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Entropy numbers of tensor products of operators

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This paper estimates the entropy numbers of tensor products of operators, mainly in a global sense. Let $S \in L_{s,w}^{(e)}(E_1, F_1)$, $T \in L_{s,w}^{(e)}(E_2, F_2)$ be operators between the Banach spaces E_i, F_i $(i=1,2)$. Here $L_{s,w}^{(e)}$ denotes the quasi-normed operator ideal consisting of the bounded linear operators with an $l_{s,w}$-summable sequence of entropy numbers for $0<s<\infty$, $0<w\le\infty$. The size of the sequence

$$(e_n(S\widehat{\otimes}_\alpha T)) \tag{0.1}$$

is studied in the scale of the Lorentz sequence spaces for tensor norms α. Upper and lower estimates for the parameters of this scale are obtained for the sequence (0.1) for operators between special Banach spaces. We determine in Section 1 the precise behaviour in the Lorentz scale under tensoring with respect to the Hilbert–Schmidt tensor product of Hilbert spaces. König [K1, Lemma 1] exhibited relative to this problem the first examples of the instability of the entropy number ideals under the projective tensor norm. In Section 3 some stability results are shown assuming cotype 2 conditions on the spaces involved. We also compute bounds in some cases for the instability in the Lorentz scale with the help of volume arguments. The corresponding "local" problem of evaluating the individual entropy numbers of $S\widehat{\otimes}_\alpha T$ in terms of the entropy numbers of S and T is subtler. We establish in Section 2 asymptotic bounds for the entropy numbers of tensored operators on the Schatten trace classes $c_p(l^2)$.

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1. Prerequisites and the Hilbert space case

The n-th (dyadic) entropy number of a bounded linear operator $S \in L(E,F)$ between the Banach spaces E and F is

$$e_n(S) = \inf\{\, \varepsilon > 0 : SB_E \subset \{x_1, \ldots, x_k\} + \varepsilon B_F, k \le 2^{n-1} \,\}, \quad n \in \mathbf{N},$$

where B_E is the closed unit ball of E. The n-th approximation number of S is

$$a_n(S) = \inf\{\, \|S - R\| : R \in L(E,F), \operatorname{rank}(R) < n \,\}.$$

The basic properties of these non-increasing sequences are contained in [P1]. It is standard to measure the degree of compactness of S by requiring that they belong to a Lorentz sequence space $l_{s,w} = \{\, x = (x_n) \in c_0 : \|x\|_{s,w} < \infty \,\}$ for $0 < s < \infty$, $0 < w \le \infty$. Here $\|x\|_{s,w} = (\sum_{n=1}^{\infty} n^{w/s-1} (x_n^*)^w)^{1/w}$ if $w < \infty$ while $\|x\|_{s,\infty} = \sup_{n \ge 1} n^{1/s} x_n^*$. The sequence (x_n^*) stands for the non-increasing positive rearrangement of (x_n). The customary notation l^s is also used instead of $l_{s,s}$. $\|\cdot\|_{s,w}$ is in general a quasi-norm on $l_{s,w}$. The entropy number ideals are thus

$$L_{s,w}^{(e)}(E,F) = \{\, S \in L(E,F) : \sigma_{s,w}^{(e)}(S) = \|(e_n(S))\|_{s,w} < \infty \,\},$$

while the approximation number ideals are

$$L_{s,w}^{(a)}(E,F) = \{\, S \in L(E,F) : \sigma_{s,w}^{(a)}(S) = \|(a_n(S))\|_{s,w} < \infty \,\}.$$

The sequence spaces $l_{s,w}$ (as well as also $L_{s,w}^{(e)}$ and $L_{s,w}^{(a)}$) are lexicographically ordered by inclusion (see [K2, p. 52]):

$$0 < s < t < \infty, 0 < u, v \le \infty \text{ imply that } l_{s,w} \subset l_{t,v} \text{ strictly,}$$
$$0 < s < \infty, 0 < w < u \le \infty \text{ imply that } l_{s,w} \subset l_{s,u} \text{ strictly.}$$

A tensor norm α is a norm defined on the algebraic tensor product $E \otimes F$ for all pairs (E,F) of Banach spaces that satisfies the additional properties

$$\alpha(x \otimes y) = \|x\|\,\|y\| \quad \text{for all } x \in E, y \in F, \tag{1.1}$$

$$\|S \otimes T : (E_1 \otimes E_2, \alpha) \to (F_1 \otimes F_2, \alpha)\| \le \|S\|\,\|T\| \tag{1.2}$$

for all operators $S \in L(E_1, F_1)$, $T \in L(E_2, F_2)$. Here $S \otimes T$ is defined by linear extension of $(S \otimes T) x \otimes y = Sx \otimes Ty$ for $x \in E_1$, $y \in E_2$ and (1.2) states that $S \otimes T$ induces a bounded linear operator $S \widehat{\otimes}_\alpha T : E_1 \widehat{\otimes}_\alpha E_2 \to F_1 \widehat{\otimes}_\alpha F_2$ between the completions.

The survey [DF] is a convenient reference for properties and examples of tensor norms. There is a large supply of tensor norms on account of the connection between finitely generated tensor norms and maximal normed operator ideals, cf. [DF, Chapter 4]. For instance, there is a family $\alpha_{p,q}$ of tensor norms associated with the ideals consisting of the (r,p,q)-integral operators. Let E and F be Banach spaces. The projective tensor norm π (which coincides with $\alpha_{1,1}$) is

$$\pi(z)=\inf\left\{\sum_{i=1}^{n}\|x_i\|\,\|y_i\|:z=\sum_{i=1}^{n}x_i\otimes y_i\in E\otimes F\right\}$$

and the injective tensor norm is

$$\varepsilon(z)=\sup\left\{\left|\sum_{i=1}^{n}x'(x_i)y'(y_i)\right|:(x',y')\in B_{E'}\times B_{F'}\right\}$$

for $z=\sum_{i=1}^{n}x_i\otimes y_i\in E\otimes F$. We also use $\|\cdot\|_\pi$ and $\|\cdot\|_\varepsilon$ instead of π and ε. It is known that $\varepsilon\le\alpha\le\pi$ for any tensor norm α.

If H and K are Hilbert spaces equipped with the respective inner-products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_K$, then $\alpha_{2,1}$ is the completion of $H\otimes K$ with respect to the inner-product obtained by the extension of

$$\langle x\otimes y,z\otimes w\rangle=\langle x,z\rangle_H\langle y,w\rangle_K\qquad\text{for } x\otimes y,z\otimes w\in H\otimes K.$$

The completion $H\widehat{\otimes}_{\mathrm{hs}}K$ is called the Hilbert–Schmidt tensor product of H and K.

This paper mainly studies the behaviour of the entropy number ideals $L^{(e)}_{s,w}$ under tensor norms α. More precisely, given $0<s<\infty$ and $0<w\le\infty$, find the minimal parameters (t,u) such that

$$S\widehat{\otimes}_\alpha T\in L^{(e)}_{t,u}(E_1\widehat{\otimes}_\alpha E_2,F_1\widehat{\otimes}_\alpha F_2)$$

for all $S\in L^{(e)}_{s,w}(E_1,F_1)$, $T\in L^{(e)}_{s,w}(E_2,F_2)$ and for the Banach spaces E_i, F_i $(i=1,2)$, usually in some restricted class of spaces. This is not always possible for all parameters of the Lorentz scale. For instance, the condition

$$S\widehat{\otimes}_\pi T\in L^{(e)}_{t,t}(l^1\widehat{\otimes}_\pi l^1,l^2\widehat{\otimes}_\pi l^2)$$

for all $S,T\in L^{(e)}_{s,s}(l^1,l^2)$ is impossible unless $1/t\le 1/s-\frac14$ [K1, Lemma 1]. In any case, one always has $t\ge s$ by $e_n(S\widehat{\otimes}_\alpha T)\ge\max\{\|T\|e_n(S),\|S\|e_n(T)\}$.

The tensor product notation is convenient in connection with the double-indexed product of the scalar-valued sequences $x=(x_n)$ and $y=(y_m)$, thus $x\otimes y=(x_ny_m)$ where $(n,m)\in\mathbf{N}^2$. The simplest possible case of our problem, the Hilbert–Schmidt tensor product of operators on l^2, reduces to an analytic problem of the Lorentz sequence spaces. Here a complete solution is available. We first state the results in terms of entropy ideals and outline the (essentially known) reduction. The resulting analytic problem is solved in Proposition 1.2.

Theorem 1.1. *Let* $0<s<\infty$, $0<w\le\infty$ *and* $S,T\in L^{(e)}_{s,w}(l^2)$.

(a) $S\widehat{\otimes}_{\mathrm{hs}}T\in L^{(e)}_{s,w}(l^2\widehat{\otimes}_{\mathrm{hs}}l^2)$ *for all* S, T *as above if and only if* $0<w\le s$.

(b) *If* $0<s<w\le 2s$, *then*

$$S\widehat{\otimes}_{\mathrm{hs}}T\in L^{(e)}_{s,u}(l^2\widehat{\otimes}_{\mathrm{hs}}l^2),$$

where u *satisfies* $1/u=2/w-1/s$ *if* $w<2s$ *and* $u=\infty$ *if* $w=2s$. *This inclusion is the best possible.*

(c) *If* $2s<w\le\infty$, *then* $L^{(e)}_{s,w}(l^2)\widehat{\otimes}_{\mathrm{hs}}L^{(e)}_{s,w}(l^2)\not\subset L^{(e)}_{s,\infty}(l^2\widehat{\otimes}_{\mathrm{hs}}l^2)$, *but* $L^{(e)}_{s,w}(l^2)\widehat{\otimes}_{\mathrm{hs}} L^{(e)}_{s,w}(l^2)\subset L^{(e)}_{v,u}(l^2\widehat{\otimes}_{\mathrm{hs}}l^2)$ *for all* $v>s$, $u>0$.

Proof. If $S\in L^{(e)}_{s,w}(l^2)$ then there are partially isometric operators $X_0,Y_0\colon l^2\to l^2$ according to the Schmidt representation theorem (see [P1, D.3.3]) with the properties that $S=Y_0D_sX_0^*$ and $D_s=Y_0^*SX_0$, where D_s is the diagonal operator on l^2 induced by the singular number sequence $s=(s_n(S))$ of S. Factorize $T\in L^{(e)}_{s,w}(l^2)$ similarly through D_t using partial isometries X_1 and Y_1 on l^2. After tensoring

$$e_n(S\widehat{\otimes}_{\mathrm{hs}}T)=e_n((Y_0\widehat{\otimes}_{\mathrm{hs}}Y_1)\circ(D_s\widehat{\otimes}_{\mathrm{hs}}D_t)\circ(X_0^*\widehat{\otimes}_{\mathrm{hs}}X_1^*))\le e_n(D_s\widehat{\otimes}_{\mathrm{hs}}D_t),$$

and conversely also $e_n(D_s\widehat{\otimes}_{\mathrm{hs}}D_t)\le e_n(S\widehat{\otimes}_{\mathrm{hs}}T)$. Hence it suffices to consider the diagonal operator $D_s\widehat{\otimes}_{\mathrm{hs}}D_t=D_{s\otimes t}$ on $l^2(\mathbf{N}^2)$ since $l^2\widehat{\otimes}_{\mathrm{hs}}l^2=l^2(\mathbf{N}^2)$ isometrically.

Recall the asymptotic formula due to Gordon, König and Schütt for the entropy numbers of diagonal operators on spaces with an unconditional basis. Let (e_n) be an orthonormal basis of l^2 and let D_σ be the diagonal operator $e_n\to\sigma_ne_n$, $n\in\mathbf{N}$, whenever $\sigma=(\sigma_n)$ is a positive non-increasing sequence. Then

$$e_{k+1}(D_\sigma)\approx\sup_{n\ge1}2^{-k/n}\Big(\prod_{j\le n}\sigma_j\Big)^{1/n},$$

for all $k\in\mathbf{N}$ [GKS, 1.7]. In particular, $D_\sigma\in L^{(e)}_{s,w}(l^2)$ if and only if $\sigma=(\sigma_n)\in l_{s,w}$ with equivalence of the corresponding quasi-norms. This is [GKS, 1.8] when $s=w$ and the general case follows in a similar fashion from a Hardy-type inequality for $l_{s,w}$:

If $0<w\le\infty$ and if $0<r<s$ then there is $d_{r,w}>0$ such that

$$\bigg\|\Big(\frac{1}{n}\sum_{k=1}^{n}\sigma_k^r\Big)_{n\in\mathbf{N}}\bigg\|_{s,w}\le d_{r,w}\|\sigma\|_{s,w}$$

for all non-increasing positive $(\sigma_n)\in l_{s,w}$ [P3, 2.1.7].

In particular, since also $e_n(S)=e_n(D_s)$ and $e_n(T)=e_n(D_t)$ for all $n\in\mathbf{N}$, one concludes that $S,T\in L^{(e)}_{u,w}(l^2)$ if and only if $s=(s_n(S))$ and $t=(s_n(T))$ belong to $l_{u,w}$

while $S\widehat{\otimes}_{\mathrm{hs}}T\in L^{(e)}_{v,x}(l^2\widehat{\otimes}_{\mathrm{hs}}l^2)$ if and only if $s\otimes t\in l_{v,x}(\mathbf{N}^2)$. The proof of Theorem 1.1 is completed by applying the following result concerning the size of the positive non-increasing rearrangement of tensor products of sequences.

We require some facts from bilinear interpolation. The standard reference for real and complex interpolation is [BL]. Let (X_0,X_1), (Y_0,Y_1) and (Z_0,Z_1) be compatible couples of quasi-Banach spaces such that Z_i is r_i-normed $(0<r_i\le 1)$ for $i=0,1$. Suppose that T defines a bounded bilinear operator $X_i\times Y_i\to Z_i$ for $i=0,1$. Let $0<\theta<1$, $0<q_1$, $q_2\le\infty$ and $1/r=(1-\theta)/r_0+\theta/r_1$. The real bilinear interpolation theorem due to Karadzov (cf. [K1, p. 89]) states that

$$T\colon (X_0,X_1)_{\theta,q_1}\times(Y_0,Y_1)_{\theta,q_2}\to(Z_0,Z_1)_{\theta,q}$$

is bounded, where $1/q=1/q_1+1/q_2-1/r$ if $\min\{q_1,q_2\}\ge r$ and $q=\max\{q_1,q_2\}$ if $\min\{q_1,q_2\}<r$. If the compatible couples consist of Banach spaces, then the complex bilinear interpolation property says that T is bounded from $(X_0,X_1)_\theta\times(Y_0,Y_1)_\theta$ to $(Z_0,Z_1)_\theta$ for any $\theta\in(0,1)$ [BL, 4.4.1]. Recall finally that the Lorentz sequence spaces form a real as well as a complex interpolation scale of quasi-normed spaces:

Suppose that $0<s_0<s_1<\infty$, $0<w_0$, $w_1\le\infty$ and that at least one of w_0, w_1 is finite. Then for any $\theta\in(0,1)$ and $0<w\le\infty$ there is up to equivalent (quasi-)norms

$$(l_{s_0},l_{s_1})_{\theta,w}=l_{s,w}, \tag{1.3}$$

$$(l_{s_0,w_0},l_{s_1,w_1})_\theta=l_{s,u}, \tag{1.4}$$

where $1/s=(1-\theta)/s_0+\theta/s_1$ and $1/u=(1-\theta)/w_0+\theta/w_1$. In the quasi-normed cases of (1.4) we consider the extension of complex interpolation explained in [CMS].

We next evaluate the size of the doubly-indexed products on the Lorentz sequence spaces $l_{s,w}$ in the unstable cases $0<s<w\le\infty$. The cases $0<w\le s<\infty$ were considered by Pietsch [P2]. The principle of uniform boundedness implies here that $l_{s,w}\otimes l_{s,w}\subset l_{t,u}(\mathbf{N}^2)$ if and only if $(x,y)\to x\otimes y$ is a bounded bilinear operator from $l_{s,w}\times l_{s,w}$ to $l_{t,u}(\mathbf{N}^2)$.

Proposition 1.2. *Let $0<s<\infty$ and $0<w\le\infty$.*

(a) *$l_{s,w}\otimes l_{s,w}\subset l_{s,w}(\mathbf{N}^2)$ if and only if $0<w\le s$.*

(b) *If $0<s<w<2s$, then $l_{s,w}\otimes l_{s,w}\subset l_{s,u}(\mathbf{N}^2)$, where $1/u=2/w-1/s$, while $l_{s,2s}\otimes l_{s,2s}\subset l_{s,\infty}(\mathbf{N}^2)$.*

These inclusions are optimal in the scale of Lorentz sequence spaces.

(c) *If $2s<w\le\infty$, then $l_{s,w}\otimes l_{s,w}\not\subset l_{s,\infty}(\mathbf{N}^2)$, but $l_{s,w}\otimes l_{s,w}\subset l_{v,u}(\mathbf{N}^2)$ for all $v>s$ and $u>0$.*

Proof. (a) is in [P2, pp. 34–35]. The proof of (b) is based on a careful application of real and complex bilinear interpolation.

The cases $0<s\leq 1$, $s<w\leq 2s$. Suppose that $0<s<1$ and choose $0<s_0<s<s_1<1$ as well as $\theta\in(0,1)$ satisfying $1/s=(1-\theta)/s_0+\theta/s_1$ (the case $s=1$ requires minor changes). Karadzov's real bilinear interpolation theorem implies that $\otimes$ is bounded from $(l_{s_0},l_{s_1})_{\theta,w}\times(l_{s_0},l_{s_1})_{\theta,w}$ to $(l_{s_0},l_{s_1})_{\theta,q}(\mathbf{N}^2)$, where $1/q=2/w-1/s$. This is admissible provided $0<w\leq 2s$. Thus (1.3) yields that $l_{s,w}\otimes l_{s,w}\subset l_{s,u}(\mathbf{N}^2)$, where $1/u=2/w-1/s$ when $s<w<2s$, and that $l_{s,2s}\otimes l_{s,2s}\subset l_{s,\infty}(\mathbf{N}^2)$ when $0<s\leq 1$.

The cases $1<s<w\leq 2s$. Observe to begin with that

$$l_{s,\infty}\otimes l_s\subset l_{s,\infty}(\mathbf{N}^2),\quad 0<s<\infty. \tag{1.5}$$

If $s=1$ then it suffices to verify that

$$\operatorname{card}\left\{(k,m):\xi_k\frac{1}{m}\geq\frac{1}{n}\right\}\leq n\quad\text{for all } n\in\mathbf{N},$$

whenever $(\xi_k)\in l^1$ is a positive non-increasing sequence with $\|(\xi_k)\|_1\leq 1$. Indeed, fix $n\in\mathbf{N}$ and let $N_m=\{k\in\mathbf{N}:\xi_k\geq m/n\}$ for $m\in\{1,\dots,n\}$. Then

$$\begin{aligned}1\geq\sum_k\xi_k&\geq\sum_{r=1}^{n}\operatorname{card}\left\{\xi_k:\frac{r+1}{n}>\xi_k\geq\frac{r}{n}\right\}\frac{r}{n}\\&=\frac{1}{n}\sum_{r=1}^{n-1}(N_r-N_{r+1})r=\frac{1}{n}\sum_{r=1}^{n-1}N_r=\frac{1}{n}\operatorname{card}\left\{(k,m):\xi_k\frac{1}{m}\geq\frac{1}{n}\right\}.\end{aligned}$$

The claim (1.5) for $0<s<\infty$ is obtained by considering (ξ_k^s).

We next claim that

$$l_{s,2s}\otimes l_{s,2s}\subset l_{s,\infty}(\mathbf{N}^2) \tag{1.6}$$

whenever $1<s<\infty$. In order to see this, take p_0, p_1 and $0<\theta<1$ satisfying $1<p_0<s<p_1<\infty$ and $1/2s=(1-\theta)/p_0=\theta/p_1$. Apply the complex bilinear interpolation result to the bounded map

$$\otimes: l_{p_0}\times l_{p_0,\infty}\to l_{p_0,\infty}(\mathbf{N}^2)\quad\text{and}\quad\otimes: l_{p_1,\infty}\times l_{p_1}\to l_{p_1,\infty}(\mathbf{N}^2)$$

obtained above in (1.5) and deduce from (1.4) and the choices of p_0, p_1 and θ that

$$\otimes: l_{s,2s}\times l_{s,2s}\to(l_{p_0,\infty},l_{p_1,\infty})_\theta(\mathbf{N}^2)$$

is bounded. Finally, the fact that $(l_{p_0,\infty},l_{p_1,\infty})_\theta\subset l_{s,\infty}$ (see the proof of [BL, 4.7.2]) establishes (1.6).

Suppose next that $1<s<w<2s$ and let $\theta=(2s-w)/s$, whence $1/s=\theta/w+(2(1-\theta))/w$. Then $\otimes$ is bounded from $l_{w/2,w}\times l_{w/2,w}$ to $l_{w/2,\infty}(\mathbf{N}^2)$ and from $l_w\times l_w$ to $l_w(\mathbf{N}^2)$ in view of (1.6). Complex bilinear interpolation implies the boundedness of

$$\otimes: l_{s,w}\times l_{s,w}\to (l_{w/2,\infty}, l_w)_\theta(\mathbf{N}^2)=l_{s,u}(\mathbf{N}^2),$$

where $1/u=\theta/w=2/w-1/s$. Note that $l_{w/2,w}$ and $l_{w/2,\infty}$ are quasi-normed spaces if $w\leq 2$. In these cases the complex bilinear interpolation property remains valid for the extension of complex interpolation considered in [CMS].

Optimality. Let $x^{(m)}=(x_k^{(m)})$, $m\in\mathbf{N}$, be the finite sequences

$$x_k^{(m)}=\begin{cases} 2^{-k/s}, & \text{if } 2^j\leq k<2^{j+1} \text{ for some natural number } j\leq m;\\ 0, & \text{otherwise.}\end{cases}$$

Pietsch [P2, p. 35] estimated that

$$\|x^{(m)}\|_{s,w}\approx m^{1/w}, \quad \|x^{(m)}\otimes x^{(m)}\|_{s,u}\geq c_0 m^{1/s+1/u}$$

with $c_0>0$ independent of m. The assumption $l_{s,w}\otimes l_{s,w}\subset l_{s,u}(\mathbf{N}^2)$ for some u with $w\leq u<\infty$ implies that there is a constant $c>0$ such that

$$\|x^{(m)}\otimes x^{(m)}\|_{s,u}\leq c\|x^{(m)}\|_{s,w}^2 \quad \text{for } m\in\mathbf{N}.$$

Hence $1/s+1/u\leq 2/w$.

(c) The sequences $x^{(m)}$ show as above that if $l_{s,w}\otimes l_{s,w}\subset l_{s,\infty}(\mathbf{N}^2)$, then there would be positive constants c and d such that

$$cm^{1/s}\leq \|x^{(m)}\otimes x^{(m)}\|_{s,\infty}\leq dm^{2/w} \quad \text{for } m\in\mathbf{N}.$$

This is impossible if $2s<w\leq\infty$. The general inclusions $l_{s,w}\otimes l_{s,w}\subset l_{t,u}(\mathbf{N}^2)$ for $t>s$ and $u>0$ are seen for instance from the proof of [K1, Proposition 1] for $w<\infty$ and from Proposition 3.1.a below for $w=\infty$.

2. Tensor norms on Hilbert spaces

The operator theoretic version of Sudakov's inequality for gaussian processes yields estimates for the entropy numbers of tensor products of operators between special tensor products, one of which is the Hilbert–Schmidt tensor product $l^2\widehat{\otimes}_{\mathrm{hs}}l^2$.

Let γ_n be the canonical gaussian probability measure on $\mathbf{R}^n$ with density function $d\gamma_n=\exp(-1/2(\sum_{i=1}^n x_i^2))\,dx_1\,...\,dx_n$ for $n\in\mathbf{N}$ and let E be a Banach space. The l-norm of the operator $u: l_2^n \to E$ is $l(u) = (\mathbf{E}\|ux\|^2\, d\gamma_n(x))^{1/2}$, while one defines $l(u) = \sup\{\, l(uv) : v \in L(l_2^n, l^2), \|v\| \le 1, n \in \mathbf{N}\,\}$ for $u \in L(l^2, E)$. The rotation-invariance of γ_n implies that

$$l(u) = \sup_{n\in\mathbf{N}} \left(\mathbf{E} \left\| \sum_{j=1}^{n} g_j u e_j \right\|^2 d\gamma_n \right)^{1/2} \tag{2.1}$$

for $u \in L(l^2, E)$, whenever (g_j) is a sequence of independent normal gaussian random variables on l_2^n and (e_j) is any orthonormal basis of l^2 (see [Pi2, p. 35]). The operator version of Sudakov's inequality (see [Kü, p. 54] or [Pi2, 5.5]) states that there is a constant c such that for all Banach spaces E and all $u \in L(l^2, E)$

$$\|(e_n(u'))\|_{2,\infty} = \sup_{n\in\mathbf{N}} n^{1/2} e_n(u') \le c\, l(u). \tag{2.2}$$

Let (r_j) be the sequence of Rademacher functions on $[0,1]$; $r_j(t) = \operatorname{sgn}\sin(2^j \pi t)$ for $t \in [0,1]$. Recall that the Banach space E is of type p for some p with $1 \le p \le 2$ if there is constant $c > 0$ such that $(\mathbf{E}\|\sum_{j=1}^n r_j(t) x_j\|^2\, dt)^{1/2} \le c(\sum_{j=1}^n \|x_j\|^p)^{1/p}$ for all $n \in \mathbf{N}$ and all $x_1, \dots, x_n$ in E. If $p > 1$ then there is also in this event $d > 0$ with

$$\left(\mathbf{E} \left\| \sum_{j=1}^{n} g_j x_j \right\|^2 dP \right)^{1/2} \le d \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}, \tag{2.3}$$

for all $n \in \mathbf{N}$ and all $x_1, \dots, x_n$ in E, whenever (g_j) is an independent sequence of normal gaussian random variables defined on a probability space (Ω, Σ, P), see [TJ3, 25.1].

If E_i and F_i are Banach spaces and if $S_i \in L(E_i, F_i)$ $(i = 1, 2)$, then the notation $S_1 {}_\alpha\widehat{\otimes}_\beta S_2$ is used for the extension of $S_1 \otimes S_2$ whenever it extends to a bounded operator from $E_1 \widehat{\otimes}_\alpha E_2$ to $F_1 \widehat{\otimes}_\beta F_2$ for given tensor norms α and β. The Schatten trace-class spaces are

$$c_p(l^2) = \{\, S \in L(l^2) : \|S\|_p = \|(s_n(S))\|_p < \infty \,\}$$

for $1 \le p < \infty$. The products $l^2 \widehat{\otimes}_{c_p} l^2$ are actually induced by the tensor norm associated with the maximal ideal consisting of the $(p, 2, 2)$-absolutely summing operators for $1 \le p < \infty$ [P1, 17.5.2]. This space equals $l^2 \widehat{\otimes}_\pi l^2$ for $p = 1$ and the Hilbert–Schmidt tensor product for $p = 2$. Suppose that p satisfies $2 < p < \infty$ and that S, T are compact operators on l^2. One obtains after a tensoring of the Schmidt decompositions of S and T that $S \otimes T$ extends to a bounded linear operator from $c_p(l^2)$ into $c_2(l^2)$ precisely when $D_s \otimes D_t$ extends similarly, where $s = (s_n(S))$ and $t = (s_n(T))$. The general Hölder inequality for the trace-class spaces [P3, 2.11.23] provides a sufficient condition for this;

$$\|(D_s \otimes D_t)a\|_2 = \|D_t \circ a \circ D_s\|_2 \le \|D_s\|_u \|D_t\|_v \|a\|_p$$

for all $a \in c_p(l^2)$ whenever u, v satisfy $1/u + 1/v + 1/p = \frac{1}{2}$.

Proposition 2.1. *Let $2<p<\infty$ and assume that $S,T\in L^{(e)}_{r,w}(l^2)$ with $0<r$, $w\le p'$. Then*

$$S_{c_p}\widehat{\otimes}_{\mathrm{hs}}T\in L^{(e)}_{t,y}(l^2\widehat{\otimes}_{c_p}l^2,l^2\widehat{\otimes}_{\mathrm{hs}}l^2),$$

whenever $S\otimes T$ admits a bounded extension, with $1/t=1/r-1/p-\frac{1}{2}$ and where $1/y=1/w-1/p'$ if $0<w\le r$ and $1/y=2/w-2/p'-1/r$ if $r<w\le 2(1/r+1/p')^{-1}$. The same statement applies to

$$S_{\mathrm{hs}}\widehat{\otimes}_{c_{p'}}T\colon l^2\widehat{\otimes}_{\mathrm{hs}}l^2\to l^2\widehat{\otimes}_{c_{p'}}l^2.$$

Proof. Schmidt decomposition of S and T as in the proof of Theorem 1.1 implies that it is enough to consider the diagonal operators D_s, D_t on l^2 induced by the singular value sequences $s=(s_n(S))$ and $t=(s_n(T))$. In order to apply Sudakov's inequality (2.2) we have to evaluate (according to (2.1))

$$l(D_{s\,\mathrm{hs}}\widehat{\otimes}_{c_{p'}}D_t)=\sup\left\{\left(E\left\|\sum_{(i,j)\in\Delta}g_{ij}s_ie_i\otimes t_je_j\right\|_{p'}^2\right)^{1/2}:\Delta\subset\mathbf{N}^2\text{ finite}\right\}.$$

Here (g_{ij}), $(i,j)\in\mathbf{N}^2$, is an independent sequence of normal gaussian random variables. One obtains from (2.3) that there is a constant $c>0$ with

$$\left(E\left\|\sum_{(i,j)\in\Delta}g_{ij}s_ie_i\otimes t_je_j\right\|_{p'}^2\right)^{1/2}\le c\left(\sum_{(i,j)\in\Delta}s_i^{p'}t_j^{p'}\right)^{1/p'}$$

for all finite $\Delta\subset\mathbf{N}^2$, since $c_{p'}(l^2)$ is of type p' by [TJ1, 3.1]. Thus (2.2) implies that $D_{s\,c_p}\widehat{\otimes}_{\mathrm{hs}}D_t$ belongs to $L^{(e)}_{2,\infty}$ whenever $s,t\in l^{p'}$.

The result extends to some other values of r and w with the help of a simple factorization trick based on the Hölder inequality. Let $0<r$, $w\le 2$. For any positive non-increasing sequence $s=(s_n)\in l_{r,w}$ there are positive sequences $s'=(s'_n)\in l^{p'}$ and $s''=(s''_n)\in l_{x,y}$ satisfying $s_n=s'_ns''_n$ for all $n\in\mathbf{N}$, $1/r=1/p'+1/x$ and $1/w=1/p'+1/y$. Let $t=t't''$ be a similar factorization of $t=(t_n)\in l_{r,w}$. In order to apply Theorem 1.1 and the preceding $l^{p'}$-case to the factorization

$$D_{s\,c_p}\widehat{\otimes}_{\mathrm{hs}}D_t=(D_{s''}\widehat{\otimes}_{\mathrm{hs}}D_{t''})\circ(D_{s'\,c_p}\widehat{\otimes}_{\mathrm{hs}}D_{t'})\tag{2.4}$$

one distinguishes between the possibilities $0<w\le r$ or $r<w\le 2(1/r+1/p')^{-1}$. If $0<w\le r$ then $0<y\le x$ and Theorem 1.1.a yields, after reordering with unitary operators if necessary, that

$$D_{s\,c_p}\widehat{\otimes}_{\mathrm{hs}}D_t\in L^{(e)}_{x,y}\circ L^{(e)}_{2,\infty}\subset L^{(e)}_{t,y},$$

where $1/t=1/r-1/p-\frac{1}{2}$ and $1/y=1/w-\frac{1}{2}$. The above inclusion follows from the multiplicativity property of the entropy numbers and [P1, 2.1.13]. On the other hand, in the case $r<w\le 2(1/r+1/p')^{-1}$ one clearly has $0<x<y\le 2x$ and hence, in view of (2.4) and Theorem 1.1.b, that

$$D_{s\,c_p}\widehat{\otimes}_{\text{hs}}D_t\in L_{x,u}^{(e)}\circ L_{2,\infty}^{(e)}\subset L_{t,u}^{(e)},$$

where t is as above and with $1/u=2/y-1/r=2/w-2/p'-1/r$. This completes the argument for $S_{c_p}\widehat{\otimes}_{\text{hs}}T$.

The statement concerning the matrix operators $S_{\text{hs}}\widehat{\otimes}_{c_{p'}}T$ is seen from the duality properties of the entropy numbers of operators with values in a Hilbert space [TJ2].

We mention an example in the direction of [C1].

Example 2.2. Let $s=(s_k)$, $t=(t_k)$ be positive non-increasing sequences. If $s,t\in l^r$, $0<r<\infty$, then

$$D_{s\,\pi}\widehat{\otimes}_{\text{hs}}D_t\in L_{u,r}^{(e)}(l^1\widehat{\otimes}_\pi l^1,l^2\widehat{\otimes}_{\text{hs}}l^2),$$

where $1/u=1/r+\frac{1}{2}$. Moreover, there are sequences $s\in c_0$ such that $D_{s\,\pi}\widehat{\otimes}_{\text{hs}}D_s\notin L_{2,1}^{(e)}(l^1\widehat{\otimes}_\pi l^1,l^2\widehat{\otimes}_{\text{hs}}l^2)$.

Proof. There is an isometric identification $l^1\widehat{\otimes}_\pi l^1=l^1(\mathbf{N}^2)$ and $D_{s\,\pi}\widehat{\otimes}_{\text{hs}}D_t$ is identified with the diagonal operator $D_{s\otimes t}\colon l^1(\mathbf{N}^2)\to l^2(\mathbf{N}^2)$ taken with respect to the natural symmetric basis $(e_i\otimes e_j)$. Let $0<w<\infty$ and $1/u=1/r+\frac{1}{2}$. Then $D_{s\otimes t}\in L_{u,w}^{(e)}(l^1(\mathbf{N}^2),l^2(\mathbf{N}^2))$ if and only if $s\otimes t\in l_{r,w}$ according to [C1, 3.1 and 3.2]. In particular, $D_{s\,\pi}\widehat{\otimes}_{\text{hs}}D_t\in L_{u,r}^{(e)}$ whenever $s,t\in l^r$.

For the second assertion consider $s=(s_i)\in c_0$, $s_i=1/\log(k+2)$ for $2^k\le i<2^{k+1}$, $k\in\mathbf{N}$. According to (2.1) and the estimate from below in Chevet's inequality [Ch, 3.1] it follows that

$$\begin{aligned}
l(D_{s\,\text{hs}}\widehat{\otimes}_\varepsilon D_s\colon l^2\widehat{\otimes}_{\text{hs}}l^2\to c_0\widehat{\otimes}_\varepsilon c_0)&\ge\sup_{n\in\mathbf{N}}\mathbf{E}\Big\|\sum_{i=1}^n\sum_{j=1}^n g_{ij}s_is_je_i\otimes e_j\Big\|_\varepsilon\\
&\ge\sup_{n\in\mathbf{N}}\sup\Big\{\Big(\sum_{i=1}^n|x'(s_ie_i)|^2\Big)^{1/2}:x'\in B_{l^1}\Big\}\mathbf{E}\Big\|\sum_{j=1}^n g_js_je_j\Big\|_{c_0}\\
&=s_1\sup_{n\in\mathbf{N}}\mathbf{E}\max_{1\le i\le n}s_i|g_i|,
\end{aligned}$$

where (g_{ij}) and (g_i) are independent normal gaussian random variables defined on some probability space (Ω,Σ,P). It follows (for instance) from Sudakov's inequality for gaussian processes (see [Pi2, 5.6]) that there is a constant $c>0$ such that

$$\mathbf{E}\max_{1\le i\le 2^{n+1}}s_i|g_i|\ge c\frac{n^{1/2}}{\log(n+2)},\quad n\in\mathbf{N},$$

since $\{ s_i g_i : 1 \le i \le 2^{n+1}-1 \}$ forms an orthogonal set in $L^2(\Omega, P)$ by independence and since $\|s_i g_i - s_j g_j\|_{L^2(\Omega)} = (s_i^2 + s_j^2)^{1/2}$ whenever $i \neq j$. Hence $l(D_{s\,\mathrm{hs}} \widehat{\otimes}_\varepsilon D_s)$ fails to be finite and thus Dudley's inequality [Pi2, 5.5],

$$l(u) \le c_1 \sum_n n^{-1/2} e_n(u')$$

($u: l^2 \to E$, E any Banach space) implies that $(D_{s\,\mathrm{hs}} \widehat{\otimes}_\varepsilon D_s)' = D_{s\,\pi} \widehat{\otimes}_{\mathrm{hs}} D_s \notin L_{2,1}^{(e)}$.

If (e_i) is the standard unit basis of l^2, then $\{ (e_i \otimes e_j) : (i,j) \in \mathbf{N}^2 \}$ constitutes a Schauder basis for $l^2 \widehat{\otimes}_{c_p} l^2$ in the usual box order. Let $D_s, D_t : l^2 \to l^2$ be the diagonal operators corresponding to the positive non-increasing sequences $s = (s_n)$ and $t = (t_n)$. In this case $D_s \widehat{\otimes}_{c_p} D_t$ is the diagonal operator $e_i \otimes e_j \to s_i t_j e_i \otimes e_j$, $(i,j) \in \mathbf{N}^2$, on $l^2 \widehat{\otimes}_{c_p} l^2$. However, $c_p(l^2)$ fails to have an unconditional basis whenever $p \neq 2$, (cf. [Pi1, 8.20]) and thus the formula due to Gordon, König and Schütt [GKS, 1.7] for the entropy numbers does not apply as such to this concrete situation.

We next establish asymptotic bounds for the single entropy numbers of $D_s \widehat{\otimes}_{c_p} D_t$. It is crucial that there are uniformly bounded sequences of finite-dimensional projections on $c_p(l^2)$ associated with the level sets of the non-increasing rearrangement of $s \otimes t$. For this purpose ideas of Kwapien and Pelczynski [KP] are required. Suppose that $s = (s_n)$ and $t = (t_n)$ are positive non-increasing 0-sequences. Set

$$\Delta_r(s,t) = \{ (i,j) \in \mathbf{N}^2 : s_i t_j \ge 1/r \} \quad \text{and} \quad M_r(s,t) = [e_i \otimes e_j : (i,j) \in \Delta_r(s,t)]$$

and let $Q_r(s,t)$ be the natural finite-dimensional projection

$$\sum_i \sum_j a_{i,j} e_i \otimes e_j \to \sum_{(i,j) \in \Delta_r(s,t)} a_{i,j} e_i \otimes e_j$$

from $c_p(l^2)$ onto $M_r(s,t)$ for any $r \in \mathbf{N}$. The matrix notation $a = \sum_i \sum_j a_{i,j} e_i \otimes e_j$ is used for $a \in c_p(l^2)$, with the summation in the box order, i.e. as $\sum_{n=1}^{\infty} \sum_{i \vee j = n} a_{i,j} e_i \otimes e_j$ where $i \vee j = \max\{i, j\}$. We will often suppress (s,t) in the interest of brevity and thus write Δ_r, M_r and Q_r.

A result due to Macaev states that the main triangle projections T_n,

$$T_n(e_i \otimes e_j) = \begin{cases} e_i \otimes e_j, & \text{if } i+j \le n+1, \\ 0, & \text{otherwise,} \end{cases}$$

$n \in \mathbf{N}$, are uniformly bounded on $c_p(l^2)$ when $1 < p < \infty$,

$$d_p = \sup_{n \in \mathbf{N}} \|T_n : c_p(l^2) \to c_p(l^2)\| < \infty, \tag{2.5}$$

cf. [GK, III.6.2].

The following lemma of a technical nature concerning the norms of irregular triangular projections on $c_p(l^2)$ has independent interest.

Lemma 2.3. *Suppose that p satisfies $1<p<\infty$. Then*

$$a_p = \sup\{\, \|Q_r(s,t)\colon c_p(l^2)\to c_p(l^2)\| : s \text{ and } t \text{ non-increasing, positive 0-sequences, } r\in\mathbf{N}\,\} < \infty. \tag{2.6}$$

Proof. If $p=2$ then evidently $a_2=1$, since $c_2(l^2)$ is isometric to $l^2(\mathbf{N}^2)$. Suppose that $r\in\mathbf{N}$ and take $n\in\mathbf{N}$ with $\max\{s_1t_n, s_nt_1\}<1/r$. Let $P_{k,m}$ stand for the contractive box-projections on $c_p(l^2)$ sending $\sum_i\sum_j a_{i,j}e_i\otimes e_j$ to $\sum_{i\le k}\sum_{j\le m} a_{i,j}e_i\otimes e_j$ when $k,m\in\mathbf{N}$. It suffices to find uniform bounds on the $k\times k$-matrices since $P_{k,k}a\to a$ in $c_p(l^2)$ for all a as $k\to\infty$. There is also no loss of generality in assuming that n is large enough in order that

$$\Delta_r(s,t)\subset D_n=\{\,(i,j)\in\mathbf{N}^2 : i+j\le n+1\,\}.$$

We indicate how the uniform boundedness of the projections $Q_r(s,t)$ is reduced with the help of uniformly bounded operations on $c_p(l^2)$ to the unconditionality of the Schauder decomposition $(P_{k+1,k+1}-P_{k,k})_{k\in\mathbf{N}}$ of $c_p(l^2)$ for $1<p<\infty$, which was established in [KP, p. 67]. It is instructive to visualize the different steps on finite matrices.

The sets $\Delta_r=\Delta_r(s,t)$ obviously enjoy the following "convexity" property: if $(i,j)\notin\Delta_r$, then $(k,l)\notin\Delta_r$ whenever $k\ge i$ and $l\ge j$. Let U_α be the isometry $U_\alpha(\sum_{i\le n} a_ie_i)=\sum_{i\le n} a_ie_{\alpha(i)}$ on l_2^n whenever α is a permutation of $\{1,\dots,n\}$. Set $\pi(k)=n+1-k$ on $\{1,\dots,n\}$, whence $\pi^{-1}=\pi$. The tensor property implies that

$$\begin{aligned}
\Big\|\sum_{(i,j)\in\Delta_r} a_{i,j}e_i\otimes e_j\Big\| &= \Big\|(\mathrm{id}\otimes U_\pi)\sum_{(i,j)\in\Delta_r} a_{i,j}e_i\otimes e_{\pi(j)}\Big\|\\
&\le \Big\|\sum_{(i,j)\in\Delta_r} a_{i,j}e_i\otimes e_{\pi(j)}\Big\|\\
&\le \Big\|\sum_{(i,j)\in\Delta_r(+)} a_{i,j}e_i\otimes e_{\pi(j)}\Big\| + \Big\|\sum_{(i,j)\in\Delta_r(-)} a_{i,j}e_i\otimes e_{\pi(j)}\Big\|.
\end{aligned}\tag{2.7}$$

Here (as well as in the proof of 2.4 below) we delete for simplicity the subscript in the norm $\|\cdot\|_p$ of $c_p(l^2)$. Above $\Delta_r(+)=\{\,(i,j)\in\Delta_r(s,t):(i,\pi(j))\in D_n\,\}$ and $\Delta_r(-)=\{\,(i,j)\in\Delta_r(s,t):i+\pi(j)>n+1\,\}$. We proceed to estimate the first term of (2.7). Put $\widehat{\Delta}_r(+)=(\mathrm{id}\times\pi)\Delta_r(+)$. The "convexity"property of $\Delta_r(s,t)\subset D_n$ implies that there are finite sequences (r_k) and (s_k) of integers satisfying:

$$\begin{aligned}
\widehat{\Delta}_r(+)=\{\,(i,j)\in\{1,\dots,n\}^2 : {}&r_k\le j\le r_{k+1}-1,\\
&s_k\le i\le n+1-j \text{ for } k=1,\dots,m\,\},
\end{aligned}$$

where $1<r_1<\ldots<r_m\le[n/2]+1$, $1\le s_1<\ldots<s_m\le n$ for some $m=m(s,t)\le[n/2]+1$, $r_k-s_k\ge 0$, $r_{k+1}-1-s_k\ge 0$ for all k and with

$$\begin{aligned}(2.8)\qquad n+1-(r_k-s_k)&>n+1-(r_{k+1}-1-s_k)\\&>n+1-(r_{k+1}-s_{k+1})\quad\text{for } k=1,\ldots,m-1.\end{aligned}$$

There exists a pair (σ,μ) of permutations of $\{1,\ldots,n\}$ with the following properties:

(2.9) μ maps the disjoint subsets $\{r_k,r_k+1,\ldots,r_{k+1}-1\}$ increasingly onto the dis joint sets (by (2.8)) $\{r_k-s_k,r_k+1-s_k,\ldots,r_{k+1}-1-s_k\}$ for $k=1,\ldots,m-1$ and

(2.10) σ maps the disjoint subsets $\{n+1-(r_{k+1}-1),\ldots,n+1-r_k\}$ increasingly onto the disjoint sets (by(2.8)) $\{n+1-(r_{k+1}-1-s_k),\ldots,n+1-(r_k-s_k)\}$ for $k=1,\ldots,m-1$.

The conditions (2.9) and (2.10) state intuitively that the pair (σ,μ) permutes any "block" of the form

$$\{\,(i,j):r_k\le j\le r_{k+1}-1,\ s_k\le i\le n+1-j\,\}$$

of $\widehat{\Delta}_r(+)$ in D_n onto the corresponding block of equal size containing $\{(1,r_k-s_k),\ldots,(1,r_{k+1}-1-s_k)\}$.

This entails that

$$\begin{aligned}\Big\|\sum_{(i,j)\in\Delta_r(+)}a_{i,j}e_i\otimes e_{\pi(j)}\Big\|&=\Big\|(U_{\sigma^{-1}}\otimes U_{\mu^{-1}})\sum_{(i,j)\in\Delta_r(+)}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\\&\le\Big\|\sum_{(i,j)\in\Delta_r(+)}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\\&=\Big\|T_n(R_{r_{k+1}-1-s_k,r_{k+1}-1-s_k}-R_{r_k-s_k,r_k-s_k})\sum_{i\le n}\sum_{j\le n}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\\&\le d_pK_p\Big\|\sum_{i\le n}\sum_{j\le n}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\le d_pK_p\Big\|\sum_{i\le n}\sum_{j\le n}a_{i,j}e_i\otimes e_j\Big\|,\end{aligned}$$

where $R_{r,r}=(\mathrm{id}\otimes U_\pi)P_{n+1-r,n+1-r}$. The above inequalities follow from (2.5), the tensor property and the unconditionality of the Schauder decomposition $(P_{k+1,k+1}-P_{k,k})_{k\in\mathbf{N}}$ for $c_p(l^2)$ [KP, p. 67]. K_p is the associated unconditional constant.

The second term $\|\sum_{(i,j)\in\Delta_r(-)}a_{i,j}e_i\otimes e_{\pi(j)}\|$ of (2.7) admits a similar bound. This completes the proof of the lemma.

Let $s=(s_n)$ and $t=(t_n)$ be positive non-increasing 0-squences. We denote

$$m_x=m_x(s,t)=\min\{\,r\in\mathbf{N}:\max\{s_rt_1,s_1t_r\}<x\,\}$$

for $x>0$ and put

$$b(n)=b(s,t)(n)=\sup_{r\in\mathbf{N}}\Bigl(2^{-n}\prod_{(i,j)\in\Delta_r(s,t)}s_it_j\Bigr)^{1/\#\Delta_r(s,t)}$$

for $n\in\mathbf{N}$. The function $x\to m_x(s,t)$ is clearly decreasing. Recall that the n-th non-dyadic entropy number of $S\in L(E,F)$ is

$$\varepsilon_n(S)=\inf\{\,\varepsilon>0:SB_E\subset\{x_1,\dots,x_n\}+\varepsilon B_F,x_1,\dots,x_n\in F\,\}.$$

Evidently $e_n(S)=\varepsilon_{2^{n-1}}(S)$.

Theorem 2.4. *Suppose that p satisfies $1<p<\infty$, $p\neq 2$. Then*

$$\begin{aligned}(2.11)\qquad \frac{1}{a_p}b(n)\le e_n(D_s\widehat{\otimes}_{c_p}D_t)&\le[3+2a_p+2\log(2m_{b(n-1)(1+2\log(2m_{b(n-1)}))})]\\&\quad\times b(n-1)(1+2\log(2m_{b(n-1)}))\end{aligned}$$

for all $n\ge 2$ and for all positive non-increasing sequences s and t (the logarithm is to the base 2). In particular,

$$e_{n+1}(D_s\widehat{\otimes}_{c_p}D_t)\le b_pb(n)\bigl(\log(m_{b(n)})\bigr)^2$$

for some uniform constants $b_p<\infty$.

Proof. A standard volume argument, which is indicated for completeness, yields the lower bound. Indeed, fix $r\in\mathbf{N}$ and consider the restriction $(D_s\otimes D_t)^{(r)}=Q_r(D_s\widehat{\otimes}_{c_p}D_t)_{|M_r}:M_r\to M_r$, for which

$$e_n((D_s\otimes D_t)^{(r)})\le\|Q_r(s,t)\|e_n(D_s\widehat{\otimes}_{c_p}D_t)\le a_pe_n(D_s\widehat{\otimes}_{c_p}D_t)$$

according to (2.6). Suppose that $\lambda>e_n((D_s\otimes D_t)^{(r)})$ and that

$$(D_s\otimes D_t)^{(r)}B_{M_r}\subset\{a_1,\dots,a_{2^n}\}+\lambda B_{M_r}$$

for some $a_1,\dots,a_{2^n}\in M_r$. The evaluation of the $\#\Delta_r(s,t)$-dimensional volume with respect to Lebesgue product-measure entails that

$$\begin{aligned}\mathrm{vol}((D_s\otimes D_t)^{(r)}B_{M_r})&=|\det((D_s\otimes D_t)^{(r)})|\,\mathrm{vol}(B_{M_r})\\&=\Bigl(\prod_{(i,j)\in\Delta_r}s_it_j\Bigr)\mathrm{vol}(B_{M_r})\le 2^n\lambda^{\#\Delta_r}\,\mathrm{vol}(B_{M_r}).\end{aligned}$$

Thus

$$\lambda \geq \left(2^{-n} \prod_{(i,j)\in\Delta_r} s_i t_j\right)^{1/\#\Delta_r}.$$

The supremum over r gives the left-hand inequality of (2.11).

We proceed to establish the right-hand inequality. It is assumed that $s_i>0$ and $t_i>0$ for all $i\in\mathbf{N}$ since the argument simplifies if s or t are finite sequences. Let $0<x<1$. We want to determine the optimal choice of x by a volume argument as in [GKS], but considerable complications arise due to the lack of unconditionality in $c_p(l^2)$. There is $r\in\mathbf{N}$ such that $1/(r+1)\leq x<1/r$. Let $\{a_1,...,a_N\}$ be a maximal set of elements of $(D_s\otimes D_t)^{(r)}B_{M_r}$ with the property that

$$\|a_i-a_j\|>2x \quad \text{for } i\neq j.$$

Consequently

$$(D_s\otimes D_t)^{(r)}B_{M_r}\subset\{a_1,...,a_N\}+2xB_{M_r}.$$

One has

$$\varepsilon_N(D_s\widehat{\otimes}_{c_p}D_t)\leq\varepsilon_N(Q_r(D_s\widehat{\otimes}_{c_p}D_t))+\|(\mathrm{id}-Q_r)D_s\widehat{\otimes}_{c_p}D_t\|. \tag{2.12}$$

The right-hand terms of (2.12) are dealt with as follows. Observe first that

$$Q_r(D_s\widehat{\otimes}_{c_p}D_t)B_{c_p(l^2)}=Q_r(D_s\widehat{\otimes}_{c_p}D_t)Q_rB_{c_p(l^2)}\subset a_p(D_s\otimes D_t)^{(r)}B_{M_r}$$

because of (2.6). Hence $\varepsilon_N(Q_r(D_s\widehat{\otimes}_{c_p}D_t))\leq 2a_px$.

The second term of (2.12) splits into 4 parts. Let $a=\sum_i\sum_j a_{i,j}e_i\otimes e_j\in c_p(l^2)$ be an operator with finite matrix. Note that $\Delta_r=\Delta_r(s,t)\subset\{1,...,m_x\}^2$ by the choice of $m_x=m_x(s,t)$ and the mononicity of s and t. Let $\Delta_r'=\{1,...,m_x\}^2-\Delta_r$. Write

$$\begin{aligned}(\mathrm{id}-Q_r)(D_s\widehat{\otimes}_{c_p}D_t)a&=\sum_{(i,j)\notin\Delta_r}s_it_ja_{i,j}e_i\otimes e_j\\ &=\sum_{i\geq m_x+1}\sum_{j\geq m_x+1}s_it_ja_{i,j}e_i\otimes e_j+\sum_I s_it_ja_{i,j}e_i\otimes e_j\\ &\quad+\sum_{II}s_it_ja_{i,j}e_i\otimes e_j+\sum_{(i,j)\in\Delta_r'}s_it_ja_{i,j}e_i\otimes e_j.\end{aligned} \tag{2.13}$$

The sum in I extends over $(i,j)\in\mathbf{N}^2$ satisfying $1\leq i\leq m_x$ and $j\geq m_x+1$, while the summation in II is over (i,j) with $i\geq m_x+1$ and $1\leq j\leq m_x$. The tensor property and the monotonicity of s and t imply that

$$\left\|\sum_{i\geq m_x+1}\sum_{j\geq m_x+1}s_it_ja_{i,j}e_i\otimes e_j\right\|\leq\max\{\,s_it_j:i,j\geq m_x+1\,\}\|a\|\leq x\|a\|,$$

while also

$$\Big\|\sum_I s_i t_j a_{i,j} e_i \otimes e_j\Big\| \leq s_1 t_{m_x+1}\|a\| \leq x\|a\|$$

by the definition of m_x. Similarly $\|\sum_{II} s_i t_j a_{i,j} e_i \otimes e_j\| \leq x\|a\|$. The preceding inequalities hold for all $a \in c_p(l^2)$ in view of the density of the finite operators.

We require the combinatorial result formulated below in Lemma 2.5 in order to estimate the remaining term of (2.13). Recall that a (finite) chain C in $\mathbf{N}^2$ has the form

$$C = \bigcup_{j \leq r(c)} A_j \times B_j,$$

for some $r(C) \in \mathbf{N}$, where $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. The disjointness of the supports of the corresponding operators leads to

$$\begin{aligned} \Big\|\sum_{(i,j)\in C} s_i t_j a_{i,j} e_i \otimes e_j\Big\| &\leq \max_{r \leq r(C)} \Big\|\sum_{(i,j)\in A_r \times B_r} s_i t_j a_{i,j} e_i \otimes e_j\Big\| \\ &\leq \max\{\, s_i t_j : (i,j) \in C \,\}\|a\| \end{aligned} \tag{2.14}$$

for $a = \sum_i \sum_j a_{i,j} e_i \otimes e_j$. The first inequality is seen from [K1, p. 87–88], while the second one follows from

$$\Big\|\sum_{(i,k)\in A_j \times B_j} s_i t_k a_{i,k} e_i \otimes e_k\Big\| \leq \max\{\, s_i t_k : (i,k) \in A_j \times B_j \,\}\|a\|,$$

which is an immediate consequence of the tensor property.

According to the combinatorial result of Lemma 2.5 below one may partition Δ_r' as $\bigcup_{m \leq k(r)} C_m$ into chains (C_m) with $k(r) \leq \log(2m_x)$. Consequently

$$\begin{aligned} \Big\|\sum_{(i,j)\in \Delta_r'} s_i t_j a_{i,j} e_i \otimes e_j\Big\| &\leq \sum_{m \leq k(r)} \Big\|\sum_{(i,j)\in C_m} s_i t_j a_{i,j} e_i \otimes e_j\Big\| \\ &\leq \log(2m_x) \max\{\, s_i t_j : (i,j) \in \Delta_r' \,\}\|a\| \\ &\leq \log(2m_x) 2x\|a\| \end{aligned} \tag{2.15}$$

by (2.14). A combination of (2.13) and (2.15) leads to

$$\varepsilon_N(D_s \widehat{\otimes}_{c_p} D_t) \leq (3 + 2a_p + 2\log(2m_x))x. \tag{2.16}$$

Next we estimate N. The sets $\{a_i + xB_{M_r}\}$, $i = 1, ..., N$, are disjoint in M_r according to the choice of $\{a_1, ..., a_N\}$ in $(D_s \otimes D_t)^{(r)} B_{M_r}$. Moreover,

$$\begin{aligned} \{a_1, ..., a_N\} + xB_{M_r} &\subset (1 + x\|(D_{s^{-1}} \otimes D_{t^{-1}})^{(r)}\|)(D_s \otimes D_t)^{(r)} B_{M_r} \\ &\subset (1 + 2\log(2m_x))(D_s \otimes D_t)^{(r)} B_{M_r}. \end{aligned} \tag{2.17}$$

Here $(D_{s^{-1}}\otimes D_{t^{-1}})^{(r)}=((D_s\otimes D_t)^{(r)})^{-1}$ stands for the diagonal operator on M_r that maps $e_i\otimes e_j$ to $(s_it_j)^{-1}e_i\otimes e_j$. We have used in (2.17) the estimate

$$\|(D_{s^{-1}}\otimes D_{t^{-1}})^{(r)}\|\le 2\log(2m_x)\frac{1}{x}. \tag{2.18}$$

This inequality is verified as follows. We have $\Delta_r\subset D_{m_x}$, where D_{m_x} partitions into a union of at most $\log(2m_x)$ chains (C_m) according to [KP, p. 46]. This enables us to argue as in the proof of Lemma 2.3. Let $a=\sum_{(i,j)\in\Delta_r} a_{i,j}e_i\otimes e_j\in M_r$. One obtains as in (2.7) that

$$\begin{aligned}\|(D_{s^{-1}}\otimes D_{t^{-1}})^{(r)}a\| \le &\Big\|\sum_{(i,j)\in\Delta_r(+)}(s_it_j)^{-1}a_{i,j}e_i\otimes e_{\pi(j)}\Big\|\\ &+\Big\|\sum_{(i,j)\in\Delta_r(-)}(s_it_j)^{-1}a_{i,j}e_i\otimes e_{\pi(j)}\Big\|,\end{aligned}$$

where the notations π, $\Delta_r(+)$ and $\Delta_r(-)$ are those of the proof of Lemma 2.3. An application of the pair (σ,μ) of permutations satisfying (2.9) and (2.10) entails that

$$\begin{aligned}\Big\|\sum_{(i,j)\in\Delta_r(+)}(s_it_j)^{-1}a_{i,j}e_i\otimes e_{\pi(j)}\Big\| &\le \Big\|\sum_{(i,j)\in\Delta_r(+)}(s_it_j)^{-1}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\\ &\le \Big\|\sum_{(\pi(i),j)\in(\sigma\times(\mu\circ\pi))\Delta_r(+)}(s_it_j)^{-1}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\\ &= \Big\|\sum_{m=1}^{\log(2m_x)}\sum_{C_m\cap(\sigma\times(\mu\circ\pi))\Delta_r(+)}(s_it_j)^{-1}a_{i,j}e_{\sigma(i)}\otimes e_{\mu(\pi(j))}\Big\|\\ &\le \log(2m_x)\max\{\,(s_it_j)^{-1}:(i,j)\in\Delta_r\,\}\Big\|\sum_{(i,j)\in\Delta_r}a_{i,j}e_i\otimes e_j\Big\|\\ &\le \frac{1}{x}\log(2m_x)\|a\|.\end{aligned}$$

In the above inequalities we have used (2.14) together with the fact that the intersections $C_m\cap(\sigma\times(\mu\circ\pi))\Delta_r(+)$ are also chains. The second term

$$\Big\|\sum_{(i,j)\in\Delta_r(-)}(s_it_j)^{-1}a_{i,j}e_i\otimes e_{\pi(j)}\Big\|$$

admits an analogous bound and thus (2.18) holds.

The application of $\#\Delta_r(s,t)$-dimensional product measure to (2.17) implies then that

$$Nx^{\#\Delta_r} \leq (1+2\log(2m_x))^{\#\Delta_r}\Big(\prod_{(i,j)\in\Delta_r} s_i t_j\Big)$$

and thus

$$N \leq \left(\frac{1+2\log(2m_x)}{x}\right)^{\#\Delta_r} \prod_{(i,j)\in\Delta_r} s_i t_j.$$

Then $N\leq 2^n$ at least if x satisfies

$$\Big(2^{-n}\prod_{(i,j)\in\Delta_r} s_i t_j\Big)^{1/\#\Delta_r} \leq b(n) \leq \frac{x}{1+2\log(2m_x)}.$$

The latter inequality is equivalent to the condition

$$x - b(n)2\log(2m_x) - b(n) \geq 0,$$

which is satisfied (at least) if $x=b(n)(1+2\log(2m_{b(n)}))$. In fact, then the condition reduces to

$$\log(2m_{b(n)}) - \log(2m_{b(n)(1+2\log(2m_{b(n)}))}) \geq 0,$$

and this holds since $x\to m_x$ is non-increasing.

The insertion of $x=b(n)(1+2\log(2m_{(b(n)}))$ into (2.16) produces the upper bound of (2.11) for $e_{n+1}(D_s\widehat{\otimes}_{c_p}D_t)=\varepsilon_{2^n}(D_s\widehat{\otimes}_{c_p}D_t)\leq\varepsilon_N(D_s\widehat{\otimes}_{c_p}D_t)$. The argument is thus completed by the combinatorial Lemma 2.5 below.

Finally, the simpler bound

$$e_{n+1}(D_s\widehat{\otimes}_{c_p}D_t) \leq b_p b(n)\big(\log(m_{b(n)})\big)^2$$

results from the monotonicity of $x\to\log(m_x)$.

Lemma 2.5. *Suppose that s and t are non-increasing positive 0-sequences, $m\in\mathbf{N}$ and let $r\in\mathbf{N}$ be such that $\Delta_r(s,t)\subset\{1,...,m\}^2$. Then it is possible to partition $\{1,...,m\}^2-\Delta_r(s,t)$ into at most $\log(2m)$ chains.*

Proof. We verify a general statement which only relies on the "convexity" of the sets $\Delta_r(s,t)$. Suppose that $m\in\mathbf{N}$ and that $\Delta\subset\{1,...,m\}^2$ satisfies the property

(2.19) if $(i,j)\in\{1,...,m\}^2-\Delta$, then $(k,n)\notin\Delta$ whenever $(k,n)\in\{1,...,m\}^2$, $k\geq i$ and $n\geq j$.

Claim. $\{1,...,m\}^2-\Delta$ partitions into at most $\log(2m)$ chains.

Let $f(m)$ be the smallest natural number so that $\{1,\dots,m\}^2-\Delta$ partitions into at most $f(m)$ chains for any $\Delta\subset\{1,\dots,m\}^2$ for which (2.19) holds. It suffices to verify that f admits the growth

$$f(m)\le 1+f\left(\left[\frac{m}{2}\right]\right) \tag{2.20}$$

for natural numbers $m\ge 2$, where $[x]$ denotes the entire part of x. Indeed, since $f(1)=1=\log(2)$ (logarithm to the base 2), one gets from (2.20) that $f(k)\le\log(2k)$ for all $k\in\mathbf{N}$.

We indicate an argument for (2.20), that also provides a procedure for obtaining a partition (not necessarily the most efficient one for a given set Δ). Suppose that $\Delta\subset\{1,\dots,m\}^2$ satisfies (2.19) for some $m\ge 2$. Pick the largest possible square contained in $\{1,\dots,m\}^2-\Delta$ with opposite corners (m,m) and (r,r). Let $C=\{r,\dots,m\}\times\{r,\dots,m\}$ be the first chain. Thus $\{1,\dots,m\}^2-\{\Delta\cup C\}=A_1\cup A_2$, where

$$A_1=(\{1,\dots,m\}^2-\Delta)\cap\{r,\dots,m\}\times\{1,\dots,r-1\},$$
$$A_2=(\{1,\dots,m\}^2-\Delta)\cap\{1,\dots,r-1\}\times\{r,\dots,m\}.$$

To continue, it suffices to partition A_1 and A_2 separately into chains, since these sets have disjoint projections in $\{1,\dots,m\}$ and thus their respective chains can be joined. We discuss the case of A_1. Observe that the length of the smaller side of the rectangle $\{r,\dots,m\}\times\{1,\dots,r-1\}$ satisfies $\min\{m-r+1,r-1\}\le[m/2]$, since otherwise $m=(m-r+1)+(r-1)\ge 2[m/2]+2\ge m+1$. Moreover, note that $\Delta_1=\Delta\cap(\{r,\dots,m\}\times\{1,\dots,r-1\})$ satisfies (2.19) in this rectangle. Hence A_1 partitions into at most $f([m/2])$ chains. In fact, by "shrinking" the sets involved if necessary, one observes that partitioning A_1 is at worst as difficult as that of partitioning inside corresponding squares having sidelength the smaller of the sides of the rectangle, that is at most $[m/2]$. Finally, repeat this for A_2 to get (2.20).

Remarks 2.6. We do not know if the upper bound of (2.11) is sharp. We stress that the sequence $(b(s,t)(n))_{n\in\mathbf{N}}$ has according to Theorem 1.1 the same behaviour in the Lorentz scale $l_{r,w}$ as the sequence $s\otimes t$, which was determined in Proposition 1.2 (see also Proposition 3.1.a below for the rate of decrease in the case $w=\infty$). In fact, the sequence $(b(s,t)(n))$ is clearly obtained from the asymptotic entropy formula [GKS, 1.7] for the diagonal operator $D_s\widehat{\otimes}_{\mathrm{hs}}D_t$ on $l^2\widehat{\otimes}_{\mathrm{hs}}l^2=l^2(\mathbf{N}^2)$, if the orthonormal basis $(e_n\otimes e_m)$ is reordered to correspond to the rearrangement of the sequence $s\otimes t$.

The argument of 2.4 breaks down for $p=1$ (or $p=\infty$), since already

$$\|T_n\colon c_1(l^2)\to c_1(l^2)\|\ge c\log n$$

by [KP, 1.2].

3. General estimates

The results of Sections 1 and 2 are based on particular geometric properties of Banach spaces not available in arbitrary tensor products. In this section we first state some general consequences of the stability under tensoring of the related approximation number ideals. Moreover, volume comparisons yield instability estimates. Better results are available for Banach spaces endowed with special structure.

The behaviour of the approximation number ideals $L_{r,w}^{(a)}$ under tensor products was studied in [P2], [K1]. These ideals are almost tensor-stable in the sense that for all tensor norms α and all Banach spaces one has $S\widehat{\otimes}_\alpha T\in L_{t,u}^{(a)}$ for all $t>r$ and all $u>0$ whenever $S,T\in L_{r,w}^{(a)}$. We formulate below a more precise statement of tensor stability up to logarithmic weights.

Let $f,g\colon(0,\infty)\times(0,\infty)\to\mathbf{R}_+$ be the functions

$$f(r,w)=\begin{cases}\dfrac{w}{r}, & \text{if } 0<w\le 1,\\ \dfrac{w}{r}-w-1, & \text{if } w\ge 1,\end{cases}$$

$$g(r,w)=\begin{cases}2\Big(\dfrac{w}{r}-1\Big), & \text{if } 0<r<w<\infty,\\ 0, & \text{otherwise.}\end{cases}$$

Put $M_n(S)=\#\{i\in\mathbf{N}:2^{-(n+1)/r}\|S\|<s_i(S)\le 2^{-n/r}\|S\|\}$ when $n\in\mathbf{N}$ and $S\in L_{r,w}^{(e)}(l^2)$, and set $f_n=f_n(S,T)=\sum_{k+m\le n}M_k(S)M_m(T)$ (with $f_0=0$) for $S,T\in L_{r,w}^{(e)}(l^2)$. Thus f_n also depends on r.

Proposition 3.1. (a) *Let $0<r<\infty$ and $0<w\le\infty$. There are $c_{r,w}>0$ such that for all Banach spaces E_i, F_i, $(i=1,2)$, all tensor norms α and all operators $S\in L_{r,w}^{(a)}(E_1,F_1)$, $T\in L_{r,w}^{(a)}(E_2,F_2)$ one has*

$$\Bigg(\sum_{n=1}^{\infty}\frac{n^{w/r-1}a_n(S\widehat{\otimes}_\alpha T)^w}{(\log(n+1))^{f(r,w)}}\Bigg)^{1/w}\le c_{r,w}\sigma_{r,w}^{(a)}(S)\sigma_{r,w}^{(a)}(T) \tag{3.1}$$

for $0<w<\infty$ and

$$\sup_{n\in\mathbf{N}}\frac{n^{1/r}}{(\log(n+1))^{1+1/r}}a_n(S\widehat{\otimes}_\alpha T)\le c_{r,\infty}\sigma_{r,\infty}^{(a)}(S)\sigma_{r,\infty}^{(a)}(T).$$

(b) *If $0<r$, $w<\infty$ then there are $d_{r,w}>0$ such that for all tensor norms α on $l^2\otimes l^2$ and all $S,T\in L_{r,w}^{(a)}(l^2)$,*

$$\Bigg(\sum_{n=1}^{\infty}\frac{1}{n^{g(r,w)}}\sum_{j=f_n+1}^{f_{n+1}}j^{w/r-1}a_j(S\widehat{\otimes}_\alpha T)^w\Bigg)^{1/w}\le d_{r,w}\sigma_{r,w}^{(a)}(S)\sigma_{r,w}^{(a)}(T). \tag{3.2}$$

Proof. The statements in (a) and (b) are straight-forward computational extensions to arbitrary values $0<r$, $w<\infty$ of [K1, Propositions 1 and 3], which is referred to for the arguments. We indicate the proof in the case $w=\infty$ not considered by König.

The numerical constants $c_0, c_1, c_2, \dots$ depend only on r in the following estimates. Suppose that $S\in L^{(a)}_{r,\infty}(E_1,F_1)$. There is according to [P3, 2.3.8] a norm convergent expansion $S=\sum_{k=0}^{\infty} S_k$ in $L(E_1,F_1)$ satisfying $rk(S_k)\le 2^k$ and

$$\sup_{k\in\mathbf{N}} 2^{k/r}\|S_k\|\le c_0\sigma^{(a)}_{r,\infty}(S).$$

Decompose $T\in L^{(a)}_{r,\infty}(E_2,F_2)$ similarly in $L(E_2,F_2)$ as $T=\sum_{k=0}^{\infty} T_k$. Thus

$$S\widehat{\otimes}_\alpha T=\sum_{n=0}^{\infty}\sum_{k+l=n} S_k\widehat{\otimes}_\alpha T_l$$

with convergence in the operator norm. Set $h(n)=n2^n$ for $n\in\mathbf{N}$. Observe that

$$rk\left(\sum_{n=0}^{m-1}\sum_{k+l=n} S_k\widehat{\otimes}_\alpha T_l\right)\le\sum_{n=0}^{m-1}(n+1)2^n<m2^m=h(m)$$

for $m\in\mathbf{N}$. The properties of the decompositions lead to

$$\begin{aligned} a_{h(m)}(S\widehat{\otimes}_\alpha T)&\le\left\|\sum_{n=m}^{\infty}\sum_{k+l=n} S_k\widehat{\otimes}_\alpha T_l\right\|\le\sum_{n=m}^{\infty}\sum_{k+l=n}\|S_k\|\,\|T_l\|\\ &\le c_0^2\sigma^{(a)}_{r,\infty}(S)\sigma^{(a)}_{r,\infty}(T)\sum_{n=m}^{\infty}(n+1)(2^{-1/r})^n. \end{aligned}$$

An elementary calculation shows for $x=2^{-1/r}$ that

$$\sum_{n=m}^{\infty}(n+1)x^n=\frac{x^{m-1}}{(1-x)^2}(m(1-x-(1-x)^2)+1).$$

Consequently monotonicity together with the previous estimates entail that

$$\begin{aligned} &\sup_{n\in\mathbf{N}}\frac{n^{1/r}}{(\log(n+1))^{1+1/r}}a_n(S\widehat{\otimes}_\alpha T)\\ &\qquad=\sup_{m\in\mathbf{N}}\ \sup_{h(m)\le k\le h(m+1)-1}\frac{k^{1/r}}{(\log(k+1))^{1+1/r}}a_k(S\widehat{\otimes}_\alpha T)\\ &\qquad\le\sup_{m\in\mathbf{N}}\frac{(h(m+1))^{1/r}}{(\log(h(m)+1))^{1+1/r}}a_{h(m)}(S\widehat{\otimes}_\alpha T)\\ &\qquad\le c_1\sigma^{(a)}_{r,\infty}(S)\sigma^{(a)}_{r,\infty}(T)\sup_{m\in\mathbf{N}}\frac{(h(m+1))^{1/r}}{(\log(h(m)+1))^{1+1/r}}m(2^{-1/r})^{m-1}\\ &\qquad\le c_2\sigma^{(a)}_{r,\infty}(S)\sigma^{(a)}_{r,\infty}(T). \end{aligned}$$

This establishes claim (a) for $w=\infty$.

Remarks 3.2. (a) $L^{(e)}_{r,w}$ is stable on $l^2\widehat{\otimes}_\alpha l^2$ for all tensor norms α whenever $0<w\le r\le\infty$ in view of (3.2). It is also evident that one cannot achieve better than the result for $l^2\widehat{\otimes}_{\mathrm{hs}}l^2$ in the cases $0<r<w$. In fact, $|\lambda_n(S)|\le 2e_{n+1}(S)$ for all $S\in L(E)$, E a (complex) Banach space, and for all $n\in\mathbf{N}$ by the Carl–Triebel inequality [CS, 4.2.1]. Here $(\lambda_n(S))$ is the sequence of eigenvalues of S ordered in decreasing magnitude and counting multiplicities. Thus one obtains at least the behaviour of $s\otimes t$ since it is contained in the sequence of eigenvalues of $D_s\widehat{\otimes}_\alpha D_t$. Clearly a similar statement also holds for tensor norms on spaces with unconditional bases.

(b) The weighted inequalities (3.1) and (3.2) contain no general information on the change under tensoring of the logarithmic parameter w in the Lorentz scale $l_{r,w}$ when $0<w<\infty$. Indeed, let $0<r$, $t<\infty$ and consider the quasi-normed weighted Lorentz sequence spaces

$$l_{r,t}(\omega)=\left\{x=(x_n)\in l^\infty : \|x\|=\left(\sum_{n=1}^{\infty}\omega_n n^{t/r-1}(x_n^*)^t\right)^{1/t}<\infty\right\}.$$

Our weights $\omega=(\omega_n(r,t))$ are $\omega_n=1/(\log(n+1))^\gamma$ with $\gamma>0$. Then the identity mapping from $l_{r,t}(\omega)$ to $l_{r,\infty}$ fails to be bounded (compare the quasi-norms of the sequence $(z^{(j)})$, $j\in\mathbf{N}$, where $z^{(j)}_k=1$ if $1\le k\le 2^{j+1}$ and 0 elsewhere).

The almost stability of the approximation number ideals is relevant under special geometric assumptions. Recall that the n-th Gelfand number of $S\in L(E,F)$ is

$$c_n(S)=\inf\{\,\|SJ_M\|: M\subset E,\operatorname{codim} M<n\,\},\quad n\in\mathbf{N},$$

and that the corresponding ideal components $L^{(c)}_{r,w}(E,F)$ consist of the operators S with $(c_n(S))\in l_{r,w}$. The Banach space E is said to be of cotype q for $2\le q<\infty$ if there is $c>0$ with $(\sum_{j=1}^n\|x_j\|^q)^{1/q}\le c(\mathbf{E}\|\sum_{j=1}^n r_j(t)x_j\|^2\,dt)^{1/2}$ for all $n\in\mathbf{N}$ and all $x_1,\dots,x_n$ in E. Here (r_j) is the sequence of Rademacher functions.

Theorem 3.3. *Assume that E_i and F_i $(i=1,2)$ are Banach spaces such that E_i is of type 2, F_i is of cotype 2 and that F_i does not contain l_1^n's uniformly. Let r and w satisfy $0<r<\infty$, $0<w\le\infty$ as well as $1/w\ge 1/r+1$. If $S_i\in L^{(e)}_{r,w}(E_i,F_i)$ $(i=1,2)$ then*

$$S_1\widehat{\otimes}_\alpha S_2\in L^{(e)}_{r,w}(E_1\widehat{\otimes}_\alpha E_2,F_1\widehat{\otimes}_\alpha F_2)$$

for any tensor norm α. Moreover, there are constants $c_{r,w}>0$ with

$$\sigma^{(e)}_{r,w}(S_1\widehat{\otimes}_\alpha S_2)\le c_{r,w}\sigma^{(e)}_{r,w}(S_1)\sigma^{(e)}_{r,w}(S_2)$$

for all $S_i \in L^{(e)}_{r,w}(E_i, F_i)$.

Proof. It is a consequence of [C2, Theorem 5] that

$$\Big(\prod_{j\leq n} c_j(S)\Big)^{1/n} \leq c(E_i,F_i)e_n(S), \quad n\in\mathbf{N},$$

for some constants $c(E_i,F_i)>0$ and for all $S\in L(E_i,F_i)$ $(i=1,2)$, since E_i and F_i' are of type 2. The fact that F_i' is of type 2 for $i=1,2$ follows from duality results, because F_i does not contain l_1^n's uniformly (see [TJ3, 12.8]). Thus $L^{(e)}_{r,w}(E_i,F_i)\subset L^{(c)}_{r,w}(E_i,F_i)$ for all r and w in view of [P3, 2.1.8]. On the other hand, $L^{(c)}_{r,w}(E,F)\subset L^{(e)}_{r,w}(E,F)$ for arbitrary Banach spaces E and F by [CS, 3.1]. Thus $L^{(e)}_{r,w}(E_i,F_i)=L^{(c)}_{r,w}(E_i,F_i)$ $(i=1,2)$ with comparable quasi-norms.

Recall next that the Gelfand and the approximation numbers of $S\in L(E_i,F_i)$ are comparable under these assumptions on E_i and F_i. In fact,

$$c_n(S)\leq a_n(S)\leq c\,c_n(S)$$

for some constant c and for all $n\in\mathbf{N}$ by Maurey's extension theorem, see [GKS, 1.4]. This entails in particular that here $L^{(e)}_{r,w}(E_i,F_i)=L^{(a)}_{r,w}(E_i,F_i)$ $(i=1,2)$ with comparable quasi-norms for all r and w.

The duality of type and cotype yields further that E_i' $(i=1,2)$ is of cotype 2 [TJ3, 12.8]. Suppose that $S_i\in L^{(e)}_{r,w}(E_i,F_i)$ $(i=1,2)$. In this case

$$S_1\widehat{\otimes}_\alpha S_2\in L^{(a)}_{r,w}(E_1\widehat{\otimes}_\alpha E_2, F_1\widehat{\otimes}_\alpha F_2)$$

for all tensor norms α whenever r and w satisfy $1/w\geq 1/r+1$ on the strength of [K1, Theorem 1]. Moreover, there is $d_{r,w}>0$ with

$$\sigma^{(a)}_{r,w}(S_1\widehat{\otimes}_\alpha S_2)\leq d_{r,w}\sigma^{(a)}_{r,w}(S_1)\sigma^{(a)}_{r,w}(S_2)$$

for all S_1 and S_2. This entails the claim since $L^{(a)}_{r,w}\subset L^{(e)}_{r,w}$ in general, and $\sigma^{(e)}_{r,w}(S)\leq b_{r,w}\sigma^{(a)}_{r,w}(S)$ for some $b_{r,w}>0$ and for all $S\in L^{(a)}_{r,w}$ [CS, 3.1].

A standard procedure associated with essentially finite-dimensional properties is to bound parameters by comparing suitable quantities. Volume estimates are related to entropy numbers and they are used to find instability in the Lorentz scale in some cases (cf. [K1, Lemma 1]). A systematic application of this idea requires precise bounds on the volumes of the unit balls of finite-dimensional tensor products. We commence by phrasing a principle of this kind.

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Recall that the Schauder basis (e_n) of the Banach space E is 1-symmetric if

$$\left\|\sum_{n=1}^{\infty} \varepsilon_n a_{\pi(n)} e_n\right\| = \left\|\sum_{n=1}^{\infty} a_n e_n\right\|$$

for all signs $\varepsilon_n = \pm 1$, all permutations π of $\mathbf{N}$ and all $\sum_{n=1}^{\infty} a_n e_n \in E$. Volumes $\operatorname{vol}(B)$ will always be taken with respect to n-dimensional Lebesgue product measure when B is a bounded subset of an n-dimensional (real) normed space. The notation $a_n \approx b_n$ for positive sequences means that (a_n) and (b_n) are uniformly comparable, that is, $c_0 b_n \le a_n \le c_1 b_n$ for all $n \in \mathbf{N}$ with constants $c_0, c_1 > 0$.

Proposition 3.4. *Let α be any tensor norm. Suppose that (e_n) and (f_n) are 1-symmetric bases of some Banach spaces and put $E_n = [e_1, ..., e_n]$ and $F_n = [f_1, ..., f_n]$. Assume moreover that there is $\beta \in [-1, 1]$ satisfying*

$$(\operatorname{vol}(B_{E_n}))^{1/n} \approx n^{\beta} (\operatorname{vol}(B_{F_n}))^{1/n}$$

and that both E_n embed into E and F_n embed into F uniformly complementedly. Then the condition

$$S \widehat{\otimes}_{\alpha} T \in L_{t,u}^{(e)}(E \widehat{\otimes}_{\alpha} E, F \widehat{\otimes}_{\alpha} F)$$

for all $S, T \in L_{r,w}^{(e)}(E, F)$, where $1/r > \max\{0, -\beta\}$, implies that there is a constant $c > 0$ satisfying

$$n^{2/t} \left(\frac{\operatorname{vol}(B_{E_n \widehat{\otimes}_{\alpha} E_n})}{\operatorname{vol}(B_{F_n \widehat{\otimes}_{\alpha} F_n})} \right)^{1/n^2} \le c n^{2/r + 2\beta} \tag{3.3}$$

for all $n \in \mathbf{N}$.

Proof. Let $P_n: E \to E_n$ be quotient maps and let $J_n: E_n \to E$ be embeddings such that $P_n J_n = \mathrm{id}_{E_n}$, $\sup_n \|P_n\| < \infty$ and $\sup_n \|J_n\| < \infty$. Let $Q_n: F \to F_n$ and $K_n: F_n \to F$ be operators similarly related to the uniformly complemented copies of F_n in F. Consider $S_n = K_n I_n P_n \in L(E, F)$, $n \in \mathbf{N}$, where $I_n: E_n \to F_n$ is the natural identity $\sum_{i=1}^n a_i e_i \to \sum_{i=1}^n a_i f_i$.

Note first that the condition

$$S \widehat{\otimes}_{\alpha} T \in L_{t,u}^{(e)}(E \widehat{\otimes}_{\alpha} E, F \widehat{\otimes}_{\alpha} F) \quad \text{for all } S, T \in L_{r,w}^{(e)}(E, F)$$

implies the existence of $c > 0$ such that

$$\sigma_{t,u}^{(e)}(S \widehat{\otimes}_{\alpha} T) \le c \sigma_{r,w}^{(e)}(S) \sigma_{r,w}^{(e)}(T) \tag{3.4}$$

for all S, T. In fact, by passing to equivalent p-norms and by employing a similar completeness argument to that of [P1, 6.1.6] it is verified that the bilinear operator $(S,T)\to S\widehat{\otimes}_\alpha T$ is separately continuous from $L^{(e)}_{r,w}\times L^{(e)}_{r,w}$ to $L^{(e)}_{t,u}$. This in turn entails the boundedness of the operator by a general version of the Banach–Steinhaus principle, see [R, 2.17].

The inequality (3.4) is tested by the sequence (S_n). Observe that $I_n=Q_nS_nJ_n$ for $n\in\mathbf{N}$. The uniform bounds on the norms of P_n, J_n, Q_n and K_n yield, after tensoring the factorizations of S_n and I_n, that there are constants $c', c''>0$ with

$$\sigma^{(e)}_{t,u}(I_n\widehat{\otimes}_\alpha I_n)\le c'\sigma^{(e)}_{t,u}(S_n\widehat{\otimes}_\alpha S_n)\le cc'(\sigma^{(e)}_{r,w}(S_n))^2\le c''(\sigma^{(e)}_{r,w}(I_n))^2 \tag{3.5}$$

for all $n\in\mathbf{N}$.

Suppose that

$$(I_n\widehat{\otimes}_\alpha I_n)B_{E_n\widehat{\otimes}_\alpha E_n}\subset\{a_1,\dots,a_{2^r}\}+\lambda B_{F_n\widehat{\otimes}_\alpha F_n}$$

for $r\in\mathbf{N}$. Comparing n^2-dimensional volumes we get

$$\operatorname{vol}(B_{E_n\widehat{\otimes}_\alpha E_n})\le 2^r\lambda^{n^2}\operatorname{vol}(B_{F_n\widehat{\otimes}_\alpha F_n})$$

and thus

$$\lambda\ge 2^{-r/n^2}\left(\frac{\operatorname{vol}(B_{E_n\widehat{\otimes}_\alpha E_n})}{\operatorname{vol}(B_{F_n\widehat{\otimes}_\alpha F_n})}\right)^{1/n^2}.$$

This lower bound for $e_r(I_n\widehat{\otimes}_\alpha I_n)$ leads to

$$\begin{aligned}\sigma^{(e)}_{t,u}(I_n\widehat{\otimes}_\alpha I_n)&\ge\left(\sum_{j\le n^2} j^{u/t-1}e_j(I_n\widehat{\otimes}_\alpha I_n)^u\right)^{1/u}\\&\ge c_0\left(\frac{\operatorname{vol}(B_{E_n\widehat{\otimes}_\alpha E_n})}{\operatorname{vol}(B_{F_n\widehat{\otimes}_\alpha F_n})}\right)^{1/n^2}\left(\sum_{j\le n^2} j^{u/t-1}\right)^{1/u}.\end{aligned}$$

It is easily checked that $(\sum_{j\le n^2} j^{u/t-1})^{1/u}\approx n^{2/t}$. On the other hand, $\sigma^{(e)}_{r,w}(I_n)\le c_1n^{1/r+\beta}$ whenever r satisfies $1/r>\max\{0,-\beta\}$ in view of [S2, Theorem 7]. The desired inequality (3.3) thus follows by combining (3.5) with the preceding estimates.

We apply (3.3) with $E_n=l^n_p$ and $F_n=l^n_q$, $1\le p<q\le\infty$. It is known that $(\operatorname{vol}(B_{l^n_p}))^{1/n}\approx n^{-1/p}$ for all $1\le p\le\infty$, cf. [S1, p. 395], and thus $\beta=1/q-1/p$ satisfies the volume condition of Proposition 3.4. Let

$$h(p)=\begin{cases}-\dfrac{1}{2}+\dfrac{2}{p}, & \text{if } 1\le p\le 2,\\[2mm] \dfrac{1}{p}, & \text{if } 2\le p\le\infty.\end{cases}$$

Corollary 3.5. *Assume that E, F are Banach spaces containing uniformly complemented copies of l_p^n, respectively of l_q^n, with $1 \le p < q \le \infty$. Suppose that*

$$S \widehat{\otimes}_\alpha T \in L_{t,u}^{(e)}(E \widehat{\otimes}_\alpha E, F \widehat{\otimes}_\alpha F)$$

for all $S, T \in L_{r,w}^{(e)}(E,F)$ where $0 < r < (1/p+1/q)^{-1}$. Then $t > r$ in the following cases:

(i) *if $\alpha = \pi$, then $1/t \le 1/r + \frac{1}{2}(1/q - 1/p)$ for $1 \le p < q \le 2$, while $1/t \le 1/r + \frac{1}{2}(\frac{1}{2} - 1/p)$ for $1 \le p < 2 \le q \le \infty$;*

(ii) *if $\alpha = \varepsilon$, then $1/t \le 1/r + \frac{1}{2}(1/q - \frac{1}{2})$ for $1 \le p \le 2 < q \le \infty$, while $1/t \le 1/r + \frac{1}{2}(1/q - 1/p)$ for $2 \le p < q \le \infty$.*

Proof. Schütt [S1, 3.2] showed that $(\mathrm{vol}(B_{l_p^n \widehat{\otimes}_\pi l_p^n}))^{1/n^2} \approx n^{h(p')-1}$ and that $(\mathrm{vol}(B_{l_p^n \widehat{\otimes}_\varepsilon l_p^n}))^{1/n^2} \approx n^{-h(p')}$ for all p satisfying $1 \le p \le \infty$. For instance, if $1 \le p < q \le 2$ and if $\alpha = \pi$, then by (3.3) there is $c > 0$ with

$$n^{2/t + 1/q - 1/p} \le c n^{2/r + 2(1/q - 1/p)}$$

for all $n \in \mathbf{N}$. Thus $1/t \le 1/r + \frac{1}{2}(1/q - 1/p)$ where $1/q - 1/p < 0$. The other cases are similar.

We finally use the argument of Proposition 3.4 in order to derive some bounds related to Propostion 2.1.

Example 3.6. Suppose that $2 < p < \infty$, $0 < t < \infty$, $0 < r < 4p/(p-2)$ and that $0 < u \le \infty$. If

$$S_{c_p} \widehat{\otimes}_{\mathrm{hs}} T \in L_{t,u}^{(e)}(l^2 \widehat{\otimes}_{c_p} l^2, l^2 \widehat{\otimes}_{\mathrm{hs}} l^2)$$

for all $S, T \in L_r^{(e)}(l^2)$, then $1/t \le 1/r + \frac{1}{2}(\frac{1}{2} - 1/p)$ and thus $t > r$.

Proof. Observe first that if $0 < r < 4p/(p-2)$, then $S \otimes T$ extends to a bounded linear operator from $c_p(l^2)$ into $c_2(l^2)$ for all $S, T \in L_r^{(e)}(l^2)$ in view of the remark prior to Proposition 2.1 and [CS, 1.3.2]. Thus an argument similar to the one in the proof of Proposition 3.4 provides a constant $c > 0$ with

$$\sigma_{t,u}^{(e)}(S_{c_p} \widehat{\otimes}_{\mathrm{hs}} T) \le c \sigma_r^{(e)}(S) \sigma_r^{(e)}(T)$$

whenever $S, T \in L_r^{(e)}(l^2)$. Let $I_n = \mathrm{id}_{l_2^n}$ be the identity map. One obtains that there is $c_0 > 0$ with

$$n^{2/t} \left(\frac{\mathrm{vol}(B_{c_p(l_2^n)})}{\mathrm{vol}(B_{l_2^{n^2}})} \right)^{1/n^2} \le c_0 n^{2/r}$$

for all $n \in \mathbf{N}$ since $\sigma_{r,w}^{(e)}(I_n) \approx n^{1/r}$. The desired inequality follows from the estimate $(\mathrm{vol}(B_{c_p(l_2^n)}))^{1/n^2} \approx n^{-1/2-1/p}$ for $2 \le p \le \infty$, see [S1, p. 399].

In particular, one obtains $0 < r \le 4p/(3p-2)$ if $t = 2$. Unfortunately we do not know whether the bounds exhibited in Proposition 2.1 are precise.

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