THE METHOD OF SUCCESSIVE APPROXIMATIONS FOR FUNCTIONAL EQUATIONS.

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In the functional analysis abstract linear spaces are considered which may have for their elements mathematical objects of a various nature: numbers, sequences of numbers, functions etc. Therefore theorems established for such abstract spaces usually can be applied to very different branches of mathematical analysis. Thus the general theory of functional equations, i. e. of such equations where the unknown quantities are elements of a linear space, comprises the theories of differential, integral and some other equations considered in analysis as well as the theory of finite and infinite systems of algebraic equations. One of the most important methods for establishing the existence of solutions and for the investigations of these solutions is the method of successive approximations. We shall give here the general theory of this method for linear and non-linear functional equations in a very wide class of spaces viz. the spaces normed with the elements of a semi-ordered space. This class comprises inter alia Banach's spaces and semi-ordered spaces.[1] The theory of this method will be based on the principle of majorants. We shall give also some applications of the general theory to the systems of algebraic equations and to the differential and integral equations.

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[1] For the case of semi-ordered spaces some theorems on these functional equations have been published already by the author of the present paper. See List of Literature, vide KANTOROVITCH, I.

Contents.

§ 1. The Spaces Considered.

1. Spaces of the Type B.

A linear (or vector) set is by definition a set $X = \{x\}$ in which the following operations are defined and satisfy the usual rules[1]: sum $x_1 + x_2$ of two elements of X, their difference $x_1 - x_2$, the product λx of $x \in X$ by a real number; and the zero element 0 of X.

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[1] Cf. S. Banach, I, p. 26.

A linear set X is called a normed space if to every element x of X a (necessarily non-negative) real number $\|x\|$ is correlated (which is called the norm of x) and the following 3 conditions are satisfied:

1). $\|x\| = 0$ if and only if $x = 0$,

2). $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$,

3). $\|\lambda x\| = |\lambda| \cdot \|x\|$.

We shall say that a sequence $\{x_n\}$ of elements of a normed space X converges towards $x\,(x \in X)$ and write $x_n \to x$ or $x_n \to x\,(b)$ if

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

A normed space X is called a space of the type B if it is complete i. e. if whatever be a sequence x_n of elements of X satisfying the condition

$$\lim_{n,\, m \to \infty} \|x_n - x_m\| = 0$$

there exists an element $x \in X$ such that $x_n \to x\,(b)$.[1]

2. Semi-ordered Spaces.[2]

A semi-ordered space is a linear set Z in which the relation $z_1 > z_2$ is defined for certain pairs of its elements.[3] This relation must satisfy the usual rules concerning the inequalities. In these spaces we can define (in the usual manner) the notions of an upper and a lower bound of a set $E \subset Z$ and also of its least upper and greatest lower bounds. We shall suppose moreover that every finite set is bounded and that any bounded subset E of Z has a least upper and a greatest lower bound. These bounds we shall denote resp. sup E and inf E. A space possessing all these properties we shall call a space of the type K_5.[4]

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[1] Ibid., p. 33.

[2] KANTOROVITCH, II, §§ 1—4.

[3] For two given elements z_1 and z_2 may subsist neither of the relations $z_1 > z_2$; $z_1 < z_2$; $z_1 = z_2$.

[4] All these conditions are fulfilled if Z satisfies the following 5 axioms

1) 0 is not > 0.

2) $z_1 > 0$ and $z_2 > 0$ implies $z_1 + z_2 > 0$.

3) For any $z \in Z$ there exists an $z_1 \geq 0$ such that $z_1 - z \geq 0$.

4) If $z > 0$, $z \in Z$ and λ is a positive real number then $\lambda z > 0$.

5) Any bounded set E possesses the least upper bound.

Cf. KANTOROVITCH, II.

The following definitions will be useful:

The positive and the negative parts of z (z_+ and z_-) are defined as follows: $z_+ = \sup(0, z)$; $z_- = \sup(-z, 0)$. It may be easily seen that $z = z_+ - z_-$.

The »absolute value» $|z|$ of z is defined as $|z| = \sup(z, -z) = z_+ + z_-$. This absolute value possesses the following properties: 1). $|0| = 0$; $|z| > 0$ if $z \neq 0$; 2). $|z_1 + z_2| \leq |z_1| + |z_2|$; 3). $|\lambda z| = |\lambda| \cdot |z|$.

A limit. If Z_n is a bounded sequence, we shall denote

$$\overline{\lim_{n\to\infty}}\, z_n = \inf_n [\sup(z_n, z_{n+1}, \ldots)]; \quad \underline{\lim_{n\to\infty}}\, z_n = \sup_n [\inf(z_n, z_{n+1}, \ldots)].$$

If $\overline{\lim\limits_{n\to\infty}}\, z_n = \underline{\lim\limits_{n\to\infty}}\, z_n = z$ we shall say that the sequence z_n converges (o) towards z and write $\lim\limits_{n\to\infty} z_n = z$ or $z_n \to z\,(o)$. This limit possesses all the elementary properties of the ordinary limits. In particular it satisfies the criterium of Cauchy, i. e. if for a given sequence $\lim\limits_{n,\, m\to\infty} |z_n - z_m| = 0$ then the sequence z_n converges (o) i. e. $z \in Z$ exists such that $z_n \to z\,(o)$, or (which is the same) $\lim |z_n - z| = 0$.[1]

3. Spaces, Normed with the Elements of a Semi-ordered Space.[2]

Let Y be a linear set and suppose that to every element $y \in Y$ is correlated an element $|y|$ of a certain semi-ordered space Z (of the type K_5), so that the following conditions are satisfied:

1). $|y| = 0$ if and only if $y = 0$;

2). $|y_1 + y_2| \leq |y_1| + |y_2|$;

3). $|\lambda y| = |\lambda|\,|y|$.

We shall say then that Y is normed with Z. From 1)—3) follows that for any $y \neq 0$ we have $|y| > 0$.

We shall write $y_n \to y\,(bs)$ if $|y_n - y| \to 0\,(o)$ in the space Z. If the space Y is complete, i. e. if any sequence y_n such that $\lim\limits_{n,\, m\to\infty} |y_n - y_m| = 0$ converges (bs) towards an element $y \in Y$, i. e. $y_n \to y\,(bs)$ then we shall call Y the space of the type B_S. This type includes both preceding types B and K_5 or more precisely any space of one of these types can be easily converted into a space of the type B_S. In fact if Y is a space B we may take for Z the space of real

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[1] KANTOROVITCH, II. Theorem XX b).

[2] These spaces were introduced in KANTOROVITCH, III, p. 272.

numbers and set $\|y\| = \|y\|$. In the case when Y is a space K_5 then we set $Z = Y$ and $\|y\| = |y|$. All the properties enumerated above of the space (B_S) are then satisfied (this follows from what has been already said in 2. and 3.). Accordingly in what follows we shall consider chiefly the spaces B_S.

4. Some Particular Spaces.[1]

1). The space of measurable functions. Consider the set S of all measurable functions defined in a given measurable set E. Two elements of the set S we shall consider as identical if their values coincide almost everywhere in E. We shall write $\varphi_1 \geq \varphi_2$ if $\varphi_1(t) \geq \varphi_2(t)$ for almost all points of E.

This last condition transforms S into a semi-ordered space; $\varphi_n \to \varphi\,(o)$ if $\varphi_n(t) \to \varphi(t)$ almost everywhere.

2). Secondly such measurable functions that $\int\limits_E |\varphi(t)|^p\, dt < +\infty$ (p is here a fixed number ≥ 1) constitute another semi-ordered space L^p; in this case $\varphi_n \to \varphi\,(o)$ means that $\varphi_n \to \varphi\,(o)$ in S and there exists a function $\varphi_0 \in L^p$ such that for any n: $|\varphi_n| \leq \varphi_0$. The set $L^{(p)}$ may be considered also as a space of the type B: if we set $\|\varphi\| = \left(\int\limits_E |\varphi(t)|^p\, dt\right)^{\frac{1}{p}}$. The convergence (b) will be in this case the convergence in mean of the order p, i. e. $\varphi_n \to \varphi\,(b)$ in L^p is equivalent to

$$\int\limits_E |\varphi_n(t) - \varphi(t)|^p\, dt \to 0.$$

3). Consider further the space s of all sequences $y = (\eta^{(1)}, \eta^{(2)}, \ldots)$ of real numbers, where $y_1 \geq y_2$ if for all i: $\eta_1^{(i)} \geq \eta_2^{(i)}$. In this space $y_n \to y\,(o)$ if for every i we have $\eta_n^{(i)} \to \eta^{(i)}$ (when $n \to \infty$). We may consider also the space (m) of bounded sequences, i. e. of such sequences that $\sup\limits_i |\eta^{(i)}| < +\infty$. This last space may be considered either as semi-ordered or as normed, with $\|y\| = \sup\limits_i |\eta^{(i)}|$. The convergence will be different in both cases.

4). The n-dimensional vector space is the set of all systems $y = (\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$ of n real numbers. This space may be considered either as semi-ordered (as above) or as a space of the type B with either

[1] Vide KANTOROVITCH II, §§ 10—12 and S. BANACH § 7.

$$\|x\| = \sup_i |\eta^{(i)}| \quad \text{or} \quad \|x\| = \left(\sum_{i=1}^{n} (\eta^{(i)})^2\right)^{\frac{1}{2}}.$$

5). The space C of continuous functions $x(t)$ defined in a segment $[a; b]$ with the norm $\|x\| = \sup_{a \leq t \leq b} |x(t)|$.

§ 2. Fundamental Theorems on Functional Equations.

5. Existence Theorems.

Theorem I. Let Z be a space of the type K_5. Consider the equation

$$z = V(z). \tag{1}$$

We shall suppose that ($z' > 0$ being a certain fixed element of Z):

1). $V(z)$ is defined for every $z \in Z$ such that $0 \leq z \leq z'$ and that for any such z we have $V(z) \in Z$.

2). If $0 \leq z_1 \leq z_2 \leq \cdots \leq z'$ and $\lim_{n \to \infty} z_n = z$ (in this case evidently $0 \leq z \leq z'$) then $\lim_{n \to \infty} V(z_n) = V(z)$.

3). $V(z)$ is monotonous for $0 \leq z \leq z'$, i. e. if $0 \leq z < z + \Delta z \leq z'$ then $V(z) \leq V(z + \Delta z)$.

4). $V(0) \geq 0$.

5). $V(z') \leq z'$.

If all these conditions are satisfied then the equation (1) has a solution z^* such that $0 \leq z^* \leq z'$ and this solution can be found by the method of successive approximations.

Proof. Let $z_0 = 0$, $z_n = V(z_{n-1})$ for any natural n; we shall prove that for any n

$$0 = z_0 \leq z_1 \leq \cdots \leq z_n \leq z'.$$

For $n = 0$ these inequalities subsist. Suppose that they are proved for $n = n_0$; we shall prove them for $n = n_0 + 1$. In fact $z_{n_0+1} = V(z_{n_0}) \geq V(z_{n_0-1}) = z_{n_0}$ (by 3). and 5).), if $n_0 > 0$ and $z_{n_0+1} = V(z_{n_0}) = V(0) \geq 0 = z_{n_0}$ (by 4).), if $n_0 = 0$; thus in all cases $z_{n_0+1} \geq z_{n_0}$; on the other hand $z_{n_0+1} = V(z_{n_0}) \leq V(z') \leq z'$ (by 3). and 5).). We have proved thus that the sequence $z_1, z_2, \ldots$ is monotonous (increasing) and bounded. Therefore it converges towards an element z^* of Z such that $0 \leq z^* \leq z'$. Hence follows by condition 2).

$$V(z^*) = \lim_{n\to\infty} V(z_n)$$

or, $V(z_n)$ being identical with z_{n+1} and $\lim V(z_n)$ being therefore equal to $\lim_{n\to\infty} z_{n+1} = z^*$, we have

$$z^* = V(z^*).$$

As besides $0 \leq z^* \leq z'$ and $z^* = \lim_{n\to\infty} z_n$, Theorem I is completely proved.

Remark. Suppose that besides conditions 1).—5)., V satisfies the following condition: 2'). if $z' \geq z_1 \geq z_2 \geq \cdots\cdots \geq 0$ and $\lim_{n\to\infty} z_n = z$, then $V(z) = \lim_{n\to\infty} V(z_n)$. Then setting $z'_0 = z'$, $z'_{n+1} = V(z'_n)$ we obtain as before a solution z'^* of the equation (1). This solution will not necessarily coincide with z^*. It is easy to see however that

$$z'^* \geq z^*.$$

Theorem II. Let Y be a space of the type B_S normed by the elements of the space Z. The space Z and the operation V, as well as the element $z' \in Z$ shall be the same as in Theorem I. We shall consider the equation

$$y = U(y) \tag{2}$$

where the operation U is such that the equation (1) is a majorant of (2).

By this we shall mean that

1). $U(y)$ is defined for every $y \in Y$ such that

$$|y| \leq z'$$

and for any such y we have $U(y) \in Y$,

2). $|U(0)| \leq V(0)$,

3). $|U(y + \Delta y) - U(y)| \leq V(z + \Delta z) - V(z)$ if only $|y| \leq z$, $|\Delta y| \leq \Delta z$; $z + \Delta z \leq z'$.

Then the equation (2) possesses a solution y^* satisfying the inequality $|y^*| \leq z'$ (or more precisely $|y^*| \leq z^*$) which can be found by the method of successive approximations.

Proof. Set $y_0 = 0$, $y_n = U(y_{n-1})$. We shall prove that for any n and m such that $m > n$

$$|y_m - y_n| \leq z_m - z_n.$$

In fact $|y_1 - y_0| = |y_1| = |U(0)| \leq V(0) = z_1 = z_1 - z_0$. Suppose that we have proved the above inequality for any m and n such that $n < m \leq m_0$ ($m_0 \geq 1$). We shall prove it for $n < m \leq m_0 + 1$; then (by 3).) if $m_0 \geq n \geq 1$

$$|y_{m_0+1} - y_n| = |U(y_{n-1} + (y_{m_0} - y_{n-1})) - U(y_{n-1})| \leq$$
$$\leq V(z_{n-1} + (z_{m_0} - z_{n-1})) - V(z_{n-1}) = z_{m_0+1} - z_n$$

because $|y_{n-1}| = |y_{n-1} - y_0| \leq z_{n-1} - z_0 = z_{n-1} (n - 1 \leq m_0)$. In the case when $n = 0$ we have

$$|y_{m_0+1} - y_n| = |y_{m_0+1}| \leq |y_{m_0+1} - y_1| + |y_1| \leq (z_{m_0+1} - z_1) + z_1 = z_{m_0+1} - z_n.$$

We have proved thus our inequality for any n, m such that $n < m$. But now as

$$z_m - z_n \underset{m,\, n \to \infty}{\to} 0 \ (o) \text{ in } Z$$

it follows that $|y_m - y_n| \to 0$ (o) (in Z) whence (by Cauchy's criterium for the space) there exists a

$$(b\, s) \lim_{n \to \infty} y_n = y^* = y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots.$$

As on the other hand $|y_n| \leq z_n$, it follows that $|y^*| \leq z^* \leq z'$.

Besides
$$|U(y^*) - U(y_n)| \leq V(z^*) - V(z_n) = z^* - z_{n+1}$$

(because $|y^* - y_n| \leq z^* - z_n$ and $|y_n| \leq z_n$). Hence follows

$$U(y^*) = (b\, s) \lim_{n \to \infty} U(y_n) = \lim_{n \to \infty} y_{n+1} = y^*.$$

So that y^* is the required solution.

Remark. We shall call these solutions z^* and y^* of resp. equations (1) and (2) obtained by the method of successive approximations and starting with the value 0, the principal solutions of these equations.

Corrolary.[1] If the inequality

$$|U(y + \Delta y) - U(y)| \leq \alpha |\Delta y|, \tag{3}$$

where $0 < \alpha < 1$, subsists for all values of y and Δy then the equation

$$y = U(y) + y_0 \tag{4}$$

has a solution whatever be y_0.

[1] In the case when $Y = Z$ is a space of the type B (and $|y| = \|y\|$) this corrolary coincides with the so-called principle of CACCIOPOLLI-BANACH; vide, e. g., S. BANACH, II, p. 161 and CACCIOPOLLI, I.

Let in fact

$$V(z) = | U(0) + y_0 | + \alpha z$$

then setting

$$z' = \frac{| U(0) + y_0 |}{1 - \alpha}$$

we have $V(z') = z'$. On the other hand $V(0) \geq 0$ and V evidently satisfies also all other conditions of Theorem I. Also the function $U(y) + y_0$ satisfies all conditions of Theorem II (cond. 3) is satisfied because of (3)). Therefore we may apply Theorem II to this function and conclude that a solution y^* of the equa tion (4) exists such that

$$| y^* | \leq \frac{| U(0) + y_0 |}{1 - \alpha}.$$

6. **Unicity Theorems.**

Generally speaking there can be more than one solution of the equation (2), however if the majorant-equation (1) has only one solution then the same is true of the equation (2), or, more exactly:

Theorem III. *If the functions U and V satisfy all conditions of Theorem II and V satisfies besides the condition 2'). (see 5. Rem.) and if moreover $z^* = z'^*$ then there exists but one solution of the equation (2) satisfying the condition $| y | \leq z'$. This solution can be obtained by the method of successive approximations starting with any value y'_0 of y satisfying the inequality $| y'_0 | \leq z'$.*

Proof. Let $y'_0 \in Y$ and suppose that $| y'_0 | \leq z'_0 = z'$. Set $y'_n = U(y'_{n-1})$. Then we have (using the notations of Theorems I and II)

$$| y'_0 - y_0 | = | y'_0 | \leq z'_0 = z'_0 - z_0.$$

We shall prove now that in general

$$| y'_n - y_n | \leq z'_n - z_n.$$

For $n = 0$ this inequality is true. Suppose that it is true for $n = n_0$; we shall prove it for $n = n_0 + 1$. In fact

$$| y'_{n_0+1} - y_{n_0+1} | = | U(y'_{n_0}) - U(y_{n_0}) | = | U(y_{n_0} + (y'_{n_0} - y_{n_0})) - U(y_{n_0}) | \leq$$
$$\leq V(z_{n_0} + (z'_{n_0} - z_{n_0})) - V(z_{n_0}) = z'_{n_0+1} - z_{n_0+1}.$$

The above inequality is thus proved inductively. But we have $\lim_{n\to\infty} z'_n = z'^* = z^* = \lim_{n\to\infty} z_n$ and consequently $\lim_{n\to\infty}(z'_n - z_n) = 0$. Hence $\lim_{n\to\infty} y'_n = \lim_{n\to\infty} y_n = y^*$; thus starting with an arbitrary y'_0 satisfying only the condition $|y'_0| \leq z'$ we arrive at the principal solution y^* of the equation (2).

Suppose in particular that y'_0 satisfies the equation (2); then $y'_n = y'_0$ and $y^* = \lim_{n\to\infty} y'_n = y'_0$. Thus we have proved that there can be no other solutions y of (2) satisfying the inequality $|y| \leq z'$ except y^*.

Corrolary. The equation (4), where U satisfies the condition (3), admits of but one solution which can be obtained by the method of successive approximations starting with an arbitrary element of y.

In fact the majorant equation

$$z = |U(0) + y_0| + \alpha z$$

has evidently one solution only.

Theorem III'. If U and V satisfy the conditions of Theorem II then there exists only one solution of the equation (2) satisfying the condition $|y| \leq z^*$. This solution can be obtained by the method of successive approximations starting with any y'_0 such that $|y'_0| \leq z^*$.

Theorem III' is proved in exactly the same manner as Theorem III with this difference only that everywhere in the proof we must substitute z^* for z'_n.

7. Continuity Questions.

We shall prove now that if an equation of type (2) depends in a continuous manner on a parameter then (if certain conditions are satisfied) its solution is also a continuous function of the parameter.

Theorem IV. If an equation is given

$$y = U(y; \lambda) \tag{5}$$

where U is a function of y and λ defined for a certain set Λ of values of λ containing $\lambda_0 (\lambda_0 \in \Lambda)$; and if the following conditions are satisfied: 1). the equation (1) is majorant to (5) for every value of λ; moreover V is defined and satisfy the above conditions for $0 \leq z \leq 3z'$ and condition 3). of Theorem II is satisfied for every y and Δy such that $|y| \leq z \leq z'$ and $|\Delta y| \leq \Delta z \leq 2z'$;

2). $U(y, \lambda)$ as function of λ is continuous in the point λ_0, i. e. $\lim\limits_{\lambda \to \lambda_0} U(y, \lambda) = U(y, \lambda_0)$; 3). V satisfies condition 2') of Remark to Theorem I; then the principal solution $y^*(\lambda)$ of the equation (5) converges towards $y^*(\lambda_0)$ when $\lambda \to \lambda_0$ i. e. $y^*(\lambda)$ regarded as a function of λ is continuous in the point λ_0.

Proof. We have (for any n)

$$y^*(\lambda) = y_n(\lambda) + (y^*(\lambda) - y_n(\lambda))$$

and besides

$$|y^*(\lambda) - y_n(\lambda)| \leq z^* - z_n.$$

We shall prove now inductively that for any n

$$\lim_{\lambda \to \lambda_0} y_n(\lambda) = y_n(\lambda_0).$$

In fact for $n = 0$ this equality is true (because then both members are 0) suppose that it is true for $n = n_0$. Then we have

$$\begin{aligned} |y_{n_0+1}(\lambda) - y_{n_0+1}(\lambda_0)| &= |U(y_{n_0}(\lambda), \lambda) - U(y_{n_0}(\lambda_0), \lambda_0)| \leq \\ &\leq |U(y_{n_0}(\lambda), \lambda) - U(y_{n_0}(\lambda_0), \lambda)| + |U(y_{n_0}(\lambda_0), \lambda) - U(y_{n_0}(\lambda_0), \lambda_0)| \leq \\ &\leq V(|y_{n_0}(\lambda_0)| + |y_{n_0}(\lambda) - y_{n_0}(\lambda_0)|) - V(|y_{n_0}(\lambda_0)|) + \\ &+ |U(y_{n_0}(\lambda_0), \lambda) - U(y_{n_0}(\lambda_0), \lambda_0)| \end{aligned}$$

because $|y_n(\lambda_0)| \leq z'$ and $|y_{n_0}(\lambda) - y_{n_0}(\lambda_0)| \leq |y_{n_0}(\lambda)| + |y_{n_0}(\lambda_0)| \leq 2z'$. The first summand in the right hand member of this inequality converges towards 0 because $y_{n_0}(\lambda) \to y_{n_0}(\lambda_0)$ $(b\,s)$ and V satisfies the condition 2'); the second summand converges towards 0 because U is continuous for $\lambda = \lambda_0$. This proves that $y_{n_0+1}(\lambda) \to y_{n_0+1}(\lambda_0)$. Thus we have proved that for any n: $y_n(\lambda) \underset{\lambda \to \lambda_0}{\to} y_n(\lambda_0)$.

Now from the inequality

$$|y^*(\lambda) - y_n(\lambda)| \leq z^* - z_n$$

follows

$$\begin{aligned} |y^*(\lambda) - y^*(\lambda_0)| &\leq |y^*(\lambda) - y_n(\lambda)| + |y^*(\lambda_0) - y_n(\lambda_0)| + \\ &+ |y_n(\lambda) - y_n(\lambda_0)| \leq 2(z^* - z_n) + |y_n(\lambda) - y_n(\lambda_0)|. \end{aligned}$$

If $\lambda \to \lambda_0$ then (according to what precedes)

$$\overline{\lim_{\lambda \to \lambda_0}} |y^*(\lambda) - y^*(\lambda_0)| \leq 2(z^* - z_n),$$

n being arbitrary there follows

$$\lim_{\lambda \to \lambda_0} y^*(\lambda) = y^*(\lambda_0). \qquad \text{q. e. d.} \tag{6}$$

§ 3. Linear Equations.

8. Operations Admitting a Majorant.

In this § we shall apply the general theorems of the preceding § to the special case of linear equations.

Let Y be a space of the type B_S. Consider an additive operation transforming the space Y into itself, i. e. such that whatever be $y \in Y$ there exist always $f(y) \in Y$ and that for any $y_1 \in Y$, $y_2 \in Y$ we have $f(y_1+y_2)=f(y_1)+f(y_2)$. Suppose that there exists an operation g transforming the space Z into itself, additive, positive (i. e. such that $z \geq 0$ implies $g(z) \geq 0$) and besides such that for any $y \in Y$ the inequality

$$\| f(y) \| \leq g(\| y \|) \tag{7}$$

subsists. We shall say then that the operation f admits a majorant and that g is a majorant for f.

In this case (i. e. if f admits a majorant) f is necessarily homogenous, i. e. whatever be $y \in Y$ and a real number λ we have always $f(\lambda y) = \lambda f(y)$. In fact in the case of rational λ this follows from the definition of an additive operation. Suppose λ irrational and let $\lambda_n (n = 1, 2, \ldots)$ be rational numbers such that $|\lambda_n - \lambda| < \frac{1}{n}$. We have then

$$\| \lambda f(y) - f(\lambda y) \| \leq \| (\lambda - \lambda_n) f(y) \| + \| f(\lambda y - \lambda_n y) \| \leq$$
$$\leq \frac{1}{n} \| f(y) \| + g(|\lambda - \lambda_n| \cdot \| y \|) \leq \frac{1}{n} \| f(y) \| + \frac{1}{n} g(\| y \|)$$

whence n being arbitrary and in a semi-ordered space $z \geq 0$ always implying $\inf \left(\frac{1}{n} z\right) = 0$ we obtain $f(\lambda y) = \lambda f(y)$.

In the particular case when $Y = Z$ and $\| y \| = |y|$ (i. e. when Z is a semi-ordered space and for the norm of an element z of Z we take its »absolute value» $|z|$) the set of all operations f admitting a majorant g, (i. e. such that for any $z \in Z$

$$|f(z)| \leq g(|z|)$$

can be considered itself as a semi-ordered space (if we write $f_2 > f_1$ when the operation $f_2 - f_1$ is positive) which we shall denote $(Z \to Z)$.[1] Accordingly for

[1] Kantorovitch, III, Theor. 3.

such operations can be defined f_+, f_-, $|f|$ (which are again operations belonging to $Z \rightarrow Z$ and moreover positive) and hold the formulae $f = f_+ - f_-$, $|f| = f_+ + f_-$ &c. We see thus that any operation belonging to $Z \rightarrow Z$ is the difference of two positive operations. On the other hand every positive operation is its own majorant so thus every positive operation and consequently every *difference of two positive operations belongs to* $Z \rightarrow Z$.

If Z satisfies some additional conditions then the class $Z \rightarrow Z$ coincides with the class of (o)-continuous additive operations, i. e. of such additive operations that $z_n \rightarrow z\,(o)$ implies $f(z_n) \rightarrow f(z)\,(o)$.[1]

We shall remark lastly that if Y is a space of the type B, i. e. if $|y| = \|y\|$ then an additive operation admits a majorant if and only if a constant C exists such that for any $y \in Y$

$$\|f(y)\| \leq C\,\|y\|.$$

The smallest possible constant here we shall define as $\|f\|$.

This class of operations coincides with the class of (b)-continuous additive operations.[2]

9. Fundamental Theorems for Linear Equations.

In the case when operations mentioned in theorems of § 2 are linear these theorems may be somewhat simplified.

Theorem V. Let f be a linear operation admitting a continuous majorant g; then if for a certain z', $z' \geq 0$ we have

$$z_0 = z' - g(z') \geq 0 \tag{8}$$

then the equation

$$y = f(y) + y_0 \tag{9}$$

has a solution for any y_0 such that $|y_0| \leq z_0$.

Proof. This theorem follows immediately from Theorem II, if we set $U(y) = f(y) + y_0$; $V(z) = g(z) + z_0$.

In fact all conditions of Theorem II are satisfied:

$$V(0) = z_0 \geq 0;\; V(z') = g(z') + z_0 = z';$$

$$V(z + \varDelta z) - V(z) = g(\varDelta z) \geq 0 \text{ if } \varDelta z \geq 0;$$

[1] As to which vide Ibid., Theor. IV.

[2] Cf. S. Banach, I, p. 54.

$V(z)$ is a continuous function,

$$|U(0)| = |y_0| \leq z_0 = V(0);$$

$$|U(y + \Delta y) - U(y)| = |f(\Delta y)| \leq g(|\Delta y|) \leq g(\Delta z) = V(z + \Delta z) - V(z),$$

if only $|\Delta y| \leq \Delta z$.

Hence follows by Theorem II that the equation $y = U(y) = f(y) + y_0$ possesses a solution which can be found by the method of successive approximations.

Theorem VI. Let z^* be the principal solution of the equation (8). Then there exists only one solution of the equation (9) satisfying the condition $|y| \leq k z^*$ (k is a real number ≥ 1), viz. the principal solution of (9). Besides if the equation (8) possesses only one solution satisfying the condition $|z| \leq z'$ then (9) has only one solution satisfying the inequality $|y| \leq k z'$.

The theorem follows immediately from Theorem III if we take $V(z) = g(z) + k z_0$ and substitute $k z'$ for z'.

Corrolary. Let Y be a space of the type B. Then if $\|f\| = \alpha < 1$ then the equation (9) has exactly one solution whatever be $y_0 \in Y$. This solution satisfies the following inequality

$$\|y\| \leq \frac{\|y_0\|}{1 - \alpha}$$

and can be found by the method of successive approximations starting with an arbitrary $y \in Y$.

This corrolary can be proved by applying to the equation (9) the corrolaries of Theorems II and III; it may be found in Banach.[1]

Theorem VII. If

$$y = f(y; \lambda) + y_0(\lambda) \tag{10}$$

is an equation depending of a parameter λ and the equation (8) is majorant to (10) for any λ and if (whatever be $y \in Y$)

$$f(y; \lambda) \underset{\lambda \to \lambda_0}{\to} f(y) \text{ and } y_0(\lambda) \underset{\lambda \to \lambda_0}{\to} y_0$$

then

$$\lim_{\lambda \to \lambda_0} y^*(\lambda) = y^*$$

where $y^*(\lambda)$ and y^* are principal solutions of (10) and (9) respectively.

[1] S. Banach, I, p. 158.

This theorem follows from Theorem IV where we set $U(y;\lambda)=f(y;\lambda)+y_0(\lambda)$, $U(y;\lambda_0)=f(y)+y_0$ and $V(z)=g(z)+z_0$. It is easy to verify that all the conditions of Theorem IV are thus fulfilled.

10. Approximate Solution of Equations.

In order to obtain an approximate solution of an equation of the type (9) we must substitute in (9) for f another simpler operation f' such that the norm of difference $f'(y)-f(y)$ be small and that the solution of the equation

$$y'=f'(y')+y_0 \tag{11}$$

be known to us. Then this solution of (11) will be the approximate value of the solution of (9). We shall give now a theorem which will enable us (under certain conditions) to estimate the error of this approximate solution and in the same time will prove (in the case the above mentioned conditions are satisfied) the existence of a solution of the equation (9).

Theorem VIII. Let the equation (11) have a solution for every $y_0\in Y$ and suppose that this solution is an additive function of y_0

$$y'=\Gamma(y_0)$$

and that the following inequality subsists for every $y\in Y$

$$|\Gamma(f(y)-f'(y))|\leq\alpha|y|. \qquad (0<\alpha<1) \tag{12}$$

Then the equation (9) has a solution y and the following inequality subsists

$$|y-y'|\leq\frac{\alpha}{1-\alpha}|y'|. \tag{13}$$

Proof. We can write the equation (9) as follows

$$y=(f-f')(y)+f'(y)+y_0. \tag{14}$$

This equation (14) is evidently equivalent (because $\Gamma(y_0)$ is the solution of (11)) with the equation

$$y=\Gamma((f-f')(y)+y_0)=T(y)+y' \tag{15}$$

where $T=\Gamma(f-f')$. To this equation we can apply the corrolary to Theorem II (condition (3) of this corrolary takes in this case the form of the inequality (12) which we have supposed to be true). Therefore the equation (15) has a solution y such that

$$|y| \leq \frac{1}{1-\alpha} |y'|.$$

This solution is in the same time a solution of the equation (9). Deducing the equation (11) from (14) we obtain the equality

$$y - y' = f'(y - y') + (f - f')(y),$$

whence

$$y - y' = \Gamma((f - f')(y))$$

and

$$|y - y'| \leq \alpha |y| \leq \frac{\alpha}{1-\alpha} |y'|$$

which prove (13).

Remark if Y is a space of the type B then the condition (12) may be written as follows

$$\|\Gamma\| \cdot \|f - f'\| \leq \alpha \qquad (0 < \alpha < 1). \tag{12 a}$$

§ 4. Systems of Algebraic Equations.

In this paragraph we shall apply the results of the preceding paragraphs to systems of algebraic equations chiefly to certain classes of infinite systems of equations.

11. A Class of Infinite Systems of Linear Equations.

Consider first the systems of equations of the form

$$\eta_i = \sum_{k=1}^{\infty} c_{i,k} \eta_k + b_i \qquad (i = 1, 2, \ldots) \tag{16}$$

where (for any i)

$$\sum_{k=1}^{\infty} |c_{i,k}| \leq \alpha < 1. \tag{17}$$

Such systems were considered, e. g. by Helge v. Koch, I.

Let Y be the space m of bounded sequences $y = (\eta_1, \eta_2, \ldots)$ where $\|y\|$ is defined as $\sup_i |\eta_i|$ and suppose that the sequence $b = (b_1, b_2, \ldots)$ belongs to Y, i. e. that

$$\|b\| = \sup_i |b_i| < +\infty. \tag{18}$$

The operation $f(y) = \left\{\sum_{k=1}^{\infty} c_{i,k}\eta_k\right\}$ transforms the space Y into itself and we have besides

$$\|f(y)\| = \sup_i \left|\sum_{k=1}^{\infty} c_{i,k}\eta_k\right| \leq \sup_i \sum_{k=1}^{\infty} |c_{i,k}| \cdot \|y\| \leq \alpha \|y\|$$

or $\|f\| \leq \alpha$.

The equation (16) is thus an equation of the type (9) satisfying the conditions of Corollary to Theorem VI. Hence follows

Theorem IX.[1] *If conditions (17) and (18) are fulfilled then the system (16) possesses one and only one bounded solution (i. e. one and only one such solution that $\sup_i |\eta_i| < +\infty$). This only solution can be found by the method of successive approximations starting with any bounded system af values of $\{\eta_i\}$ and it satisfies the following condition*

$$\sup_i |\eta_i| \leq \frac{1}{1-\alpha} \sup_i |b_i|.$$

Remark 1. If instead of condition (17) our system satisfies the following (weaker) conditions:

$$\left.\begin{aligned} &1) \qquad \sum_{k=N+1}^{\infty} |c_{i,k}| \leq \alpha < 1 \qquad (i = N+1, N+2, \ldots) \\ &2) \qquad \sum_{k=N+1}^{\infty} |c_{i,k}| < +\infty \qquad (i = 1, 2, \ldots, N) \end{aligned}\right. \tag{17'}$$

and there exists an upper bound of all numbers $|c_{i,k}|$, then we can apply the preceding theorem to the system

$$\eta_i = \sum_{k=N+1}^{\infty} c_{i,k}\eta_k + \left(b_i + \sum_{k=1}^{N} c_{i,k}\eta_k\right) \qquad (i = N+1, N+2, \ldots) \tag{16'}$$

and express the unknown quantities $\eta_{N+1}, \ldots$ as functions of $\eta_1, \ldots, \eta_N$. Then after substituting these functions for $\eta_{N+1}, \ldots$ into the first N equations of the system (16) we shall obtain a finite system of equations. Hence, e. g., such a system does not possess more than N linearly independent solutions etc.

---

[1] V. Koch, I, Kantorovitch & Kryloff, p. 28—31.

Mark that conditions (17') will be always satisfied when the system satisfies any of the following two conditions (which were considered by Helge v. Koch)

a) $$\sum_{i,k=1}^{\infty} |c_{i,k}| < +\infty$$ [1]

b) $$|c_{i,k}| < \varkappa_i \qquad (k = 1, 2, \ldots)$$

where $\sum_{i=1}^{\infty} \varkappa_i < +\infty$.[2]

Remark 2. If instead of the space of bounded sequences we shall consider the space l^p (where $p \geq 1$) of such sequences $y = (\eta_1, \eta_2, \ldots)$ that

$$\|y\| = \left(\sum_{i=1}^{\infty} |\eta_i|^p\right)^{\frac{1}{p}}$$

is finite then we can arrive to the analogous conclusions.

In this case the expression of $\|f\|$ is rather complicated but it is easy to see that in any case

$$\|f\| \leq \left[\sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |c_{i,k}|^{\frac{p}{p-1}}\right)^{p-1}\right]^{\frac{1}{p}} = H.$$ [3]

Consequently if $H < 1$ then the system (16) possesses a single solution satisfying the condition

$$\sum_{i=1}^{\infty} |\eta_i|^p < +\infty$$

if only

$$\sum |b_i|^p < +\infty .$$

---

[1] v. Koch, II.

[2] v. Koch, III.

[3] In fact

$$\|f(x)\| = \left\| \left\{\sum_{k=1}^{\infty} c_{i,k} x_k\right\}_{i=1,2,\ldots} \right\| = \left\{\sum_{i=1}^{\infty} \left|\sum_{k=1}^{\infty} a_{i,k} x_k\right|^p\right\}^{\frac{1}{p}} \leq$$

$$\leq \left\{\sum_{i=1}^{\infty} \left[\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |a_{i,k}|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right]^p\right\}^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \left\{\sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{i,k}|^{\frac{p}{p-1}}\right)^{p-1}\right\}^{\frac{1}{p}} =$$

$$= H \cdot \|x\|.$$

Vide Kantorovitch, IV, Theorem 6.

If instead of the inequality $H < 1$ the (weaker) condition $H < +\infty$ is satisfied then for the system (16′) the corresponding number H' will be < 1 provided that N is sufficiently great. Consequently we may argue as in Remark 1 and reduce the system (16) to a finite system. In particular when $p = 2$ this condition takes the form $\sum c_{i,k}^2 < +\infty$. This case ($p = 2$) has been considered by Helge von Koch.[1]

Consider again systems of the type (16) but instead of (17) we shall suppose the following weaker condition to be fulfilled

$$\sum_{k=1}^{\infty} |c_{i,k}| = 1 - \varrho_i < 1. \tag{19}$$

The set of all bounded sequences we shall consider now as a semi-ordered space.

I say that the system

$$\eta_i = \sum_{k=1}^{\infty} |c_{i,k}| \eta_k + K \varrho_i \tag{20}$$

is a majorant for the system (16) provided only that for any i we have

$$b_i| \leq K \varrho_i. \tag{21}$$

In fact if $f^*(y) = \left\{\sum_{k=1}^{\infty} |c_{i,k}| \eta_k\right\}$ then

$$f|(y)| = \left\{\left|\sum_{k=1}^{\infty} c_{i,k} \eta_k\right|\right\}_i \leq \left\{\sum_{k=1}^{\infty} |c_{i,k}||\eta_k|\right\}_i = f^*(|y|).$$

It is also evident that the operation f^* is continuous, i. e. that $y \to 0$ implies $f^*(y) \to 0$. The system (20) possesses a positive solution, viz. the solution $\eta_1 = \eta_2 = \cdots = K$. Hence Theorems V and VI of § 3 may be applied and we obtain the following theorems.

Theorem X. The system (16) satisfying the conditions (19) and (21) possesses a solution satisfying the inequalities $|\eta_i| \leq K$ ($i = 1, 2, \ldots$); the solution can be found by the method of successive approximations.[2]

---

[1] See H. v. Koch, IV.

[2] Dixon, I; Pellet, III; Wintner, II; Koyalovitch, I; Kuzmin, I; Kantorovitch & Kryloff.

Remark. Note that condition (21) is always satisfying in the case when all b_i, with the exception of a finite number of them, are 0.

Theorem XI.[1] If $\{\eta_i^*\}$ is the principal solution of the system (20) then there exists one only solution of (16) satisfying the inequality $|\eta_i| \leq \lambda \eta_i^*$. In particular if the lower bound h of the sequence η_i^* is not 0 (i. e. if for any i: $\eta_i^* \geq h > 0$) then each of the systems (16) and (20) possesses a single bounded solution.

So, e. g., we find that the system

$$\sum_{k=1}^{\infty} \frac{\eta_k}{(2i+1-2k)(2i-1-2k)} + b_i = 0 \qquad (i = 1, 2, \ldots)$$

possesses exactly one solution if $|b_i| \leq \frac{K}{i}$. On the other hand the system

$$\eta_i = \frac{1}{(i+1)^2}\eta_{i+1} + \frac{1}{(i+1)^2} \qquad (i = 1, 2, \ldots)$$

has two bounded solutions, viz. $\eta_i = 1$ and $\eta_i = \frac{1}{i+1}$.

We shall remark in conclusion that from the next No. will follow that the principal solution of the system (16) may be obtained not only by the method of successive approximations but also as the limit of the solutions of (finite) »reduced» systems.

12. A Class of Non-linear Systems.

Consider two systems of equations

$$\eta_i = b_i + \sum_{k_1=1}^{\infty} c_{k_1}^{(i)} \eta_{k_1} + \sum_{k_1, k_2=1}^{\infty} c_{k_1, k_2}^{(i)} \eta_{k_1} \eta_{k_2} + \cdots = f_i(\eta_1, \eta_2, \ldots) \quad (i=1, 2, \ldots) \tag{22}$$

and

$$\zeta_i = B_i + \sum_{k_1=1}^{\infty} C_{k_1}^{(i)} \zeta_{k_1} + \sum_{k_1, k_2=1}^{\infty} C_{k_1, k_2}^{(i)} \zeta_{k_1} \zeta_{k_2} + \cdots = \bar{f}_i(\zeta_1, \zeta_2, \ldots) \quad (i=1, 2, \ldots) \tag{23}$$

where we suppose that the second system is a majorant for the first, i. e. that for any i and $k_1, \ldots, k_j$

[1] Kantorovitch & Kryloff.

$$|b_i| \leq B_i \quad \text{and} \quad \left| c^{(i)}_{k_1, \ldots, k_j} \right| \leq C^{(i)}_{k_1, \ldots, k_j}. \tag{24}$$

Suppose moreover that the second system possesses a positive solution, i. e. there exists such sequence of positive numbers $z' = (\zeta'_1, \zeta'_2, \ldots)$ that if we substitute in (23) $\zeta'_1, \zeta'_2, \ldots$ for $\zeta_1, \zeta_2, \ldots$ then (for every i) the series in the right hand member of (23) will converge towards the left hand member of (23).[1] Then the equations (22) and (23) we may consider as the equations of the form (2) and (1) respectively (see above, Th. II) in the space $Y = Z = s$ (see above, p. 67) i. e. the semi-ordered space of all sequences of real numbers. We shall prove that all conditions of Theorems I and II are satisfied.

1. We know that $V(z')$ exists. It follows at once that $V(z)$ exists for any $z = (\zeta_1, \zeta_2, \ldots)$ such that $0 \leq z \leq z'$.

2. It is evident that in the region $0 \leq z \leq z'$ each of the series $\bar{f}_i(\zeta_1, \zeta_2, \ldots)$ converges uniformly and consequently if for every i we have $\lim\limits_{n \to \infty} \zeta_i^{(n)} = \zeta_i$ and $0 \leq \zeta_i^{(n)} \leq \zeta'_i$ then $\lim \bar{f}_i(\zeta_1^{(n)}, \zeta_2^{(n)}, \ldots) = \bar{f}_i(\zeta_1, \zeta_2, \ldots)$. Hence follows the continuity of the operation V (in the considered region $0 \leq z \leq z'$) and in particular the conditions 2) and 2').

3.—5. Are evident

$$[V(0) = \{B_i\}; \quad V(z') = z'].$$

As to operation U

1). $U(y)$ evidently exists for any y such that $|y| \leq z'$, i. e. that for every i: $|\eta_i| \leq \zeta'_i$.

2). $|U(0)| = \{|b_i|\} \leq \{B_i\}$.

3). We must prove that if $|y| \leq z$, $|\varDelta y| \leq \varDelta z$ and $z + \varDelta z \leq z'$ then

$$|U(y + \varDelta y) - U(y)| \leq V(z + \varDelta z) - V(z)$$

or in other words, if

$$|\eta_k| \leq \zeta_k \quad \text{and} \quad |\varDelta \eta_k| \leq \varDelta \zeta_k \qquad (k = 1, 2, \ldots)$$

then

$$\left| \sum_{j=1}^{\infty} \sum_{k_1, k_2, \ldots, k_j = 1}^{\infty} c^{(i)}_{k_1, \ldots, k_j} \left[(\eta_{k_1} + \varDelta \eta_{k_1}) \ldots (\eta_{k_j} + \varDelta \eta_{k_j}) - \eta_{k_1} \ldots \eta_{k_j} \right] \right| \leq$$

$$\leq \sum_{j=1}^{\infty} \sum_{k_1, \ldots, k_j = 1}^{\infty} C^{(i)}_{k_1, \ldots, k_j} \left[(\zeta_{k_1} + \varDelta \zeta_{k_1}) \ldots (\zeta_{k_j} + \varDelta \zeta_{k_j}) - \zeta_{k_1} \ldots \zeta_{k_j} \right].$$

But this last inequality is almost evident.

---

[1] Finite systems of this form were considered by Pellet (cf. I and II).

Thus all conditions of Theorem II (see § 2) are satisfied; its conclusion is therefore also true and we have thus

Theorem XII. Under the above conditions:

1). the system (22) possesses a solution satisfying the condition

$$|\eta_i| \leq \zeta'_i \tag{25}$$

which can be found by the method of successive approximations;

2). if $\{\zeta'_i\}$ is the principal solution of the system (23) then there exists only one solution of the system (22) satisfying the condition (25) viz. its principal solution;

3). if coefficients of the system (22) depend in a continuous manner on a certain parameter λ (so that $b_i = b_i(\lambda)$, $c^{(i)}_{k_1, \ldots, k_j} = c^{(i)}_{k_1, \ldots, k_j}(\lambda)$) and satisfy the inequality (24) for all values of λ, and if $\{\eta_i(\lambda)\}$ is the principal solution of (22) then the functions η_i are continuous.

This theorem follows immediately from Theorems II—IV. But from Theorem IV follows besides another interesting property of the systems of the type (22) (satisfying (24)).

Theorem XIII. Consider together with the system (22) the following »reduced» system (finite)

$$\eta_i = \sum_{j=1}^{N} \sum_{k_1, \ldots, k_j = 1}^{N} c^{(i)}_{k_1, \ldots, k_j} \eta_{k_1} \ldots \eta_{k_j} + b_i \qquad (i = 1, 2, \ldots, N) \tag{26}$$

where N is a natural number, and let $\{\eta_i^{(N)}\}$ be the principal solution of the system (26). Then

$$\lim_{N \to \infty} \eta_i^{(N)} = \eta_i^{(0)} \tag{27}$$

where $\{\eta_i^{(0)}\}$ is the principal solution of the system (25). In fact if we add to the system (26) the equations $\eta_i = 0\,(i = N + 1, \ldots)$ we obtain an infinite system of equations with coefficients depending on N. When $N \to \infty$ these coefficients converge towards coefficients of (22). Besides the system (23) is a common majorant of all these systems. Consequently, by Theorem IV the principal solution $\{\eta_i^{(N)}\}$ of the system (26) converges towards the principal solution $\{\eta_i^{(0)}\}$ of (22) which proves (27).

The following theorem is a corollary of Theorem XII.

Theorem XIV.[1] If there exists such positive numbers a and b that (f_i and $\bar{f}_i$ denoting the same operations as above, see (22) and (23))

$$f_i(a, b, b, \ldots) \leq M \tag{28}$$

for every i, then the system of equations

$$\eta_i = \lambda f_i(\lambda, \eta_1, \eta_2, \ldots) \qquad (i = 1, 2, \ldots) \tag{29}$$

possesses a solution for all λ's such that $|\lambda| \leq \lambda_0$ where λ_0 denotes Min $\left(a, \frac{b}{M}\right)$; this solution $\{\eta_i(\lambda)\}$ depends on λ in a continuous manner and vanishes for $\lambda = 0$.

Proof. The system (29) is evidently a system of the type (22) with coefficients depending continuously of λ. The following system will be its majorant

$$\zeta_i = \lambda_0 \bar{f}_i(\lambda_0, \zeta_1, \zeta_2, \ldots) + (b - \lambda_0 \bar{f}_i(\lambda_0, b, b, \ldots))$$

(here the differences $b - \lambda_0 \bar{f}_i(\lambda_0, b, b, \ldots)$ are non-negative because of (28) and of the definition of λ_0).

This majorant system possesses a positive solution $\zeta_1 = \zeta_2 = \cdots = b$.

Applying now Theorem XII we arrive at once at the conclusion of our theorem.

13. A Theorem on Finite Non-linear Systems.

Theorem XV.[2] Let 1). $f_i(\zeta_1, \zeta_2, \ldots, \zeta_n)$ $(i = 1, 2, \ldots, n)$ be n increasing continuous functions of their arguments; 2). $f_i(0, 0, \ldots, 0) = 0$ $(i = 1, \ldots, n)$ and

3). $$\lim_{x \to \infty} \frac{1}{x} \cdot f_i(x, x, \ldots, x) = 0 \text{ for } i = 1, \ldots, n \tag{30}$$

then the system of equations

$$\zeta_i = f_i(\zeta_1, \ldots, \zeta_n) + a_i \qquad (i = 1, \ldots, n) \tag{31}$$

has a solution whatever be $a_i \geq 0$.

Proof. If we denote by Z the n-dimensional vector space considered as a semi-order space, $z' = (H, H, \ldots, H)$ where H is a positive number such that for $(i = 1, \ldots, n)$

[1] A. Wintner, I.

[2] This theorem follows also from the Brower's theorem on »Fixpunkten».

$$H - f_i(H, H, \ldots, H) - a_i > 0$$

(according to (30) this inequality will be always satisfied provided that H is sufficiently great) and $V(z) = V(\zeta_1, \ldots, \zeta_n) = \{f_i(\zeta_1, \ldots, \zeta_n) + a_i\}$ then all conditions of Theorem I are satisfied and therefore its conclusion is true q. e. d.

14. The Existence Theorem for Implicit Functions.

We are going to show that the classical existence theorem for implicit functions[1] can be obtained as a particular case of the general theorems of § 2.

Theorem XVI. If a system of equations is given

$$F_i(\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_m) = 0 \qquad (i = 1, 2, \ldots, n) \tag{32}$$

where 1). F_i as well as $\frac{\partial F_i}{\partial \eta_k}$ are continuous functions of $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_m$ in a neighbourhood of a certain point $(y_0, x_0) = (\eta_1^0, \ldots, \eta_n^0, \xi_1^0, \ldots, \xi_m^0)$; 2). in the point (y_0, x_0) the equations (32) are satisfied; 3). in the same point the Jacobian $\frac{D(F_1, \ldots, F_n)}{D(\eta_1, \ldots, \eta_n)}$ does not vanish, then this system possesses a continuous solution in a neighbourhood of the point (y_0, x_0)

$$\eta_i = f_i(\xi_1, \ldots, \xi_m) \qquad (i = 1, 2, \ldots, n) \tag{33}$$

where

$$f_i(\xi_1^{(0)}, \ldots, \xi_m^{(0)}) = \eta_i^{(0)}. \qquad (i = 1, 2, \ldots, n) \tag{34}$$

Proof. Consider the vector space of the points $y = (\eta_1, \ldots, \eta_n)$ with the norm

$$\|y\| = \sup(|\eta_1|, \ldots, |\eta_n|).$$

The system (32) may be then written as

$$F(y, x) = 0$$

where x is a system $(\xi_1, \ldots, \xi_m)$ of m real numbers. Besides

$$F(y^0, x^0) = 0.$$

Set

$$A(y, x) = \frac{d F(y, x)}{d y}.$$

[1] Cf. Vallée Poussin, Ch. IV, § 1.

Then $A(y, x)$ is a matrix depending in a continuous manner on y and x; if $D(A)$ is the determinant of the matrix A then we have $D(A_0) = D(A(y^0, x^0)) \neq 0$. Let $y = y^0 + A_0^{-1} z$, then we shall have the following equation for z

$$z = [A_0(A_0^{-1} z) - F(y^0 + A_0^{-1} z, x)] = U(z, x) \ldots \tag{35}$$

But $\left[\frac{d\, U(z, x)}{d z}\right]_{z=0} = 0$ and consequently in a neighbourhood of the point $z = 0$ the Lipschitz's condition (with an arbitrarily small coefficient) is fulfilled. In particular there exists such δ that $\|z\| + \|\Delta z\| < \delta$ implies (for x sufficiently near to x_0)

$$\|U(z + \Delta z, x) - U(z, x)\| \leq \alpha \|\Delta z\|. \qquad (0 < \alpha < 1)$$

Thus the condition (3) of the Corollary to Theorem II is fulfilled, whence we conclude that the considered equation in a neighbourhood of the point $z = 0$ possesses a continuous solution

$$z = \theta(x) \quad \text{where} \quad \theta(x^0) = 0.$$

But then

$$y = y^0 + A_0^{-1} \theta(x)$$

and

$$\eta_i = f_i(\xi_1, \ldots, \xi_m) \qquad (i = 1, 2, \ldots, n)$$

where $f_i (i = 1, \ldots, n)$ are continuous functions and

$$f_i(\xi_1^0, \xi_2^0, \ldots, \xi_m^0) = \eta_i^0. \qquad (i = 1, 2, \ldots, n)$$

15. The Convergence of Newton's Method.

An approximate solution of a system of equations

$$F_i(\eta_1, \ldots, \eta_n) = 0 \qquad (i = 1, \ldots, n) \tag{36}$$

which can be written shorter as

$$F(y) = 0 \tag{37}$$

often may be found by the Newtonian method of successive approximations. These successive approximations are expressed[1] by the following formulae

$$y_n = y_{n-1} - A^{-1}(y_{n-1}) F(y_{n-1}) \tag{38}$$

where

[1] Scarborough, No. 63; Stenin, I; Ostrowski.

$$A(y) = \frac{d\,F(y)}{d\,y} = \left\| \frac{\partial F_i}{\partial \eta_k} \right\|_{\substack{i=1,\ldots,n\\k=1,\ldots,n}}.$$

We shall establish now (subject to certain conditions) the convergence of this process towards the solution of (36). Without detracting of the genarality of our reasoning we can suppose that $y_0 = 0 = (0, 0, \ldots, 0)$.

Theorem XVII. If all functions F_i as well as all their first and second derivatives are continuous then the Newtonian process converges towards a solution provided that $\|F(0)\|$ be sufficiently small.

Proof. Consider the equation

$$y = A^{-1}(y)\,[A(y)\,y - F(y)] = U(y). \tag{39}$$

This equation is equivalent to the equation (37) and the expression (38) gives the usual process of successive approximations for it. Consequently the process will surely converge towards a solution if the condition (3) is fulfilled. But this condition will evidently be fulfilled with $\alpha = \frac{1}{2}$ if $\|F(y)\|$ is so small that

$$\left\| \frac{d\,U(y)}{d\,y} \right\| = \left\| \frac{d\,A^{-1}(y)}{d\,y} F(y) \right\| \leq \frac{1}{2}\,[1] \tag{40}$$

for such values of y that $\|y\| < 2\,\|F(y_0)\|$.

§ 5. Integral Equations.

16. Linear Integral Equations of the Second Kind.

Consider an equation of Volterra's type

$$y(t) - \int_a^t K(s, t)\,y(s)\,d\,s = f(t) \tag{41}$$

and suppose that in a certain interval $(a; b)$ we have: $|K(s, t)| \leq M$, $|f(t)| \leq N$. Then the equation

[1] Here $\frac{d\,U(y)}{d\,y}$ is a matrix (depending on y) i. e. an operation transforming the space y into itself. Its norm is the norm of an operation.

$$z(t) - \int_a^t M z(s)\, ds = N \tag{42}$$

will be a majorant equation for (41).

But the equation (42) evidently possesses a positive solution which can be found e. g. by the method of successive approximations

$$z(t) = N + \frac{MN(t-a)}{1} + \frac{M^2 N(t-a)^2}{2} + \cdots = N e^{M(t-a)}. \tag{43}$$

Hence follows (if we apply Theorems V and VI) that Volterra's equation has in an interval (a, b) one and only one bounded solution and that this solution can be found by the method of successive approximations; this single bounded solution will be also the only integrable solution because it is evident that any integrable solution of Volterra's equation is in the same time bounded in (a, b).

In a similar way we can make analogical conclusions concerning Fredholm's equation

$$y(t) - \lambda \int_a^b K(s, t)\, y(s)\, ds = f(t). \tag{44}$$

In this case we shall obtain different boundaries for λ by using different spaces of functions (i. e. by giving different definitions to $\|y\|$). E. g. if we set $\|y\| = \sup\limits_s |y(s)|$ then we can prove that the solution of Fredholm's equation exists when

$$|\lambda| \sup_t \int_a^b |K(s, t)|\, ds < 1. \tag{45}$$

Theorem VII will allow us to make several conclusions as to the continuity of the solution regarded as a function of parameter and as to lawfulness of passing to a limit in the solution when the nucleis converge towards a »limit nucleus». In particular the solution of an integral equation can be obtained as a limit of solutions of a system of algebraic linear equations.

17. Systems of Fredholm's Equations.

Consider a system

$$a_i \leq t \leq b_i;\ \eta_i(t) - \sum_{j=1}^{n} \int_{a_j}^{b_j} K_{i,j}(s, t)\, \eta_j(s)\, d s = \varphi_i(t) \qquad (i = 1, 2, \ldots, n). \tag{46}$$

Define in any way the norm of a function $\eta_i(t)$ e. g. let $\|\eta_i\| = \sup\limits_{a_i \leq t \leq b_i} |\eta_i(t)|$, and take for y the set of systems of functions $y = (\eta_1, \ldots, \eta_n)$ and for Z the semi-ordered space of systems of real numbers $z = (\zeta_1, \ldots, \zeta_n)$ and set $\|y\| = (\|\eta_1\|, \ldots, \|\eta_n\|)$. Denote, besides,

$$\sup_{a_i \leq t \leq b_i} \int_{a_j}^{b_j} |K_{i,j}(s, t)|\, d s = c_{i,j}.$$

Then if we consider the system (46) as a single equation of the type (9) the following equation of the type (8)

$$\zeta_i - \sum_{j=1}^{n} c_{i,j}\, \zeta_j = \|\varphi_i\| \qquad (i = 1, \ldots, n) \tag{47}$$

will be its majorant.

Hence follows:

Theorem XVIII. If the algebraic system of equations (47) has a positive solution $z = (\zeta_1, \ldots, \zeta_n)$ then the system (46) also has a solution.

This theorem follows immediately from Theorem V.

18. Approximate Solution of Integral Equations.

We shall now apply Theorem VIII which will allow us to estimate the error of an approximate solution of an integral equation.

Theorem XIX.[1] Let $k(s, t)$ and $K(s, t)$ be two nuclei such that the resolvent $\gamma(s, t, \lambda)$ of the nucleus $k(s, t)$ be known and let

$$\sup_{a \leq t \leq b} \int_a^b |K(s, t) - k(s, t)|\, d s = h$$

[1] Kantorovitch & Kryloff, p. 159.

$$\sup_{a \leq t \leq b} \int_a^b |\gamma(s, t, \lambda)| \, ds = B \tag{48}$$

$$\sup_{a \leq t \leq b} |g(t)| = N.$$

In this case if $1 - |\lambda| h(1 + |\lambda| B) > 0$ then the equation

$$\varphi(t) - \lambda \int_a^b K(s, t) \varphi(s) \, ds = g(t) \tag{49}$$

possesses one and only one solution which differs from the solution of the equation

$$\tilde{\varphi}(t) - \int_a^b k(s, t) \tilde{\varphi}(s) \, ds = g(t) \tag{50}$$

not more than by the following quantity

$$|\varphi(t) - \tilde{\varphi}(t)| \leq \frac{|\lambda| h(1 + |\lambda| B) \tilde{M}}{1 - h(1 + |\lambda| B)} \leq \frac{h |\lambda| (1 + |\lambda| B)^2 N}{1 - h |\lambda| (1 + |\lambda| B)} \tag{51}$$

where $\tilde{M} = \sup_{a \leq t \leq b} |\tilde{\varphi}(t)|$.

Proof. Apply Theorem VIII. In the place of the relation $y' = \Gamma(y_0)$ we have the following one

$$\tilde{\varphi}(t) = g(t) + \lambda \int_a^b \gamma(s, t, \lambda) g(s) \, ds. \tag{52}$$

Hence considering all the functions as elements of the space of bounded function M, i. e. defining norme f as $\|f\| = \sup_{a \leq t \leq b} |f(t)|$ we obtain the inequality

$$\|\tilde{\varphi}\| \leq \|g\| + |\lambda| B \|g\|. \tag{53}$$

We see hence that in the present case

$$\|\Gamma\| \leq 1 + |\lambda| B.$$

Similarly (cf. Theorem VIII as to notations): $\|f - f'\| \leq h |\lambda|$. Hence follows that

$$\|\Gamma\| \cdot \|f - f'\| \leq h(1 + |\lambda| B)|\lambda| = \alpha < 1$$

because we have supposed that $h|\lambda|(1 + |\lambda| B) < 1$ and consequently according to (13)

$$\|\varphi - \tilde{\varphi}\| \leq \frac{h|\lambda|(1 + |\lambda| B)}{1 - h|\lambda|(1 + |\lambda| B)} \tilde{M} \leq \frac{h|\lambda|(1 + |\lambda| B)^2 N}{1 - h|\lambda|(1 + |\lambda| B)}.$$

Defining differently the norm of a function we shall obtain (from the same Theorem VIII) different other estimates for the difference $|\varphi - \tilde{\varphi}|$. In particular we can obtain thus two theorems of Akbergenoff giving such estimates.[1]

19. A Theorem on Fredholm's Equations on the Infinite Interval.

Theorem XX. Let

$$\int_{-\infty}^{+\infty} |K(s, t)| \, ds = h(t) < 1. \tag{54}$$

Then the integral equation

$$\varphi(t) - \int_{-\infty}^{\infty} K(s, t) \varphi(s) \, ds = f(s) \tag{55}$$

possesses a solution if its right-hand member $f(t)$ satisfies the inequality $|f(t)| \leq \leq C(1 - h(t))$, where C is a constant; this solution is bounded, viz. $|\varphi(t)| \leq C$.

Proof. This theorem is an immediate consequence of Theorem V because the equation

$$\tilde{\varphi}(t) - \int_{-\infty}^{+\infty} |K(s, t)| \tilde{\varphi}(s) \, ds = C(1 - h(t)) \tag{56}$$

which is a majorant for the equation (55), possesses evidently a positive solution $\tilde{\varphi}(t) = C$.

20. Non-linear Integral Equation.

Applying theorems of § 2 to the case of non-linear integral equation we may obtain several theorems of which we shall prove only one.

Theorem XXI. Let the function $K(s, t, y)$ be defined and continuous for (s, t) lying in a certain region G and for y such that $|y| \leq y_0(s)$ and suppose that

[1] Akbergenoff, I, pp. 681 and 689.

there exists a function $\Phi(s, t, y)$ of the same variables, defined and continuous in the same region (as K) and satisfying the following inequalities

1) $|K(s, t, 0)| \leq \Phi(s, t, 0)$.

2) $\Phi'_y(s, t, y) \geq 0$; $\Phi''_{y^2}(s, t, y) \geq 0$.

3) $|K(s, t, y + \Delta y) - K(s, t, y)| \leq \Phi(s, t, |y| + |\Delta y|) - \Phi(s, t, |y|)$ provided that $|y| + |\Delta y| \leq y_0(s)$.

In this case if

$$y_0(t) \geq \int_G \Phi(s, t, y_0(s))\, d s \tag{57}$$

then the integral equation

$$y(t) = \int_G K(s, t, y(s))\, d s \tag{58}$$

has a solution $y(t)$ such that $|y(t)| \leq y_0(s)$. If besides there does not exist more than one solution of the equation

$$y(t) = \int_G \Phi(s, t, y(s))\, d s \tag{59}$$

satisfying the condition $|y(t)| \leq y_0(t)$ (one such solution always existing) then the equation (58) also possesses one only solution satisfying this condition and this solution can be found by the method of successive approximations starting with any function $y_1(s)$ such that $|y_1(s)| \leq y_0(s)$.

Proof. This theorem follows immediately from Theorems II and III. We must only ascertain that the conditions of these theorems are satisfied. It is sufficient to prove condition 3) (all other conditions being evident). This condition is fulfilled because if $|y(t)| \leq z(t)$ and $|\Delta y(t)| \leq \Delta z(t)$

$$\left| \int_G K(s, t, y(s) + \Delta y(s))\, d s - \int_G K(s, t, y(s))\, d s \right| \leq$$

$$\leq \int_G [\Phi(s, t, |y(s)| + |\Delta y(s)|) - \Phi(s, t, |y(s)|)]\, d s \leq$$

$$\leq \int_G [\Phi(s, t, z(s) + \Delta z(s))\, d s - \Phi(s, t, z(s))]\, d s.$$

Corollary.[1] If in particular the function $K(s, t, y)$ satisfies the condition $|K(s, t, y + \varDelta y) - K(s, t, y)| \leq C|\varDelta y|$ if $|y| + |\varDelta y| \leq L$ then the equation (58) has one and only one solution, satisfying the condition $|y(s)| \leq L$ provided only that the following conditions be satisfied

$$C m G < 1, \quad S m G + L C m G \leq L \tag{60}$$

where

$$S = \sup |K(s, t, 0)|.$$

This Corollary follows immediately from Theorem XXI if we set

$$\Phi(s, t, y) = S + C y \quad \text{and} \quad y_0(s) = L.$$

Remark. If the second inequality (60) only holds then the satisfying condition $|y(s)| \leq L$ nevertheless exists but there may be in this case several such solutions.

§ 6. Differential Equations.

21. The Convergence of Picard's Method.

We shall prove now the classical theorem on the convergence of Picard's method.

Theorem XXII. Let a system of differential equations be given

$$\frac{d\eta_i}{dt} = f_i(\eta_1, \ldots, \eta_n, t) \qquad (i = 1, 2, \ldots, n) \tag{61}$$

and suppose that the functions f_i are continuous in t and satisfy the condition of Lipschitz rel. $\eta_1, \ldots, \eta_n$ for $|t|$ and $|\eta_i|$ sufficiently small i. e. that for $|t| \leq t_0$ and $|\eta_i| + |\varDelta \eta_i| \leq h$ $(i = 1, 2, \ldots, n)$ we have

$$|f_i(\eta_1 + \varDelta \eta_1, \ldots, \eta_n + \varDelta \eta_n, t) - f(\eta_1, \ldots, \eta_n, t)| \leq \\ \leq C(|\varDelta \eta_1| + \cdots + |\varDelta \eta_n|). \tag{62}$$

Then the system of equations (61) possesses a solution $\eta_i = \eta_i(t)$ where the functions $\eta_i(t)$ are defined for t sufficiently small ($|t| \leq \delta$) and satisfy the following initial conditions

$$\eta_i(0) = 0. \qquad (i = 1, \ldots, n) \tag{63}$$

[1] Niemytzki, I, p. 656.

Proof. The system (61) we may write in the following form

$$\eta_i(t) = \int_0^t f_i(\eta_1(t), \ldots, \eta_n(t), t)\, dt. \tag{64}$$

Then regarding the system of functions $(\eta_1(t), \ldots, \eta_n(t))$ as an element y of the abstract (semi-ordered) space Y of systems of functions we may consider the system (64) as an equation of the type (2) $y = U(y)$. As a majorant equation $z = V(z)$ we can take the following system

$$\zeta_i(t) = \int_0^t C(\zeta_1(t) + \cdots + \zeta_n(t))\, dt + \int_0^t |f_i(0, \ldots, 0, t)|\, dt \quad (i = 1, 2, \ldots, n). \tag{65}$$

(Here $Z = Y$ which is considered as a semi-ordered space.) All conditions of Theorems I and II are fulfilled. In fact 1)—4) of Theorem I are evident; 5) is satisfied if we take for z' the system of functions $\zeta_i'(t) = \alpha e^{nCt}$ $(i = 1, \ldots, n)$ provided that δ and α are such that the following two inequalities subsist

$$\int_0^\delta |f_i(0, \ldots, 0, t)|\, dt \leq \alpha \qquad (i = 1, \ldots, n) \tag{66}$$

$$\alpha e^{nC\delta} \leq h$$

(the second inequality (66) is not necessary in order that 5) of Theorem I be satisfied but it will be needed for proving property 3) of Th. II).

1) and 2) of Th. II are evident. 3) is also true. In fact we have for

$$|\varDelta \eta_i(t)| \leq \varDelta \zeta_i(t); \ |\eta_i(t)| \leq \zeta_i(t) \text{ and } \zeta_i(t) + \varDelta \zeta_i(t) \leq \alpha e^{nCt} \qquad (i = 1, 2, \ldots, n)$$

$$\left| \int_0^t f_i(\eta_1(t) + \varDelta \eta_1(t), \ldots, \eta_n(t) + \varDelta \eta_n(t), t)\, dt - \right.$$

$$\left. - \int_0^t f_i(\eta_1(t), \ldots, \eta_n(t), t)\, dt \right| \leq$$

$$\leq \int_0^t C(\varDelta \zeta_1(t) + \cdots + \varDelta \zeta_n(t))\, dt.$$

We have proved thus that all conditions of Theorem II are satisfied; therefore its conclusion is also true and thus Theorem XXII is proved.

22. Cauchy's Method of Majorants.

The classical method of proving the existence of the analytic solution of a differential equation can be also shown as a particular case of theorems of § 2; viz. we can prove the following theorem.

Theorem XXIII. Let a system of equations be given

$$\frac{d\eta_i}{dt} = f_i(\eta_1, \ldots, \eta_n, t) = \sum_{k_1, \ldots, k_n, l=0}^{\infty} c_{k_1, \ldots, k_n, l}\, \eta_1^{k_1} \ldots \eta_n^{k_n} t^l \qquad (i = 1, 2, \ldots, n). \tag{67}$$

Then if the system

$$\frac{d\zeta_i}{dt} = F_i(\zeta_1, \ldots, \zeta_n, t) = \sum_{k_1, \ldots, k_n, l=0}^{\infty} C_{k_1, \ldots, k_n, l}\, \zeta_1^{k_1} \ldots \zeta_n^{k_n} t^l \tag{68}$$

where $|c_{k_1, \ldots, k_n, l}| \leq C_{k_1, \ldots, k_n, l}$ possesses a solution

$$\zeta_i(t) = \sum_{s=0}^{\infty} h_s^{(i)} t^s, \text{ where all } h_s^{(i)} \geq 0 \qquad (i = 1, 2, \ldots, n)$$

then the system (67) possesses a solution satisfying the initial conditions $\eta_i(0) = \eta_0^{(i)}$ $i = 1, \ldots, n)$ and having the form

$$\eta_i(t) = \eta_0^{(i)} + \sum_{s=1}^{\infty} g_s^{(i)} t^s$$

where $|g_s^{(i)}| \leq h_s^{(i)}$ provided only that $|\eta_0^{(i)}| \leq h_0^{(i)}$ for $i = 1, \ldots, n$.

Proof. Substituting in (67) and (68) for η_i and ζ_i the corresponding series we shall obtain in left hand and right hand members of these equations formal series by comparing coefficients of which we arrive at the equations of the types (22) and (23). Applying to these equations Theorem XII we obtain the desired result.

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