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Extremal rational elliptic surfaces in characteristic p. II: Surfaces with three or fewer singular fibres

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Introduction

In this paper, we complete the classification of extremal rational elliptic surfaces in characteristic p, begun in [3].

In [4], Miranda and Persson classified all rational elliptic surfaces over the complex numbers such that the Mordell–Weil group of the generic fibre is finite. They called these surfaces *extremal* rational elliptic surfaces. They found 16 families of such surfaces. All but one of these families have only one member, and the exceptional family depends on one parameter.

In our first paper on extremal rational elliptic surfaces in characteristic p, we classified those where the singular fibres are semi-stable. These are the characteristic p analogues of the surfaces studied by Beauville in [1], and we called them Beauville surfaces.

In this paper, we classify all other extremal rational elliptic surfaces. The classification is identical to the classification in characteristic zero in all characteristics except two and three. (There is one exceptional case in characteristic five.) The classification in characteristics two and three looks quite different. This is due to the presence of a wild ramification term in the formula of Neron–Ogg–Shararevich, which appears only in these characteristics.

Here is a plan of the paper. In Section 1, we give the preliminary results on extremal rational elliptic surfaces that we need. (Almost all of these results appeared in [3].) In Section 2A, we classify all possible singular fibres on rational elliptic surfaces with section in characteristic two, taking into account the extra term in the Neron–Ogg–Shararevich formula. It is hoped that this list may be useful for other purposes. In Section 2B, we use the results of Section 2A (together with the material of Section 1) to classify extremal rational elliptic surfaces in characteristic two. Sections 3A and 3B carry out the same program in characteristic three. Section 4 carries out the classification in all characteristics not equal to two or three.

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1. Preliminaries

Most of this section is repeated from [3].

Definition. (Miranda–Persson) Let $f: X \to C$ be an elliptic surface with a section over C. We will say that X is an *extremal* elliptic surface if the rank $\varrho(X)$ of the Neron–Severi group is equal to $h^{1,1}(X)$ and if the rank of the Mordell–Weil group of the generic fibre (we will denote this group by $MW(X_g)$) is zero.

If $f: X \to C$ is an elliptic surface with a section over an algebraically closed field of characteristic p, the definition of extremal remains the same except that we replace the condition $\varrho = h^{1,1}$ by the condition $\varrho = B_2$.

We will assume that all elliptic surfaces are relatively minimal and have a section.

Definition. A Beauville surface is an extremal rational elliptic surface such that all singular fibres are semi-stable.

Let $f: X \to C$ be an elliptic surface over an algebraically closed field k. Following [4], we assign three numerical invariants to each singular fibre F of X. The first is δ_F, the order of vanishing of the discriminant Δ of the Weierstrass equation for the point of the base under F. The second is r_F, which is the number of components of F which do not meet the zero section. Finally, we consider the lattice in $NS(X)$ of rank r_F spanned by the components of fibres not meeting the zero section, and we let d_F be the discriminant of this lattice. (If $r_F = 0$, we adopt the convention that $d_F = 1$.)

Lemma 1.1. *If X is an extremal rational elliptic surface, then $\prod d_F$ is a perfect square, and the order of the Mordell–Weil group of the generic fibre is the square root of $\prod d_F$.*

Proof. See [4], Corollary 2.6.

Lemma 1.2. *Let F be a singular fibre on an extremal rational elliptic surface. Then $\delta_F - r_F = 1$ if F is a fibre of multiplicative type, and $\delta_F - r_F = 2 + f_F$, where $f_F \geq 0$ if F is a fibre of additive type. Moreover, $f_F = 0$ unless the characteristic is two or three.*

Proof. This is a consequence of the formula of Neron–Ogg–Shararevich. See [5] or [6, p. 361].

Now suppose $f: X \to P^1$ is an extremal rational elliptic surface. Since X is rational, $\sum \delta_F = 12$, and since X is extremal, $\sum r_F = 8$. Therefore $\sum(\delta_F - r_F) = 4$. Using Lemma 1.2, we obtain

Lemma 1.3. *Any extremal rational elliptic surface has 4 or fewer singular fibres. The surface has 4 singular fibres if and only if it is a Beauville surface.*

We classified Beauville surfaces in [3]. Therefore we need only deal with surfaces with three or fewer singular fibres in this paper.

2. Characteristic two

2A. Classification of singular fibres of additive type on rational elliptic surfaces in characteristic two

Throughout this section, we will assume $f: X \to P^1$ is a rational elliptic surface with section over an algebraically closed field of characteristic two.

We begin by listing the possible types of singular fibres of additive type that can appear on a rational elliptic surface in characteristic two. We put the Weierstrass equation for each type into a normal form, and compute Δ, δ_F, and the Kodaira type of each fibre. The Kodaira type determines r_F.

The proof is a straightforward exercise in applying Tate's algorithm for determining the type of a singular fibre in an elliptic pencil [7]. We start by writing the Weierstrass equation for our surface

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where the a_i are polynomials in t of degree $\leq i$. We locate our singular fibre of additive type at $t=0$. We may change coordinates so that $t | a_3, a_4$, and a_6. We may now write the equation in the form

$$y^2 + a_1xy + tc_2y = x^3 + a_2x^2 + atc_3x + tc_5.$$

Then since our fibre is of additive type, $t | a_1$. We have two possibilities:

(1) $a_1 \neq 0$, in which case we may scale and assume $a_1 = t$;

(2) $a_1 = 0$. If $a_1 = 0$, the j-invariant of the surface is identically zero.

We will handle these possibilities separately. Also, by making a substitution $y=y+kx$, we may assume $t|a_2$ also. Now we simply work through the algorithm. The calculations are straightforward, and will not be given in detail. Notations such as a_i, c_i, d_i, etc. represent polynomials in t of degree ≤ 1.

$$\text{Case 1A} \quad \begin{cases} y^2+txy+tc_2y=x^3+tc_1x^2+tc_3x+tc_5, \ t\nmid c_5, t\nmid c_2, \\ \Delta=t^4(t^3c_5+t^3c_2c_3+t^3c_1c_2^2+t^2c_3^2)+t^4c_2^4+t^6c_2^3, \\ \delta=4, \ r=0 \quad \text{Type II.} \end{cases}$$

$$\text{Case 1B} \quad \begin{cases} y^2+txy+t^2d_1y=x^3+tc_1x^2+tc_3x+tc_5, \ t\nmid c_5, t\nmid c_3, \\ \Delta=t^4(t^3c_5+t^4d_1c_3+t^5c_1d_1^2+t^2c_3^2)+t^8d_1^4+t^9d_1^3, \\ \delta=6, \ r=0 \quad \text{Type II.} \end{cases}$$

$$\text{Case 1C} \quad \begin{cases} y^2+txy+t^2d_1y=x^3+tc_1x^2+t^2c_2x+tc_5, \ t\nmid c_5, \\ \Delta=t^4(t^3c_5+t^5d_1c_2+t^5c_1d_1^2+t^4c_2^2)+t^8d_1^4+t^9d_1^3, \\ \delta=7, \ r=0 \quad \text{Type II.} \end{cases}$$

$$\text{Case 2A} \quad \begin{cases} y^2+txy+tc_2y=x^3+tc_1x^2+tc_3x+t^2c_4, \ t\nmid c_3, t\nmid c_2, \\ \Delta=t^4(t^4c_4+t^3c_2c_3+t^3c_1c_2^2+t^2c_3^2)+t^4c_2^4+t^6c_2^3, \\ \delta=4, \ r=1 \quad \text{Type III.} \end{cases}$$

$$\text{Case 2B} \quad \begin{cases} y^2+txy+t^2d_1y=x^3+tc_1x^2+tc_3x+t^2c_4, \ t\nmid c_3, \\ \Delta=t^4(t^4c_4+t^4d_1c_3+t^5c_1d_1^2+t^2c_3^2)+t^8d_1^4+t^9d_1^3, \\ \delta=6, \ r=1 \quad \text{Type III.} \end{cases}$$

$$\text{Case 3} \quad \begin{cases} y^2+txy+tc_2y=x^3+tc_1x^2+t^2d_2x+t^2c_4, \ t\nmid c_2, \\ \Delta=t^4(t^4c_4+t^4c_2d_2+t^3c_1c_2^2+t^4d_2^2)+t^4c_2^4+t^6c_2^3, \\ \delta=4, \ r=2 \quad \text{Type IV.} \end{cases}$$

$$\text{Case 4A} \quad \begin{cases} y^2+txy+t^2c_1y=x^3+td_1x^2+t^2d_2x+t^2c_4. \\ \text{By substituting } y=y+kt, x=x+lt, \text{this becomes} \\ y^2+txy+t^2c_1y=x^3+td_1x^2+t^3e_1x+t^3c_3, \ t\nmid c_3, t\nmid c_1, \\ \Delta=t^4(t^5c_3+t^6c_1e_1+t^5d_1c_1^2+t^6e_1^2)+t^8c_1^4+t^9c_1^3, \\ \delta=8, \ r=4 \quad \text{Type I}_0^*. \end{cases}$$

$$\text{Case 4B} \quad \begin{cases} y^2+txy+t^3c_0y=x^3+td_1x^2+t^3e_1x+t^3c_3, \ t\nmid c_3, \\ \Delta=t^4(t^5c_3+t^7c_0e_1+t^7d_1c_0+t^6e_1^2)+t^{12}c_0^4+t^{12}c_0^3, \\ \delta=9, \ r=4 \quad \text{Type I}_0^*. \end{cases}$$

Case 5A $\begin{cases} y^2+txy+t^2c_1y=x^3+td_1x^2+t^3e_1x+t^4c_2, \ t\nmid d_1, t\nmid c_1, \\ \Delta=t^4(t^6c_2+t^6c_1e_1+t^5d_1c_1^2+t^6e_1^2)+t^8c_1^4+t^9c_1^3, \\ \delta=8, \ r=5 \quad \text{Type } \mathrm{I}_1^*. \end{cases}$

Case 5B $\begin{cases} y^2+txy+t^3c_0y=x^3+td_1x^2+t^3e_1x+t^4c_2. \\ \text{Replace } y \text{ by } y+kt^2. \text{ For suitable } k, \text{ we have} \\ y^2+txy+t^3c_0y=x^3+td_1x^2+t^3e_1x+t^5c_1, \ t\nmid d_1, t\nmid e_1. \\ \Delta=t^4(t^7c_1+t^7c_0e_1+t^7c_0^2d_1+t^6e_1^2)+t^{12}c_0^4+t^{12}c_0^3, \\ \delta=10, \ r=6 \quad \text{Type } \mathrm{I}_2^*. \end{cases}$

Case 5C $\begin{cases} y^2+txy+t^3c_0y=x^3+td_1x^2+t^4e_0x+t^5e_1. \\ \text{Replace } x \text{ by } x+kt^2. \text{ For suitable } k, \text{ we have} \\ y^2+txy+t^3c_0y=x^3+td_1x^2+t^4e_0x+t^6d_0, \ t\nmid d_1, t\nmid c_0. \\ \Delta=t^4(t^8d_0+t^8c_0e_0+t^7d_1c_0^2+t^8e_0^2)+t^{12}c_0^4+t^{12}c_0^3, \\ \delta=11, \ r=7 \quad \text{Type } \mathrm{I}_3^*. \end{cases}$

Case 5D $\begin{cases} y^2+txy=x^3+td_1x^2+t^4e_0x+t^6e_0. \\ \text{Replace } y \text{ by } y+kt^3. \text{ For suitable } k, \text{ we have} \\ y^2+txy=x^3+td_1x^2+t^4e_0x, \ t\nmid d_1, t\nmid e_0. \\ \Delta=t^{12}e_0^3, \\ \delta=12, \ r=8 \quad \text{Type } \mathrm{I}_4^*. \end{cases}$

Case 6 $\begin{cases} y^2+txy+t^2c_1y=x^3+t^2d_0x^2+t^3e_1x+t^4c_2, \ t\nmid c_1, \\ \Delta=t^4(t^6c_2+t^6c_1e_1+t^6d_0c_1^2+t^6e_1^2)+t^8c_1^4+t^9c_1^3, \\ \delta=8, \ r=6 \quad \text{Type } IV^*. \end{cases}$

Case 7 $\begin{cases} y^2+txy+t^3c_0y=x^3+t^2d_0x^2+t^3e_1x+t^4c_2. \\ \text{Replace } y \text{ by } y+kt^2. \text{ For suitable } k, \text{ we have} \\ y^2+txy+t^3c_0y=x^3+t^2d_0x^2+t^3e_1x+t^5f_1, \ t\nmid e_1. \\ \Delta=t^4(t^7f_1+t^7c_0e_1+t^8c_0^2d_0+t^6e_1^2)+t^{12}c_0^4+t^{12}c_0^3, \\ \delta=10, \ r=7 \quad \text{Type III}^*. \end{cases}$

Case 8 $\begin{cases} y^2+txy+t^3c_0y=x^3+t^2d_0x^2+t^4e_0x+t^5e_1, \ t\nmid e_1, \\ \Delta=t^4(t^7e_1+t^8c_0e_0+t^8c_0^2d_0+t^8e_0^2)+t^{12}c_0^4+t^{12}c_0^3, \\ \delta=11, \ r=8 \quad \text{Type II}^*. \end{cases}$

Cases 1–8 exhaust all possibilities with $a_1 \neq 0$. In cases 9–16, we will have $a_1=0$, which forces $j=0$.

Case 9A
$$\begin{cases} y^2+tc_2y=x^3+tc_1x^2+tc_3x+tc_5, \ t\nmid c_5, t\nmid c_2, \\ \Delta=t^4c_2^4, \\ \delta=4, \ r=0 \quad \text{Type II.} \end{cases}$$

Case 9B
$$\begin{cases} y^2+t^2d_1y=x^3+tc_1x^2+tc_3x+tc_5, \ t\nmid c_5, t\nmid d_1, \\ \Delta=t^8d_1^4, \\ \delta=8, \ r=0 \quad \text{Type II.} \end{cases}$$

Case 9C
$$\begin{cases} y^2+t^3c_0y=x^3+tc_1x^2+tc_3x+tc_5, \ t\nmid c_5, t\nmid c_0, \\ \Delta=t^{12}c_0^4, \\ \delta=12, \ r=0 \quad \text{Type II.} \end{cases}$$

Case 10A
$$\begin{cases} y^2+tc_2y=x^3+tc_1x^2+tc_3x+t^2c_4, \ t\nmid c_3, t\nmid c_2, \\ \Delta=t^4c_2^4, \\ \delta=4, \ r=1 \quad \text{Type III.} \end{cases}$$

Case 10B
$$\begin{cases} y^2+t^2d_1y=x^3+tc_1x^2+tc_3x+t^2c_4, \ t\nmid c_3, t\nmid d_1, \\ \Delta=t^8d_1^4, \\ \delta=8, \ r=1 \quad \text{Type III.} \end{cases}$$

Case 10C
$$\begin{cases} y^2+t^3d_0y=x^3+tc_1x^2+tc_3x+t^2c_4, \ t\nmid c_3, t\nmid d_0, \\ \Delta=t^{12}d_0^4, \\ \delta=12, \ r=1 \quad \text{Type III.} \end{cases}$$

Case 11
$$\begin{cases} y^2+tc_2y=x^3+tc_1x^2+t^2d_2x+t^2c_4, \ t\nmid c_2, \\ \Delta=t^4c_2^4, \\ \delta=4, \ r=2 \quad \text{Type IV.} \end{cases}$$

Case 12A
$$\begin{cases} y^2+t^2d_1y=x^3+tc_1x^2+t^2d_2x+t^2c_4. \\ \text{We may change coordinates so this becomes} \\ y^2+t^2c_1y=x^3+td_1x^2+t^3c_1x+t^3c_3, \ t\nmid c_3, t\nmid c_1, \\ \Delta=t^8c_1^4, \\ \delta=8, \ r=4 \quad \text{Type I}_0^*. \end{cases}$$

Case 12B
$$\begin{cases} y^2+t^3c_0y=x^3+td_1x^2+t^3c_1x+t^3c_3, \ t\nmid c_3, t\nmid c_0, \\ \Delta=t^{12}c_0^4, \\ \delta=12, \ r=4 \quad \text{Type } \mathrm{I}_0^*. \end{cases}$$

Case 13A
$$\begin{cases} y^2+t^2c_1y=x^3+td_1x^2+t^3e_1x+t^4d_2, \ t\nmid d_1, t\nmid c_1, \\ \Delta=t^8c_1^4, \\ \delta=8, \ r=5 \quad \text{Type } \mathrm{I}_1^*. \end{cases}$$

Case 13B
$$\begin{cases} y^2+t^3c_0y=x^3+td_1x^2+t^3e_1x+t^4d_2, \ t\nmid d_1, t\nmid c_0. \\ \text{Replace } y \text{ by } y+kt^2. \text{ For suitable } k, \text{ we obtain} \\ y^2+t^3c_0y=x^3+td_1x^2+t^3e_1x+t^5f_1, \ t\nmid e_1, t\nmid c_0, \\ \Delta=t^{12}c_0^4. \\ \delta=12, \ r=6 \quad \text{Type } \mathrm{I}_2^*. \end{cases}$$

Case 13C
$$\begin{cases} y^2+t^3c_0y=x^3+td_1x^2+t^4d_0x+t^5e_1. \\ \text{Replace } x \text{ by } x+kt^2. \text{ For suitable } k, \text{ we obtain} \\ y^2+t^3c_0y=x^3+td_1x^2+t^4d_0x+t^6e_0, \ t\nmid d_1, t\nmid c_0. \\ \Delta=t^{12}c_0^4, \\ \delta=12, \ r=7 \quad \text{Type } \mathrm{I}_3^*. \end{cases}$$

Case 14
$$\begin{cases} y^2+t^2c_1y=x^3+t^2d_0x^2+t^3e_1x+t^4d_2, \ t\nmid c_1, \\ \Delta=t^8c_1^4, \\ \delta=8, \ r=6 \quad \text{Type IV}^*. \end{cases}$$

Case 15
$$\begin{cases} y^2+t^3c_0y=x^3+t^2d_0x^2+t^3e_1x+t^4d_2. \\ \text{Replace } y \text{ by } y+kt^2. \text{ For suitable } k, \text{ we obtain} \\ y^2+t^3c_0y=x^3+t^2d_0x^2+t^3e_1x+t^5d_1, \ t\nmid e_1, t\nmid c_0, \\ \Delta=t^{12}c_0^4, \\ \delta=12, \ r=7 \quad \text{Type III}^*. \end{cases}$$

Case 16
$$\begin{cases} y^2+t^3c_0y=x^3+t^2d_0x^2+t^4e_0x+t^5d_1, \ t\nmid d_1, t\nmid c_0, \\ \Delta=t^{12}c_0^4, \\ \delta=12, \ r=8 \quad \text{Type II}^*. \end{cases}$$

We now make a table of the possibilities obtained.

	δ	r	$\delta - r$
Case 1A	4	0	4
Case 1B	6	0	6
Case 1C	7	0	7
Case 2A	4	1	3
Case 2B	6	1	5
Case 3	4	2	2
Case 4A	8	4	4
Case 4B	9	4	5
Case 5A	8	5	3
Case 5B	10	6	4
Case 5C	11	7	4
Case 5D	12	8	4
Case 6	8	6	2
Case 7	10	7	3
Case 8	11	8	3

In cases 9–16, the j-invariant is identically zero

Case 9A	4	0	4
Case 9B	8	0	8
Case 9C	12	0	12
Case 10A	4	1	3
Case 10B	8	1	7
Case 10C	12	1	11
Case 11	4	2	2
Case 12A	8	4	4
Case 12B	12	4	8
Case 13A	8	5	3
Case 13B	12	6	6
Case 13C	12	7	5
Case 14	8	6	2
Case 15	12	7	5
Case 16	12	8	4

We now list the possibilities for extremal rational elliptic surfaces permitted by the table. Recall from Section 1 that $\sum \delta_F = 12$, $\sum r_F = 8$, $\sum(\delta_F - r_F) = 4$. Also, since cases 9–16 have j identically zero, we cannot have fibres from cases 9–16 appearing with either a fibre of multiplicative type or with a fibre from cases 1–8.

Possibilities with one singular fibre

I	Case 5D,
II	Case 16.

Possibilities with two singular fibres

III	Case 2A, I_8,
IV	Case 5A, I_4,
V	Case 7, I_2,
VI	Case 8, I_1,
VII	Case 1_1, Case 1_4.

Possibilities with three singular fibres

VIII	Case 3, I_2, I_6,
IX	Case 6, I_1, I_3.

The possibility of Case 3 and Case 6 appearing together is excluded, since both of these require $a_1 \neq 0$ and that a_1 have a zero below the fibre in question. Since a_1 has degree ≤ 1, it has at most one zero.

2B. Existence and uniqueness of surfaces with prescribed fibre types

In this section we will show that surfaces with fibres of each type listed at the end of Section 2A exist. We will examine the uniqueness of each type.

Before beginning our case-by-case analysis, we remark that in the cases where j is not identically zero, the Weierstrass equation may be written in the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

By appropriate changes of variable, this can be put into the following form.

$$y^2 + txy + \lambda y = x^3 + a_2x^2 + \mu x + a_6,$$

where λ and μ are constants. We refer to this as the $\lambda\mu$-form of the Weierstrass equation. For the $\lambda\mu$-form,

$$\Delta = t^4(t^2a_6 + t\lambda\mu + \lambda^2a_2 + \mu^2) + \lambda^4 + \lambda^3t^3.$$

Case I. This surface has one singular fibre, which we locate at the origin. Putting the equation into $\lambda\mu$-form, we must have $\lambda = \mu = 0$, $a_6 = kt^6$, k a constant. Our equation becomes

$$y^2 + txy = x^3 + a_2x^2 + kt^6.$$

By making a substitution of the form $y=y+(at+b)x$, we may assume $a_2=dt$, d a constant. We now have

$$y^2+txy=x^3+dtx^2+kt^6.$$

We cannot have $d=0$, since if $d=0$, the Weierstrass equation would not be minimal. We may scale x, y, and t so that $d=1$. The equation is now

$$y^2+txy=x^3+tx^2+kt^6.$$

Then $\Delta=kt^{12}$, so we cannot have $k=0$. Computing, we find $j=1/k$, so different choices of k lead to non-isomorphic surfaces. Thus, the surfaces in Case I form a 1-parameter family.

Case II. Our equation is

$$y^2+t^3c_0y=x^3+t^2d_0x^2+t^4e_0x+t^5d_1.$$

Scaling, we may assume $c_0=1$. A substitution of the form $y=y+ktx$ allows us to assume $d_0=0$. We now have

$$y^2+t^3y=x^3+t^4e_0x+t^5d_1.$$

We can force $e_0=0$ by a substitution of the form $x=x+kt^2$, $y=y+lt$, $l^2=k$. This gives

$$y^2+t^3y=x^3+t^5d_1.$$

Finally, by making a substitution $y=y+kt^3$, we may assume $d_1=d$, a constant. Clearly $d\neq 0$, and we can scale so that $d=1$. The final equation is

$$y^2+t^3y=x^3+t^5,$$

and this surface is unique.

Case III. In this case, we locate the fibre of type 2A at $t=0$ and the fibre of type I_8 at $t=\infty$. This forces $\Delta=ct^4$, c a constant. Writing the Weierstrass equation in $\lambda\mu$-form, we see $\lambda=a_6=0$. The equation becomes

$$y^2+txy=x^3+a_2x^2+\mu x.$$

By replacing y by $y+ax+btx$, we may assume $a_2=dt$, d a constant. Now we run through Tate's algorithm to determine if we have a fibre of the desired type. For this, make a substitution $x=x+c$, $c^2=\mu$. Our equation becomes

$$y^2+txy+cty=x^3+(c+dt)x^2+dtc^2.$$

Note that if $c=0$, $\Delta=0$. So $c\neq 0$. To have a fibre of the desired type at the origin, we must have $d=0$. Our equation becomes

$$y^2+txy+cty=x^3+cx^2, \quad c\neq 0.$$

Scaling x, y, and t, we may force $c=1$. The equation becomes

$$y^2+txy+ty=x^3+x^2,$$

and our surface is unique.

Case IV. We locate the fibre of type 5A at $t=0$ and the fibre of type I_4 at $t=\infty$. We must have $\Delta=kt^8$, k a constant. Writing the Weierstrass equation in $\lambda\mu$-form, we see $\lambda=\mu=0$, $a_6=ct^2$, c a non-zero constant. As in the preceding case, we may assume $a_2=dt$. The equation becomes

$$y^2+txy=x^3+dtx^2+ct^2.$$

Substitute $y=y+et$, $e^2=c$. We get

$$y^2+txy=x^3+dtx^2+et^2x, \quad e\neq 0.$$

In order to have a fibre of type 5A at $t=0$, Tate's algorithm tells us the cubic polynomial z^3+dz^2+ez must have a double root. This forces $d=0$. By scaling, we may assume $e=1$. This gives the equation

$$y^2+txy=x^3+t^2x.$$

This surface is unique.

Case V. We locate the fibre of type 7 at $t=0$ and the fibre of type I_2 at $t=\infty$. This gives $\Delta=kt^{10}$, k a constant. Using the $\lambda\mu$-form, this leads to the equation

$$y^2+txy=x^3+dtx^2+ct^4.$$

Tate's algorithm tells us that $d=0$. Scaling, we may assume $c=1$. Our equation becomes

$$y^2+txy=x^3+t^4.$$

This surface is unique.

Case VI. This is similar to the preceding case. The equation works out to be

$$y^2+txy=x^3+t^5.$$

This surface is unique.

Case VII. We locate the fibre of type 14 at 0, and the fibre of type 11 at ∞. Our equation has the form

$$y^2+t^2c_1y=x^3+t^2d_0x^2+t^3e_1x+t^4d_2, \quad t\nmid c_1.$$

Since the singular fibres are at 0 and ∞, c_1 must be a non-zero constant, and by scaling, we may assume $c_1=1$. By suitable changes of coordinates, we may reduce this to

$$y^2+t^2y=x^3+t^3ex+t^5d_1.$$

Let $u=1/t$. Then at ∞, our equation becomes

$$y^2+uy=x^3+uex+ud.$$

To have a fibre of type 11, Tate's algorithm tells us $d=e=0$. Our final equation is

$$y^2+t^2y=x^3.$$

This surface is unique.

Case VIII. We locate the fibre of type 3 at $t=0$, the fibre of type I_2 at $t=1$, and the fibre of type I_6 at $t=\infty$. This means that $\Delta=ct^4+ct^6$, c a non-zero constant. Writing the Weierstrass equation in $\lambda\mu$-form gives us $\lambda=0$, $\mu^2=c$, $a_6=c$. As before, we may assume $a_2=dt$. Our equation becomes

$$y^2+txy=x^3+dtx^2+\mu x+c.$$

Make a substitution $x=x+e$, $e^2=\mu$. We get

$$y^2+txy+ety=x^3+(e+dt)x^2+\mu x+c+dte.$$

Now substitute $y=y+f$, $f^2=c$. Get

$$y^2+txy+ety=x^3+(e+dt)x^2+ftx+(ef+ed)t.$$

In order to have a fibre as in Case 3, we must have $ef+ed=0$, which forces $f=d$, since $e\neq 0$. Our equation now becomes

$$y^2+txy+ety=x^3+(e+ft)x^2+ftx.$$

Continuing with Tate's algorithm, we find that $e^3+f^2=0$. But $e^2=\mu$, $\mu^2=c$, $f^2=c$. This forces $f=\mu$, $e^2=f$, so $e^3+e^4=0$. Since $e\neq 0$, $e=f=1$. Our equation is now

$$y^2+txy+ty=x^3+(1+t)x^2+tx.$$

Substituting $y=y+x$, we get

$$y^2+txy+ty=x^3.$$

This surface is unique.

Case IX. We locate the fibre of type 6 at 0, the fibre of type I_1 at 1, and the fibre of type I_3 at ∞. This gives $\Delta=ct^8+ct^9$, c a non-zero constant. Writing the Weierstrass equation in $\lambda\mu$-form we get,

$$y^2+txy=x^3+dtx^2+c(t^2+t^3).$$

Make a substitution of the form $y=y+et$, $e^2=c$ to get

$$y^2+txy=x^3+dtx^2+et^2x+ct^3.$$

In order to have a fibre of the desired type at the origin, the polynomial z^3+dz^2+ez+c must have a triple root. This forces $e=d^2$, $c=d^3$. Since $e^2=c$, we find $d^4=d^3$. Since $c\neq 0$, we see $d\neq 0$, so $d=c=e=1$. The equation is now

$$y^2+txy=x^3+tx^2+t^2x+t^3.$$

Substituting $x=x+t$ gives

$$y^2+txy+t^2y=x^3.$$

This surface is unique.

To summarize, we have found nine types of extremal rational elliptic surfaces with three or fewer singular fibres. There is a unique surface of each type except for Type I, where there is a 1-parameter family of surfaces.

We list the Weierstrass equations and Kodaira fibre types in each case, for the convenience of the reader.

I	$y^2+txy=x^3+tx^2+kt^6, \quad k\neq 0$	I_4^*,
II	$y^2+t^3y=x^3+t^5$	II^*,
III	$y^2+txy+ty=x^3+x^2$	III, I_8,
IV	$y^2+txy=x^3+t^2x$	I_1^*, I_4,
V	$y^2+txy=x^3+t^4$	III^*, I_2,
VI	$y^2+txy=x^3+t^5$	II^*, I_1,
VII	$y^2+t^2y=x^3$	IV, IV^*,
VIII	$y^2+txy+ty=x^3$	IV, I_2, I_6,
IX	$y^2+txy+t^2y=x^3$	IV^*, I_1, I_3.

3. Characteristic three

3A. Classification of singular fibres of additive type on rational elliptic surfaces in characteristic three

Throughout this section, we assume $f: X \to P^1$ is a rational elliptic surface with section over an algebraically closed field of characteristic three.

We repeat the program of the previous section. We start by listing the possible types of singular fibres of additive type that can appear on a rational elliptic surface in characteristic three.

We begin by writing the Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_4x + a_6,$$

where the a_i are polynomials in t of degree $\leq i$. We locate our singular fibre of additive type at $t=0$. We may change coordinates so that $t | a_3, a_4$ and a_6. The equation now becomes

$$y^2 = x^3 + tc_1x^2 + tc_3x + tc_5.$$

We find it more convenient to work with $-\Delta$ instead of Δ in characteristic three.

Case 1A $\begin{cases} y^2 = x^3 + tc_1x^2 + tc_3x + tc_5, \ t \nmid c_5, t \nmid c_3, \\ -\Delta = t^2c_1^2(t^2c_1c_5 - t^2c_3^2) + t^3c_3^3, \\ \delta = 3, \ r = 0 \quad \text{Type II.} \end{cases}$

Case 1B $\begin{cases} y^2 = x^3 + tc_1x^2 + t^2c_2x + tc_5, \ t \nmid c_5, t \nmid c_1, \\ -\Delta = t^2c_1^2(t^2c_1c_5 - t^4c_2^2) + t^6c_2^3, \\ \delta = 4, \ r = 0 \quad \text{Type II.} \end{cases}$

Case 1C $\begin{cases} y^2 = x^3 + t^2c_0x^2 + t^2c_2x + tc_5, \ t \nmid c_5, t \nmid c_2, \\ -\Delta = t^4c_0^2(t^3c_0c_5 - t^4c_2^2) + t^6c_2^3, \\ \delta = 6, \ r = 0 \quad \text{Type II.} \end{cases}$

Case 1D $\begin{cases} y^2 = x^3 + t^2c_0x^2 + t^3c_1x + tc_5, \ t \nmid c_5, t \nmid c_0, \\ -\Delta = t^4c_0^2(t^3c_0c_5 - t^6c_1^2) + t^9c_1^3, \\ \delta = 7, \ r = 0 \quad \text{Type II.} \end{cases}$

Case 1E $\begin{cases} y^2 = x^3 + t^3c_1x + tc_5, \ t \nmid c_5, t \nmid c_1, \\ -\Delta = t^9c_1^3, \\ \delta = 9, \ r = 0 \quad \text{Type II.} \end{cases}$

Case 1F
$$\begin{cases} y^2=x^3+t^4c_0x+tc_5, \ t\nmid c_5, t\nmid c_0, \\ -\Delta=t^{12}c_0^3, \\ \delta=12, \ r=0 \quad \text{Type II.} \end{cases}$$

Case 2
$$\begin{cases} y^2=x^3+tc_1x^2+tc_3x+t^2c_4, \ t\nmid c_3, \\ -\Delta=t^2c_1^2(t^3c_1c_4-t^2c_3^2)+t^3c_3^3, \\ \delta=3, \ r=1 \quad \text{Type III.} \end{cases}$$

Case 3A
$$\begin{cases} y^2=x^3+tc_1x^2+t^2c_2x+t^2c_4, \ t\nmid c_4, t\nmid c_1, \\ -\Delta=t^2c_1^2(t^3c_1c_4-t^4c_2^2)+t^6c_2^3, \\ \delta=5, \ r=2 \quad \text{Type IV.} \end{cases}$$

Case 3B
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^2c_2x+t^2c_4, \ t\nmid c_4, t\nmid c_2 \\ -\Delta=t^4c_0^2(t^4c_0c_4-t^4c_2^2)+t^6c_2^3, \\ \delta=6, \ r=2 \quad \text{Type IV.} \end{cases}$$

Case 3C
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^3c_1x+t^2c_4, \ t\nmid c_4, t\nmid c_0, \\ -\Delta=t^4c_0^2(t^4c_0c_4-t^6c_1^2)+t^9c_1^3, \\ \delta=8, \ r=2 \quad \text{Type IV.} \end{cases}$$

Case 3D
$$\begin{cases} y^2=x^3+t^3c_1x+t^2c_4, \ t\nmid c_4, t\nmid c_1, \\ -\Delta=t^9c_1^3, \\ \delta=9, \ r=2 \quad \text{Type IV.} \end{cases}$$

Case 3E
$$\begin{cases} y^2=x^3+t^4c_0x+t^2c_4, \ t\nmid c_4, t\nmid c_0, \\ -\Delta=t^{12}c_0^3, \\ \delta=12, \ r=2 \quad \text{Type IV.} \end{cases}$$

Case 4A
$$\begin{cases} y^2=x^3+tc_1x^2+t^2c_2x+t^3c_3. \\ \text{Assume } t\nmid c_1. \text{ Then we can make a substitution} \\ x=x+kt+lt^2 \text{ and put the equation into the form} \\ y^2=x^3+tc_1x^2+t^4c_0x+t^3c_3, \ t\nmid c_1, t\nmid c_3. \\ -\Delta=t^2c_1^2(t^4c_1c_3-t^8c_0^2)+t^{12}c_0^3, \\ \delta=6, \ r=4 \quad \text{Type I}_0^*. \end{cases}$$

Case 4B
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^2c_2x+t^3c_3, \ t\nmid c_2, \\ -\Delta=t^4c_0^2(t^5c_0c_3-t^4c_2^2)+t^6c_2^3, \\ \delta=6, \ r=4 \quad \text{Type I}_0^*. \end{cases}$$

Case 5A
$$\begin{cases} y^2=x^3+tc_1x^2+t^4c_0x+t^4c_2,\ t\nmid c_1, t\nmid c_2, \\ -\Delta=t^2c_1^2(t^5c_1c_2-t^8c_0^2)+t^{12}c_0^3, \\ \delta=7,\ r=5 \quad \text{Type } \mathrm{I}_1^*. \end{cases}$$

Case 5B
$$\begin{cases} y^2=x^3+tc_1x^2+t^4c_0x+t^5d_1,\ t\nmid c_1, t\nmid d_1, \\ -\Delta=t^2c_1^2(t^6c_1d_1-t^8c_0^2)+t^{12}c_0^3, \\ \delta=8,\ r=6 \quad \text{Type } \mathrm{I}_2^*. \end{cases}$$

Case 5C
$$\begin{cases} y^2=x^3+tc_1x^2+t^4c_0x+t^6d_0,\ t\nmid c_1, t\nmid d_0, \\ -\Delta=t^2c_1^2(t^7c_1d_0-t^8c_0^2)+t^{12}c_0^3, \\ \delta=9,\ r=7 \quad \text{Type } \mathrm{I}_3^*. \end{cases}$$

Case 5D
$$\begin{cases} y^2=x^3+tc_1x^2+t^4c_0x,\ t\nmid c_1, t\nmid c_0, \\ -\Delta=-t^{10}c_1^2c_0^2+t^{12}c_0^3, \\ \delta=10,\ r=8 \quad \text{Type } \mathrm{I}_4^*. \end{cases}$$

Case 6A
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^3c_1x+t^3c_3. \\ \text{Make a substitution of the form } x=x+kt \\ \text{and put the equation into the form} \\ y^2=x^3+t^2c_0x^2+t^3c_1x+t^4c_2,\ t\nmid c_2, t\nmid c_1. \\ -\Delta=t_4c_0^2(t^6c_0c_2-t^6c_1^2)+t^9c_1^3, \\ \delta=9,\ r=6 \quad \text{Type IV}^*. \end{cases}$$

Case 6B
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^4d_0x+t^4c_2,\ t\nmid c_2, t\nmid c_0. \\ -\Delta=t^4c_0^2(t^6c_0c_2-t^8d_0^2)+t^{12}d_0^3, \\ \delta=10,\ r=6 \quad \text{Type IV}^*. \end{cases}$$

Case 6C
$$\begin{cases} y^2=x^3+t^4d_0x+t^4c_2,\ t\nmid c_2, t\nmid d_0. \\ -\Delta=t^{12}d_0^3, \\ \delta=12,\ r=6 \quad \text{Type IV}^*. \end{cases}$$

Case 7
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^3c_1x+t^5d_1,\ t\nmid c_1. \\ -\Delta=t^4c_0^2(t^7c_0d_1-t^6c_1^2)+t^9c_1^3, \\ \delta=9,\ r=7 \quad \text{Type III}^*. \end{cases}$$

Case 8A
$$\begin{cases} y^2=x^3+t^2c_0x^2+t^4d_0x+t^5d_1,\ t\nmid d_1, t\nmid c_0. \\ -\Delta=t^4c_0^2(t^7c_0d_1-t^8d_0^2)+t^{12}d_0^3, \\ \delta=11,\ r=8 \quad \text{Type II}^*. \end{cases}$$

Case 8B $\begin{cases} y^2=x^3+t^4d_0x+t^5d_1, \ t\nmid d_1, t\nmid d_0. \\ -\Delta=t^{12}d_0^3, \\ \delta=12, \ r=8 \quad \text{Type II}^*. \end{cases}$

We list the possibilities obtained in a table.

	δ	r	$\delta-r$
Case 1A	3	0	3
Case 1B	4	0	4
Case 1C	6	0	6
Case 1D	7	0	7
Case 1E	9	0	9
Case 1F	12	0	12
Case 2	3	1	2
Case 3A	5	2	3
Case 3B	6	2	4
Case 3C	8	2	6
Case 3D	9	2	7
Case 3E	12	2	10
Case 4A	6	4	2
Case 4B	6	4	2
Case 5A	7	5	2
Case 5B	8	6	2
Case 5C	9	7	2
Case 5D	10	8	2
Case 6A	9	6	3
Case 6B	10	6	4
Case 6C	12	6	6
Case 7	9	7	2
Case 8A	11	8	3
Case 8B	12	8	4

We now list the possibilities for extremal rational elliptic surfaces permitted by this table and the results of Section 1.

Possibilities with one singular fibre

I Case 8B.

Possibilities with two singular fibres

II Case 1A, I_9,
III Case 6A, I_3,
IV Case 8A, I_1,

V	Case 2, Case 7,
VI	Case 4A, Case 4A,
VII	Case 4A, Case 4B,
VIII	Case 4B, Case 4B.

Possibilities with three singular fibres

IX	Case 2, I_1, I_8,
X	Case 2, I_3, I_6,
XI	Case 4A, I_3, I_3,
XII	Case 4B, I_3, I_3,
XIII	Case 5A, I_1, I_4,
XIV	Case 5B, I_2, I_2,
XV	Case 5D, I_1, I_1,
XVI	Case 7, I_1, I_2.

3B. Existence and uniqueness of surfaces with prescribed fibre types

We now examine each of the above possibilities separately to determine whether a surface of the given type exists. In each case where it does exist (except for Case VI), we will show that it is unique.

Case I. We want the fibre at $t=0$ to be as in Case 8B. This gives us a Weierstrass equation

$$y^2=x^3+t^4d_0x+t^5d_1,\quad d_0\neq 0.$$

By scaling, we may assume $d_0=1$. Making a substitution of the form $x=x+kt^2$, we may reduce the equation to

$$y^2=x^3+t^4x+t^5e,$$

where e is a non-zero constant. Finally, by scaling x, y, and t, we may force $e=1$. Thus, the surface exists and is unique. The Weierstrass equation is

$$y^2=x^3+t^4x+t^5.$$

Case II. We assume the fibre of type 1A is at 0, and that the fibre of type I_9 is at ∞. The Weierstrass equation has the form

$$y^2=x^3+tc_1x^2+tc_3x+tc_5,\quad t\nmid c_5,\ t\nmid c_3.$$

We know $-\Delta=t^2c_1^2(t^2c_1c_5-t^2c_3^2)+t^3c_3^3$, and we want $-\Delta=ct^3$, c a constant. Now if $c_1=0$, then $-\Delta=t^3c_3^3$, so we must have $c_3=d$, d a non-zero constant, and we may assume $d=1$. The equation now becomes

$$y^2=x^3+tx+tc_5.$$

Making a substitution of the type $x=x+kt^2$, we may assume $\deg c_5 \leq 4$. But then we find that the fibre at ∞ is of additive type, which is not what we want. So we cannot have $c_1=0$. We now break the proof into subcases.

Subcase A. $t|c_1$. We now have

$$y^2=x^3+t^2c_0x^2+tc_3x+tc_5, \quad c_0\neq 0,\ t\nmid c_5,\ t\nmid c_3.$$

We may assume $c_0=1$. Making a substitution of the form $x=x+kt+lt^2$, we may assume $c_3=a+bt$. Computing, we find

$$-\Delta=t^7c_5-t^6(a^2+2abt+b^2t^2)+a^3t^3+b^6t^6.$$

Scaling, we may assume $a=1$. Then our requirement that $-\Delta=ct^3$ forces $b=1$ and $-\Delta=t^3-2t^7-t^8+t^7c_5$. This forces $c_5=t+2$. Thus, in this subcase, we have the unique possibility

$$y^2=x^3+t^2x^2+t(t+1)x+t(t+2).$$

Subcase B. $t\nmid c_1$. We may locate the zero of c_1 at $t=-1$, and then scale so that we have

$$y^2=x^3+t(t+1)x^2+tc_3x+tc_5.$$

By substituting $x=x+kt+lt^2$, we may assume $c_3=a+bt^3$. Computing, we find

$$-\Delta=t^7+t^4c_5+t^3+b^3t^{12}-t^4(a^2+2a^2t+a^2t^2+2abt^3 +abt^4+2abt^5+b^2t^2+2b^2t^7+b^2t^8).$$

This forces $c_5=a^2+2a^2t+a^2t^2+ct^3+dt^4+et^5$ with

$$c=2a^2+2ab,$$
$$d=a^2+ab,$$
$$e=2a^2+2ab.$$

Setting the t^{10},t^{11}, and t^{12} terms to 0, we get

$$2a^2+2ab=0,$$
$$a^2+ab=0,$$
$$2a^2+2ab+b^3=0.$$

This forces $a=b=0$. Since $t\nmid c_3$, this is impossible. Hence no surface exists under this subcase. Hence the surface in Case II exists and is unique, with Weierstrass equation

$$y^2=x^3+t^2x^2+t(t+1)x+t(t+2).$$

Case III. We assume the fibre of type 6A is at 0, and the fibre of type I_3 is at ∞. The Weierstrass equation is

$$y^2=x^3+t^2c_0x^2+t^3c_1x+t^4c_2, \quad t\nmid c_2,\ t\nmid c_1.$$

As in the previous case, $c_0\neq 0$, so we may assume $c_0=1$. By making a substitution of the form $x=x+kt^2$, we may assume $c_1=e$, a non-zero constant. Our equation is now

$$y^2=x^3+t^2x^2+t^3ex+t^4c_2.$$

Computing, we find $-\Delta=t^{10}c_2-t^{10}e^2+t^9e^3$. Since we want $-\Delta=ct^9$, we must have $c_2=e^2$. By scaling, we may assume $e=1$. Therefore the surface

$$y^2=x^3+t^2x^2+t^3x+t^4$$

is the unique surface in Case III.

Case IV. We locate the fibre of type 8A at 0, the fibre of type I_1 at ∞. Our equation becomes

$$y^2=x^3+t^2c_0x^2+t^4d_0x+t^5d_1, \quad t\nmid d_1,\ t\nmid c_0.$$

By scaling, we may assume $c_0=1$. Substituting $x=x+kt^2$ allows us to assume $d_0=0$. Then $-\Delta=t^{11}d_1$. We must have $d_1=e$, e a non-zero constant. Scaling, we may assume $e=1$. So the unique surface in this case is

$$y^2=x^3+t^2x^2+t^5.$$

Case V. We locate the Case 7 fibre at 0 and the Case 2 fibre at ∞. In order to have this configuration, we must have $a_2=0$. Our equation is

$$y^2=x^3+t^3c_1x+t^5d_1, \quad t\nmid c_1.$$

In order to have a Case 2 fibre at ∞, we must have $c_1=e$, a non-zero constant. We may assume $e=1$. We may make a substitution of the form $x=x+kt^2$ to get $d_1=f$, a constant. Let $u=t^{-1}$. Then at ∞, our equation becomes

$$y^2=x^3+ux+fu.$$

In order to have a Case 2 fibre at ∞, we must have $f=0$. Thus, the surface exists and is unique. The Weierstrass equation is

$$y^2=x^3+t^3x.$$

Case VI. We want fibres of type 4A at both 0 and ∞. Our equation is

$$y^2=x^3+tc_1x^2+t^4c_0x+t^3c_3.$$

In order to have additive reduction at both 0 and ∞, we must have $c_1=$ constant. Since we want fibres of type 4A, we must have $c_1\neq 0$, so we may assume $c_1=1$. Calculating, we find

$$-\Delta=t^2(t^4c_3-t^8c_0^2)+t^{12}c_0^3.$$

Since we want $-\Delta=kt^6$, k a non-zero constant, this forces $c_0=0$, c_3 a non-zero constant. We get a 1-parameter family of surfaces in this case. The equation is

$$y^2=x^3+tx^2+kt^3, \quad k \text{ a non-zero constant.}$$

Case VII. This case does not exist, since we see by examining the forms of the equations that we cannot have fibres of types 4A and 4B together.

Case VIII. We want fibres of type 4B at both 0 and ∞. This forces the equation into the form

$$y^2=x^3+t^2c_2x+t^3c_3, \quad t\nmid c_2.$$

We have $-\Delta=t^6c_2^3$, so we must have $c_2=e$, a non-zero constant. We may assume $e=1$. Looking at ∞, we find $c_3=f$, a constant. The equation is now

$$y^2=x^3+t^2x+ft^3.$$

By making a substitution of the form $x=x+kt$, we may force $f=0$. This surface exists and is unique. It has equation

$$y^2=x^3+t^2x.$$

Case IX. We claim that this surface does not exist. For if it did, the Mordell–Weil group of the generic fibre would have order 4. It cannot be $\mathbf{Z}/2\times\mathbf{Z}/2$, for if it were, all the roots of the cubic on the right-hand side of the Weierstrass equation would be in $k[t]$, and therefore Δ would be a square in $k[t]$, contradicting the requirement that Δ have a simple zero. So the Mordell–Weil group must be $\mathbf{Z}/4$. So the j-map for our surface must factor through the j-map $X_1(4)\to P^1$, where $X_1(4)$ is the modular curve. But looking at ∞, we see the j-map for our surface has degree 9, while the j-map for $X_1(4)$ has degree 6. So this surface does not exist, as claimed.

Case X. We locate the fibre as in Case 2 at $t=0$. The Weierstrass equation is of the form

$$y^2=x^3+tc_1x^2+tc_3x+t^2c_4, \quad t\nmid c_3.$$

Subcase A. $t|c_1$. In this subcase, we may assume our equation is of the form

$$y^2=x^3+t^2x^2+tc_3x+t^2c_4.$$

Since the Mordell–Weil group is of order 6, we have a 2-torsion point. This means the cubic in x on the right-hand side of the Weierstrass equation has a root in $k[t]$. It is easy to see that this root has degree ≤ 2 in t and no constant term. Hence we may make a substitution $x=x+kt+lt^2$ and put the equation in the form

$$y^2=x^3+t^2x^2+tc_3x.$$

Now $-\Delta=t^6c_3^2+t^3c_3^3$. If we locate the I_6 fibre at ∞, then $-\Delta$ must have the form $-\Delta=t^3(A+Bt)^3$. Write $c_3=a+bt+ct^2+dt^3$, $a\neq 0$, and compute. We find $b=c=d=0$, and we can scale so that $a=1$. Thus, we have a unique surface in this subcase, with equation

$$y^2=x^3+t^2x^2+tx.$$

Subcase B. $t\nmid c_1$. We locate the I_6 fibre at ∞, and the zero of c_1 at $t=-1$. Scaling, we may assume the Weierstrass equation has the form

$$y^2=x^3+t(t+1)x^2+tc_3x+t^2c_4, \quad t\nmid c_3.$$

Making a substitution $x=x+kt+lt^2$, we may assume $c_3=a+bt^3$. Then

$$-\Delta=t^5(t+1)^3c_4-t^4(t+1)^2(a+bt^3)^2+t^3(a+bt^3)^3.$$

We want this to be of the form $-\Delta=At^3+Bt^6$. Considering the t^4 term, this forces $a=0$, which is impossible. So there is no surface in this subcase.

Hence the unique surface in Case X has the equation

$$y^2=x^3+t^2x^2+tx.$$

Case XI. In this case, we want a fibre of type 4A at 0, a fibre of type I_3 at ∞, and one other singular fibre of type I_3. We write the Weierstrass equation in the form

$$y^2=x^3+tc_1x^2+t^2c_2x+t^3c_3, \quad t\nmid c_1.$$

Since the Mordell–Weil group is of order 6, we can eliminate c_3 as in the preceding case to get

$$y^2=x^3+tc_1x^2+t^2c_2x.$$

Locate the zero of c_1 at $t=-1$ and scale to get

$$y^2=x^3+t(t+1)x^2+t^2c_2x.$$

Then $-\Delta=t^6(c_2^3-(t+1)^2c_2^2)$. We want the term in parentheses to be a perfect cube, which forces $c_2=d(t+1)^2$. Then $-\Delta=(d^3-d^2)t^6(t+1)^6$, which is inconsistent with having a I_3 fibre at ∞. Hence, this surface does not exist.

Case XII. We want a fibre of type 4B at 0, a fibre of type I_3 at ∞, and one other singular fibre of type I_3. The equation has the form

$$y^2=x^3+t^2c_0x^2+t^2c_2x+t^3c_3, \quad t\nmid c_2.$$

Since we want fibres of multiplicative type, we cannot have $c_0=0$, so we may assume $c_0=1$. Since we have a 2-torsion point, we may put the equation in the form

$$y^2=x^3+t^2x^2+t^2c_2x.$$

Then $-\Delta=-t^8c_2^2+t^6c_2^3$. Since $t\nmid c_2$, we get a non-zero t^8 term. Since $-\Delta$ must be a perfect cube, we see this surface does not exist.

Case XIII. We have a fibre of type 5A at $t=0$, and fibres of types I_1 and I_4. The order of the Mordell–Weil group is 4. Since Δ has a simple zero, it cannot be a square, and hence the 2-torsion points cannot all be rational over $k(t)$. Hence the Mordell–Weil group is $\mathbf{Z}/4$. Since the degree of the j-map is 6 for both our surface and the elliptic modular surface $E_1(4)$, this surface must be $E_1(4)$ and so is unique. One computes easily that the Weierstrass equation is

$$y^2=x^3+t(t+1)x^2+t^2x.$$

Case XIV. We want a fibre of type 5B at $t=0$, a fibre of type I_2 at ∞, and another singular fibre of type I_2. We may assume our equation is

$$y^2=x^3+t(t+1)x^2+t^4c_0x+t^5d_1, \quad t\nmid d_1.$$

Write $d_1=A+Bt$, and compute $-\Delta$. We find $B=c_0^2-c_0^3$, $A=2c_0^2$, and $-\Delta=2c_0^2t^8\times(t^2+(c_0+2)t+1)$. We want $(t^2+(c_0+2)t+1)$ to be a perfect square, which forces $c_0=0$ or $c_0=-1$. If $c_0=0$, then $\Delta=0$, so $c_0=-1$. This surface is unique, with equation

$$y^2=x^3+t(t+1)x^2-t^4x+t^5(2t+2).$$

Case XV. We want a fibre of type 5D at 0, a fibre of type I_1 at ∞, and another fibre of type I_1. The equation is

$$y^2 = x^3 + tc_1x^2 + t^4c_0x, \quad t \nmid c_1, \ t \nmid c_0.$$

As before, we may assume $c_1 = t+1$. Computing, we find $-\Delta = t^{10}(t+1)^2c_0^2 - t^{12}c_0^3$. We want the t^{12} term to be zero, which forces $c_0^2 - c_0^3 = 0$, $c_0 = 1$. Hence we have the unique surface

$$y^2 = x^3 + t(t+1)x^2 + t^4x.$$

Case XVI. This time, we have a fibre as in Case 7 at $t=0$, a fibre of type I_2 at $t=\infty$, and a fibre of type I_1. The equation is

$$y^2 = x^3 + t^2c_0x^2 + t^3c_1x + t^5d_1, \quad t \nmid c_1.$$

Since we have fibres of multiplicative type, $c_0 \neq 0$, and we may assume $c_0 = 1$. Make a substitution of the form $x = x + kt^2$ to get $c_1 = e$, a non-zero constant. Scaling, we may assume $e = 1$. Then $-\Delta = t^4(t^7d_1 - t^6) + t^9$, so in order to have a fibre of type I_2 at ∞, we must have $d_1 = 0$. Hence this surface is unique, with equation

$$y^2 = x^3 + t^2x^2 + t^3x.$$

To summarize, we list the extremal rational elliptic surfaces found, together with the Weierstrass equations, and the types of singular fibres. The numbering does not correspond to the numbering above.

I	$y^2=x^3+t^4x+t^5$	II^*,
II	$y^2=x^3+t^2x^2+t(t+1)x+t(t+2)$	II, I_9,
III	$y^2=x^3+t^2x^2+t^3x+t^4$	IV^*, I_3,
IV	$y^2=x^3+t^2x^2+t^5$	II^*, I_1,
V	$y^2=x^3+t^3x$	III^*, III,
VI	$y^2=x^3+tx^2+kt^3, \ k\neq 0$	I_0^*, I_0^*,
VI bis	$y^2=x^3+t^2x$	I_0^*, I_0^*,
VII	$y^2=x^3+t^2x^2+tx$	III, I_3, I_6,
VIII	$y^2=x^3+t(t+1)x^2+t^2x$	I_1^*, I_1, I_4,
IX	$y^2=x^3+t(t+1)x^2-t^4x+t^5(2t+2)$	I_2^*, I_2, I_2,
X	$y^2=x^3+t(t+1)x^2+t^4x$	I_4^*, I_1, I_1,
XI	$y^2=x^3+t^2x^2+t^3x$	III^*, I_1, I_2.

4. All characteristics $\neq 2, 3$

Theorem 4.1. *The classification of extremal rational elliptic surfaces with three or fewer singular fibres over an algebraically closed field of characteristic* $p > 5$

is identical to the classification of such surfaces in characteristic zero. The classification of such surfaces in characteristic five is identical to the above except that there exists a unique surface with three singular fibres of type II, I_5, I_5.

Proof. We merely sketch the proof of this theorem, which is essentially a straightforward application of the techniques of Miranda–Persson. First, we note that Table 1.1 of [4] is valid in characteristic p, $p \neq 2, 3$. Next, we consider all the types of surfaces permitted by the results of Section 1. For each type, we work out the degree of the j-map and the ramification over 0 and ∞. Then we apply the Hurwitz genus formula, and find that the possibilities excluded in [4] are also excluded in our case. The reader will check that in each case, no problems are caused by wild ramification or inseparability of the j-map except in one case in characteristic 5.

This exceptional case is the one where we have singular fibres of type II, I_5, I_5. In this case, the j-map has degree 10, and using the Hurwitz formula, we see that this can only exist in characteristic five, with the j-map inseparable.

Let us assume that we have such a surface with the fibre of type II at $t=0$. The Weierstrass equation has the form $y^2=x^3+a_4x+a_6$, with the a_i polynomials in t of degree ≤ 1. Since the j-map is inseparable, all multiplicities of zeroes of j must be divisible by 5. Hence by Table 1.1 of [4], we see that $t^4 | a_4$, and we may scale so that $a_4=t^4$. Also, from the same table, $t | a_6$. Computing, we see $-\Delta/16=4t^{12}+27a_6^2$. We want Δ to have a zero of order 5 at ∞ and one zero of order 5 at finite distance, and we want the j-map to be inseparable. This forces $a_6=at+bt^6$. Plugging this into Δ, and setting the t^{12} term to 0, we get $b=\sqrt{3}$. Making j inseparable forces $a=2\sqrt{3}$. So our surface is unique, with equation

$$y^2=x^3+t^4x+2\sqrt{3}t+\sqrt{3}t^6.$$

To eliminate the square roots, make a substitution $x=\sqrt{3}x$, $y=3^{3/4}y$. Our equation becomes

$$y^2=x^3+2t^4x+4t+2t^6.$$

The existence and uniqueness of the remaining cases can be deduced exactly as in [4], and the Weierstrass equations are the same as those listed there. This completes our sketch of the proof of Theorem 4.1.

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