ON WARING'S PROBLEM FOR CUBES.

By

H. DAVENPORT

of MANCHESTER.

Introduction.

The object of this paper is to give a proof of the following

Theorem. *Almost all positive integers are representable as the sum of four positive integral cubes. More precisely, if $E(N)$ denotes the number of positive integers less than N that are not so representable, then*

$$E(N) = O\left(N^{1-\frac{1}{30}+\varepsilon}\right)$$

as $N \to \infty$, for any $\varepsilon > 0$.

It was proved by Hardy and Littlewood[1] that almost all positive integers are representable as the sum of five positive integral cubes. The new weapon which is necessary in order to improve on this is provided by Lemma 1 below.

It is not true that almost all positive integers are sums of three positive integral cubes. This can be seen in two different ways. Firstly, since any cube is congruent to 0, 1, or $-1 \pmod 9$, the sum of three cubes cannot be congruent to 4 or 5 (mod 9). Secondly, the number of integral solutions of

$$x^3 + y^3 + z^3 \leqq N,$$

$$x \geqq y \geqq z > 0,$$

is easily found to be asymptotically

$$\frac{1}{6}\left(\Gamma\left(\frac{4}{3}\right)\right)^3 N$$

[1] Partitio Numerorum VI, *Math. Zeitschrift*, 23 (1925), 1—37.

as $N \to \infty$. Hence not more than $0 \cdot 134 \ldots N$ positive integers less than N are representable as the sum of three positive integral cubes.

Lemma 1. *The number of solutions of*

$$(1) \qquad x_1^3 + y_1^3 + z_1^3 = x_2^3 + y_2^3 + z_2^3$$

in integers $x_1, y_1, z_1, x_2, y_2, z_2$ *subject to*

$$(2) \qquad P \leqq x_1, x_2 \leqq 2P, \quad P^{\frac{4}{5}} \leqq y_1, z_1, y_2, z_2 \leqq 2P^{\frac{4}{5}}$$

is $O\left(P^{\frac{13}{5}+\varepsilon}\right)$ *as* $P \to \infty$, *for any* $\varepsilon > 0$.

Proof.[1] The number of solutions with $x_1 = x_2$ is $O\left(P^{\frac{13}{5}+\varepsilon}\right)$, since the number of choices for x_1, y_1, z_1 is $O\left(P^{\frac{13}{5}}\right)$, and the equation $m = y_2^3 + z_2^3$ has only $O(m^\varepsilon)$ solutions for given m.

Hence we consider only solutions with $x_1 > x_2$. Writing $x_2 = x$, $x_1 = x + t$, (1) becomes

$$(3) \qquad 3tx^2 + 3t^2x + t^3 + y_1^3 + z_1^3 = y_2^3 + z_2^3.$$

Since the left-hand side is greater than $3P^2t$, and the right-hand side is at most $16P^{\frac{12}{5}}$, we have

$$(4) \qquad 0 < t < 6P^{\frac{2}{5}}.$$

For any t satisfying (4), denote by $r(t, m)$ the number of representations of an integer m by the left-hand side of (3), subject to (2) (where $x_2 = x$, $x_1 = x + t$). Denote by $r(m)$ the number of representations of m by the right-hand side of (3), subject to (2). The number of solutions of (3) is

$$\sum_t \sum_m r(m)\, r(t, m) \leqq \left\{\sum_t \sum_m r^2(m)\right\}^{\frac{1}{2}} \left\{\sum_t \sum_m r^2(t, m)\right\}^{\frac{1}{2}}.$$

The first factor on the right is

$$O\left(\left\{P^{\frac{2}{5}} P^{\frac{8}{5}+\varepsilon}\right\}^{\frac{1}{2}}\right) = O(P^{1+\varepsilon}).$$

[1] Expositions of the general method (of which the proof of this Lemma is a particular case) will appear in the Proc. Royal Soc., and in Acta Arithmetica.

Also

$$\sum_t \sum_m r^2(t, m) \leqq M,$$

where M is the number of solutions of

(5) $$3\,tx_1^2 + 3\,t^2 x_1 + y_1^3 + z_1^3 = 3\,tx_2^2 + 3\,t^2 x_2 + y_2^3 + z_2^3$$

in all the variables, subject to (2) and (4).

The number of solutions of (5) with $x_1 = x_2$ is

$$O\left(P^{\frac{2}{5}+1+\frac{8}{5}+\varepsilon}\right) = O(P^{3+\varepsilon}).$$

As for the solutions with $x_1 \neq x_2$, given y_1, z_1, y_2, z_2 with $y_2^3 + z_2^3 - y_1^3 - z_1^3 \neq 0$, the equation (5) determines t and $x_1 - x_2$ with only $O(P^\varepsilon)$ possibilities, as factors of this number. These then determine x_1 and x_2 uniquely. Hence the number of solutions of (5) with $x_1 \neq x_2$ is

$$O\left(P^{\frac{16}{5}+\varepsilon}\right).$$

Hence

$$M = O\left(P^{\frac{16}{5}+\varepsilon}\right),$$

and so the number of solutions of (1) subject to (2) is

$$O\left(P^{\frac{13}{5}+\varepsilon}\right) + O\left(P^{1+\varepsilon}\left(P^{\frac{16}{5}+\varepsilon}\right)^{\frac{1}{2}}\right)$$

$$= O\left(P^{\frac{13}{5}+2\varepsilon}\right),$$

which proves the Lemma.

Notation.

Let δ be a fixed small positive number. We shall use ε to denote an arbitrarily small positive number, not the same throughout. The constants implied by the symbol O depend only on δ and ε. $c_1, \ldots$ denote positive absolute constants. $c(r, h)$, $c_1(h)$ denote numbers which depend only on the variables specified.

We use the abbreviations

$$e(\alpha) = e^{2\pi i \alpha}, \quad e_q(b) = e\left(\frac{b}{q}\right).$$

For any large positive real number P we define

$$T(\alpha) = \sum_{P \leqq x \leqq 2P} e(\alpha x^3),$$

$$T_1(\alpha) = \sum_{P^{\frac{4}{5}} \leqq x \leqq 2P^{\frac{4}{5}}} e(\alpha x^3),$$

$$V(\alpha) = T^2(\alpha)\, T_1^2(\alpha) = \sum_n \varrho(n)\, e(n\alpha),$$

so that $\varrho(n)$ denotes the number of representations of n as

$$w^3 + x^3 + y^3 + z^3,$$

where

$$P \leqq w,\ x \leqq 2P, \quad P^{\frac{4}{5}} \leqq y,\ z \leqq 2P^{\frac{4}{5}}.$$

Throughout the paper, q and a denote positive integers satisfying $a \leqq q$, $(a, q) = 1$. We define

$$S_{a,q} = \sum_{x=1}^{q} e_q(ax^3),$$

$$I(\beta) = \sum_{P^3 \leqq n \leqq (2P)^3} \frac{1}{3} n^{-\frac{2}{3}} e(\beta n),$$

$$I_1(\beta) = \sum_{\left(P^{\frac{4}{5}}\right)^3 \leqq n \leqq \left(2P^{\frac{4}{5}}\right)^3} \frac{1}{3} n^{-\frac{2}{3}} e(\beta n),$$

$$T^*(\alpha, a, q) = q^{-1} S_{a,q}\, I\left(\alpha - \frac{a}{q}\right),$$

$$T_1^*(\alpha, a, q) = q^{-1} S_{a,q}\, I_1\left(\alpha - \frac{a}{q}\right),$$

$$V^*(\alpha, a, q) = (T^*(\alpha, a, q)\, T_1^*(\alpha, a, q))^2,$$

$$A(n, q) = q^{-4} \sum_a (S_{a,q})^4 e_q(-na),$$

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q),$$

$$\mathfrak{S}(R, n) = \sum_{q=1}^{R} A(n, q).$$

The functions $V(\alpha)$ and $V^*(\alpha, a, q)$ (for fixed a, q) are periodic in α with period 1.

Inequality for $T^*(\alpha, a, q)$.

Lemma 2. $|S_{a,q}| < c_1 q^{\frac{2}{3}}$.

Proof. Landau, Satz 315.[1]

Lemma 3. *If* $|\beta| \leqq \frac{1}{2}$, *then*

$$I(\beta) = O(\min(P, P^{-2}|\beta|^{-1})),$$

$$I_1(\beta) = O(\min(P^{\frac{4}{5}}, P^{-\frac{8}{5}}|\beta|^{-1})).$$

Proof. The inequality $I(\beta) = O(P)$ is trivial. Also, if $|\beta| \leqq \frac{1}{2}$, we have

$$\sum_{n_1}^{n_2} e(\beta n) = O(|\beta|^{-1})$$

for any n_1, n_2. Hence, by Abel's Lemma,

$$I(\beta) = \frac{1}{3} \sum_{P^3 \leqq n \leqq (2P)^3} n^{-\frac{2}{3}} e(\beta n)$$

$$= O(P^{-2}|\beta|^{-1}).$$

Lemma 4. *If* $\alpha = \frac{a}{q} + \beta$, *where* $|\beta| \leqq \frac{1}{2}$, *then*

$$T^*(\alpha, a, q) = O\left(q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1})\right),$$

$$T_1^*(\alpha, a, q) = O\left(q^{-\frac{1}{3}} \min\left(P^{\frac{4}{5}}, P^{-\frac{8}{5}}|\beta|^{-1}\right)\right).$$

Proof. Lemmas 2, 3.

Approximation to $T(\alpha)$.

Lemma 5. *If* $n \neq 0$ *is any integer, then*

$$S_{a,n,q} = \sum_{x=1}^{q} e_q(ax^3 + nx) = O\left(q^{\frac{2}{3}+\varepsilon}(n, q)\right).$$

Proof. Lemma 2 of Davenport-Heilbronn, *Proc. London Math. Soc.*, (2), 43 (1937), 73—104.

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[1] References to Landau are to *Vorlesungen über Zahlentheorie*, (Leipzig 1927), volume 1.

Lemma 6. *Suppose that*

$$H \geqq 1,\ q \leqq H^{1-\delta},\ \beta = O(q^{-1} H^{-2-\delta}),$$

and let $n \neq 0$ be any integer. Then

$$\int_0^H e\left(\beta x^3 - \frac{nx}{q}\right) dx = -\frac{q}{2\pi i n}\left(e\left(\beta H^3 - \frac{nH}{q}\right) - 1\right) + O(qn^{-2} H^{-\delta}).$$

Proof. After integration by parts l times, the integral becomes

$$-\frac{q}{2\pi i n}\left(e\left(\beta H^3 - \frac{nH}{q}\right) - 1\right) - \sum_{h=1}^{l-1}\left(\frac{q}{2\pi i n}\right)^{h+1}\left[e\left(-\frac{nx}{q}\right) D^h(e(\beta x^3))\right]_0^H$$

$$+ \left(\frac{q}{2\pi i n}\right)^l \int_0^H e\left(-\frac{nx}{q}\right) D^l(e(\beta x^3))\, dx,$$

where D^h denotes the h-th differential coefficient, and $[f(x)]_0^H = f(H) - f(0)$. It is easily verified that

$$D^h(e(\beta x^3)) = \sum_{\frac{1}{3}h \leqq r \leqq h} c(r, h)\beta^r x^{3r-h} e(\beta x^3).$$

For $0 \leqq x \leqq H$, $\frac{1}{3}h \leqq r \leqq h$, we have

$$\begin{aligned}\beta^r x^{3r-h} &= O(q^{-r} H^{-r(2+\delta)} H^{3r-h}) \\ &= O(q^{-h}(qH^{-1+\delta})^{h-r} H^{-h\delta}) \\ &= O(q^{-h} H^{-h\delta}).\end{aligned}$$

Hence, for $0 \leqq x \leqq H$,

$$D^h(e(\beta x^3)) = O(c_1(h)\, q^{-h} H^{-h\delta}).$$

Using this in the above expression, the error term becomes

$$O\left(\sum_{h=1}^{l-1}\left(\frac{q}{|n|}\right)^{h+1} c_1(h)\, q^{-h} H^{-h\delta} + \left(\frac{q}{|n|}\right)^l c_1(l)\, q^{-l} H^{-l\delta} H\right).$$

Choose l to be the least integer for which $1 - l\delta \leqq -\delta$. Then this error term is

$$O(qn^{-2} H^{-\delta}).$$

Lemma 7[1]. *If* $\alpha = \frac{a}{q} + \beta$, *where* $q \leqq P^{1-\delta}$ *and* $|\beta| \leqq q^{-1} P^{-2-\delta}$, *then*

$$T(\alpha) = T^*(\alpha, a, q) + O\left(q^{\frac{2}{3}+\varepsilon}\right).$$

Proof. We have

$$T(\alpha) = \sum_{h=1}^{q} \sum_{\frac{P-h}{q} \leqq m \leqq \frac{2P-h}{q}} e\left(\left(\frac{a}{q} + \beta\right)(mq + h)^3\right)$$

$$= \sum_{h=1}^{q} e_q(ah^3) \sum_{\frac{P-h}{q} \leqq m \leqq \frac{2P-h}{q}} e(\beta(mq + h)^3).$$

By Poisson's summation formula,

$$\sum'_{\frac{P-h}{q} \leqq m \leqq \frac{2P-h}{q}} e(\beta(mq + h)^3) = \int_{\frac{P-h}{q}}^{\frac{2P-h}{q}} e(\beta(xq + h)^3)\,dx + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \int_{\frac{P-h}{q}}^{\frac{2P-h}{q}} e(\beta(xq + h)^3 - nx)\,dx$$

$$= q^{-1} \int_{P}^{2P} e(\beta x^3)\,dx + q^{-1} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e_q(nh) \int_{P}^{2P} e\left(\beta x^3 - \frac{nx}{q}\right) dx,$$

where Σ' denotes that any term with $m = \frac{P-h}{q}$ or $\frac{2P-h}{q}$ is counted with a factor $\frac{1}{2}$. There are at most two values of h for which such terms exist, so the presence of the dash only introduces an error $O(1)$ in $T(\alpha)$.

Since

$$y^{-\frac{2}{3}} e(\beta y) = n^{-\frac{2}{3}} e(\beta n) + O\left(n^{-\frac{5}{3}}\right) + O\left(n^{-\frac{2}{3}} |\beta|\right)$$

$$= n^{-\frac{2}{3}} e(\beta n) + O(P^{-5}) + O(P^{-2} |\beta|)$$

$$= n^{-\frac{2}{3}} e(\beta n) + O(P^{-3})$$

for $n \leqq y \leqq n + 1$, $P^3 \leqq n \leqq (2P)^3$, we have

[1] An alternative method of proof is given in Landau, Satz 329 to Satz 337. The less precise result obtainable by a single partial summation would in fact also suffice, but then the proof of Lemma 11 would be much more complicated.

$$\int_P^{2P} e(\beta x^3)\,dx = \frac{1}{3}\int_{P^3}^{(2P)^3} y^{-\frac{2}{3}} e(\beta y)\,dy$$

$$= I(\beta) + O(1).$$

Hence

$$T(\alpha) = T^*(\alpha, a, q) + O(1) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} q^{-1} S_{a,n,q} \int_P^{2P} e\left(\beta x^3 - \frac{nx}{q}\right) dx$$

$$= T^*(\alpha, a, q) + O(1) + \Sigma,$$

say.

The conditions of Lemma 6 are satisfied for $\int_0^P$ and $\int_0^{2P}$, hence

$$\Sigma = -\frac{1}{2\pi i}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} n^{-1} S_{a,n,q}\left(e\left(\beta(2P)^3 - \frac{2nP}{q}\right) - e\left(\beta P^3 - \frac{nP}{q}\right)\right)$$

$$+ O\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} q^{-1} |S_{a,n,q}|\, q n^{-2} P^{-\delta}\right)$$

$$= -\frac{1}{2\pi i}\Sigma_1 + \Sigma_2,$$

say.

By Lemma 5,

$$\Sigma_2 = O\left(P^{-\delta}\sum_{n=1}^{\infty} q^{\frac{2}{3}+\varepsilon} n^{-2}(n, q)\right)$$

$$= O\left(P^{-\delta} q^{\frac{2}{3}+\varepsilon}\sum_{d\mid q} d \sum_{m=1}^{\infty}(md)^{-2}\right)$$

$$= O\left(q^{\frac{2}{3}+2\varepsilon}\right).$$

Also

$$\Sigma_1 = e(\beta(2P)^3)\sum_{|n|>q^2} n^{-1} S_{a,n,q}\, e_q(-2nP)$$

$$- e(\beta P^3)\sum_{|n|>q^2} n^{-1} S_{a,n,q}\, e_q(-nP)$$

$$+ O\left(\sum_{n=1}^{q^2} n^{-1} |S_{a,n,q}|\right)$$

$$= e(\beta(2P)^3)\sum_{h=1}^{q} e_q(ah^3) \sum_{n=q^2+1}^{\infty} \frac{2i}{n}\sin\frac{n(h-2P)}{q}2\pi$$

$$- e(\beta P^3)\sum_{h=1}^{q} e_q(ah^3) \sum_{n=q^2+1}^{\infty} \frac{2i}{n}\sin\frac{n(h-P)}{q}2\pi$$

$$+ O\left(\sum_{n=1}^{q^2} n^{-1} q^{\frac{2}{3}+\varepsilon}(n, q)\right)$$

$$= O\left\{\sum_{h=1}^{q} \min\left(1, \frac{1}{q^2\left\|\frac{h-2P}{q}\right\|}\right) + \sum_{h=1}^{q} \min\left(1, \frac{1}{q^2\left\|\frac{h-P}{q}\right\|}\right)\right.$$

$$\left. + q^{\frac{2}{3}+\varepsilon} \sum_{d|q} d \sum_{m \leqq \frac{q^2}{d}} \frac{1}{md}\right\}$$

$$= O\left(q^{\frac{2}{3}+2\varepsilon}\right),$$

where $\|x\|$ denotes the minimum of $|x-n|$ for all integral n. This proves Lemma 7.

Lemma 8. *If* $\alpha = \frac{a}{q}+\beta$, *where* $q \leqq P^{\frac{4}{5}(1-\delta)}$ *and* $|\beta| \leqq q^{-1}P^{-\frac{4}{5}(2+\delta)}$, *then*

$$T_1(\alpha) = T_1^*(\alpha, a, q) + O\left(q^{\frac{2}{3}+\varepsilon}\right).$$

Lemma 9. *If* $q \leqq P^{1-\delta}$, *and* $|\beta| \leqq q^{-1}P^{-2-\delta}$, *then*

$$T(\alpha) = O\left(q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1})\right).$$

Proof. By Lemmas 7, 4,

$$T(\alpha) = O\left(q^{\frac{2}{3}+\varepsilon}\right) + O\left(q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1})\right).$$

Also

$$q^{1+\varepsilon} < P, \quad q^{1+\varepsilon} < P^{-2}|\beta|^{-1}.$$

Lemma 10. *If* $q \leqq P^{\frac{4}{5}(1-\delta)}$, *and* $|\beta| \leqq q^{-1}P^{-\frac{4}{5}(2+\delta)}$, *then*

$$T_1(\alpha) = O\left(q^{-\frac{1}{3}} \min\left(P^{\frac{4}{5}}, P^{-\frac{8}{5}}|\beta|^{-1}\right)\right).$$

The Farey Dissection.

We divide the interval $0 \leqq \alpha \leqq 1$ into "Farey arcs", that is, into intervals surrounding each rational point $\frac{a}{q}$ with $q \leqq P^{2+\delta}$. The Farey arc surrounding $\frac{a}{q}$

extends from $\frac{a+a_1}{q+q_1}$ to $\frac{a+a_2}{q+q_2}$, where $\frac{a_1}{q_1}$ and $\frac{a_2}{q_2}$ are the Farey fractions of order $P^{2+\delta}$ next less than and next greater than $\frac{a}{q}$. These Farey arcs cover precisely the interval $0 \leqq \alpha \leqq 1$, when we consider the arc surrounding $\frac{1}{1}$ to have period 1, and so to project into the left-hand side of the interval. The points of the Farey arc surrounding $\frac{a}{q}$ have the form

$$\alpha = \frac{a}{q} + \beta,$$

$$-\vartheta_1 q^{-1} P^{-2-\delta} \leqq \beta \leqq \vartheta_2 q^{-1} P^{-2-\delta},$$

where $\frac{1}{2} \leqq \vartheta_1 \leqq 1$, $\frac{1}{2} \leqq \vartheta_2 \leqq 1$.

If $q \leqq P^{\frac{4}{5}(1-\delta)}$, we call the arc a major arc[1], and denote it by $\mathfrak{M}_{a,q}$. If $P^{\frac{4}{5}(1-\delta)} < q \leqq P^{2-\delta}$, we call the arc a minor arc, and denote it by $\mathfrak{m}_{a,q}$. The aggregate of the major arcs is denoted by $\mathfrak{M}$, and that of the minor arcs by $\mathfrak{m}$.

The Major Arcs.

Lemma 11. $\sum_{\mathfrak{M}} \int_{\mathfrak{M}} | V(\alpha) - V^*(\alpha, a, q)|^2 \, d\alpha = O\left(P^{3+\frac{2}{3}}\right).$

Proof. The conditions of Lemmas 4, 7, 9 are satisfied on $\mathfrak{M}_{a,q}$, since $P^{\frac{4}{5}(1-\delta)} < P^{1-\delta}$, hence

$$T(\alpha) - T^*(\alpha, a, q) = O\left(q^{\frac{2}{3}+\varepsilon}\right), \tag{6}$$

$$T(\alpha),\ T^*(\alpha, a, q) = O\left(q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1})\right). \tag{7}$$

The conditions of Lemmas 4, 8, 10 are satisfied on $\mathfrak{M}_{a,q}$, since $q^{-1}P^{-2-\delta} < q^{-1}P^{-\frac{4}{5}(2+\delta)}$, hence

$$T_1(\alpha) - T_1^*(\alpha, a, q) = O\left(q^{\frac{2}{3}+\varepsilon}\right), \tag{8}$$

$$T_1(\alpha),\ T_1^*(\alpha, a, q) = O\left(q^{-\frac{1}{3}} P^{\frac{4}{5}}\right). \tag{9}$$

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[1] This violates the usual convention connecting the upper bound for the denominators of the major arcs with the order of the Farey dissection. It corresponds, however, to the fundamentally different ways in which the arcs are treated.

By (6), (7), (8), (9),

$$T^2 T_1^2 - T^{*2} T_1^{*2} = O\Big(q^{\frac{2}{3}+\varepsilon} q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1}) q^{-\frac{2}{3}} P^{\frac{8}{5}} + q^{\frac{2}{3}+\varepsilon} q^{-\frac{1}{3}} P^{\frac{4}{5}} q^{-\frac{2}{3}} \min(P^2, P^{-4}|\beta|^{-2})\Big).$$

Hence

$$\int\limits_{\mathfrak{M}_{a,q}} |V(\alpha) - V^*(\alpha, a, q)|^2 d\alpha$$

$$= O\Big(q^{-\frac{2}{3}+2\varepsilon} P^{\frac{16}{5}} \int\limits_0^\infty \min(P^2, P^{-4}\beta^{-2})\, d\beta + q^{-\frac{2}{3}+2\varepsilon} P^{\frac{8}{5}} \int\limits_0^\infty \min(P^4, P^{-8}\beta^{-4})\, d\beta\Big)$$

$$= O\Big(q^{-\frac{2}{3}+2\varepsilon} P^{\frac{11}{5}} + q^{-\frac{2}{3}+2\varepsilon} P^{\frac{13}{5}}\Big).$$

Hence

$$\sum_{\mathfrak{M}} \int\limits_{\mathfrak{M}} |V(\alpha) - V^*(\alpha, a, q)|^2 d\alpha = O\Big(P^{\frac{13}{5}} \sum_{q \leq P^{\frac{4}{5}(1-\delta)}} q \cdot q^{-\frac{2}{3}+2\varepsilon}\Big)$$

$$= O\Big(P^{\frac{13}{5}+\frac{4}{3}\frac{4}{5}}\Big),$$

whence the result.

Weyl's Inequality.

Lemma 12. $\sum_{x=1}^{m} e_q(ax^3) = O\Big(m^\varepsilon q^\varepsilon \Big(m^{\frac{3}{4}} + m q^{-\frac{1}{4}} + m^{\frac{1}{4}} q^{\frac{1}{4}}\Big)\Big).$

Proof. Landau, Satz 267, with $k = 3$ $(K = 4)$.

Lemma 13. *If* $P^{1-\delta} < q \leqq P^{2+\delta}$, *and* $|\beta| \leqq q^{-1} P^{-2-\delta}$, *then*

$$T(\alpha) = O\Big(P^{\frac{3}{4}+\delta}\Big).$$

Proof. Let

$$S_m = \sum_{1 \leqq x \leqq m^{\frac{1}{3}}} e_q(ax^3).$$

By Lemma 12, if $m \leqq (2P)^3$,

$$S_m = O\Big(P^\varepsilon q^\varepsilon \Big(P^{\frac{3}{4}} + P q^{-\frac{1}{4}} + P^{\frac{1}{4}} q^{\frac{1}{4}}\Big)\Big)$$

$$= O\left(P^{4\varepsilon}\left(P^{\frac{3}{4}} + P^{\frac{3}{4}+\frac{1}{4}\delta}\right)\right)$$

$$= O\left(P^{\frac{3}{4}+\frac{1}{4}\delta+4\varepsilon}\right).$$

By partial summation, if $P_2 = [(2P)^3]$ and $P_1 = -[-P^3]$,

$$T(\alpha) = \sum_{n=P_1}^{P_2} (S_n - S_{n-1})\, e(\beta n)$$

$$= \sum_{n=P_1}^{P_2-1} S_n\left(e(\beta n) - e(\beta(n+1))\right) - S_{P_1-1}\, e(\beta(P_1-1)) + S_{P_2}\, e(\beta P_2)$$

$$= O\left(P^{\frac{3}{4}+\frac{1}{4}\delta+4\varepsilon}(P^3|\beta| + 1)\right)$$

$$= O\left(P^{\frac{3}{4}+\frac{1}{4}\delta+4\varepsilon}\right),$$

since $P^3|\beta| \leqq q^{-1}P^{1-\delta} \leqq 1$.

Lemma 14. *If* $P^{\frac{4}{5}(1-\delta)} < q \leqq P^{\frac{4}{5}(2+\delta)}$, *and* $|\beta| \leqq q^{-1}P^{-\frac{4}{5}(2+\delta)}$, *then*

$$T_1(\alpha) = O\left(P^{\frac{4}{5}\left(\frac{3}{4}+\delta\right)}\right).$$

Minor Arcs.

Lemma 15. $\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |V(\alpha)|^2\, d\alpha = O\left(P^{4+\frac{1}{10}+3\delta}\right).$

Proof. (1) Consider arcs $\mathfrak{m}_{a,q}$ for which

$$P^{\frac{4}{5}(1-\delta)} < q \leqq P^{1-\delta}.$$

The conditions of Lemma 9 are satisfied, hence

$$T(\alpha) = O\left(q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1})\right).$$

The conditions of Lemma 14 are satisfied, since $q^{-1}P^{-2-\delta} < q^{-1}P^{-\frac{4}{5}(2+\delta)}$, hence

$$T_1(\alpha) = O\left(P^{\frac{3}{5}+\delta}\right).$$

Thus, for the values of q under consideration,

$$\int_{\mathfrak{m}_{a,q}} |V(\alpha)|^2\,d\alpha = O\left(q^{-\frac{4}{3}} P^{4\left(\frac{3}{5}+\delta\right)} \int_0^\infty \min\left(P^4, P^{-8}\beta^{-4}\right) d\beta\right)$$

$$= O\left(q^{-\frac{4}{3}} P^{\frac{12}{5}+4\delta+1}\right),$$

and the total contribution of the arcs corresponding to these q's is

$$O\left(\sum_{q \leqq P^{1-\delta}} \sum_a q^{-\frac{4}{3}} P^{3+\frac{2}{5}+4\delta}\right)$$

$$= O\left(P^{3+\frac{2}{5}+4\delta+\frac{2}{3}}\right)$$

$$= O\left(P^{4+\frac{1}{15}+4\delta}\right).$$

(2) Consider now values of q satisfying $P^{1-\delta} < q \leqq P^{2+\delta}$. On such arcs,

$$T(\alpha) = O\left(P^{\frac{3}{4}+\delta}\right),$$

by Lemma 13. Hence the contribution of these arcs is

$$O\left(P^{2\left(\frac{3}{4}+\delta\right)} \int_0^1 |T(\alpha)\,T_1^2(\alpha)|^2\,d\alpha\right).$$

The integral is precisely equal to the number of solutions of (1), subject to (2). Hence by Lemma 1, it is $O\left(P^{\frac{13}{5}+\varepsilon}\right)$, and the above contribution is

$$O\left(P^{\frac{3}{2}+2\delta+\frac{13}{5}+\varepsilon}\right)$$

$$= O\left(P^{4+\frac{1}{10}+3\delta}\right).$$

Completion of the Analytical Argument.

Let $R = \left[P^{\frac{4}{5}(1-\delta)}\right]$, so that the major arcs are those for which $q \leqq R$. In what follows, A, Q (and similarly A_1, Q_1 and A_2, Q_2) denote positive integers satisfying $Q \leqq R$, $A \leqq Q$, $(A, Q) = 1$.

Lemma 16. $\int_0^1 \left| \sum_Q \sum_A{}' V^*(\alpha, A, Q) \right|^2 d\alpha = O\left(P^{4-\frac{8}{15}}\right)$, *where, if* α *is on a major arc* $\mathfrak{M}_{a,q}$, *the term* $Q = q$, $A = a$ *is to be omitted from the sum.*

Proof. By Lemma 4, and the periodicity of $V^*(\alpha, A, Q)$, we have, for any α, A, Q,

$$V^*(\alpha, A, Q) = O\left(Q^{-\frac{4}{3}} P^{-4} \left\| \alpha - \frac{A}{Q} \right\|^{-2} P^{\frac{8}{5}}\right).$$

Hence the integral of the Lemma is

$$O\left(\sum_Q \sum_A P^{-\frac{24}{5}} Q^{-\frac{8}{3}} \int \left\| \alpha - \frac{A}{Q} \right\|^{-4} d\alpha \right.$$
$$\left. + \sum_{Q_1} \sum_{A_1} \sum_{Q_2} \sum_{A_2} P^{-\frac{24}{5}} Q_1^{-\frac{4}{3}} Q_2^{-\frac{4}{3}} \int \left\| \alpha - \frac{A_1}{Q_1} \right\|^{-2} \left\| \alpha - \frac{A_2}{Q_2} \right\|^{-2} d\alpha \right),$$
$$Q_1, A_1 \neq Q_2, A_2$$

where the first integral is over $(0, 1)$ excluding $\mathfrak{M}_{A,Q}$, and the second is over $(0, 1)$ excluding $\mathfrak{M}_{A_1,Q_1}$ and $\mathfrak{M}_{A_2,Q_2}$.

The first sum is

$$O\left(P^{-\frac{24}{5}} \sum_Q \sum_A Q^{-\frac{8}{3}} \int_{\frac{1}{2}Q^{-1}P^{-2-\delta}}^{\infty} \beta^{-4} d\beta \right)$$
$$= O\left(P^{-\frac{24}{5}} \sum_Q Q\, Q^{-\frac{8}{3}} Q^3 P^{3(2+\delta)}\right)$$
$$= O\left(P^{-\frac{24}{5} + 6 + 3\delta + \frac{7}{3}\frac{4}{5}(1-\delta)}\right)$$
$$= O\left(P^{3 + \frac{1}{15} + 3\delta}\right).$$

For the second sum, we observe that, for any α, either

$$\left\| \alpha - \frac{A_1}{Q_1} \right\| \geqq \frac{1}{2} \left\| \frac{A_1}{Q_1} - \frac{A_2}{Q_2} \right\| \text{ or } \left\| \alpha - \frac{A_2}{Q_2} \right\| \geqq \frac{1}{2} \left\| \frac{A_1}{Q_1} - \frac{A_2}{Q_2} \right\|.$$

We can suppose without loss of generality that the former holds. Since α is not on $\mathfrak{M}_{A_2,Q_2}$, we have

$$\left\| \alpha - \frac{A_2}{Q_2} \right\| \geqq \frac{1}{2} Q_2^{-1} P^{-2-\delta}.$$

Hence

$$\int \left\| \alpha - \frac{A_1}{Q_1} \right\|^{-2} \left\| \alpha - \frac{A_2}{Q_2} \right\|^{-2} d\alpha = O\left(P^{4+2\delta} Q_2^2 \frac{1}{\left\| \frac{A_1}{Q_1} - \frac{A_2}{Q_2} \right\|^2} \right),$$

and the sum is

$$O\left(P^{-\frac{24}{5}+4+2\delta} \sum_{\substack{Q_1 \\ Q_1,}} \sum_{\substack{A_1 \\ A_1 \neq Q_2,}} \sum_{\substack{Q_2 \\ Q_2,}} \sum_{\substack{A_2 \\ A_2}} \frac{Q_1^{\frac{2}{3}} Q_2^{\frac{8}{3}}}{\langle A_1 Q_2 - A_2 Q_1 \rangle^2} \right),$$

where $\langle A_1 Q_2 - A_2 Q_1 \rangle$ denotes the absolutely least residue of $A_1 Q_2 - A_2 Q_1$ (mod $Q_1 Q_2$). Given Q_1, Q_2, n, the relation $\langle A_1 Q_2 - A_2 Q_1 \rangle = n$ determines A_1, A_2 uniquely. Hence the above expression is

$$O\left(P^{-\frac{24}{5}+4+2\delta} \sum_{Q_1} Q_1^{\frac{2}{3}} \sum_{Q_2} Q_2^{\frac{8}{3}} \sum_{n=1}^{\infty} n^{-2} \right)$$

$$= O\left(P^{-\frac{24}{5}+4+2\delta+\frac{5}{3}\frac{4}{5}(1-\delta)+\frac{11}{3}\frac{4}{5}(1-\delta)} \right)$$

$$= O\left(P^{4-\frac{8}{15}} \right).$$

Lemma 17. $\displaystyle\int_0^1 \left| V(\alpha) - \sum_Q \sum_A V^*(\alpha, A, Q) \right|^2 d\alpha = O\left(P^{4+\frac{1}{10}+3\delta} \right).$

Proof. If α is on a major arc $\mathfrak{M}_{a,q}$,

$$\left| V(\alpha) - \sum_Q \sum_A V^*(\alpha, A, Q) \right| \leqq \left| V(\alpha) - V^*(\alpha, a, q) \right| + \left| \sum_Q \sum_A{}' V^*(\alpha, A, Q) \right|,$$

and if α is on a minor arc

$$\left| V(\alpha) - \sum_Q \sum_A V^*(\alpha, A, Q) \right| \leqq \left| V(\alpha) \right| + \left| \sum_Q \sum_A{}' V^*(\alpha, A, Q) \right|,$$

where the dash has the same meaning as in the enunciation of Lemma 16. Hence the result follows from Lemmas 11, 15, 16.

Lemma 18. $\displaystyle\sum_n (\varrho(n) - \psi(n) \mathfrak{S}(R, n))^2 = O\left(P^{4+\frac{1}{10}+3\delta} \right),$

where, for $3P^3 \leqq n \leqq 15P^3$, $\psi(n)$ *satisfies*

$$c_2 P^{\frac{3}{5}} < \psi(n) < c_3 P^{\frac{3}{5}}.$$

Proof. Let

$$\psi(n) = \frac{1}{81} \sum_{n_1, n_2, n_3, n_4} (n_1 n_2 n_3 n_4)^{-\frac{2}{3}},$$

where n_1, n_2, n_3, n_4 are summed over all representations of n as $n_1 + n_2 + n_3 + n_4$ subject to

$$P^3 \leqq n_1,\ n_2 \leqq (2P)^3,\ \left(P^{\frac{4}{5}}\right)^3 \leqq n_3,\ n_4 \leqq \left(2P^{\frac{4}{5}}\right)^3.$$

By the definition of I, I_1,

$$I^2\left(\alpha - \frac{A}{Q}\right) I_1^2\left(\alpha - \frac{A}{Q}\right) = \sum_n \psi(n)\, e\left(\left(\alpha - \frac{A}{Q}\right) n\right).$$

Hence

$$V^*(\alpha, A, Q) = \sum_n Q^{-4} (S_{A,Q})^4\, \psi(n)\, e\left(\left(\alpha - \frac{A}{Q}\right) n\right),$$

and

$$\sum_Q \sum_A V^*(\alpha, A, Q) = \sum_n \psi(n)\, \mathfrak{S}(R, n)\, e(n\alpha).$$

Thus

$$V(\alpha) - \sum_Q \sum_A V^*(\alpha, A, Q) = \sum_n (\varrho(n) - \psi(n)\, \mathfrak{S}(R, n))\, e(n\alpha).$$

This shows that the first assertion of Lemma 18 is a consequence of Lemma 17.

As regards the magnitude of $\psi(n)$, it is obvious that if $3P^3 \leqq n \leqq 15P^3$, n_3 and n_4 can be chosen arbitrarily, and that the number of representations of $n - n_3 - n_4$ as $n_1 + n_2$ lies between $c_4(n - n_3 - n_4)$ and $c_5(n - n_3 - n_4)$, that is, between $c_6 P^3$ and $c_7 P^3$. Hence

$$\psi(n) > \frac{1}{81} c_6 P^3 \left\{(2P)^3 (2P)^3 \left(2P^{\frac{4}{5}}\right)^3 \left(2P^{\frac{4}{5}}\right)^3\right\}^{-\frac{2}{3}} \left(P^{\frac{12}{5}}\right)^2$$

$$> c_2 P^{\frac{3}{5}},$$

and similarly,

$$\psi(n) < c_3 P^{\frac{3}{5}}.$$

The Singular Series.

Lemma 19. *For all* n, q, $|A(n, q)| < c_8 q^{-\frac{1}{3}}$.

Proof. By Lemma 2,

$$|A(n, q)| < q^{-4} \sum_a \left(c_1 q^{\frac{2}{3}}\right)^4 < c_8 q^{-\frac{1}{3}}.$$

Lemma 20. *If* $(q_1, q_2) = 1$, *then* $A(n, q_1 q_2) = A(n, q_1) A(n, q_2)$.

Proof. Landau, Satz 282.

The following notation corresponds to that of Landau, pp. 280—302, in the case $k = 3$, $s = 4$.

For any prime p and any positive integer l, let $N(p^l, n)$ denote the number of solutions of

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \equiv n \pmod{p^l}, \quad 0 \leqq x_i < p^l,$$

in which not all of the x's are divisible by p.

For any prime p let $\gamma = 1$ if $p > 3$ and $\gamma = 2$ if $p \leqq 3$. Let $P_0 = p^\gamma$.

Lemma 21. *Let* $3\varrho + \sigma$ *be the exact power to which* p *divides* n, *where* $0 \leqq \sigma \leqq 2$. *Let*

$$l_0 = \max(3\varrho + \sigma + 1, \ 3\varrho + \gamma).$$

Then

$$A(n, p^l) = 0 \quad \text{if} \quad l > l_0,$$

and

$$\chi_p(n) = \sum_{\nu=0}^{\infty} A(n, p^\nu) = P_0^{-3} N(P_0, 0) \sum_{v=0}^{\varrho-1} p^{-v} + p^{-\varrho} P_0^{-3} N\left(P_0, \frac{n}{p^{3\varrho}}\right),$$

where, if $\varrho = 0$, *the sum over* v *is to be read as zero.*

Proof. This is the case $k = 3$, $s = 4$ of Landau's Satz 293.

Corollary. If $p \nmid 6n$ *then* $A(n, p^\nu) = 0$ *for* $\nu > 1$.

Lemma 22. *If* $p \neq 3$ *then* $N(P_0, n) > 0$ *for all* n.

Proof. By Landau, Satz 300 and Satz 301 (with $s = 4$), it suffices to prove that

$$4 \geqq \frac{P_0 - 1}{p - 1}(3, p - 1) + 1.$$

If $p = 2$,

$$\frac{P_0 - 1}{p - 1}(3, p - 1) = \frac{2^2 - 1}{2 - 1}(3, 1) = 3.$$

If $p > 3$,

$$\frac{P_0 - 1}{p - 1}(3, p - 1) = (3, p - 1) \leqq 3.$$

Lemma 23. *If* $p = 3$ *(so that* $P_0 = 9$*) then* $N(P_0, n) > 0$ *for all* n.

Proof.
$$1^3 + 8^3 + 0^3 + 0^3 \equiv 0 \pmod 9,$$
$$m \cdot 1^3 + (4 - m) \cdot 0^3 \equiv m \pmod 9), \text{ for } 1 \leqq m \leqq 4,$$
$$m \cdot 8^3 + (4 - m) \cdot 0^3 \equiv -m \pmod 9), \text{ for } 1 \leqq m \leqq 4.$$

Lemma 24. *For any prime* p *and any* n,
$$\chi_p(n) \geqq p^{-6}.$$

Proof. (1) Suppose $p^3 \nmid n$, so that $\varrho = 0$. By Lemmas 21, 22, 23,
$$\chi_p(n) = P_0^{-3} N(P_0, n)$$
$$\geqq P_0^{-3}$$
$$\geqq p^{-6}.$$

(2) Suppose $p^3 | n$, so that $\varrho \geqq 1$. By Lemmas 21, 22, 23,
$$\chi_p(n) \geqq P_0^{-3} N(P_0, 0)$$
$$\geqq P_0^{-3}$$
$$\geqq p^{-6}.$$

Lemma 25. *For any prime* p,
$$|A(n, p)| < c_9 p^{-\frac{3}{2}} \text{ if } p \nmid n,$$
$$|A(n, p)| < c_9 p^{-1} \text{ if } p | n.$$

Proof. Landau, Sätze 317, 318.

Lemma 26. *For any prime* p,
$$|\chi_p(n) - 1| < c_{10} p^{-\frac{3}{2}} \text{ if } p \nmid n,$$
$$\chi_p(n) > 1 - c_{10} p^{-1} \text{ if } p | n.$$

Proof. Landau, Sätze 320, 322.

Lemma 27. *The series*
$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q)$$

is absolutely convergent, and

$$\mathfrak{S}(n) > c_{11} (\log \log n)^{-c_{12}}.$$

Proof. By the first half of Lemma 26, the product

$$\prod_p \chi_p(n)$$

is absolutely convergent; hence $\sum_{q=1}^{\infty} A(n, q)$ is absolutely convergent, and $\mathfrak{S}(n) = \prod_p \chi_p(n)$. By Lemmas 24, 26,

$$\prod_p \chi_p(n) > \Big(\prod_{p \leqq 2c_{10}} p^{-6}\Big)\Big(\prod_{p > 2c_{10}} (1 - c_{10} p^{-\frac{3}{2}})\Big)\Big(\prod_{\substack{p > 2c_{10} \\ p \mid n}} (1 - c_{10} p^{-1})\Big)$$

$$> c_{13} \prod_{\substack{p > 2c_{10} \\ p \mid n}} (1 - c_{10} p^{-1})$$

$$> c_{13} \prod_{p \mid n} (1 - p^{-1})^{c_{14}}$$

$$> c_{11} (\log \log n)^{-c_{12}}.$$

Lemma 28. *For* $\eta \geqq 1$,

$$\sum_{q \geqq \eta} A(n, q) = O\left(\eta^{-\frac{1}{6}} n^{\varepsilon}\right).$$

Proof. Any positive integer q is expressible as $q_1 q_2 q_3$, where
(1) q_1 is quadratfrei, and $(q_1, 6) = 1$,
(2) q_2 is composed of prime powers with exponents $\geqq 2$, and $(q_2, 6) = 1$,
(3) q_3 is of the form $2^{\lambda} 3^{\varkappa}$,
(4) $(q_1, q_2) = (q_2, q_3) = (q_3, q_1) = 1$.
By Lemma 21, if $p \mid q_2$ (so that $p > 3$),

$$A(n, p^l) = 0 \text{ if } l > l_0 = \max(3\varrho + \sigma + 1,\ 3\varrho + 1)$$
$$= 3\varrho + \sigma + 1.$$

Hence, if p^l is one of the prime powers composing q_2, and $A(n, p^l) \neq 0$, we have

$$l \leqq 3\varrho + \sigma + 1,$$

whence $p^{l-1}|n$, whence (since $l \geqq 2$), $p^l|n^2$. Hence, if $A(n, q_2) \neq 0$, we must have $q_2|n^2$.

By Lemma 19,

$$A(n, q_2) = O\left(q_2^{-\frac{1}{3}}\right),$$

$$A(n, q_3) = O\left(q_3^{-\frac{1}{3}}\right).$$

By Lemma 25,

$$A(n, q_1) < \left(\prod_{p|q_1} c_9 p^{-\frac{3}{2}}\right)\left(\prod_{\substack{p|q_1 \\ p|n}} p^{\frac{1}{2}}\right) = O\left(q_1^{-\frac{3}{2}+\varepsilon}(n, q_1)^{\frac{1}{2}}\right).$$

Hence

$$\sum_{q \geqq \eta} A(n, q) = O\left(\sum_{\substack{q_1, q_2, q_3 \\ q_1 q_2 q_3 \geqq \eta \\ q_2 | n^2}} q_1^{-\frac{3}{2}+\varepsilon}(n, q_1)^{\frac{1}{2}} q_2^{-\frac{1}{3}} q_3^{-\frac{1}{3}}\right)$$

$$= O\left(\eta^{-\frac{1}{6}} \sum_{\substack{q_1, q_2, q_3 \\ q_2 | n^2}} q_1^{-\frac{4}{3}+\varepsilon}(n, q_1)^{\frac{1}{2}} q_2^{-\frac{1}{6}} q_3^{-\frac{1}{6}}\right).$$

Now

$$\sum_{q_2 | n^2} q_2^{-\frac{1}{6}} \leqq \sum_{q_2 | n^2} 1 = O(n^{\varepsilon}),$$

$$\sum_{q_3} q_3^{-\frac{1}{6}} \leqq \left(1 - 2^{-\frac{1}{6}}\right)^{-1}\left(1 - 3^{-\frac{1}{6}}\right)^{-1} = O(1),$$

$$\sum_{q_1} q_1^{-\frac{4}{3}+\varepsilon}(n, q_1)^{\frac{1}{2}} \leqq \sum_{d|n} d^{\frac{1}{2}} \sum_{r=1}^{\infty} (rd)^{-\frac{4}{3}+\varepsilon}$$

$$= O\left(\sum_{d|n} 1\right)$$

$$= O(n^{\varepsilon}).$$

Hence the result.

Lemma 29. $$\sum_{3P^3 \leqq n \leqq 15 P^3} (\psi(n)\mathfrak{S}(R, n) - \psi(n)\mathfrak{S}(n))^2 = O(P^4).$$

Proof. By Lemma 18, $\psi(n) = O\left(P^{\frac{3}{5}}\right)$. By Lemma 28,

$$\mathfrak{S}(n) - \mathfrak{S}(R, n) = \sum_{q > R} A(n, q)$$

$$= O\left(R^{-\frac{1}{6}} P^{\varepsilon}\right)$$

$$= O\left(P^{-\frac{2}{15}+\delta}\right).$$

Hence the result.

Proof of the Theorem.

If $E(N)$ denotes the number of positive integers not exceeding N that are not representable as the sum of four positive integral cubes, we have to prove that

$$E(N) = O\left(N^{1-\frac{1}{30}+\delta_1}\right)$$

as $N \to \infty$, for any $\delta_1 > 0$.

Choose $\delta = \frac{1}{2}\delta_1$, and choose $P = \left(\frac{1}{5}N\right)^{\frac{1}{3}}$. Then

$$3P^3 < N < 2N < 15P^3.$$

By Lemmas 18, 29,

$$\sum_{N<n\leqq 2N} (\varrho(n) - \psi(n)\mathfrak{S}(n))^2 = O\left(P^{4+\frac{1}{10}+3\delta}\right).$$

For any n in this range which is not representable as the sum of four positive integral cubes we have $\varrho(n) = 0$, whence, by Lemmas 18, 27,

$$(\varrho(n) - \psi(n)\mathfrak{S}(n))^2 > \left(c_2 P^{\frac{3}{5}} c_{11} (\log\log 15P^3)^{-c_{12}}\right)^2$$
$$> P^{\frac{6}{5}-\varepsilon}.$$

Thus we have

$$E(2N) - E(N) = O\left(P^{4+\frac{1}{10}+3\delta-\frac{6}{5}+\varepsilon}\right)$$
$$= O\left(P^{3-\frac{1}{10}+3\delta+\varepsilon}\right)$$
$$= O\left(N^{1-\frac{1}{30}+\delta_1}\right)$$

for $N > N_0 = N_0(\delta_1)$. Hence, if $2^{r_0+1} < \frac{N}{N_0}$,

$$E\left(\frac{N}{2^r}\right) - E\left(\frac{N}{2^{r+1}}\right) = O\left(\left(\frac{N}{2^{r+1}}\right)^{1-\frac{1}{30}+\delta_1}\right) \text{ for } 0 \leqq r \leqq r_0,$$

$$E(N) = O\left(\frac{N}{2^{r_0+1}}\right) + O\left(\sum_{r=0}^{r_0}\left(\frac{N}{2^{r+1}}\right)^{1-\frac{1}{30}+\delta_1}\right).$$

Choose r_0 so that $2^{r_0} \leqq N^{\frac{1}{30}} < 2^{r_0+1}$. The condition $2^{r_0+1} < \frac{N}{N_0}$ is satisfied for $N > N_1 = N_1(\delta_1)$. Hence, for $N > N_1$,

$$E(N) = O\left(N^{1-\frac{1}{30}+\delta_1}\right).$$