{1}{2}\varepsilon$ in (6.2) we obtain $P_0u_2\in H'_{(0,-1)}$ and $D_tP_0u_2\in H'_{(0,-2)}$ in $\pi_0\Omega$. In order to get an equation for P_0u_2, we use the fact that $q_{22}\sim P_{11}P_{22} \mod C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$ to get

$$(6.6)\qquad [q_{22},P_0]\sim P_{11}[P_{22},P_0]+[P_{11},P_0]P_{22} \mod C^\infty(\mathbf{R},\operatorname{Op}\Psi^{-1}_{\mathrm{phg}}).$$

By equation (6.4) we find $[P_{22},P_0]\sim -C_1P_0-C_0P_{12}$ modulo $C^\infty(\mathbf{R},\Psi^{-1,1}_{\mathrm{phg}})$ with $C_j\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$, since the principal symbols are equal on $\{\xi_1=0\}$. Here, the symbol classes $\Psi^{m,k}_{\mathrm{phg}}$ are defined in Appendix B. Thus, if $a\in S^0$ is homogeneous and vanishes at $\{\xi_1=0\}$ then $a\in\Psi^{-1,1}_{\mathrm{phg}}$. We obtain

$$(6.7)\qquad [q_{22},P_0]\sim [P_{11},P_0]P_{22}-P_{11}C_1P_0-P_{11}C_0P_{12}$$

modulo $C^\infty(\mathbf{R},\Psi^{-1,2}_{\mathrm{phg}})$ and $C^\infty(\mathbf{R},\Psi^{-1,1}_{\mathrm{phg}})D_t$. Since $u\in H'_{(-1)}$, $D_tu\in H'_{(-1,-1)}$, Qu and $D_tPu\in H'_{(0,-1)}$ in $\pi_0(U\cap\Sigma_2)$, this gives

$$(6.8)\quad (q_{22}+P_{11}C_1)P_0u_2\sim (P_0[P_{21},P_{11}]-[P_{11},P_0]P_{21}+P_{11}C_0P_{11})u_1 \mod H'_{(0,-2)}$$

in $\pi_0(U\cap\Sigma_2)$. As in the proof of Theorem 6.1 we find that

$$(6.9)\qquad (u_1,D_tu_1,P_0u_2,D_tP_0u_2)\in H'_{(0)}\times H'_{(0,-1)}\times H'_{(0,-1)}\times H'_{(0,-2)}$$

in $\pi_0\Omega$. This gives by Fubini's theorem

$$(6.10)\qquad (u_1,D_tu_1,P_0u_2,D_tP_0u_2)|_{t=r}\in H_{(0)}\times H_{(0,-1)}\times H_{(0,-1)}\times H_{(0,-2)}$$

at $i_r(\Omega\cap\{t=r\})$, for almost all $r<0$.

By using equation (6.5) to eliminate the term $C_0D_t^2u_1$ in equation (6.8), we find that (6.5), (6.8) and (6.10) form a Cauchy problem for (u_1,P_0u_2), with a 2×2 system Q on the form (B.6)–(B.8). Then Proposition B.2 with $r=s=0$ gives $(u_1,P_0u_2)\in H'_{(0)}\times H'_{(0,-1)}$ at π_0w_0, proving the result by Lemma A.3. $\square$

Next, we shall consider the complex polarization case. Then we need no transport equation like (6.4), but the conclusion is weaker than in the limit Hamilton orbit case.

Theorem 6.3. *Let $P\in\Psi^m_{\mathrm{phg}}$ be an $N\times N$ system of transversal type at $w_0\in\Sigma_2$, and let $A\in\Psi^0_{\mathrm{phg}}$ be a $1\times N$ system, such that the dimension of the fiber of $\mathcal{N}_A\cap\mathcal{N}_P$ is equal to 1 over w_0. Assume that $\mathcal{N}_A\cap\mathcal{N}_P\subset\mathcal{N}_{\mathbf{C}}$ at w_0. Let $u\in\mathcal{D}'(X,\mathbf{C}^N)$ satisfy*

$\min(s^*_{Pu}+m-1, s^*_u+1)>s$ *at* w_0. *If* $s^*_{Au}>s$ *in one of the propagation cones* $C^j_{w_0}\setminus w_0$ *microlocally near* w_0, *then* $\mathrm{WF}^s_{\mathrm{pol}}(u)\subseteq\mathcal{N}_A\cap\mathcal{N}_P$ *at* w_0.

Proof. As in the proof of Theorem 6.1, we may assume that $m=1$, $s=0$, $N=2$, and $P=Q$ is on the form in Proposition 3.1. Also, we may assume that $Pu\in H_{(\varepsilon)}$ and $u\in H_{(\varepsilon-1)}$ in a conical neighborhood U of $w_0\in\Sigma_2$, $Au\in H_{(\varepsilon)}$ in the propagation cone $\Omega=(C^j_{w_0}\setminus w_0)\cap U$ for some $\varepsilon>0$, and we shall prove that $\mathrm{WF}^0_{\mathrm{pol}}(u)\subset\mathcal{N}_A\cap\mathcal{N}_P$ at w_0. As before, we may assume that the coordinates are chosen so that $t<0$ in Ω and $t=0$ at w_0, and $A\in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ is independent of τ. We have $\sigma(A)=(a_1,a_2)\neq 0$, since $\mathcal{N}_A\cap\mathcal{N}_P$ is one-dimensional over w_0. In order to avoid that $a_1^2+a_2^2=0$ at w_0, we may conjugate P by a constant diagonal matrix with different, non-vanishing diagonal elements, which preserves the normal form (3.4). Thus we may assume $\sigma(A)\equiv e\cdot{}^t(\cos(\theta),\sin(\theta))$, $0\neq e\in S^0$, where $2\theta(w)/\pi\notin\mathbf{Z}$ in U after shrinking the neighborhood, since $\mathcal{N}_A\cap\mathcal{N}_P\subset\mathcal{N}_{\mathbf{C}}$ over w_0.

Now we shall use the base change (4.2), which transforms $\sigma(A)$ into ${}^t(e,0)$, $0\neq e\in S^0$. Since $u\in H_{(\varepsilon-1)}$ in U, we find that $u_1\in H_{(\varepsilon)}$ in Ω. Let $Q={}^tP^{\mathrm{co}}P$ be the 2×2 system defined by (4.10), then $Qu\in H_{(\varepsilon,-1)}$ in U. As in the proof of Theorem 6.1, we may assume that u, Pu and Qu satisfies condition (6.2). By choosing $\delta\leq\varepsilon$ in (6.2) we obtain $u_1\in H'_{(0)}$, $D_tu_1\in H'_{(0,-1)}$ in $\pi_0\Omega$, and $Pu\in H'_{(0)}$, $u\in H'_{(-1)}$ and $Qu\in H'_{(0,-1)}$ in $\pi_0(U\cap\Sigma_2)$. Now, by Proposition 4.3 we can find $A_j\in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$, so that

$$(6.12)\qquad [P_{12},P_{22}]-A_1P_{12}-A_2P_{22}=K_0\in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$$

since $2\theta/\pi\notin\mathbf{Z}$ in U. We find that

$$(6.13)\qquad q_{11}u_1+A_1P_{11}u_1+A_2P_{21}u_1\sim K_0u_2 \mod H'_{(0,-1)}$$

in $\pi_0(U\cap\Sigma_2)$.

We shall first consider the case when $K_0\in C^\infty(\mathbf{R}, \Psi^{-1,1}_{\mathrm{phg}})$ in U, i.e., (4.15) is identically zero on $\Sigma_2\cap U$. Since $u\in H'_{(-1)}$ in $\pi_0(U\cap\Sigma_2)$, we then obtain $K_0u_2\in H'_{(0,-1)}$ there. Thus we obtain

$$(6.14)\qquad q_{11}u_1+A_1P_{11}u_1+A_2P_{21}u_1\sim 0 \mod H'_{(0,-1)} \quad\text{on } \pi_0(U\cap\Sigma_2)$$

with initial data

$$(6.15)\qquad (u_1,D_tu_1)|_{t=r}\in H_{(0)}\times H_{(0,-1)}$$

at $i_r(\Omega\cap\{t=r\})$ for almost all $r<0$ close to 0. Then Proposition B.1 with $N=1$ gives $u_1\in H'_{(0)}$ at π_0w_0, proving the result in this case by Lemma A.3.

Next, we consider the case when $K_0 \notin C^\infty(\mathbf{R}, \Psi^{-1,1}_{\mathrm{phg}})$ in U, i.e., (4.15) is not identically zero on $\Sigma_2 \cap U$. Then, we shall make the base change (4.2) for a different θ making (4.15) equal to zero on Σ_2. Let $r<0$ and θ_r be a solution of $\sigma(K_0)=0$ on Σ_2 for $t \geq r$, such that

$$\left\{\begin{aligned} &\theta_r = \theta, \\ &D_t\theta_r = D_t\theta \end{aligned}\right. \qquad \text{on } \{t=r\}\cap\Omega. \tag{6.16}$$

By (4.15), the equation $\sigma(K_0)=0$ on Σ_2 is a homogeneous semilinear strictly hyperbolic equation in the variables (t, x_1) when $2\theta/\pi \notin \mathbf{Z}$. It follows that there exists a homogeneous C^∞ solution θ_r in a fixed conical neighborhood of $t=r$ in Ω for r close to 0, since the Cauchy data and the coefficients are uniformly bounded in C^∞. For $r<0$ close enough to 0, this implies that θ_r is defined at w_0, and by continuity $2\theta_r/\pi \notin \mathbf{Z}$ in $\Omega \cap \{r \leq t \leq 0\}$. It is clear that the initial data $(u_1, D_t u_1)$ only depend on the values of θ and $D_t\theta$ at $\{t=r\}$. Thus condition (6.15) is preserved, and we also have (6.14). This gives $u_1^r = \cos(\theta_r)u_1 + \sin(\theta_r)u_2 \in H'_{(0)}$ at $\pi_0 w_0$, thus $u_1^r \in H_{(0)}$ at w_0 as before. Since $\theta_r(w_0) \to \theta(w_0)$ when $r \to 0$ and $\mathrm{WF}^s_{\mathrm{pol}}(u)$ is closed, we obtain the result. $\square$

Finally, we consider the case when $\mathcal{N}_A \cap \mathcal{N}_P \subset \mathcal{N}_{\mathbf{R}}$ at $w_0 \in \Sigma_2$, but it is not tangent to the limit Hamilton orbit through $\mathcal{N}_A \cap \mathcal{N}_P$.

Theorem 6.4. *Let $P \in \Psi^m_{\mathrm{phg}}$ be an $N \times N$ system of transversal type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{\mathrm{phg}}$ be a $1 \times N$ system, such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0. Assume that $\mathcal{N}_A \cap \mathcal{N}_P \subset \mathcal{N}_{\mathbf{R}}$ at w_0, and assume that $\mathcal{N}_A \cap \mathcal{N}_P|_{\Sigma_2}$ is not tangent to the limit Hamilton orbit through $\mathcal{N}_A \cap \mathcal{N}_P$ at w_0. Let $u \in \mathcal{D}'(X, \mathbf{C}^N)$ satisfy $\min(s^*_{Pu} + m - 1, s^*_u + 1) > s$ at w_0. If $s^*_{Au} > s$ in one of the propagation cones $C^j_{w_0} \setminus w_0$ microlocally near w_0, then $\mathrm{WF}^s_{\mathrm{pol}}(u) \subseteq \mathcal{N}_A \cap \mathcal{N}_P$ at w_0.*

Proof. As in the proof of Theorem 6.3, we may assume that $m=1$, $s=0$, $N=2$, and $P=Q$ is on the form in Proposition 3.1. Also, we may assume that $Pu \in H_{(\varepsilon)}$ and $u \in H_{(\varepsilon-1)}$ in a conical neighborhood U of $w_0 \in \Sigma_2$, and $Au \in H_{(\varepsilon)}$ in the propagation cone $\Omega = (C^j_{w_0} \setminus w_0) \cap U$ for some $\varepsilon > 0$, and we shall prove that $\mathrm{WF}^0_{\mathrm{pol}}(u) \subset \mathcal{N}_A \cap \mathcal{N}_P$ at w_0. As before, it is no restriction to assume that $t=0$ at w_0 and $t<0$ in Ω, and we may assume that $A \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ is independent of τ. Since $\mathcal{N}_A \cap \mathcal{N}_P \subset \mathcal{N}_{\mathbf{R}}$ at w_0, we may (after shrinking the neighborhood) assume that $\sigma(A) \equiv e \cdot {}^t(\cos(\theta), \sin(\theta))$ in U, where $0 \neq e \in S^0$ and $2\theta/\pi \in \mathbf{Z}$ at w_0.

Again, we shall use the base change (4.2), which changes $\sigma(A)$ into ${}^t(e, 0)$, $e \neq 0$. Since $u \in H_{(\varepsilon-1)}$ in U, we find that $u_1 \in H_{(\varepsilon)}$ in Ω. Let Q be the 2×2 system defined by (4.10), then $Qu \in H_{(\varepsilon,-1)}$ in U. Since $\mathcal{N}_A \cap \mathcal{N}_P$ is not tangent to a limit Hamilton orbit, we can find $A_j \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ so that

$$[P_{12}, P_{22}] - A_1 P_{12} - A_2 P_{22} = BP_0 \in C^\infty(\mathbf{R}, \Psi^1_{\mathrm{phg}}) \tag{6.17}$$

by Proposition 4.2. Here the principal symbol of $B \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ is non-zero, $P_0 = P_{11} + F$, with $F \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$. By Proposition C.1 and the ellipticity of B, we find $P_0 u_2 \in H_{(\varrho,-1)}$ for $\varrho < \varepsilon$ in Ω, which implies $D_t P_0 u_2 \in H_{(\varrho,-2)}$ there. As in the proof of Theorem 6.3, we may assume that u, Pu, $P_0 u_2$ and Qu satisfies condition (6.2). By choosing $\delta < \frac{1}{2}\varepsilon$ in (6.2), we find $u \in H'_{(-1)}$, $Pu \in H'_{(0)}$, $Qu \in H'_{(0,-1)}$ and $D_t Qu \in H'_{(0,-2)}$ in $\pi_0(U \cap \Sigma_2)$, and $u_1 \in H'_{(0)}$, $D_t u_1 \in H'_{(0,-1)}$, $P_0 u_2 \in H'_{(0,-1)}$ and $D_t P_0 u_2 \in H'_{(0,-2)}$ in $\pi_0 \Omega$. We find that

$$q_{11} u_1 + A_1 P_{11} u_1 + A_2 P_{21} u_1 \sim B P_0 u_2 \mod H'_{(0,-1)} \tag{6.18}$$

in $\pi_0(U \cap \Sigma_2)$.

Next, we need an equation for $P_0 u_2$. We have by Proposition 4.6 that

$$[q_{22}, P_0] - A_0 q_{22} - B_0 P_0 - B_1 P_{12} - B_2 P_{22} \sim 0 \quad \text{in } U \tag{6.19}$$

modulo $C^\infty(\mathbf{R}, \Psi^{-1,1}_{\mathrm{phg}})$ if (4.28) is equal to zero on $\Sigma_2 \cap U$. Here $B_0 = C_0 D_t + C_1$, A_0, $C_0 \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ and C_1, B_1, $B_2 \in C^\infty(\mathbf{R}, \Psi^{0,1}_{\mathrm{phg}})$.

First, we assume that (4.28) is equal to zero on $\Sigma_2 \cap U$. Then, since $u \in H'_{(-1)}$, we obtain by (6.19)

$$(q_{22} - B_0) P_0 u_2 \sim ((P_0 + A_0)[P_{21}, P_{11}] - B_1 P_{11} - B_2 P_{21}) u_1 \mod H'_{(0,-2)} \tag{6.20}$$

on $\pi_0 \Omega$. By the Fubini theorem we get

$$(u_1, D_t u_1, P_0 u_2, D_t P_0 u_2)|_{t=r} \in H_{(0)} \times H_{(0,-1)} \times H_{(0,-1)} \times H_{(0,-2)} \tag{6.21}$$

in $i_r(\Omega \cap \{t=r\})$, for almost all r. Then Proposition B.2 gives $u_1 \in H'_{(0)}$ at $\pi_0 w_0$, which proves the result by Lemma A.3 in this case.

When (4.28) is not equal to zero on Σ_2, we make the base change (4.2) with θ_r solving the system (4.31) with initial data

$$D_t^j \theta_r = D_t^j \theta \quad \text{at } \{t=r\} \cap \Omega \quad \text{for } j \le 3. \tag{6.22}$$

This does not change the initial data (6.21) by Proposition 4.2 and Remark 4.9. Proposition 4.8 implies that (4.31) has a homogeneous C^∞ solution in a fixed conical neighborhood of the initial surface $\{t=r\} \cap \Omega$ for $-c < r < 0$. Thus θ_r is defined at w_0, and by continuity we have $B \ne 0$ in $\{r \le t \le 0\} \cap \Omega$, for r close enough to zero. The result above implies that $u_1^r = \cos(\theta_r) u_1 + \sin(\theta_r) u_2 \in H_{(0)}$ at w_0. Since $\theta_r(w_0) \to \theta(w_0)$ as $r \to 0$ and $\mathrm{WF}^s_{\mathrm{pol}}(u)$ is closed, we obtain the result. $\square$

7. The distribution of polarization

Next, we are going to consider the distribution of polarization sets over Σ_2, when we have a polarization condition on the solution. First we consider the limit Hamilton orbit case. As before, we let $\varkappa$ be the trace of the limit Hamilton orbits over Σ_2 as defined by Definition 5.3, and let π_1 be the projection $T^*\mathbf{R}^n\times\mathbf{C}^N\to T^*\mathbf{R}^n$.

Theorem 7.1. *Let $P\in\Psi^m_{\mathrm{phg}}$ be an $N\times N$ system of transversal type at $w_0\in\Sigma_2$, and let $A\in\Psi^0_{\mathrm{phg}}$ be a $1\times N$ system such that the dimension of the fiber of $\mathcal{N}_A\cap\mathcal{N}_P$ is equal to 1 at w_0, and $M_A=\pi_1(\mathcal{N}_A\cap\mathcal{N}_P\setminus 0)$ is a hypersurface near w_0. Assume that $u\in\mathcal{D}'(X,\mathbf{C}^N)$ such that $Pu\in H_{(s-m+1)}$ and $Au\in H_{(s)}$ at w_0. Then $\mathrm{WF}^s_{\mathrm{pol}}(u)$ is a union of (limit) Hamilton orbits in $\mathcal{N}_A\cap\mathcal{N}_P$, and $\mathrm{WF}^{s-1}_{\mathrm{pol}}(u)|_{\Sigma_2}$ is a union of limit Hamilton orbits in $\mathcal{N}_A\cap\mathcal{N}_P$ on which the trace $\varkappa$ of $\mathcal{N}_A\cap\mathcal{N}_P$ over Σ_2 vanishes identically.*

Proof. As in the proof of Theorem 6.3 earlier, we may assume that $m=1$, $s=0$, $N=2$, $P=\begin{pmatrix} P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}$ is on the form in Proposition 3.1, Pu and $Au\in H_{(0)}$ in a conical neighborhood U of $w_0\in\Sigma_2$. Also, we may assume that

$$M_A=S_1=\{\tau=-\alpha\xi_1\}$$

and $A\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$ is independent of τ. Then it follows that $\sigma(A)\equiv{}^t(e,0)$, $0\neq e\in S^0$, but since we do not assume that $u\in H_{(-1)}$, lower order terms in A cannot be ignored. By conjugating P with $B\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$ having $\sigma(B)=\mathbf{I}$ and suitable lower order terms, we may assume $Au\equiv Eu_1$ in U, $\sigma(E)=e$. Then, it is clear that $\pi_1(\mathrm{WF}^\varrho_{\mathrm{pol}}(u)\setminus 0)=\mathrm{WF}_{(\varrho)}(u_2)$ near w_0 for $\varrho\leq 0$.

We find from (3.4) that $P_{21}\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$, which implies that $P_{22}u_2\in H_{(0)}$. Thus, $\mathrm{WF}_{(\varrho)}(u_2)$ is a union of bicharacteristics of S_1 for $\varrho\leq 0$. By the invariance given by Proposition 5.2 we find that $\sigma(P_{12})\in C^\infty(\mathbf{R},\Psi^0_{\mathrm{phg}})$ is proportional to the trace of $\mathcal{N}_A\cap\mathcal{N}_P$ over Σ_2. Since $P_{12}u_2\in H_{(-1)}$ in U we get the result, because the (limit) Hamilton orbits in $\mathcal{N}_A\cap\mathcal{N}_P$ are the unique liftings of the bicharacteristics of $S_1=M_A$. $\square$

Finally, we shall consider the case when polarization is not tangent to a limit Hamilton orbit. Then the polarization is contained in complex and coherent Hamilton orbits according to the following theorem.

Theorem 7.2. *Let $P\in\Psi^m_{\mathrm{phg}}$ be an $N\times N$ system of transversal type at $w_0\in\Sigma_2$, and let $A\in\Psi^0_{\mathrm{phg}}$ be a $1\times N$ system such that the dimension of the fiber of $\mathcal{N}_A\cap\mathcal{N}_P$ is equal to 1 at w_0. Assume that $\mathcal{N}_A\cap\mathcal{N}_P$ is not tangent to any limit Hamilton orbit* *over* w_0*, and that* $u \in \mathcal{D}'(X, \mathbf{C}^N)$ *satisfies* $\min(s^*_{Pu}+m-1, s^*_{Au}) > s$ *at* w_0*. Then* $\mathrm{WF}^s_{\mathrm{pol}}(u)$ *is a union of complex and coherent Hamilton orbits in* $\mathcal{N}_A \cap \mathcal{N}_P$*.*

This condition means that either $\mathcal{N}_A \cap \mathcal{N}_P|_{w_0} \subset \mathcal{N}_{\mathbf{C}}$ or $\mathcal{N}_A \cap \mathcal{N}_P$ is transversal to the limit Hamilton orbit through $\mathcal{N}_A \cap \mathcal{N}_P$ over w_0. The complex and coherent Hamilton orbits were defined in Definitions 5.6 and 5.8.

Proof. As in the proof of Theorems 6.3 and 7.1, we may assume that $m=1$, $s=0$, $N=2$, P is on the form in Lemma 4.1, Pu and $Au \sim u_1 \in H_{(\varepsilon)}$ in a conical neighborhood U of $w_0 \in \Sigma_2$ for some $\varepsilon>0$. Then we find that $\pi_1(\mathrm{WF}^0_{\mathrm{pol}}(u)) = \mathrm{WF}_{(0)}(u_2)$ in U. By cutting off where $|\tau| \le C|\xi|$ we may assume $P \in \Psi^0_{\mathrm{phg}}$.

Now we have $P_{j2}u_2 \in H_{(\varepsilon,-1)}$ for $j=1,2$, and Proposition C.1 gives $P_0 u_2 = [P_{12}, P_{22}]u_2 \in H_{(\varrho,-1)}$, $\forall\, \varrho<\varepsilon$, in U. If $2\theta\pi \notin \mathbf{Z}$ at w_0, then dp_{12} and dp_{22} span $N^*_{w_0}\Sigma_2$. If $2\theta/\pi \in \mathbf{Z}$ at w_0, then by assumption we have $D_{22}\theta = B \ne 0$, which by Proposition 4.2 implies that dp_{22} and dp_0 span $N^*_{w_0}\Sigma_2$. In any case, there exists $P_j \in \Psi^1_{\mathrm{phg}}$ such that $\sigma(P_j) = p_j = 0$ on Σ_2, $P_j u_2 \in H_{(\varrho,-1)}$ for $\varrho<\varepsilon$, $j=1,2$, and $|p_1|^2+|p_2|^2+1 \ge c(|\tau|^2+|\xi_1|^2)$, $c>0$.

It follows that $\psi u_2 \in H_{(0)}$ in U, if $\psi \in S^0_{\delta,0}$ has support where $|\tau|+|\xi_1| \ge C\langle \tau,\xi\rangle^\delta$ for C large enough and $\delta>0$. In fact, then we have $P_1^*P_1 + P_2^*P_2 = M \in \mathrm{Op}\, S(m_\delta^2, g_\delta)$ with principal symbol $|p_1|^2+|p_2|^2 \ge cm_\delta^2$ in a g_δ neighborhood of $\mathrm{supp}\,\psi$. Here m_δ and g_δ are defined by (A.4) and (A.5), so that $S(1,g_\delta) \subset S^0_{\delta,0}$. Thus, we can construct $E \in \mathrm{Op}\, S(m_\delta^{-2}, g_\delta)$ with support where $|\tau|+|\xi_1| \ge C\langle\tau,\xi\rangle^\delta$, such that $EM \sim \psi$ in U modulo $S^{-\infty}$. Since $\langle \tau, \xi_1\rangle^2 \sigma(E) \in S^0_{\delta,0}$ and $Mu_2 \in H_{(0,-2)}$ in U, we find $\psi u_2 \in H_{(0)}$ there.

Let $\phi(t) \in C_0^\infty$ satisfy $\phi(t)=1$ when $|t|<1$, and $\chi(\tau,\xi) = \phi((|\tau|+|\xi_1|)/C\langle\tau,\xi\rangle^\delta) \in S(1,g_\delta)$ for $\delta>0$. Then $u_2 \sim \chi u_2 = v$ modulo $H_{(0)}$ for C large enough, by the argument above. Thus we only have to consider v in what follows. We have

$$P_j v = [P_j, \chi]u_2 + \chi P_j u_2 \in H_{(0)} \quad \text{in } U \tag{7.1}$$

when $\delta<\varepsilon$ and C is large enough, since $P_j u_2 \in H_{(\varrho,-1)}$, for all $\varrho<\varepsilon$, and $[P_j,\chi] \in \mathrm{Op}\, S(1,g_\delta)$ is supported modulo $S^{-\infty}$ where $|\tau|+|\xi_1| \approx C\langle\tau,\xi\rangle^\delta$. Thus, we find that $\mathrm{WF}_{(0)}(v)$ is a union of leaves of the foliation of Σ_2.

By Definition 5.8, it remains to prove that $v \in H_{(0)}$ when $\mathcal{N}_A \cap \mathcal{N}_P \subset \mathcal{N}_{\mathbf{C}}$ and the curvature k of $\mathcal{N}_A \cap \mathcal{N}_P$ over Σ_2 is non-zero. By Proposition 5.4, the curvature is proportional to $\sigma(K_0)|_{\Sigma_2}$, where

$$K_0 = [P_{12}, P_{22}] - A_1 P_{12} - A_2 P_{22} \in S^0_{1,0} \tag{7.2}$$

for suitable $A_j \in \Psi^0_{\mathrm{phg}}$. Since $K_0 u_2 \in H_{(\varrho,-1)}$, we find as in (7.1) that $K_0 v \in H_{(0)}$ when $\delta<\varepsilon$. When $\sigma(K_0) \ne 0$ we obtain $v \in H_{(0)}$. This completes the proof. $\square$

Appendix A. Some technical lemmas

We also need some technical preparation. The following is more or less an adaptation of some propositions in [3] but we repeat the proofs here, since they are short. Let $H_{(r,s)}$ be the space of $u\in\mathcal{S}'$ satisfying

$$\text{(A.1)} \qquad \|u\|^2_{(r,s)} = (2\pi)^{-n}\int |\hat{u}(\tau,\xi)|^2 \langle(\tau,\xi)\rangle^{2r}\langle(\tau,\xi_1)\rangle^{2s}\,d\tau\,d\xi < \infty,$$

where $\langle\varrho\rangle=(1+|\varrho|^2)^{1/2}$. We say that $u\in H_{(r,s)}$ at $w\in T^*\mathbf{R}^n\setminus 0$, i.e., $w\notin \mathrm{WF}_{(r,s)}(u)$, if $u=u_1+u_2$ where $u_1\in H_{(r,s)}$ and $w\notin\mathrm{WF}(u_2)$. The following result is a modification of [3, Proposition 2.15].

Proposition A.1. *Assume that P is a 2×2 system of pseudo-differential operators of order 1 on $\mathbf{R}^n$, on the form (3.4) near $w_0\in\Sigma_2$. Let $u\in\mathcal{S}'(\mathbf{R}^n,\mathbf{C}^2)$ and assume $Pu\in H_{(r,s)}$ at w_0. Then for every $\delta>0$ we can write $u=u_\delta+v_\delta$, where $v_\delta\in H_{(r,s+1)}$ at w_0, and*

$$\text{(A.2)} \qquad |\hat{u}_\delta(\tau,\xi)| \le C_{\delta,N}\langle(\tau,\xi)\rangle^{-N} \quad \forall N,$$

when $|\tau|>c_\delta(\langle\xi\rangle^\delta+\langle\xi_1\rangle)$ for some c_δ and $C_{\delta,N}>0$.

Proof. Clearly, it is no restriction to assume that $\delta\le 1$ is fixed. Let $\chi\in C_0^\infty(\mathbf{R})$ satisfy $\chi(r)=1$ when $|r|\le 1$. Then for $\varepsilon>0$ we have

$$\text{(A.3)} \qquad \phi_{\varepsilon,\delta}(\tau,\xi) = \chi(\varepsilon|\tau|/(\langle\xi\rangle^\delta+\langle\xi_1\rangle)) \in S^0_{\delta,0},$$

since $d\phi_{\varepsilon,\delta}$ is supported where $|\tau|\approx\langle\xi\rangle^\delta+\langle\xi_1\rangle$. Put $v_\delta=(1-\phi_{\varepsilon,\delta})(D)u$, then obviously $u_\delta=\phi_{\varepsilon,\delta}(D)u$ satisfies (A.2).

In the support of $1-\phi_{\varepsilon,\delta}$ we find $|\det p|>cm_\delta^2$ for small enough ε, where

$$\text{(A.4)} \qquad m_\delta = \langle(\tau,\xi)\rangle^\delta + \langle(\tau,\xi_1)\rangle$$

is a weight for the metric

$$\text{(A.5)} \qquad g_\delta = |dt|^2+|dx|^2+(|d\tau|^2+|d\xi|^2)/\langle(\tau,\xi)\rangle^{2\delta}.$$

Since $\delta\le 1$, we find $P\in\mathrm{Op}\,S(m_\delta,g_\delta)$ when $|\tau|\le C|\xi|$. For small enough $\varepsilon>0$, we may construct $E\in\mathrm{Op}\,S(m_\delta^{-1},g_\delta)\subseteq\Psi^{-\delta}_{\delta,0}$ with support where $|\tau|>C(\langle\xi\rangle^\delta+\langle\xi_1\rangle)$, such that $EP\sim(1-\phi_{\varepsilon,\delta}(D))\mathbf{I} \mod C^\infty$, microlocally near w_0. Since E preserves wave front sets, and $\langle(\tau,\xi_1)\rangle\sigma(E)\in S^0_{\delta,0}$, we find $v_\delta\sim EPu\in H_{(r,s+1)} \mod C^\infty$ at w_0. $\square$

Let $H'_{(r,s)}$ be the Banach space of $u \in \mathcal{S}'$, satisfying

$$(\|u\|'_{(r,s)})^2 = (2\pi)^{-n} \int |\widehat{u}(\tau,\xi)|^2 \langle\xi\rangle^{2r} \langle\xi_1\rangle^{2s} \, d\tau \, d\xi < \infty. \tag{A.6}$$

Clearly, $u \in H'_{(r,s)}$ implies $u|_{t=\varrho} \in H_{(r,s)}$ for almost all ϱ, by Fubini's theorem. If $u \in \mathcal{S}'$ satisfies (A.2), then

$$\|u\|'_{(r-\delta s_-,s)} \le C_{r,s}(\|u\|_{(r,s)}+1) \le C'_{r,s}(\|u\|'_{(r+\delta s_+,s)}+1) \quad \forall\, r,s \in \mathbf{R}, \tag{A.7}$$

where $s_\pm = \max(\pm s, 0)$. Thus we lose only $O(\delta)$ derivatives when taking restriction of such $u \in H_{(r,s)}$ to $\{t=\varrho\}$, for almost all ϱ. We shall next define wave front sets corresponding to the spaces $H'_{(r,s)}$.

Definition A.2. Let $u \in \mathcal{S}'(\mathbf{R}^n)$, and assume that $\xi \neq 0$ in $\mathrm{WF}(u)$. We say that $u \in H'_{(r,s)}$ at (t_0, x_0, ξ_0), i.e., $(t_0, x_0, \xi_0) \notin \mathrm{WF}'_{(r,s)}(u)$, if there exists $\phi(t,x,\xi) \in C^\infty(\mathbf{R}, S^0_{1,0})$ such that $\phi(t,x,D_x)u \in H'_{(r,s)}$ and $\underline{\lim}_{\lambda\to\infty} |\phi(t_0,x_0,\lambda\xi_0)| \neq 0$.

This definition gives

$$(t_0, x_0, \xi_0) \notin \mathrm{WF}'_{(r,s)}(u) \implies (x_0, \xi_0) \notin \mathrm{WF}_{(r,s)}(u_\varrho), \tag{A.8}$$

for almost all ϱ close to t_0, where $u_\varrho = u|_{t=\varrho}$. If $\xi \neq 0$ in $\mathrm{WF}(u)$, then it follows from [1, Lemma 2.3] that

$$\pi_0(\mathrm{WF}_{(r,0)}(u)) = \mathrm{WF}'_{(r,0)}(u), \tag{A.9}$$

where $\pi_0(t,x;\tau,\xi) = (t,x,\xi)$. For the more general wave front sets, we have the following result.

Lemma A.3. *Assume that* $u \in \mathcal{S}'(\mathbf{R}^n)$ *satisfies* (A.2). *Then* Au *satisfies* (A.2), *if* $A \in C^\infty(\mathbf{R}, \Psi^\nu_{\delta,0})$, $\forall\,\nu$ *and* $\forall\,\delta > 0$. *We also obtain*

$$\mathrm{WF}'_{(r-\delta s_-,s)}(u) \subseteq \pi_0(\mathrm{WF}_{(r,s)}(u)) \subseteq \mathrm{WF}'_{(r+\delta s_+,s)}(u), \tag{A.10}$$

where $s_\pm = \max(\pm s, 0)$ *and* $\pi_0(t,x;\tau,\xi) = (t,x,\xi)$. *Since* $u \in C^\infty$ *in* $\pi_0^{-1}(\Sigma_2) \setminus \Sigma_2$ *by* (A.2), *we find*

$$\iota_0(\mathrm{WF}'_{(r-\delta s_-,s)}(u)) \subseteq \mathrm{WF}_{(r,s)}(u) \subseteq \iota_0(\mathrm{WF}'_{(r+\delta s_+,s)}(u)) \quad \text{on } \Sigma_2, \tag{A.11}$$

where $\iota_0(t,x,\xi) = (t,x;0,\xi)$.

Proof. By the proof of [3, Lemma 2.18], we find that the composition of operators in $\Psi^0_{\delta,0}$ having symbols supported where $|\tau| \le C|\xi|$, and operators in

$C^\infty(\mathbf{R}, \Psi^\nu_{\delta,0})$, is well-defined and given by the formal asymptotic expansion when $\delta>0$. Let $\phi=\phi_{\varepsilon,\delta}\in S^0_{\delta,0}$ be defined by (A.3), with ε small enough to make $\phi=1$ where (A.2) does not hold, thus $(1-\phi)u\in C^\infty$. Then, if $A\in C^\infty(\mathbf{R}, \Psi^\nu_{\delta,0})$, we find $(1-\phi)Au\in C^\infty$. In fact, the symbol of the commutator $[\phi, A]$ is supported, modulo $S^{-\infty}$, where (A.2) holds. This proves that Au satisfies (A.2).

Now, if $(t_0,x_0,\xi_0)\notin\pi_0(\mathrm{WF}_{(r,s)}(u))$ then we find $\phi\psi u\in H_{(r,s)}$, for any $\psi\in C^\infty(\mathbf{R}, S^0)$ supported in a sufficiently small conical neighborhood of (t_0,x_0,ξ_0). By (A.7) we find $\phi\psi u\in H'_{(r-\delta s_-,s)}$, implying $(t_0,x_0,\xi_0)\notin \mathrm{WF}'_{(r-\delta s_-,s)}(u)$ since $(1-\phi)\psi u\in C^\infty$. Finally, if $(t_0,x_0,\xi_0)\notin\mathrm{WF}'_{(r,s)}(u)$, then we find $\psi u\in H'_{(r,s)}$ for some $\psi\in C^\infty(\mathbf{R}, S^0)$ satisfying $\psi(t_0,x_0,\varrho\xi_0)\neq 0$ when $\varrho\gg 1$. Thus, we obtain $\phi\psi u\in H_{(r-\delta s_+,s)}$, which implies $(t_0,x_0,\xi_0)\notin\pi_0(\mathrm{WF}_{(r-\delta s_+,s)}(u))$. The last statement follows directly from (A.10). $\square$

Appendix B. The energy estimates

We are going to prove propagation of $H'_{(r,s)}$ regularity for the wave systems we are going to use, where $H'_{(r,s)}$ is defined in Appendix A. In what follows, we shall consider pseudo-differential operators in $x=(x_1,x'')$, depending C^∞ on t. As in Section 4, we shall suppress the t dependence, and write Ψ^0_{phg} instead of $C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ for example. We shall use the symbol classes $S^{r,s}=S(\langle\xi\rangle^r h^{-s}, g)$, where $h^{-2}=1+\xi_1^2$ and $\langle\xi\rangle$ are weights for the metric g defined by

$$g_{x,\xi}(dx,d\xi)=|dx|^2+|d\xi|^2h^2. \tag{B.1}$$

It is easy to see that g is σ temperate, $g/g^\sigma=h^2\le 1$, and that $S^0\subseteq S(1,g)$. Let $\Psi^{r,s}=\mathrm{Op}\,S^{r,s}$ be the corresponding pseudo-differential operators, which map $H'_{(r,s)}$ into L^2. We shall mainly use the polyhomogeneous symbol classes $S^{r,s}_{\mathrm{phg}}$ and the corresponding operators $\Psi^{r,s}_{\mathrm{phg}}$.

First we are going to consider the following $N\times N$ system:

$$Q=q\,\mathrm{Id}_N+Q_1+Q_0. \tag{B.2}$$

Here q is a scalar operator with symbol

$$q(t,x;\tau,\xi)=\tau^2-\alpha^2(t,x,\xi)\xi_1^2, \tag{B.3}$$

where $0<\alpha(t,x,\xi)\in S^0$ is homogeneous. We shall also assume $Q_0\in\Psi^{0,0}_{\mathrm{phg}}$ and

$$Q_1=A_0D_t+A_1, \tag{B.4}$$

with $A_j \in \Psi^{0,j}_{\mathrm{phg}}$. This means that Q satisfies the microlocal Levi condition $\sigma_{\mathrm{sub}}(Q) = 0$ on Σ_2. We shall study the following Cauchy problem:

$$\begin{cases} Qu = f, \\ (u, D_t u)|_{t=0} = (u_0, u_1). \end{cases} \tag{B.5}$$

Since we are going to assume that $\xi \neq 0$ in $\mathrm{WF}(u)$, the restrictions are well defined. Now Σ_2 is involutive, thus it has a natural foliation given by the Hamilton fields of the normal vectors. On the leaves of this foliation, the Hessian of q gives an invariant Lorentz structure by the natural identification of the tangent space of the leaves with the normal bundle via the symplectic form (see [9, Lemma 2.5]). Let $C^2_{w_0}$, $w_0 \in \Sigma_2$, be the closure of the set of $w \in \Sigma_2$ such that w lies in the same leaf of Σ_2 as w_0, and is in the backward (with respect to t) propagation cone emanating from w_0. Now, we assume that $t > 0$ at w_0. Let $i_0 : T^*_{t=0}\mathbf{R}^n \ni (0, x; 0, \xi) \mapsto (x, \xi) \in T^*\mathbf{R}^{n-1}$, and let $\pi_0(t, x; \tau, \xi) = (t, x, \xi)$.

Proposition B.1. *Assume that* $u \in \mathcal{D}'(\mathbf{R}^n, \mathbf{C}^N)$ *satisfies* (B.5), $w \in \Sigma_2$, *and* $\xi \neq 0$ *in* $\mathrm{WF}(u)$. *If* $u_0 \in H_{(r,s)}$, $u_1 \in H_{(r,s-1)}$ *in* $i_0(C^2_w \cap \{t=0\})$, *and* $f \in H'_{(r,s-1)}$ *in* $\pi_0(C^2_w \cap \{t \geq 0\})$, *then* $u \in H'_{(r,s)}$ *at* $\pi_0 w$.

Proof. This result follows by modifying the proof of Proposition 5.1 in [3], replacing $x' = (x_1, x_2)$ with x_1 and changing the spaces $H_{(r,s)}$ and the operators $\Psi^{r,s}_{\mathrm{phg}}$ accordingly. □

We shall also consider the following 2×2 system

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \tag{B.6}$$

where

$$q_{jj} = q + b_j D_t + c_j, \quad j = 1, 2, \tag{B.7}$$

q is given by (B.3), $b_j \in C^\infty(\mathbf{R}, \Psi^0_{\mathrm{phg}})$ and $c_j \in C^\infty(\mathbf{R}, \Psi^{0,1}_{\mathrm{phg}})$. We also assume

$$\begin{cases} q_{12} = a_{-1} D_t + a_0, \\ q_{21} = a_1 D_t + a_2, \end{cases} \tag{B.8}$$

where $a_j \in C^\infty(\mathbf{R}, \Psi^{0,j}_{\mathrm{phg}})$. We shall study the following Cauchy problem

$$\begin{cases} Qu = f, \\ u|_{t=0} = u_0, \\ D_t u|_{t=0} = u_1. \end{cases} \tag{B.9}$$

As before, we let $C^2_{w_0}$ be the closure of the backward wave cone emanating from $w_0 \in \Sigma_2$. By modifying the proof of Proposition 5.6 in [3] in the same way as in the proof of Proposition B.1, we obtain the following result.

Proposition B.2. *Assume that* $u\in\mathcal{D}'(\mathbf{R}^n,\mathbf{C}^2)$ *satisfies* (B.9), $\xi\neq 0$ *in* $\mathrm{WF}(u)$, *and* $w\in\Sigma_2$. *If* $u_0\in H_{(r,s)}\times H_{(r,s-1)}$, $u_1\in H_{(r,s-1)}\times H_{(r,s-2)}$ *in* $i_0(C^2_w\cap\{t=0\})$, *and* $f\in H'_{(r,s-1)}\times H'_{(r,s-2)}$ *in* $\pi_0(C^2_w\cap\{t\geq 0\})$, *then* $u\in H'_{(r,s)}\times H'_{(r,s-1)}$ *at* $\pi_0 w$.

Appendix C. Regularity of the coupling term

We shall now estimate the regularity of the term $[P_{12},P_{22}]u_2$, which is important when we decouple the system Q given by (4.10). We change notation, and put $x_0=t$, $x''=(x_2,\dots,x_{d_0})$, which gives $x=(x_0,x_1,x'')\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}^{n-2}$. We are going to use the spaces $H_{(r,s)}$ defined by (A.1) with τ replaced with ξ_0, and also the usual Sobolev spaces $H_{(r)}=H_{(r,0)}$. We shall use the symbol classes $\Psi^{r,s}=\operatorname{Op}S^{r,s}$ defined in Appendix B, where $S^{r,s}=S(\langle\xi\rangle^r h^{-s},g)$, but now with $h^{-2}=1+\xi_0^2+\xi_1^2$. Assume that

$$P=\begin{pmatrix} P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}\in\Psi^{0,1} \tag{C.1}$$

has principal symbol p satisfying $\det p=\xi_0^2-\beta^2$, where $\beta(x,\xi_1,\xi'')=\alpha(x,\xi_1,\xi'')\xi_1$ is independent of ξ_0 and homogeneous of degree 1 in ξ.

Proposition C.1. *Assume that* $u\in\mathcal{D}'(\mathbf{R}^n,\mathbf{C}^2)$, *with the property that* Pu *and* $u_1\in H_{(r,s)}$ *at* $w_0\in\Sigma_2$. *Then we find* $[P_{12},P_{22}]u_2\in H_{(r-\delta,s-1)}$ *at* w_0, $\forall\,\delta>0$.

Proof. Let $\delta>0$ be fixed in what follows, clearly it is no restriction to assume that $\delta\leq 1$. As in [5, Proposition 6.1], we shall modify the proof of [3, Proposition 4.1], replacing the weight $\langle\xi'\rangle$ with h^{-1}. By conjugating with an elliptic operator in $\Psi^{r-\delta,s}$, we may assume that $r=\delta$ and $s=0$. Let

$$Q=\begin{pmatrix} q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}={}^tP^{\mathrm{co}}P\in\Psi^{0,2}, \tag{C.2}$$

then $\sigma(q_{jj})=\xi_0^2-\beta^2$ is real, and $q_{12}=[P_{22},P_{12}]\in\Psi^{0,1}$. Let $\lambda\in\Psi^{0,-1}$ have symbol equal to h, put $q=\lambda q_{22}\in\Psi^{0,1}$ and $m=\lambda q_{12}\in\Psi^{0,0}$. Since $Qu\in H_{(\delta,-1)}$, $u_1\in H_{(\delta)}$ and $q_{21}\in\Psi^{0,1}$, we find qu_1+mu_2 and $qu_2\in H_{(\delta)}$ at w_0. Choose cut-off functions φ, $\psi\in S^0_{1,0}$ such that $\varphi=1$ in a conical neighborhood of w_0, $\psi=1$ on $\operatorname{supp}\varphi$, and u_1, $Pu\in H_{(\delta)}$ on $\operatorname{supp}\psi$. Then $\psi qu_2\in H_{(\delta)}$ and $\psi qu_1+\psi mu_2=f\in H_{(\delta)}$, which gives $\psi mu_2\in H_{(\delta,-1)}$. The result follows if we prove that $\varphi mu_2\in H_{(0)}$.

Next, we localize in $h^{-1}\gtrless c\langle\xi\rangle^\delta$. Let $\phi(s)\in C_0^\infty(\mathbf{R})$ such that $\phi(s)=1$ when $|s|\leq 1$, then

$$\chi(\xi)=\phi(h^{-1}/\langle\xi\rangle^\delta)\in S(1,g)\cap S^0_{\delta,0}, \tag{C.3}$$

since $d\chi$ is supported where $h^{-1}\approx\langle\xi\rangle^\delta$. Clearly, $\chi\varphi mu_2\in H_{(0)}$ since $\varphi mu_2\in H_{(\delta,-1)}$, thus it remains to consider $(1-\chi)\varphi mu_2$.

Now, we may normalize so that $|q|\le h^{-1}$ and $|q|_1^g\le h^{-1}$, where $|q|_j^g$ is the jth seminorm of q in $S(h^{-1},g)$. We introduce the Beals–Fefferman type of metric $G=Hg/h$ defined by

$$H^{-1}=\max(h^{-\delta},|q|,(|q|_1^g)^2h), \tag{C.4}$$

which implies $h^{-\delta}\le H^{-1}\le h^{-1}$. Then G is σ temperate, and $G/G^\sigma\le H^2$. Also, we have $S(1,g)\subseteq S(1,G)$, and $q\in S(H^{-1},G)$. (See the proof of [3, Proposition 4.1].) Choose $\{\varphi_k\}$, $\{\psi_k\}$ and $\{\Phi_k\}\in S(1,G)$ (with values in l^2) such that $\operatorname{supp}\Phi_k\subseteq U_{w_k}=\{w;G_{w_k}(w-w_k)<\varepsilon^2\}$, $\Phi_k=1$ on $\operatorname{supp}\psi_k$, and $\psi_k=1$ on $\operatorname{supp}\varphi_k$. Also, we may assume that $h\approx h_k=h(w_k)$ in $\operatorname{supp}\Phi_k$, $\sum_k|\varphi_k|^2=1$ on $\operatorname{supp}(1-\chi)\varphi$, and $\{\Phi_k\}$ are supported where $\psi=1$ and $h^{-1}\ge c\langle\xi\rangle^\delta$. Since $\{\varphi_k\}$ is elliptic on $\operatorname{supp}(1-\chi)\varphi$ and $H\le h^\delta$, we find

$$\|(1-\chi)\varphi mu_2\|^2\le C\Big(\sum_k\|m_ku_2\|^2+\|\varphi mu_2\|^2_{(0,-1)}\Big) \tag{C.5}$$

where $m_k=\varphi_k(1-\chi)\varphi m$.

Now we can use Lemma 4.2 in [3], which only uses the general properties of the Beals–Fefferman metric G. By substituting $\Phi_ku=v$ in this Lemma, we obtain for small enough $\varepsilon>0$

$$\|m_ku_2\|\le C_N(h_k^{-\delta}\|\psi_ku_1\|+\|\varphi_kf\|+\|\psi_kqu_2\|+\|R_{N,k}u\|)\quad\forall N,k. \tag{C.6}$$

Here $f=\psi qu_1+\psi mu_2\in H_{(\delta)}$, $h_k=h(w_k)$, and $\{R_{N,k}\}_k\in S(H^N,G)$ (with values in l^2) is bounded by any power of the G^σ distance to $\operatorname{supp}\{\Phi_k\}$. Since $h^{-1}/\langle\xi\rangle^\delta$ is a weight for g, we obtain that $h^{-1}\le\varrho\langle\xi\rangle^\delta$ implies that the g distance to $\operatorname{supp}\{\Phi_k\}$ is bounded from below by a positive constant, for small enough ϱ. Since $G^\sigma\ge h^{-1-\delta}g$ and $h^{-1}\ge c\langle\xi\rangle^\delta$ in $\operatorname{supp}\{\Phi_k\}$, we find $\{R_{N,k}\}_k\in S(\langle\xi\rangle^{-N_1},G)\ \forall N_1$ when $h^{-1}\le\varrho\langle\xi\rangle^\delta$, for small enough ϱ. When $h^{-1}\ge\varrho\langle\xi\rangle^\delta$, we find $H\le h^\delta\le c_\varrho\langle\xi\rangle^{-\delta^2}$ making $\{R_{N,k}\}_k\in S^{-\delta^2N}_{\delta^2,0}$ there. Since $\{\psi_k\}$ is supported where $\psi=1$, we may replace ψ_k by $\psi_k\psi$ in (C.6). By summing up, we obtain the result. $\square$

References

1. DENCKER, N., On the propagation of singularities for pseudo-differential operators of principal type, *Ark. Mat.* **20** (1982), 23–60.

2. DENCKER, N., On the propagation of polarization sets for systems of real principal type, *J. Funct. Anal.* **46** (1982), 351–372.
3. DENCKER, N., On the propagation of polarization in conical refraction, *Duke Math. J.* **57** (1988), 85–134.
4. DENCKER, N., The propagation of singularities for pseudo-differential operators with self-tangential characteristics, *Ark. Mat.* **27** (1989), 65–88.
5. DENCKER, N., The propagation of polarization in double refraction, *J. Funct. Anal.* **104** (1992), 414–468.
6. DENCKER, N., Preparation theorems for matrix valued functions, *Ann. Inst. Fourier (Grenoble)* **43** (1993), 865–892.
7. HÖRMANDER, L., *The Analysis of Linear Partial Differential Operators* I–IV, Springer-Verlag, Berlin–Heidelberg, 1983–1985.
8. IVRII, V. YA., Wave fronts of solutions of symmetric pseudodifferential systems, *Sibirsk. Mat. Zh.* **20** (1979), 557–578 (Russian). English transl.: *Siberian Math. J.* **20** (1979), 390–405.
9. MELROSE, R. B. and UHLMANN, G. A., Microlocal structure of involutive conical refraction, *Duke Math. J.* **46** (1979), 571–582.
10. TAYLOR, M. E., *Pseudodifferential Operators*, Princeton Univ. Press, Princeton, N. J., 1981.
11. UHLMANN, G. A., Pseudo-differential operators with involutive double characteristics, *Comm. Partial Differential Equations* **2** (1977), 713–779.

Received June 20, 1994

Nils Dencker
Department of Mathematics
University of Lund
Box 118
S-221 00 Lund
Sweden
email: nils.dencker@math.lu.se