

# Some results connected with Brennan's conjecture

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## 1. Introduction and results

### 1.1. An estimate of harmonic measure

In this paper we shall study some problems concerning large values of harmonic measure on the boundary of a simply connected domain in the complex plane. It is well-known that, under a proper normalization, harmonic measure of any disc of radius  $\rho$  does not exceed  $\sqrt{\rho}$ , and so the value of harmonic measure is considered large if it is close to this bound. The following theorem provides an estimate of the number of discs with large harmonic measure.

**Theorem 1.** *There exist absolute constants  $K$  and  $A$  such that for every simply connected domain  $\Omega$  satisfying*

$$\infty \in \Omega, \quad \text{diam } \partial\Omega = 1$$

*and any numbers  $\varepsilon > 0$  and  $\rho > 0$ , the maximal number of disjoint discs of radius  $\rho$  and harmonic measure (evaluated at  $\infty$ ) greater than  $\rho^{1/2+\varepsilon}$  is at most  $A\rho^{-K\varepsilon}$ .*

This result has some consequences for univalent functions and conformal mappings. We will indicate a couple of applications.

### 1.2. Integral means

Let  $f$  be a univalent function in the unit disc  $\mathbf{D}$  and  $t$  be a real number.

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A classical problem is to find a bound for the integral means of the derivative:

$$I_t(r, f') = \int_{\partial\mathbf{D}} |f'(r\zeta)|^t |d\zeta|, \quad 0 < r < 1.$$

**Corollary 1.** *There exists a number  $t_0 < 0$  such that for every univalent function  $f$  and  $t \leq t_0$ ,*

$$I_t(r, f') = O\left(\left(\frac{1}{1-r}\right)^{|t|-1}\right) \quad \text{as } r \rightarrow 1.$$

(One can actually take  $t_0 = -\frac{1}{2}K$ , where  $K$  is the constant of Theorem 1).

Let us denote

$$B(t) = \inf\{\beta : I_t(r, f') = O((1-r)^{-\beta}) \text{ for every univalent function } f\}.$$

In other words,  $B(t)$  is the exact universal bound for the rate of growth of integral means of order  $t$ . The example of the Koebe function  $z(1-z)^{-2}$  shows that  $B(t) \geq |t|-1$ , and so our result implies

$$B(t) \equiv |t|-1 \quad \text{for } t \leq t_0.$$

Previously it has been known (cf. [Br]) that the trivial bound  $B(t) \leq |t|$  is never sharp (for negative  $t$ 's). On the other hand, we shall show in Section 5 that for every  $t \in (-2, 0)$  there is a domain with a fractal boundary such that

$$\liminf_{r \rightarrow 1} \frac{\log I_t(r, f')}{|\log(1-r)|} > |t|-1.$$

Thus the situation can be described as follows. Let  $t_0$  be the best constant for the statement of Corollary 1. Then for  $t_0 < t < 0$ , the extremal growth of  $I_t$  corresponds to some sort of “stochastic” distribution of singularities on the boundary, while for  $t < t_0$ , the extremal growth corresponds to the case of “isolated” singularities. In a sense,  $t_0$  is a “phase transition” point for the universal spectrum  $B(t)$ : the function  $B(t)$  is linear on  $\{t \leq t_0\}$  and strictly convex on  $[t_0, 0]$ .

### 1.3. Boundary distortion

Let  $f: \mathbf{D} \rightarrow \Omega$  be a conformal map onto a Jordan domain. Another well-known problem is to compare the Hausdorff dimensions of the sets on  $\partial\mathbf{D}$  and their images on  $\partial\Omega$ .

**Corollary 2.** *Let  $E \subset \partial\mathbf{D}$  and  $\dim E = d$ . Then*

$$\dim fE \geq \frac{d}{2-2K^{-1}d} \underset{(d \rightarrow 0)}{=} \frac{d}{2} + \frac{d^2}{2K} + \dots .$$

The estimate  $\dim fE \geq d/2$  has been long known, and in [M1] it was shown that one can always improve upon this estimate.

In the class of close-to-convex (in particular, starlike) functions, one has the following sharp bound:

$$\dim fE \geq \frac{d}{2-d} = \frac{d}{2} + \frac{d^2}{4} + \dots ,$$

but for general domains there are examples with

$$\dim fE \leq \frac{d}{2} + \frac{d^2}{8} + \dots .$$

The proof of the results in [Br], [M1] mentioned above is based on an argument from the paper [C] in which the question about the dimension of harmonic measure (the case  $d=1$  in Corollary 2) was studied. A more careful analysis leads in fact to the following estimates:

$$(\text{number of discs}) \leq \varrho^{-C\varepsilon \log 1/\varepsilon}, \quad (\text{in Theorem 1}),$$

$$B(t) \leq |t| - 1 + e^{-C|t|}, \quad (\text{in Corollary 1}),$$

which are quite close to the present results but fall short of revealing the phase transition phenomenon.

#### 1.4. Brennan's conjecture

Our proof of Theorem 1 also relies on the method of [C]. We use more intricate combinatorics but the way we estimate harmonic measure remains the same—it is based on a bound for the extremal length in terms of the Ahlfors–Beurling integral. Since the latter involves only the angles and does not reflect the whole geometry of a domain, it is almost certain that this method cannot be further improved to produce the best constants.

Determining the best values of the constants  $K$  and  $t_0$  is an interesting and perhaps difficult problem. As was mentioned, we must have

$$K \geq 4, \quad t_0 \leq -2.$$

The question of whether we can actually take

$$K = 4, \quad t_0 = -2$$

is more or less equivalent to a well-known conjecture of J. Brennan.

In the second part of the paper we are trying to better understand the shape of the boundary with the extremal behavior of harmonic measure at the points in a given finite set.

### 1.5. Reduced extremal length

For a simply connected domain  $\Omega$  and any pair of points  $a, b \in \widehat{\mathbb{C}}$  we consider the following conformal invariant.

For  $\varepsilon > 0$  let  $\Gamma_\varepsilon$  denote the family of all curves joining the  $\varepsilon$ -neighborhoods of  $a$  and  $b$  in  $\Omega$ , and  $\widetilde{\Gamma}_\varepsilon$  the corresponding family in  $\widehat{\mathbb{C}}$ . (The  $\varepsilon$ -neighborhood of  $\infty$  is  $\{|z| > \varepsilon^{-1}\}$ .) Define

$$\beta = \beta(\Omega; a, b) = \lim_{\varepsilon \rightarrow 0} \exp\{2\pi[\lambda(\widetilde{\Gamma}_\varepsilon) - \lambda(\Gamma_\varepsilon)]\},$$

where  $\lambda$  denotes extremal length; the existence of the limit is a standard property of  $\lambda$ .

Let us now fix one of the points, say  $b$ , and consider  $m$  distinct points  $a_1, \dots, a_m$  on  $\partial\Omega$ . Denote

$$\beta_j = \beta(\Omega; a_j, b).$$

We will show in Section 4 that the best constant in Theorem 1 is exactly twice the minimal value of  $p$  such that

$$\sum_{j=1}^m \beta_j^p \leq 1$$

for any  $m$  and every configuration  $(\Omega; \{a_j\}, b)$ . In particular, Brennan's conjecture is equivalent to the statement

$$(*) \quad \sum_{j=1}^m \beta_j^2 \leq 1.$$

The advantage of this new approach is that  $(*)$  looks more like a standard problem in the classical geometric function theory. So far we have been able to prove  $(*)$  only for  $m=2$ .

**Theorem 2.** *Let  $\Omega$  be a simply connected domain and let  $a_1, a_2, b \in \partial\Omega$ ,  $\beta_j = \beta(\Omega; a_j, b)$ , ( $j=1, 2$ ).*

*Then*

$$\beta_1^2 + \beta_2^2 \leq 1.$$

## 2. Two lemmas

In this section we establish some facts which will be used in the proof of Theorem 1. We refer to [B] and [O] for the definition and properties of extremal length.

### 2.1 Notation

First we fix a sufficiently large number  $A > 0$ . For  $z \in \mathbb{C}$  and  $\nu \in \mathbb{N}$  denote

$$R_\nu(z) = \{\zeta : \frac{1}{2}e^{-\nu A} < |\zeta - z| < 2e^{(1-\nu)A}\}.$$

Then

$$\lambda = (2\pi)^{-1}(A + \log 4)$$

is the extremal distance between the boundary of the annulus.

Next we fix a simply connected domain  $\Omega$ ,  $\infty \in \Omega$ , and let  $\omega$  denote the harmonic measure at  $\infty$ . For technical reasons we assume that  $\text{diam } \partial\Omega = 5$ .

For  $z \in \partial\Omega$  and  $\nu \in \mathbb{N}$  we define  $X_\nu(z)$ , the "excess" of the extremal length in  $\Omega \cap R_\nu(z)$ , by the equalities

$$X_\nu(z) = \lambda_\nu(z) - \lambda,$$

$$\lambda_\nu(z) = \inf\{d_\Omega(l_+, l_-) : l_\pm \text{ arcs on } \partial_\pm \cap \Omega\},$$

where  $\partial_\pm$  denote the inner and outer boundaries of  $R_\nu(z)$  and  $d_\Omega$  the extremal distance in  $\Omega$ . Thus  $X_\nu(z) \geq 0$  and is zero if and only if the part of  $\partial\Omega$  in  $R_\nu(z)$  lies on a segment with endpoint  $z$ .

**2.2. Lemma.** *Let  $\varepsilon > 0$ ,  $\varrho = e^{-nA}$ ,  $n \geq n_0 = n_0(\varepsilon)$ . If*

$$\omega B(z, \varrho) \geq \varrho^{1/2+\varepsilon},$$

then

$$\sum_{\nu=1}^{n-1} X_\nu(z) \leq C_1 A \varepsilon n,$$

where  $C_1$  is an absolute constant. (In fact, we can take any  $C_1 > \pi^{-1}$ .)

*Proof.* Let  $L$  denote a circle of radius 10 centered at some point of  $\partial\Omega$ . Then the inequality  $\omega B(z, \varrho) \geq \varrho^{1/2+\varepsilon}$  implies that there is an arc  $l$  on  $\partial\Omega \cap \partial B(z, \varrho)$  satisfying

$$d_\Omega(l, L) \leq \pi^{-1} \left( \frac{1}{2} + 2\varepsilon \right) \log \frac{1}{\varrho},$$

(cf. [C], [M1, Corollary 1.4]). Consider the annuli

$$\begin{aligned}\tilde{R}_1(z) &= \{\zeta : e^{-(3/2)A} \leq |z-\zeta| \leq 2\} \\ \tilde{R}_\nu(z) &= \{\zeta : e^{-(\nu+1/2)A} \leq |z-\zeta| \leq e^{-(\nu-3/2)A}\}, \quad (2 \leq \nu \leq n-1),\end{aligned}$$

and define  $\tilde{\lambda}_\nu(z), \tilde{X}_\nu(z)$  with respect to  $\tilde{R}_\nu(z)$  as in Section 2.1. Then

$$\frac{1+4\varepsilon}{2\pi}An \geq d_\Omega(l, L) \geq \frac{1}{2} \left( \sum_{\nu \text{ odd}} \tilde{\lambda}_\nu(z) + \sum_{\nu \text{ even}} \tilde{\lambda}_\nu(z) \right) \geq \frac{1}{2} \sum_{\nu=1}^{n-1} \tilde{X}_\nu(z) + \frac{n-\frac{5}{4}}{2\pi}A,$$

where the second inequality follows from the subadditivity property of the extremal lengths. It remains to note that  $\tilde{X}_\nu(z) \geq X_\nu(z)$ .  $\square$

**2.3. Lemma.** *There are absolute constants  $\sigma > 0$  and  $\eta > 0$  such that if  $z \in \partial\Omega$ ,  $k \geq 2$  and*

$$X_1(z) \leq e^{-\sigma k A},$$

then for any  $z' \in \partial\Omega \cap R_1(z)$  we have

$$X_2(z'), \dots, X_k(z') \geq \eta A.$$

*Proof.* Since  $\text{diam } \partial\Omega = 5$  and  $z \in \partial\Omega$ , the curve  $\partial\Omega$  intersects both boundaries of  $R_1(z)$ . Let  $\theta$  be the minimal angle of a sector of  $R_1(z)$  containing  $\partial\Omega \cap R_\nu(z)$ . Then an easy extremal length estimate shows that

$$X_1(z) \geq \frac{c}{\lambda} \theta^3,$$

where  $c$  is an absolute constant,  $\lambda$  as in Section 2.1. Hence if  $X_1(z) \leq e^{-\delta k A}$  with  $\delta > 3$ , and if  $A$  is sufficiently large, then we have

$$\theta \ll e^{-kA},$$

which means that the curve  $\partial\Omega$  passes through a very narrow corridor in  $R_1(z)$ . This gives the lemma with  $\eta \approx 2/3\pi - 1/2\pi = 1/6\pi$  (consider the “worst” case:  $z' \in \partial\Omega \cap \partial R_1(z)$ ).

*Remark.* In fact the lemma is true for every  $\delta > 2$ . This follows from an argument in [BJ]. Also we can consider  $\eta$  essentially equal to  $1/2\pi$ , disregarding the rare cases when the point  $z'$  is very close to  $\partial R_\nu(z)$ .

### 3. Proof of Theorem 1

#### 3.1. Notation and claim

For a simply connected domain  $\Omega$ , integer  $n$  and  $x > 0$ , we define  $N(\Omega; n, x)$  to be the maximal number of points  $z_j \in \mathbf{D} \cap \partial\Omega$  satisfying

- (i)  $|z_i - z_j| \geq e^{-nA}$ ,
- (ii)  $\sum_{\nu=1}^{n-1} X_\nu(z_j) \leq x$  for all  $j$ .

We will also define

$$N(n, x) = \sup N(\Omega; n, x),$$

the supremum being taken over all domains  $\Omega$  with  $\text{diam } \Omega \geq 5$ .

**Claim.** *There are constants  $C_2, C_3 > 0$  depending only on  $A$  such that*

$$N(n, x) \leq C_3 e^{C_2 x}.$$

This inequality together with Lemma 2.2 implies Theorem 1 with the constant  $K = C_1 C_2$ .

We shall prove the claim by induction. It is clear that

$$N \leq \text{const } e^{2nA},$$

therefore the statement is true for  $n=1, 2$  by the choice of  $C_3$ . Assume now that  $n \geq 3$ , and that the inequality is true for  $n:=1, 2, \dots, n-1$  and any  $x$ . Given  $\Omega$  and the points  $z_1, \dots, z_n$  satisfying (i), (ii), we consider the following three cases:

$$\min_{z \in \Omega \cap \mathbf{D}} X_1(z) \begin{cases} \geq \tau^2, \\ \leq \tau^n, \\ \in (\tau^n, \tau^2) \end{cases}$$

where  $\tau = e^{-\sigma A}$  ( $\sigma$  is the constant in Lemma 2.3).

#### 3.2. Case: $\min \geq \tau^2$

Observe that for any  $z_j$  we have

$$\sum_{\nu=2}^{n-1} X_\nu(z_j) \leq x - \tau^2.$$

Cover the set  $\partial\Omega \cap \mathbf{D}$  with  $\asymp e^{2A}$  discs  $\mathcal{D}$  of radius  $e^{-A}$ . Fix one of these discs and assume that it contains no points  $\zeta_1, \dots, \zeta_m$  from the family  $\{z_j\}$ . By rescaling

we can view  $\mathcal{D}$  as the unit disc, so the points  $\zeta_1, \dots, \zeta_m$  satisfy (i) and (ii) with parameters  $(n-1)$  and  $(x-\tau^2)$ . By the inductive hypothesis,

$$m \leq N(n-1, x-\tau^2) \leq C_3 e^{C_2(x-\tau^2)},$$

and hence

$$N(\Omega; n, x) \leq \text{const } e^{2A} e^{-C_2\tau^2} C_3 e^{C_2x} \leq C_3 e^{C_2x},$$

provided that  $C_2$  is greater than some constant depending on  $A$ .

### 3.3. Case: $\min \leq \tau^n$

Let  $z \in \mathbf{D} \cap \partial\Omega$  satisfy  $X_1(z) \leq \tau^n$ . By Lemma 2.3,

$$X_2, \dots, X_n \geq A\eta \quad \text{on } \partial\Omega \cap R_1(z).$$

Therefore, if  $x < A\eta(n-1)$ , all points  $z_j$  lie in  $\mathbf{D} \setminus R_1(z) \subset B(z, e^{-A})$ . By rescaling, we have  $N(\Omega; n, x) \leq N(n-1, x) \leq C_3 e^{C_2x}$ . On the other hand, if  $x \geq A\eta(n-1)$ , then

$$N(\Omega; n, x) \leq \text{const } e^{2nA} \leq \text{const } e^{2nx/\eta(n-1)} \leq C_3 e^{C_2x}$$

provided that  $C_2$  and  $C_3$  are greater than some absolute constant.

### 3.4. Case: $\tau^n < \min < \tau^2$

Assume that the minimum is attained at a point  $z \in \mathbf{D} \cap \partial\Omega$ , and

$$X_1(z) \in (\tau^{k+1}, \tau^k), \quad 2 < k < n.$$

By Lemma 2.3,

$$X_1, \dots, X_k \geq A\eta \quad \text{on } \partial\Omega \cap R_1(z).$$

We can cover  $\mathbf{D} \cap \partial\Omega$  with the disc  $B(z, e^{-A})$  and  $\asymp e^{kA}$  discs of radius  $e^{-kA}$  lying in  $R_1(z)$ . By rescaling and the inductive hypothesis, we have

$$\begin{aligned} N(\Omega; n, x) &\leq N(n-1, x-\tau^{k+1}) + e^{kA} N(n-k, x-(k-1)A\eta) \\ &\leq C_3 e^{C_2x} - C_3 e^{C_2x} \left[ \frac{1}{2} C_2 \tau^{k+1} - e^{kA - C_2(k-1)A\eta} \right] \\ &= C_3 e^{C_2x} - C_3 e^{C_2(x-(k-1)A\eta)} \tau^{k+1} \left[ \frac{1}{2} C_2 e^{C_2(k-1)A\eta} - e^{kA + (k+1)\sigma A} \right] \\ &\leq C_3 e^{C_2x}, \end{aligned}$$

provided  $C_2$  is greater than some absolute constant.

This completes the proof of Theorem 1.



### 3.5. Proof of Corollary 1

Let  $f$  be a univalent function on the unit disc. Fix a small number  $y > 0$  and let  $r = 1 - y$ . Subdivide the circle  $\{|z| = r\}$  into disjoint intervals  $I$  of length  $\asymp y$ . Denote the center of  $I$  by  $z_I$ . For  $\alpha > 0$  we define  $N(\alpha)$  as the number of  $I$ 's satisfying

$$|f'(z_I)| \leq y^\alpha.$$

Observe that  $\text{diam } fI \lesssim \rho = y^{1+\alpha}$ . Applying the theorem to the domain  $f(r\mathbf{D})$ , we have

$$N(\alpha) \lesssim \left(\frac{1}{\rho}\right)^{K(1/(1+\alpha)-1/2)} = \left(\frac{1}{y}\right)^{K(1-\alpha)/2}.$$

Since, for negative  $t$ ,

$$\int_{\partial\mathbf{D}} |f'(r\zeta)|^t |d\zeta| = \text{const} \int_0^1 y^{1+\alpha t} dN(\alpha) + O(1),$$

we have

$$B(t) \leq \max_{\alpha \leq 1} \left[ \frac{K}{2}(1-\alpha) - 1 - \alpha t \right] = -t - 1, \quad \text{provided } t + \frac{K}{2} \leq 0. \quad \square$$

### 3.6. Proof of Corollary 2

From the previous result on integral means it follows that for any univalent function  $f$  and any  $\alpha \in (0, 1)$ ,

$$\lim_{t \rightarrow 1} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = \infty$$

except for a set of  $\zeta$  of dimension  $(1-\alpha)K/2$  (cf. [M1, Lemma 5.1]). Applying [M1, Theorem 0.5] we have

$$\dim E = d \quad \Rightarrow \quad \dim fE \leq \frac{d}{2 - 2K^{-1}d}. \quad \square$$

## 4. Reduced extremal lengths and dandelions

### 4.1. $\beta$ -numbers

Let  $\Omega$  be a simply connected domain and  $b, a_1, \dots, a_m \in \partial\Omega$ . The quantities

$$\beta_j = \beta(\Omega; a_j, b)$$

were defined in Section 1.5. We emphasize that they are Möbius-invariant. These numbers can be expressed in terms of conformal mappings.

(i) Suppose  $b=\infty$  and let  $f_j$  be a conformal map from  $\mathbf{C}\setminus\{x:x\leq 0\}$  onto  $\Omega$  satisfying

$$f_j(\infty) = \infty, \quad f_j(0) = a_j.$$

Suppose also that  $f$  has angular derivatives at 0 and  $\infty$ . Then

$$\beta_j = \frac{|f'_j(0)|}{|f'(\infty)|}.$$

In such a form,  $\beta$ -numbers appeared in [Ba].

(ii) Let again  $b=\infty$  and let  $f$  map the upper halfplane onto  $\Omega$  and satisfy

$$f(z) \sim z^2 \quad \text{as } z \rightarrow \infty.$$

Suppose the points  $x_j \in \mathbf{R}$  are such that

$$f(x_j) = a_j, \quad f'(x_j) = 0, \quad \exists f''(x_j).$$

Then

$$\beta_j = \frac{2}{|f''(x_j)|}.$$

Indeed, if  $\varepsilon$  is small, the preimage of  $\{|w - a_j| = \varepsilon\}$  is like a semicircle of radius

$$\sqrt{\frac{2\varepsilon}{|f''(x_j)|}},$$

and the preimage of  $\{|w| = 1/\varepsilon\}$  like a semicircle of radius  $\varepsilon^{-1/2}$ . Therefore, we have

$$\begin{aligned} \lambda(\tilde{\Gamma}_\varepsilon) &\approx \frac{1}{2\pi} \log \frac{1}{\varepsilon^2}, \\ \lambda(\Gamma_\varepsilon) &\approx \frac{1}{\pi} \log \left( \frac{1}{\varepsilon} \sqrt{\frac{|f''(x_j)|}{2}} \right), \end{aligned}$$

and

$$\beta = \lim_{\varepsilon \rightarrow 0} e^{2\pi(\lambda(\tilde{\Gamma}_\varepsilon) - \lambda(\Gamma_\varepsilon))} = \frac{2}{|f''(x_j)|}.$$

(iii) Let  $a_j$  and  $b$  be finite, and let  $f$  be a conformal map from the unit disc onto  $\Omega$ . Suppose the points  $\zeta_j, \zeta_0 \in \partial\mathbf{D}$  are such that

$$f(\zeta_0) = b, \quad f(\zeta_j) = a_j,$$

$$f'(\zeta_0) = f'(\zeta_j) = 0,$$

$$\exists f''(\zeta_0), \quad \exists f''(\zeta_j).$$

Then

$$\beta_j = \frac{4|a_j - b|^2}{|\zeta_j - \zeta_0|^4} \frac{1}{|f''(\zeta_0)| |f''(\zeta_j)|}.$$

*Proof.* Applying the transformation  $w \mapsto (w - b)^{-1}$ , we compute

$$\lambda(\tilde{\Gamma}_\varepsilon) \approx \frac{1}{2\pi} \log \frac{|a_j - b|^2}{\varepsilon^2}.$$

To compute  $\lambda(\Gamma_\varepsilon)$  we observe that the preimages of the  $\varepsilon$ -neighborhoods of  $a_j$  have "radius"

$$r_j \approx \sqrt{\frac{2\varepsilon}{|f''(\zeta_j)|}}.$$

Applying the symmetry principle for extremal lengths, we have

$$\lambda(\Gamma_\varepsilon) \approx 2 \cdot \frac{1}{2\pi} \log \frac{|\zeta_j - \zeta_0|^2}{r_0 r_j},$$

and

$$\lambda(\tilde{\Gamma}_\varepsilon) - \lambda(\Gamma_\varepsilon) \approx \frac{1}{\pi} \log \frac{2|a_j - b|}{|\zeta_j - \zeta_0|^2 |f''(\zeta_j)|^{1/2} |f''(\zeta_0)|^{1/2}}.$$

## 4.2. Dandelions

We describe now a class of fractal sets relevant to our study (cf. [Ba]).

Let  $\Omega_0$  be a simply connected domain such that  $\infty \in \Omega_0$  and the boundary  $\Gamma_0 = \partial\Omega_0$  consists of a finite number of straight line segments. Let  $b, a_1, \dots, a_m$  be the extreme points of  $\Gamma_0$ , i.e. the points at which  $\Omega_0$  makes the full angle. We assume that  $b=0$ ,  $\{x: x < 0\} \subset \Omega_0$ , and that the segment  $L$  of  $\Gamma_0$  with the endpoint  $b$  lies on the real axis. Then for every sufficiently small  $\varkappa > 0$  we can construct a fractal set, a *dandelion*,  $\Gamma(\varkappa)$  as follows.

For  $j=1, \dots, m$  let  $l_j$  denote the segment of length  $\varkappa$  lying on  $\Gamma_0$  and having  $a_j$  as an endpoint. Define the polygon  $\Gamma_1 = \Gamma_1(\varkappa)$  by replacing each  $l_j$  by a rescaled copy of  $\Gamma_0$  so that under rescaling the segment  $L$  corresponds to  $l_j$ . The polygon  $\Gamma_1$  has  $m^2$  extreme points other than  $b$ .

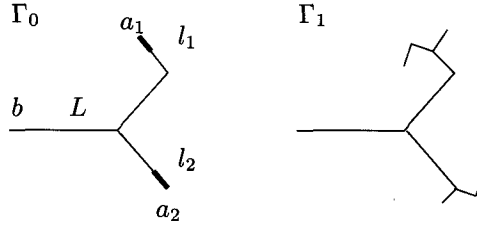


Figure 1.

To obtain  $\Gamma_2 = \Gamma_2(\varkappa)$  we repeat the above procedure with the scale  $\varkappa^2$ . Proceeding with this construction we define polygons  $\Gamma_3, \Gamma_4, \dots$  which converge to some fractal set  $\Gamma = \Gamma(\varkappa)$ . (Observe that if  $\varkappa$  is small enough, then no intersections occur at any step of the construction.) We will call the polygon  $\Gamma_0$  the initiator of the fractal set  $\Gamma$ .

There is some relationship between the properties of the harmonic measure on  $\Gamma$  and the  $\beta$ -numbers of the initiator. For a domain  $\Omega$  and  $\varepsilon > 0$  let us denote

$$\Upsilon_\Omega(\varepsilon) = \limsup_{\varrho \rightarrow 0} \frac{\log N(\varrho, \varepsilon)}{|\log \varrho|},$$

where  $N(\varrho, \varepsilon)$  is the maximal number of disjoint discs of radius  $\varrho$  and harmonic measure at least  $\varrho^{1/2+\varepsilon}$ . By Theorem 1 we have

$$\Upsilon_\Omega(\varepsilon) \leq K\varepsilon.$$

**Proposition.** (1) Let  $\{\Gamma_\varkappa\}_{\varkappa>0}$  be the family of dandelions with initiator  $\Gamma_0$ ,  $\Omega_\varkappa = \widehat{\mathbb{C}} \setminus \Gamma_\varkappa$ , and  $\{\beta_j\}$  be the  $\beta$ -numbers of  $\Gamma_0$ . If

$$\sum \beta_j^p \geq 1$$

for some  $p > 0$ , then

$$\liminf_{\varkappa \rightarrow 0} \sup_{\varepsilon > 0} \varepsilon^{-1} \Upsilon_{\Omega_\varkappa}(\varepsilon) \geq 2p.$$

(2) In the opposite direction, let

$$\Upsilon_\Omega(\varepsilon) > K\varepsilon$$

for some simply connected domain  $\Omega$  and  $\varepsilon > 0$ . Then there exists a polygon  $\Gamma_0$  satisfying

$$\sum \beta_j^{K/2} \geq 1.$$

**Corollary 3.** *There exists an absolute constant  $p$  such that*

$$\sum \beta_j^p \leq 1$$

for every configuration  $(\Omega; b, \{a_j\})$ .

The best value of this constant is exactly one-half of the best constant in Theorem 1.

4.3. *Proof of Proposition.* (1) At the  $(n-1)$ th step of the construction, we get a polygon  $\Gamma_{n-1}$  with  $m^n$  extreme points other than  $b$ . Let us code these points in a natural manner with the sequences  $X=(x_1 \dots x_n)$  of the symbols  $1, \dots, m$ , and denote them by  $a_X$ . Also let us use  $X$  to denote the segment on  $\Gamma_{n-1}$  of length  $\varrho(X)=\varkappa^n$  with endpoint  $a_X$ . For  $X=(x_1 \dots x_n)$  and  $j=1, \dots, m$  define

$$n_j = \#\{\nu : x_\nu = j\},$$

and consider only those  $X$ 's for which

$$\frac{n_j}{n} \sim \alpha_j \stackrel{\text{def}}{=} \frac{\beta_j^p}{\beta_1^p + \dots + \beta_m^p}.$$

By assumption,  $\alpha_j \leq \beta_j^p$ . The total number of such segments is

$$N = \frac{n!}{n_1! \dots n_m!} \approx \left( \prod \alpha_j^{-\alpha_j} \right)^n.$$

Now we estimate the harmonic measure of  $X$ . Since similar estimates have been made many times, we will only outline the idea. For a detailed proof one may use, for instance, the argument from [M2, §7.D].

Consider the conformal mapping

$$\varphi_X: \widehat{\mathbb{C}} \setminus \Gamma_{n-1} \rightarrow \widehat{\mathbb{C}} \setminus [0, 1]$$

satisfying

$$\varphi_X(\infty) = \infty, \quad \varphi_X(b) = 0, \quad \varphi_X(a_X) = 1.$$

Then, assuming  $\varkappa$  small, we have

$$|\varphi'_X(a_X)| \approx \prod_{j=1}^n \beta_j^{n_j},$$

and

$$\omega(X) \approx \sqrt{\varrho(X) \prod_{j=1}^m \beta_j^{n_j}} = \varrho(X)^{1/2+\varepsilon},$$

where  $\varepsilon = \varepsilon(\varkappa)$  does not depend on  $n$  and has the order of  $|\log \varkappa|^{-1}$ . Hence

$$\Upsilon_{\varkappa}(\varepsilon(\varkappa)) \geq \limsup_{n \rightarrow \infty} \frac{\log N}{|\log \varrho(x)|} \underset{\varkappa \rightarrow 0}{\sim} 2\varepsilon \frac{\log \prod \alpha_j^{-\alpha_j}}{\log \prod \beta_j^{-\alpha_j}} \geq 2\varepsilon p.$$

(2) We will again only indicate the idea (cf. [CJ], Section 6, for the details). Suppose, for a given harmonic measure  $\omega$  and some  $K' > K$ , that we have  $N \geq (1/\varrho)^{K'\varepsilon}$  disjoint discs  $B_j = B(z_j; \varrho)$  satisfying  $\omega(B_j) \sim \varrho^{1/2+\varepsilon}$ . Then we modify the boundary inside each  $B_j$  by replacing it by a suitable radius of  $B_j$ . Denote the new harmonic measure by  $\tilde{\omega}$ . Then for  $\delta \ll \varrho$ , we have

$$\tilde{\omega}B(z_j, \delta) \sim \tilde{\omega}(B_j) \sqrt{\frac{\delta}{\varrho}} \sim \varrho^\varepsilon \sqrt{\delta}.$$

Hence  $\beta_j \sim \varrho^{2\varepsilon}$  for the obtained polygon, and

$$\sum \beta_j^{K/2} \sim N \cdot (\varrho^{2\varepsilon})^{K/2}. \quad \square$$

4.4. *Remark.* There is a natural version of  $\beta$ -numbers based on the interior choice of the point  $b$ . Let

$$b \in \Omega \quad \text{and} \quad a_1, \dots, a_n \in \partial\Omega.$$

Then we can define  $\tilde{\beta}$ -numbers by the same formula:

$$\tilde{\beta}_j = \beta(\Omega; a_j, b), \quad (\text{see Section 1.5}).$$

Let, for example,  $b = \infty \in \Omega$ , and  $f$  be a conformal map from the unit disc onto  $\Omega$  such that

$$\begin{aligned} f(0) &= \infty, \\ f(\zeta_j) &= a_j, \quad f'(\zeta_j) = 0, \quad \exists f''(\zeta_j) \end{aligned}$$

for some points  $\zeta_j \in \partial\mathbf{D}$ . Then

$$\tilde{\beta}_j = \frac{2 \operatorname{cap} \partial\Omega}{|f''(\zeta_j)|}.$$

**Corollary 4.** *There exist an absolute constant  $p$  (same as in Theorem 1) and an absolute constant  $C$  such that*

$$\sum \tilde{\beta}_j^p \leq C.$$

*Proof.* Consider the numbers  $\tilde{\beta}_j = 2/|f''(\zeta_j)|$  defined for some univalent function

$$g = \frac{1}{z} + \dots \quad (\text{at } \infty),$$

and points  $\zeta_j \in \partial\mathbf{D}$ . At the expense of  $C$ , we can assume that all  $\zeta_j$  lie on a semicircle, say  $\partial\mathbf{D} \cap \{\operatorname{Re} \zeta < 0\}$ . Let  $\varphi$  be a conformal map from  $\mathbf{D}$  onto the slit disc  $\mathbf{D} \setminus [\frac{1}{2}, 1]$ , and  $\varphi(0) = 0$ . Consider the domain  $(f \circ \varphi)(\mathbf{D})$  and the points  $b = f(\frac{1}{2})$  and  $a_j = f(\zeta_j)$  on its boundary. Let  $\beta_j$  denote the corresponding  $\beta$ -numbers. Comparing the expressions for  $\tilde{\beta}_j$  and  $\beta_j$  (cf. (iii) in Section 4.1), and applying some elementary distortion results, we see that

$$\beta_j \geq (\text{abs. constant}) \tilde{\beta}_j.$$

The assertion now follows from Theorem 1.  $\square$

## 5. Examples

In this section we indicate some examples of explicit computation and also describe some further applications of  $\beta$ -numbers.

### 5.1. Explicit computations

(i) Consider the generalized Koebe function

$$f(z) = z^{-1}(1-z^n)^{2/n}, \quad z \in \mathbf{D}.$$

The boundary of the domain  $\Omega = f\mathbf{D}$  is the union of  $n$  straight line segments  $[0, a_j]$ , where

$$a_j = 2^{2/n} \exp\left\{2\pi i \cdot \frac{j}{n}\right\}, \quad (0 \leq j \leq n-1).$$

If we take  $b = a_0$ , we have  $(n-1)$   $\beta$ -numbers  $\beta_j = \beta(\Omega; a_j, a_0)$ ,  $1 \leq j \leq n-1$ , and the application of the formula (iii) of Section 4.1 gives

$$\beta_j = \frac{16}{n^2} |1 - e^{2\pi i \cdot j/n}|^{-2}.$$

In particular,

$$\sum \beta_j^2 = \begin{cases} 1, & n = 2, \\ \left(\frac{8}{9}\right)^3, & n = 3, \\ \dots, & \\ \frac{16}{45}, & n = \infty. \end{cases}$$

We see that adding new branches at a given branch point only decreases the amount of  $\sum \beta_j^2$ .

If we take  $b = \infty \in \Omega$ , then we get  $n$   $\tilde{\beta}$ -numbers  $\tilde{\beta}_j = \beta(\Omega; a_j, \infty)$  all equal to

$$\frac{4}{n} 2^{-2/n},$$

see Section 4.4. It follows that

$$\sum \tilde{\beta}_j^2 = \begin{cases} 1, & n = 1, \\ 2, & n = 2, \\ \frac{8}{3} 2^{-1/3} > 2, & n = 3, \end{cases}$$

which shows that in contrast to the case of  $\beta$ -numbers, there is no obvious conjecture about the constant  $C$  in Corollary 4.

(ii) Let  $\{t_k\}$ ,  $0 < k < n$ , be any set of real numbers and let positive numbers  $\alpha_k$  satisfy

$$\sum \alpha_k = 2.$$

Then the function

$$f(z) = \prod_{k=0}^n (z - t_k)^{\alpha_k}$$

is univalent in the upper halfplane and maps it to the domain  $\Omega$  whose boundary consists of the positive real axis and  $n$  straight line segments emanating from the origin. Apart from  $\infty$ , there are  $n$  points  $a_j \in \partial\Omega$  at which the domain makes a full angle. Applying (ii) of Section 4.1, we have

$$\beta_j = \beta(\Omega; a_j, \infty) = 2|f''(x_j)|^{-1} = 2 \left( \prod_{k=0}^n |x_j - t_k|^{\alpha_k} \sum_{k=0}^n \frac{\alpha_k}{|x_j - t_k|^2} \right)^{-1},$$

where the points  $x_j = f^{-1}a_j \in \mathbf{R}$ ,  $1 \leq j \leq n$ , are the roots of the equation

$$\sum_{k=0}^n \frac{\alpha_k}{x - t_k} = 0.$$



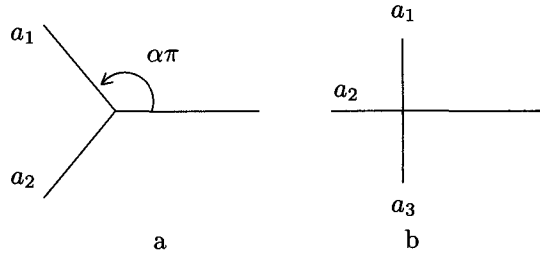


Figure 2.

This equation, as well as the expression for  $\beta_j$ , is “real variable”, but this fact does not seem to be of much help. We were unable to prove the inequality  $\sum \beta_j^2 \leq 1$  for this particular example.

On the other hand, some special cases are easy to compute. For instance, in the symmetric,  $n=2$  case (see Figure 2a), we have

$$\beta_1 = \beta_2 = \frac{1}{2} \left( \frac{\alpha}{1-\alpha} \right)^{1-\alpha} \leq 0.660 \dots < \frac{1}{\sqrt{2}}.$$

In the rectangular, symmetric,  $n=3$  case (see Figure 2b), we have

$$f(z) = \sqrt{(z^2-1)(z^2-t^2)}$$

for some  $t \in (0, 1)$ , and

$$\begin{aligned} \beta_1 = \beta_3 &= \frac{1}{2} \frac{1-t^2}{1+t^2}, \\ \beta_2 &= \frac{2t}{1+t^2}, \\ \sum \beta_j^2 &= 1 - \frac{1}{2} \left( \frac{1-t^2}{1+t^2} \right)^2 < 1, \end{aligned}$$

but for any  $p < 2$  we have

$$\sum \beta_j^p = \frac{2^p t^p + 2^{1-p} (1-t^2)^p}{(1+t^2)^p} = 1 + 2^{1-p} (1-t)^p - p^2 (1-t)^2 + \dots > 1$$

provided  $t$  is close to one. Together with the Proposition in Section 4.2 this proves that the constant  $K$  in Theorem 1 must be at least 4. This example also provides a partial motivation for Brennan's conjecture: if  $p < 2$ , then we can always increase

the amount of  $\sum \beta_j^p$  by adding a short perpendicular segment at a regular point of the boundary while this is no longer true for  $p \geq 2$ .

## 5.2. Variation of logarithmic capacity, Schwarz derivative

It is sometimes useful to look at  $\tilde{\beta}$ -numbers as the derivative of logarithmic capacity with respect to the arclength. More precisely, let  $E$  be a connected compact set,  $a \in E$ , and assume that near  $a$ ,  $E$  represents a smooth slit with the endpoint  $a$ . Let us now extend this slit beyond the endpoint by adding length  $l$  and keeping the slit smooth. Denote the new set by  $E(l)$ . (We can as well define  $E(l)$  for small negative  $l$  by cutting off the slit.)

**Proposition.**

$$\left. \frac{d}{dl} \right|_{l=0} \text{cap } E(l) = \frac{1}{4} \tilde{\beta}(E^c; a, \infty).$$

*Proof.* Let  $g(z) = cz^{-1} + \dots$  map  $\mathbf{D}$  onto the complement of  $E$  and  $g(\zeta) = a$ . Then

$$\tilde{\beta} = \frac{2|c|}{|g''(\zeta)|}.$$

Consider also the mappings

$$g_l(z) = \frac{c(l)}{z} + \dots : \mathbf{D} \rightarrow \widehat{\mathbf{C}} \setminus E(l).$$

Then

$$g_l = g \circ \varphi,$$

where  $\varphi$  is a conformal map from the unit disc onto itself minus a short slit at the point  $\zeta$ . This slit is almost a straight line radial segment of length  $h$ , where  $h$  and  $l$  are related by the equation

$$l \approx \frac{1}{2} |g''(\zeta)| h^2 = \frac{|c|h^2}{\tilde{\beta}}.$$

Without loss of generality, let  $\zeta = -1$ . Then

$$\varphi(z) \approx k^{-1}((1-\varepsilon)k(z)) = (1-\varepsilon)z + \dots,$$

where  $k(z) = z(1-z)^{-2}$  is the Koebe function and

$$\varepsilon = \frac{1}{4} h^2.$$

Therefore,

$$\text{cap } E(l) = |c(l)| = \frac{|c|}{l-\varepsilon} \approx |c| \left(1 + \frac{h^2}{4}\right) = |c| + \frac{1}{4}\tilde{\beta}l$$

and

$$\left. \frac{d|c(l)|}{dl} \right|_{l=0} = \frac{1}{4}\tilde{\beta}. \quad \square$$

**Corollary 5.** *Let  $E$  be a connected compact set,  $\gamma_j$  be some curves of lengths  $L_j$  intersecting  $E$ , and*

$$\tilde{E} = E \cup \bigcup_j \gamma_j.$$

Then

$$\text{cap } \tilde{E} \leq \text{cap } E + C \left( \sum L_j^q \right)^{1/q},$$

where  $C > 0$  and  $q > 1$  are some absolute constants.

*Proof.* We can assume that  $\gamma_j$  are smooth arcs with one of the endpoints, say  $a_j$ , lying on  $E$ . For  $x \in (0, 1)$  let  $\gamma_j(x)$  denote the subarc of  $\gamma_j$  from  $a_j$  to  $a_j(x)$  such that  $\gamma_j(x)$  has length  $xL_j$ . Denote

$$\begin{aligned} E(x) &= E \cup \bigcup_j \gamma_j(x), \\ \tilde{\beta}_j(x) &= \beta(\hat{\mathbf{C}} \setminus E(x); a_j(x), \infty), \\ c(x) &= \text{cap } E(x). \end{aligned}$$

Thus by Corollary 4 we have

$$c'(x) = \frac{1}{4} \sum_j \tilde{\beta}_j(x) L_j \leq \frac{1}{4} \left( \sum_j \tilde{\beta}_j^p(x) \right)^{1/p} \left( \sum L_j^q \right)^{1/q} \leq \text{const} \left( \sum L_j^q \right)^{1/q}$$

with  $q^{-1} + p^{-1} = 1$ .  $\square$

$\beta$ -numbers, in contrast to  $\tilde{\beta}$ , are more related to the Schwarz derivative than to capacity. Consider, for instance, a simply connected domain  $G$ ,  $G \subset \mathbf{C}_+ = \{\text{Im } w > 0\}$ , and assume that  $G$  contains  $\mathbf{C}_+ \cap \{|w| > R\}$  for some large  $R$ . Then there is a unique conformal map  $g: \mathbf{C}_+ \rightarrow G$  such that

$$g(x) = z - \frac{T}{z} + \dots \quad \text{at } \infty.$$

The parameter  $T$  is positive and equals one-sixth of the Schwarz derivative of the function  $(g(z^{-1}))^{-1}$  at the origin.

Let now

$$g(x_j) = a_j, \quad g'(x_j) = 0, \quad \exists g''(x_j)$$

for some  $x_j \in \mathbf{R}$ . Then we can define  $\beta$ -numbers at the points  $a_j$  by

$$\beta_j = \frac{1}{|a_j| |g''(x_j)|} = \beta(G^2; a_j^2, \infty),$$

where  $G^2$  is the transform of  $G$  under  $w \mapsto w^2$ .

Let us now add some short slits  $\gamma_j$  to the boundary of  $G$  at the points  $a_j$ . Then we obtain new, smaller domains  $G(\lambda_1, \lambda_2, \dots)$  which we parametrize with the quantities

$$\lambda_j = \int_{l_j} |z| |dz|.$$

Consequently, we have a function

$$T = T(\lambda_1, \lambda_2, \dots),$$

and reasoning as above we show that

$$\left. \frac{\partial T}{\partial \lambda_j} \right|_{\lambda_1=0, \lambda_2=0, \dots} = \beta_j.$$

**Corollary 6.**

$$T(\lambda_1, \lambda_2, \dots) \leq T + \left( \sum \lambda_j^q \right)^{1/q}$$

for some absolute constant  $q > 1$ .  $\square$

Observe that Brennan's conjecture  $D$  is equivalent to the statement

$$\|\nabla T\| \leq 1,$$

or to the statement of Corollary 6 with  $q=2$ .

### 5.3. Dirichlet norm of the argument

Let  $\Omega$  be a simply connected domain and  $0, \infty \in \partial\Omega$ . Let  $\arg z$  denote any branch of the argument which is continuous on  $\partial\Omega \setminus \{0\}$ . Such a branch has to exist if  $\beta(\Omega; 0, \infty) > 0$ .

**Proposition.**

$$\beta(\Omega; 0, \infty) = \exp \left\{ -\frac{1}{2\pi} D[\widehat{\arg z}] \right\},$$

where  $\widehat{\arg z}$  denotes the harmonic extension to  $\Omega$  of the boundary function  $\arg z$ , and

$$D[u] = \iint_{\Omega} |\nabla u|^2 dm_2.$$

*Proof.* By definition,

$$\beta = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} e^{-2\alpha\lambda(\varepsilon)},$$

where  $\lambda(\varepsilon)$  is the extremal distance between  $|z|=\varepsilon$  and  $|z|=1/\varepsilon$  in  $\Omega$ . On the other hand,

$$\lambda(\varepsilon) = \sup D[u]^{-1},$$

the supremum being taken over all functions  $u$  satisfying

$$u = \begin{cases} \frac{1}{2}, & |z| = \varepsilon, \\ -\frac{1}{2}, & |z| = \frac{1}{\varepsilon}. \end{cases}$$

Let us express  $u$  in the form

$$u(z) = \frac{1}{2 \log 1/\varepsilon} \left[ \log \frac{1}{|z|} + v \right],$$

where  $v=0$  on  $|z|=\varepsilon, 1/\varepsilon$ . Then we have

$$D[u] = \frac{\pi}{\log 1/\varepsilon} \left[ 1 + \frac{X[v]}{4\pi \log 1/\varepsilon} \right],$$

where

$$X[v] = 2D \left[ \log \frac{1}{|z|}, v \right] + D[v] = -2 \iint \frac{\partial v}{\partial r} dr d\theta + \int_{\partial\Omega} v \frac{\partial v}{\partial n} |dz|,$$

$(r, \theta)$  are the polar coordinates and  $n$  denotes the inner normal. Consequently,

$$2\pi\lambda(\varepsilon) = \sup_v \frac{2 \log 1/\varepsilon}{1 + (X[v])/(4\pi \log 1/\varepsilon)} = 2 \log \frac{1}{\varepsilon} - \frac{1}{2\pi} \inf_v X[v] + o(1) \quad \text{as } \varepsilon \rightarrow 0,$$

and

$$\beta = \exp \left\{ \frac{1}{2\pi} \inf X[v] \right\},$$

the infimum now is taken over all harmonic functions  $v$  in  $\Omega$ .

To solve the latter variational problem we first apply Newton–Leibnitz to the double integral in the formula for  $X[v]$  and rewrite it as an integral over the boundary:

$$-\iint \left( \frac{\partial v}{\partial r} dr \right) d\theta = \int_{\partial\Omega} v(z) p(z) |dz|,$$

where

$$(*) \quad p(z) = \pm \frac{d\theta}{|dz|}$$

with  $+$  or  $-$  according as the angle between the inner normal at  $z$  and the direction  $[0, z]$  is less or greater than  $\pi/2$ .

Thus we have

$$X[v] = \int_{\partial\Omega} v(z) \left[ \frac{\partial v}{\partial n} + 2p(v) \right] |dz|.$$

Varying the function  $v$  in this integral, we find the following condition for the extremal case:

$$\frac{\partial v_{\text{extr}}}{\partial n} = -p(z).$$

Therefore,

$$X[v_{\text{extr}}] = -D[v_{\text{extr}}] = -D[\tilde{v}_{\text{extr}}],$$

where  $\tilde{v}_{\text{extr}}$  denotes the conjugate function. But for  $\tilde{v}_{\text{extr}}$  we have

$$\frac{\partial \tilde{v}_{\text{extr}}}{\partial s} = -\frac{\partial v_{\text{extr}}}{\partial n} = p(z)$$

and comparing this with  $(*)$  we see that  $\tilde{v}_{\text{extr}}|_{\partial\Omega}$  is a continuous branch of the argument.  $\square$

Similarly, in the case of finite  $a, b \in \partial\Omega$ , we get the expression

$$\beta(\Omega; a, b) = \exp \left\{ -\frac{1}{2\pi} D \left[ \widehat{\arg} \frac{z-a}{z-b} \right] \right\}.$$

Applying Theorem 1 we obtain the following inequality which slightly resembles the well-known Goluzin inequality in the theory of univalent functions.

**Corollary 7.** *Let  $f$  be a univalent function in the upper halfplane and  $\{x_j\}$  be distinct points on the real axis. Then*

$$\sum \lambda_j \left\| \log \frac{f(z) - f(x_j)}{(z - x_j)^2} \right\|_{\mathcal{D}}^2 \leq \frac{1}{p} \sum \lambda_j \log \frac{1}{\lambda_j}$$

for some absolute constant  $p$  and for all collections of positive numbers  $\{\lambda_j\}$  satisfying  $\sum \lambda_j = 1$ , where  $\|\cdot\|_{\mathcal{D}} = (D[\cdot]/2\pi)^{1/2}$  denotes the Dirichlet norm.

*Proof.* We can assume that  $f(z) \sim \text{const } z^2$  at  $\infty$  for otherwise the left hand side is infinite. If  $x_j \in \mathbf{R}$ , then

$$\widehat{\text{arg}}(f - f(x_j))(z) = \arg \frac{f(z) - f(x_j)}{(z - x_j)^2}, \quad z \in \mathbf{C}_+,$$

and by the conformal invariance of the Dirichlet norm and the statement above, we have

$$d_j \equiv \left\| \log \frac{f(z) - f(x_j)}{(z - x_j)^2} \right\|_{\mathcal{D}}^2 = \log \frac{1}{\beta_j},$$

where  $\beta_j = \beta(f\mathbf{C}_+; f(x_j), \infty)$ . By Theorem 1,

$$\sum e^{-d_j p} = \sum \beta_j^p \leq 1,$$

which is equivalent to the fact that

$$\lambda_j > 0, \quad \sum \lambda_j = 1 \Rightarrow \sum \lambda_j (d_j p) \geq \sum \lambda_j \log \frac{1}{\lambda_j}.$$

## 6. Proof of Theorem 2

In this section we shall prove the inequality

$$\beta_1^2 + \beta_2^2 \leq 1.$$

A natural guess would be to look for the extremal case in the class of "symmetric" domains satisfying  $\beta_1 = \beta_2$ . But this turns out to be wrong. In fact we shall find the extremal configuration for the problem

$$\max\{\beta : \beta = \beta_1 = \beta_2\}$$

and will show that this maximum is equal to

$$8(3+2\sqrt{2})^{-\sqrt{2}} \approx 0.661 \dots < \frac{1}{\sqrt{2}} = 0.707 \dots,$$

(which is only slightly better than what we had in example (ii), Section 5.1).

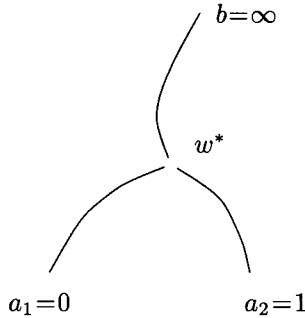


Figure 3.

On the other hand, in the case  $\beta_1 \ll \beta_2$  we have  $\beta_1^2 + \beta_2^2 \leq 1 + \varepsilon$ , and it is clear that if there is a domain satisfying  $\beta_1^2 + \beta_2^2 > 1$ , then there should exist an extremal configuration with respect to the quantity  $\beta_1^2 + \beta_2^2$ . Our strategy will be to rule out the latter possibility. From now on, we will assume that the extremal configuration  $(\Omega; a_1, a_2, b)$  exists and will eventually arrive at a contradiction. Let us fix the notation

$$A = \beta_1^{-1} \beta_2$$

for the extremal domain. Since  $\beta$ -numbers are Möbius invariant, we can assume  $b = \infty$ ,  $a_1 = 0$ ,  $a_2 = 1$ . We also assume that the boundary  $\partial\Omega$  has at least one point in the upper halfplane  $\mathbf{C}_+$ . The first step will be to apply

### 5.1. Schiffer's variation

**Lemma 1.** *The boundary  $\Gamma = \partial\Omega$  of the extremal domain  $\Omega$  is the union of the trajectories of the quadratic differential*

$$Q(w) dw^2 = \left( \frac{1}{w^2} + \frac{A^2}{(w-1)^2} \right) dw^2$$

that join the points  $\infty, 0, 1$  in  $\mathbf{C}_+$ .

This provides a complete description of the domain: the boundary consists of three analytic arcs joining the triple point

$$w^* = \frac{1+iA}{1-A^2},$$

(a zero of  $Q$  in  $\mathbf{C}_+$ ), with the points  $0, 1, \infty$  respectively, see Figure 3.



*Proof of Lemma 1.* Let  $f$  map  $\mathbf{C}_+$  onto  $\Omega$ ,  $f(x) \sim z^2$  at  $\infty$ , and  $f(x_j) = a_j$ . Then

$$\beta_j = \frac{2}{|f''(x_j)|}.$$

Fix a point  $w_0 \in \partial\Omega$ ,  $w_0 \neq a_1, a_2$ , and let  $\Upsilon$  be some conformal mapping defined on the complement to the part of  $\Gamma$  lying in a small neighborhood of  $w_0$ , and  $\Upsilon(w) \sim w$  as  $w \rightarrow \infty$ . Denote

$$\begin{aligned}\tilde{f} &= f \circ \Upsilon, \\ \tilde{\Omega} &= \Upsilon(\Omega), \quad \tilde{a}_j = \Upsilon(a_j), \\ \tilde{\beta}_j &= \beta(\tilde{\Omega}; \tilde{a}_j, \infty).\end{aligned}$$

Then

$$\beta_j = \frac{2}{|\tilde{f}''(x_j)|} = \frac{2}{|\Upsilon'(a_j)| |f''(x_j)|} = \frac{\beta_j}{|\Upsilon'(a_j)|}.$$

Suppose now that we have a family of such mappings  $\{\Upsilon_r\}$ ,  $r \rightarrow 0$  satisfying

$$\Upsilon_r(w) = w + \lambda(r) \frac{1}{(w - w_0)} + O(r^3)$$

uniformly on each set  $\{|w - w_0| > \varepsilon\}$  as  $r \rightarrow 0$ , where  $\lambda(r) = O(r^2)$ . Then we have

$$\Upsilon_r'(a_j) = 1 - \lambda(r) \frac{1}{(a_j - w_0)^2} + O(r^3),$$

$$|\Upsilon_r'(a_j)|^{-2} = 1 + 2 \operatorname{Re} \frac{\lambda(r)}{(a_j - w_0)^2} + O(r^3)$$

and

$$\begin{aligned}0 &\geq \sum \tilde{\beta}_j^2 - \sum \beta_j^2 = \sum \beta_j^2 [|\Upsilon_r'(a_j)|^{-2} - 1] \\ &= 2 \operatorname{Re} \lambda(r) \sum_j \frac{\beta_j^2}{(a_j - w_0)^2} + O(r^3).\end{aligned}$$

Since this is true for any choice of the family  $\{\Upsilon_r\}$  and any point  $w_0$ , by Schiffer's theorem (see [D], p. 297)  $\Gamma$  is a union of analytic arcs  $w = w(t)$  satisfying

$$Q(w(t)) \left( \frac{dw}{dt} \right)^2 > 0.$$

The differential  $Q(w)dw^2$  has three poles of order two and two simple zeros in  $\widehat{\mathbf{C}}$ ; so the structure of the trajectories is easily described and simple considerations complete the proof of the lemma.  $\square$

## 6.2. Conformal map

Our next step is to find an explicit expression for the conformal map  $f: \mathbf{C}_+ \rightarrow \Omega$ .

Since  $\Gamma$  is almost a straight line in a neighborhood of  $\infty$ , we can normalize  $f$  so that

$$(1) \quad f(z) = cz^2 + c_0 + \dots \quad \text{as } z \rightarrow \infty.$$

Let  $x_1, x_2 \in \mathbf{R}$  be the points such that

$$f(x_j) = a_j.$$

Without loss of generality we can choose

$$(2) \quad x_2 = -1,$$

and this choice, together with (1), determines  $f$  completely.

**Lemma 2.** *If  $f$  satisfies (1), (2), then*

$$x_1 = A^2$$

and

$$[f'(z)]^2 \left[ \frac{1}{f^2(z)} + \frac{A^2}{(f(z)-1)^2} \right] = \frac{4}{(z-A^2)^2} + \frac{4A^2}{(z+1)^2}.$$

*Proof.* Denote

$$W(z) = [f'(z)]^2 Q(f(z)).$$

This function is defined in  $\mathbf{C}_+$ , continuous and positive on the real line except, probably, at the points  $x_1, x_2$  and also at the points  $y_1, y_2, y_3$  which  $f$  takes to  $w^*$ , a zero of  $Q$  and the branch point of  $\Gamma$ . Extending  $W$  to the whole plane by symmetry;

$$W(\bar{z}) = \overline{W(z)},$$

we obtain an analytic function on

$$\widehat{\mathbf{C}} \setminus \{\infty, x_1, x_2, y_1, y_2, y_3\}.$$

Let us study the behavior of  $W$  at the singular points.

The singularities at  $y_j$  are removable. The quadratic differential  $Q(w)dw^2$  is, indeed, conformally equivalent to  $w dw^2$  in a neighborhood of  $w^*$ . Therefore,

$$\begin{aligned} |f(z) - w^*| &\asymp |z - y_j|^{2/3}, \\ |f'(z)| &\asymp |z - y_j|^{-1/3} \end{aligned}$$

as  $z \rightarrow y_j$ , and hence  $W$  is bounded at  $y_j$ .

The infinity point is also removable. In fact we have

$$(3) \quad W(z) = (2cz + O(z^{-2}))^2 \cdot \frac{1 + A^2}{c^2 z^4 + O(z^2)} = 4(1 + A^2) \frac{1}{z^2} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \rightarrow \infty.$$

Finally, we check that the points  $x_j$  are second order poles. It follows from the local structure of trajectories that  $f$  is infinitely differentiable at  $x_j$ , and therefore

$$f(z) = a_j + \frac{f''(x_j)}{2}(z - x_j)^2 + \dots \quad \text{as } z \rightarrow x_j.$$

Hence

$$\begin{aligned} W(z) &= ((f''(x_j))^2(z - x_j)^2 + \dots) A_j^2 \left[ \left( \frac{f''(x_j)}{2} \right) (z - x_j)^2 + \dots + O(1) \right]^{-2} \\ &= \frac{4A_j^2}{(z - x_j)^2} + O\left(\frac{1}{z - x_j}\right) \end{aligned}$$

where  $A_1 = 1$ ,  $A_2 = A$ .

From all the above it follows that

$$(4) \quad W(z) = \frac{4}{(z - x_1)^2} + \frac{4A^2}{(z - x_2)^2} + \frac{\delta}{z - x_1} - \frac{\delta}{z - x_2}$$

for some  $\delta \in \mathbf{R}$ . (Recall that  $W$  should have zero of order two at  $\infty$ .) Comparing the development of the right hand side of (4) with (3), we have

$$0 = \frac{\delta(x_1 - x_2)}{z^2} + \frac{\delta(x_1^2 - x_2^2)}{z^3} + 8 \frac{z_1 + x_2 A^2}{z^3},$$

and hence  $\delta = 0$  and

$$x_1 = -x_2 A^2 = A^2. \quad \square$$

Observe also that  $f$  must take the zero  $z_0$  of  $W$  lying in  $\mathbf{C}_+$  to the zero  $w_0$  of  $Q(w)$  other than  $w^*$ . Thus we have the following formula for the conformal map  $w = f(z)$ :

$$\int_{w_0} \frac{\sqrt{(w-1)^2 + A^2 w^2}}{w(w-1)} dw = 2 \int_{z_0} \frac{\sqrt{(z+1)^2 + A^2(z-A^2)}}{(z-A^2)(z+1)} dz,$$

where

$$w_0 = \frac{1 - iA}{1 + A^2}, \quad z_0 = A^2 - 1 + iA.$$

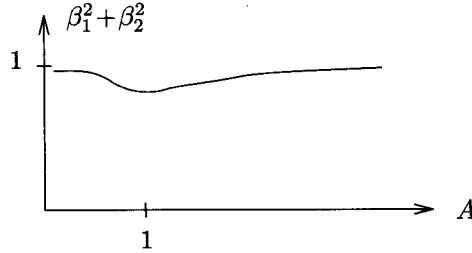


Figure 4.

### 6.3. Computation

**Lemma 3.**

$$(5) \quad |f''(x_1)| = \left( \frac{1 + \sqrt{1 + A^2}}{A} \right)^{\sqrt{1 + A^2}} \frac{A}{(1 + A^2)^2},$$

$$(6) \quad |f''(x_2)| = \left( A + \sqrt{1 + A^2} \right)^{\sqrt{1 + A^2}/A} \frac{1}{A(1 + A^2)^2},$$

$$(7) \quad |c| = \lim_{z \rightarrow \infty} \left| \frac{f(z)}{z^2} \right| = \frac{2}{A} \frac{1}{1 + A^2} \left( A + \sqrt{1 + A^2} \right)^{-A/\sqrt{1 + A^2}} \left( \frac{1 + \sqrt{1 + A^2}}{A} \right)^{-1/\sqrt{1 + A^2}}.$$

This gives the values of the numbers  $\beta_j = 2|c|/|f''(x_j)|$ , and we have

$$\begin{aligned} \beta_1^2 + \beta_2^2 &= 16 \left( A + \frac{1}{A} \right)^2 \left( A + \sqrt{1 + A^2} \right)^{-2/\sqrt{1 + A^2}} \left( A^{-1} + \sqrt{1 + A^{-2}} \right)^{-2/\sqrt{1 + A^2}} \\ &\quad \times \left[ A^2 \left( A + \sqrt{1 + A^2} \right)^{-2\sqrt{1 + A^{-2}}} + A^{-2} \left( A^{-1} + \sqrt{1 + A^{-2}} \right)^{-2\sqrt{1 + A^2}} \right]. \end{aligned}$$

The right hand side, as a function of  $A$ , is obviously invariant under the substitution  $A \rightarrow A^{-1}$ . The minimal value corresponds to the case  $A=1$  and is equal to

$$128(1 + \sqrt{2})^{-4\sqrt{2}} = .8747898 \dots,$$

(which implies (1)). The function is increasing on  $[1, \infty)$  and tends to 1 as  $A \rightarrow \infty$  (see Figure 4). It follows that for all  $A$ ,  $\beta_1^2 + \beta_2^2 < 1$ .

It remains to check (5)–(7). This is done by direct computation of the integrals. With an appropriate choice of the square root branches, we have (using the notation  $R(w)=(w-1)^2+A^2w^2$ )

$$\begin{aligned} \Upsilon(w) &= \int_{w_0} \frac{R(w)}{w(w-1)} dw = \sqrt{1+A^2} \log \frac{i}{A} \left( \sqrt{1+A^2} \sqrt{R} + A^2w + w - 1 \right) \\ &\quad - A \log \frac{1}{Ai} \left( \frac{A^2w + A\sqrt{R}}{w-1} \right) - \log \frac{1}{Ai} \frac{1-w-\sqrt{R}}{w}; \end{aligned}$$

and, (using the notation  $S(z)=(z+1)^2+A^2(z-A^2)$ ),

$$\begin{aligned} \Psi(z) &= \int_{z_0} \frac{\sqrt{S(z)}}{(z-A^2)(z+1)} dz = \sqrt{1+A^2} \log \frac{\sqrt{1+A^2}\sqrt{S} + (z+1) + A^2(z-A^2)}{iA(1+A^2)} \\ &\quad - A \log \frac{i}{z+1} \left( A(A^2-z) - \sqrt{S} \right) - \log \frac{i}{A(z-A^2)} (z+1+\sqrt{S}). \end{aligned}$$

Now as  $z \rightarrow x_1 = A^2$  and  $w \rightarrow a_1 = 0$ , we have

$$\operatorname{Re} \Psi(z) = \log |z-A^2| + \sqrt{1+A^2} \log \frac{1+\sqrt{1+A^2}}{A} - \log \frac{2}{A} (1+A^2) + o(1),$$

$$\operatorname{Re} \Upsilon(w) = \log |w| + \sqrt{1+A^2} \log \frac{1+\sqrt{1+A^2}}{A} + \log \frac{A}{2} + o(1).$$

Since

$$\Upsilon(f(z)) = 2\Psi(z),$$

we have

$$\log |f(z) - a_1| = 2 \log |z - w_1| + \sqrt{1+A^2} \log \frac{1+\sqrt{1+A^2}}{A} + \log \frac{A}{2(1+A^2)^2} + o(1),$$

which implies (5).

The relations (6) and (7) are derived in an analogous way.

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