NECESSARY CONDITIONS IN THE CALCULUS OF VARIATIONS.

BY

L. C. YOUNG

of CAMBRIDGE, England.

§ 1. **Introduction.** The object of this note is to introduce new variational methods, which belong to the order of ideas of MINKOWSKI's theorem on the existence of a flat support for a convex figure. We here apply these methods to the inhomogeneous form of the »simplest problem» of the Calculus of Variations, and we establish, by means of them, a necessary condition, of a very general form, for an attained minimum. In the proof, an important part is also played by theorems of measurability which belong to the theory of Analytic and Projective Sets.

Our necessary condition generalizes at the same time the necessary condition of WEIERSTRASS, the equations to an extremal of EULER and DU BOIS REYMOND, and the equation $\Omega(x, y) = 0$ to certain limiting solutions of CARATHEODORY.

Our methods enable us to dispense with many classical restrictions on the integrand $f(x, y, y')$. The important restrictions of TONELLI on the existence and order of magnitude of the partial derivative f_y are replaced by weaker restrictions on the behaviour of the corresponding partial finite-difference ratio. Apart from these weakened restrictions which concern only the dependence of f on the variable y, our integrand may be any function measurable (B).

We use integration consistently in the general DENJOY sense. This corresponds to an enlargement of the class of admissible curves. We consider also a still further enlargement obtained by admitting what we term *generalized curves*.

In variational problems such as we treat here, in which no restriction is imposed on the order of magnitude of f_y, these enlargements of the class of admissible curves do not necessarily lead to corresponding generalizations of the classical problems. It is easily seen, however, that our methods apply also — and are indeed slightly simplified — when classical restrictions are imposed on the admissible curves, in particular when integration is understood in the LEBESGUE sense instead of in the general Denjoy sense.

Finally, although throughout the present note we limit ourselves to the simplest problem of the Calculus of Variations, it may be observed that our methods do not demand such a limitation and that further applications, particularly to problems with subsidiary relations and to isoperimetric problems, are possible.

§ 2. **Admissible Curves.** We begin with some elementary remarks. The simplest problem of the Calculus of Variations consists in determining the minimum of an integral

$$\int_{x_0}^{x_1} f(x, y(x), y'(x))\, dx$$

for a given $f(x, y, y')$ and fixed ends $y(x_0) = y_0$, $y(x_1) = y_1$ when $y(x)$ belongs to a certain class of »admissible curves» which must be explicitely defined, and which certainly includes all the functions $y(x)$ with continuous derivatives of all orders which fulfil the boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$.

The classical theory of the Calculus of Variations was concerned with the case in which the admissible curves consisted of these analytic curves only. Modern researches have shown that in order to treat the classical problem satisfactorily it is convenient to enlarge the class of admissible curves. This is because the variational methods so far known depend on the existence of a »*minimizing*» *curve* in the class of admissible curves, i. e. a function $y(x)$ for which the minimum is attained. To ensure, as far as possible, that such a minimizing curve should exist, the class of admissible curves has, since the days of Weierstrass, been enlarged, successively, by the inclusion of functions $y(x)$ with bounded, piecewise continuous, derivative $y'(x)$ (CARATHEODORY [2]) by the inclusion of absolutely continuous functions $y(x)$ for which $y'(x)$ then denotes the derivative almost everywhere (TONELLI [13]), by the inclusion of certain non-rectifiable curves (MENGER [10, 11]), and by the inclusion of »generalized curves» (below §§ 8—11). Another interesting extension, due to MC SHANE [8], consists in the inclusion of certain admissible curves of a corresponding *parametric* problem.

Apart from their bearing on the classical form of the problem, these extensions have an interest of their own owing to the fact that modern Analysis is now chiefly concerned with the more general processes of integration. In the investigations of the following §§, we shall limit ourselves to two extensive classes of admissible curves, the »ordinary» and the »generalized» curves as we term them, and we shall interpret integration in the general Denjoy sense. We do not concern ourselves with the corresponding results for the more restrictive classes of admissible curves. It has already been remarked (§ 1) that such results may be derived by applying our methods in an appropriately specialized form.

Actually the relations between the problems which arise from the various interpretations of the class of admissible curves (and which have been studied by LAVRENTIEFF [6] and TONELLI [15]) are still rather obscure. It is easy to construct examples in which the minimum of our integral has different values for certain of these classes. This state of affairs is of frequent occurrence in problems with subsidiary relations [cf. (a) below], and the fact that it occurs also in the simplest problem in ordinary form [cf. (b) below] may perhaps be accounted for by the similarity of these two types of problems.

(a) Let (A) denote the class of pairs of real functions $y(x)$, $z(x)$ with continuous derivatives $y'(x)$, $z'(x)$ such that $[z'(x)]^2 = 1 + [y'(x)]^2$ and (B) the corresponding class of $y(x)$, $z(x)$ with bounded, piecewise continuous $y'(x)$, $z'(x)$ fulfilling the same relation. The minimum of the length

$$\int_{x_0}^{x_1} \{1 + [y'(x)]^2 + [z'(x)]^2\}^{1/2} dx$$

of the curves of (A) such that $y(x_0)$, $z(x_0) = 0, 0$ and $y(x_1)$, $z(x_1) = 1, 0$, is clearly $+\infty$, since no curve of (A) can have these ends. The corresponding minimum in (B) is evidently $\sqrt{2} \cdot (x_1 - x_0)$.

By slight modification of the subsidiary relation $z'^2 = 1 + y'^2$, we can arrange to have a finite, but extremely large, minimum length for the class (A), and a minimum length not exceeding $2 \cdot (x_1 - x_0)$ for the class (B).

(b) Let now D_* be the domain of the (x, y)-plane for which

$$0 \leq x \leq 1 \quad \textit{and} \quad 0 \leq y \log\log (e^e/x) \leq 1 + l\sqrt{x};$$

and let

$$f(x, y, y') = x^l [\log (e^e/x)]^{y'} \exp\{-y/[x \log (e^e/x)]\}$$

for all y' and all (x, y) in D_*, except when $x = 0$, in which case we have also $y = 0$ if (x, y) lies in D_*, and we choose at this point $f(x, y, y') = 0$ for all y'. Here e is the natural logarithmic base and l is a number exceeding 2 but otherwise arbitrary. By choosing l large enough, the partial derivatives of f up to any assigned order can be made continuous in D_*.

This being so, we denote by (A_*) the class of functions $y(x)$ with bounded derivative $y'(x)$, such that the point $(x, y(x))$ lies in D_* for each x of the interval

$0 \leq x \leq 1$, and that $y(0) = 0$, $y(1) = 1$; we denote by (B_*) the corresponding class of absolutely continuous $y(x)$. By the change of variable $z = y \log \log (e^e/x)$, we easily see that the minimum of the integral

$$\int_0^1 f(x, y(x), \; y'(x))dx$$

has the value e^{-l} for the class (B_*) and a value not exceeding $e^{-(l-1)}$ for the class (A_*).

[In fact, when we change our variable, the problem becomes

$$\int_0^1 x^l e^{z'(x)} dx = \text{Min. } \textit{subject to } z(0) = z_0, \; z(1) = 1,$$

where the relevant value of z_0 is 0 for the case of the class (A_*) and 1 for that of the class (B_*), and the admissible curves may be taken to consist of all the absolutely continuous $z(x)$ defined for $0 \leq x \leq 1$ which have the assigned end-values and fulfil the inequalities $0 \leq z(x) \leq 1 + l\sqrt{x}$. The minimum $e^{-(z_0 + l - 1)}$ is attained for the curve $z(x) = (1 - z_0)x + z_0 - lx \log x$. In fact, without appeal to general theory, if we write

$$g = g(x, z) = l + (z + l \log x - 1)/(1 - x),$$

we see that

$$x^l e^{z'} = -\frac{d}{dx}\{(1 - x)x^l e^{-g}\} + x^l e^{-g} \cdot \{e^{z'+g} - 1 - (z' + g)\},$$

and the result follows, in view of the inequality $e^\alpha - 1 - \alpha > 0$ which holds for all real $\alpha \neq 0$.]

§ 3. **Classical Form of the Condition** (W. E.). Let us suppose for the moment that $f(x, y, y')$ has continuous partial derivatives up to those of the second order, and that $y(x)$, defined for $x_0 \leq x \leq x_1$, is restricted to have a continuous derivative $y'(x)$ and to assume the values $y(x_0) = y_0$ and $y(x_1) = y_1$ at the ends.

The classical theorems of Euler and Weierstrass assert that if $y(x)$ is a minimizing curve for the integral of f, we must have, for all x, firstly

$$(3.1) \qquad f_{y'}(x, y(x), \; y'(x)) = c + \int_{x_0}^{x} f_y(t, y(t), \; y'(t)dt,$$

where c is a constant, and secondly

$$(3.2) \qquad f(x, y(x), y'(x) + \zeta) - f(x, y(x), y'(x)) - \zeta f_{y'}(x, y(x), y'(x)) \geq 0,$$

whatever be the real number ζ. Hence, substituting from the first relation, we conclude that there exists a constant c such that, for every x of $[x_0, x_1]$ and for all ζ,

$$(3.3) \quad f(x, y(x), y'(x) + \zeta) - f(x, y(x), y'(x)) - \zeta \cdot \left\{ c + \int_{x_0}^{x} f_y(t, y(t), y'(t) dt \right\} \geq 0.$$

The most elementary properties of a derivative show, conversely, that this condition, if it holds for all ζ, implies both the preceding ones. We call it *the condition (W. E.) in classical form.* The study of its extensions is the main object of the following §§.

The intimate connection of this condition with the theorem of MINKOWSKI on the existence of a flat support is at once clear when we consider the case in which $f(x, y, y') = F(y')$ is independent of both x and y. If we limit ourselves to a minimizing curve of the special form $y(x) = ax + b$, $y'(x) = a$, where a and b are constants, and if we suppose, for simplicity, that $x_0 = 0$, $x_1 = 1$, the necessary condition (W. E.) in order that

$$(3.4) \qquad F(a) \left(= \int_0^1 F(a) dx \right) \leq \int_0^1 F(a + \eta'(x)) dx \ \textit{whenever} \int_0^1 \eta'(x) dx = 0,$$

becomes: there exists a constant c such that

$$(3.5) \qquad F(a + \zeta) \geq F(a) + c\zeta \textit{ for all } \zeta.$$

The classical theory of the Calculus of Variations thus shows that (3. 4) implies (3. 5) when $F(\zeta)$ has a continuous second derivative. Evidently (3. 5) always implies (3. 4).

The equivalence of (3. 4) and (3. 5) is known as Minkowski's theorem. This theorem, which remains valid when we remove the restrictions of existence and continuity of a second derivative, is one of the most powerful tools of modern analysis (cf., for instance, HARDY, LITTLEWOOD and POLYA [5] p. 94, Th. 112, and, in abstract form, BANACH [1] p. 27, Th. 1).

Actually, our whole study of the condition (W. E.) of the Calculus of Variations may be regarded as the study of the variational generalizations of Minkowski's theorem. It would clearly be absurd to re-introduce into these generalizations the classical restrictions of differentiability, etc. of the integrand $f(x, y, y')$. In the sequel, we therefore dispense with these classical restrictions.

§ 4. **Problems Independent of the Variable** y. We shall suppose in this § that $f(x, y, y')$ has the special form $\varphi(x, y')$. This special case will serve as a basis for our later results.

In the classical Calculus of Variations, a still more special case occurs in the trivial lemma, often termed »the fundamental lemma» of the Calculus of Variations, which is of use in establishing the Du Bois Reymond condition (3. 1). This trivial lemma asserts that, *if, for a given function* $f(x)$, *we have*

$$\int_{x_0}^{x_1} y'(x) f(x)\, dx = 0 \quad \textit{whenever} \quad \int_{x_0}^{x_1} y'(x)\, dx = 0,$$

then there exists a constant c *such that* $f(x) = c$ *for almost all* x. Clearly this lemma is a consequence of the condition (W. E.) for the cases $f(x, y, y') = y' f(x)$ and $f(x, y, y') = -y' f(x)$. The place of this lemma is here taken by the more general result that *the condition (W. E.) is valid as it stands, when* $f(x, y, y')$ *has the form* $\varphi(x, y')$.

This result will follow from the main theorem of this §. It evidently includes Minkowski's theorem (*vide* the remarks of § 3). It's usefulness lies in the fact that it enables us to derive a conclusion which is, in appearance, stronger than the hypotheses. If in (3. 3) we choose for ζ any integrable function $\eta'(x)$ and denote by $\eta(x)$ the integral of the latter from x_0 to x, we obtain from (3. 3), by integration, the inequality

$$\int_{x_0}^{x_1} \varphi(x, y'(x) + \eta'(x))\, dx \geq \int_{x_0}^{x_1} \varphi(x, y'(x))\, dx + c\,\eta(x_1),$$

and this inequality now applies to comparisom curves of a much more general kind than were originally admitted, since they need not pass through the end y_1 at $x = x_1$.

It is convenient to state the main result of this § in the following form:

(4. 1) **Theorem.** *Let* $\Phi(x, \zeta)$ *be a function measurable* (B) *which is defined for all real* ζ *and for all* x *of a set* E *throughout which* $\Phi(x, 0) \leq 0$, *and suppose that the subsets of* E *in which* $\Phi(x, 1)$ *and* $\Phi(x, -1)$ *are finite below respectively, are of positive measure. Suppose further that to each real number* c *there exists a subset of* E *of positive measure at every point* x *of which we have* $\Phi(x, \zeta) + c\zeta > 0$ *for some* ζ *depending on* x.

Then there exist a bounded measurable function $\zeta_0(x)$ *and constants* c_0 *and* γ_0, *such that both the subsets of* E

$$E_1 = \underset{x}{\mathrm{E}}\,[\zeta_0(x) > 0] \quad \textit{and} \quad E_2 = \underset{x}{\mathrm{E}}\,[\zeta_0(x) < 0]$$

have positive measure, and so that, in $E_1 + E_2$, *we have*

$$\Phi(x, \zeta_0(x)) + c_0\,\zeta_0(x) > \gamma_0.$$

We shall see, when we deduce Theorem (4. 4), below, as an immediate corollary, that this statement differs only in minor details from the assertion of the necessity of the condition (W. E.) for problems independent of y.

It may be remarked further that Theorem (4. 1) generalizes Minkowski's theorem: to obtain the latter, simply choose $\Phi(x, \zeta)$ to be of the form $F(a) - F(a+\zeta)$, independent of x.

Proof of Theorem (4. 1). We denote by O' the set of the values of c for each of which there exists a bounded *non-negative* measurable function $\zeta_0(x)$ and a positive rational number γ such that

$$\Phi(x, \zeta_0(x)) + c\,\zeta_0(x) > \gamma \tag{4. 2}$$

for all x of a subset of E of positive measure. We denote by O'' the set of the values of c for each of which there exists a bounded *non-positive* measurable function $\zeta_0(x)$ and a positive rational number γ with the same property. It is clearly sufficient to prove that O' and O'' contain a common value c_0.

Now we see at once that the sets O' and O'' are open. In fact, if c belongs to the set O', for instance, then so does the interval $(c-h, c+h)$ provided that $0 < h < \gamma/K$, where γ is the rational number, and K the upper bound of the function $\zeta_0(x)$, associated with the particular value of c. Moreover, for sufficiently large n_0, the sets O' and O'' include respectively the values $c = n_0$ and $c = -n_0$; we have, in fact, for large n_0, $\Phi(x, 1) + n_0 > 1$ and $\Phi(x, -1) + n_0 > 1$ in subsets of E of positive measure, so that we can choose $\zeta_0(x) = 1$ when $c = n_0$, and $\zeta_0(x) = -1$ when $c = -n_0$.

The sets O' and O'' are thus non-empty open sets. To prove that their common part is non-empty, it is sufficient to show that *their sum is a continuum.*

Let therefore c be any real number. We have to show that there exist a bounded measurable function $\zeta_0(x)$ of constant sign and a positive rational number γ, such that the inequality (4. 2) is valid for all x of a subset of E of positive measure. Keeping c fixed, let $E_{\gamma, K}$ be the subset of E at each point x of which there exists a value of ζ such that

$$|\zeta| \leq K \quad \textit{and} \quad \Phi(x, \zeta) + c\,\zeta \geq \gamma > 0, \tag{4. 3}$$

the numbers γ and K being *rational*. The sets $E_{\gamma, K}$, as projections of plane Borel sets, are measurable (cf., below, lemma (5. 1) or, for instance, SAKS [12], Chap. II, § 5) and their sum for all rational γ and K has positive measure by hypothesis. Hence, one at least of the sets $E_{\gamma, K}$ has positive measure. It clearly contains a subset of positive measure at every point x of which there exists a value ζ of constant sign for which the inequalities (4. 3) are satisfied. In this last subset we can, finally, by lemma (5. 2) below, locate another subset of positive measure in which there is a measurable function $\zeta = \zeta_0(x)$ of constant sign for which the inequalities (4. 3) are satisfied. This shows that any real number c belongs to the set $O' + O''$ and so completes the proof (subject to the measurability considerations of lemmas (5. 1) and (5. 2) of § 5).

As an immediate corollary, we establish, for problems independent of y, the condition (W. E.) in the following form:

(4. 4) **Theorem.** *Let $\varphi(x, \zeta)$ be any finite function of the variables x, ζ, and $y'(x)$ a finite function of x (not necessarily integrable in any sense), both measurable (B), such that $\varphi(x, y'(x))$ is integrable (Denjoy) and that the inequality*

$$\int_{x_0}^{x_1} \varphi(x, y'(x) + \eta'(x))\, dx \geq \int_{x_0}^{x_1} \varphi(x, y'(x))\, dx$$

holds[1] *whenever $\eta'(x)$ is a bounded function measurable (B) such that*

$$\int_{x_0}^{x_1} \eta'(x)\, dx = 0.$$

Then there exists a constant c such that, for almost every x of (x_0, x_1), the inequality

$$\varphi(x, y'(x) + \zeta) - \varphi(x, y'(x)) - c\zeta \geq 0 \tag{4. 5}$$

holds for all ζ. In particular, we therefore have[2]

$$\int_{x_0}^{x_1} \varphi(x, y'(x) + \eta'(x))\, dx \geq \int_{x_0}^{x_1} \varphi(x, y'(x))\, dx + c \int_{x_0}^{x} \eta'(x)\, dx$$

for all Denjoy integrable $\eta'(x)$.

To obtain this theorem, it is sufficient to apply Theorem (4. 1) to the function $\Phi(x, \zeta) = \varphi(x, y'(x)) - \varphi(x, y'(x) + \zeta)$ — which is measurable (B) by lemma (5. 3)

[1] We suppose this only when the left-hand side exists.

[2] Asserting at the same time the existence of the left-hand side.

below — on the assumption that there is no constant c for which (4. 5) holds for all ζ at almost every x, an assumption which has to be shown to be untenable. Clearly, the function $\zeta_0(x)$ whose existence is established in Theorem (4. 1) may be supposed (by replacing it by 0 in part of E_1 or E_2 if necessary) to fulfil the condition

$$\int_{x_0}^{x_1} \zeta_0(x)\, dx = 0.$$

Therefore, choosing $\eta'(x) = \zeta_0(x)$, we obtain

$$\int_{x_0}^{x_1} \{\varphi(x, y'(x) + \eta'(x)) - \varphi(x, y'(x))\}\, dx < -\gamma_0 \int_{E_1+E_2} dx < 0$$

and this contradicts our hypotheses.

§ 5. **Lemmas Concerning Measurability.** We now come to the measurability considerations needed in § 4. We denote by $[E]_x$ the projection of any set E, in the plane or in space, on to the x-axis. We denote further by $|E|$ the Lebesgue measure of any linear set E.

(5. 1) **Lemma.** *Let E be a plane Borel set. Then $[E]_x$ is measurable and E contains a closed set H such that $|[E]_x - [H]_x| < \varepsilon$.*

This is proved almost explicitely in Lusin [7, p. 152]. The set E is, in fact, the projection on a plane, of a so-called »elementary set» E^* in three dimensions, and E^* contains a closed set H^* such that $|[E^*]_x - [H^*]_x| < \varepsilon$.

(5. 2) **Lemma.** *If E is a plane Borel set and $|[E]_x| > 0$, then E contains the graph of a measurable function $f(x)$ defined on a subset of $[E]_x$ of positive measure.*

We may suppose E closed, by lemma (5. 1), and bounded. The function $f(x)$ defined as the least value of y such that $(x, y) < E$ for fixed x, is then lower semicontinuous and so measurable.

(5. 3) **Lemma.** *Let $\varphi(x, \zeta)$ and $f(x)$ be measurable (B). Then so is the function $\varphi(x, f(x) + \zeta)$.*

This is obvious if φ and f are continuous, and by passage to the limit with f it remains true if φ is continuous and f measurable (B). Finally, keeping f fixed and passing to the limit with φ, we have the desired result.

§ 6. **Lemmas Concerning Denjoy Integration.** We state here, for convenience, the few, not quite trivial, relevant results concerning Denjoy integrals in the general sense. They are among the most immediate consequences of the definitions.

(6. 1) **Lemma.** *Let $f(x)$ be Denjoy integrable in (x_0, x_1). Then its indefinite integral is a continuous function $F(x)$ such that, given any non-empty perfect set P, there exists a non-empty portion of P (common part of P with an open interval) on which $f(x)$ is absolutely (Lebesgue) integrable and for whose complementary intervals (a_n, b_n) the series $\sum_n |F(b_n) - F(a_n)|$ converges.*

The property expressed by the above lemma, may be regarded as a definition of what we mean by an indefinite integral in the Denjoy sense.

(6. 2) **Lemma.** *A function $f(x)$ which is Denjoy integrable in (x_0, x_1), is almost everywhere in (x_0, x_1) the approximate derivative of its indefinite integral $F(x)$.*

We give the proof of this well-known result as it is short. Suppose, if possible, that a set E of positive measure consists of points at which there is no approximate derivative. Then E contains a non-empty perfect subset P, every non-empty portion of which has positive measure. In P we locate a portion ϖ to which the conclusion of lemma (6. 1) applies. The function $g(x) = f(x)$ in ϖ and constant with the value $\{F(b_n) - F(a_n)\}/(b_n - a_n)$ in each complementary interval (a_n, b_n) of ϖ, is then absolutely integrable and therefore almost everywhere in ϖ the derivative of its integral G. At almost any point of density of ϖ, f is thus the approximate derivative of F, since in ϖ we have $f = g$ and also $F = G$. This is a contradiction and the lemma is proved.

(6. 3) **Lemma.** *If $f(x)$ is Denjoy integrable in (a, b) and $\varphi(x)$ bounded and measurable, and if $F(x)$ and $\Phi(x)$ are their indefinite integrals, then the integral*

$$\int_a^b f(x)\, \Phi(x)\, dx$$

exists and its value is

$$F(b)\, \Phi(b) - F(a)\, \Phi(a) - \int_a^b F(x)\, \varphi(x)\, dx.$$

This can be proved by the same method, and is indeed only a special case of general theorems on integration by parts. For other proofs and for the deeper properties of Denjoy integrals, cf for instance Saks [12] Chap. VIII.

§ 7. **The Condition (W. E.) on an Ordinary Curve.** We shall now extend the condition (W. E.) to problems no longer restricted to be independent of y. We denote by $f(x, y, y')$ a real function, measurable (B) and everywhere finite, of three variables. We write

$$(7.1) \qquad \mathcal{M}_h f = \mathcal{M}_h f(x, y, y') = \{f(x, y+h, y') - f(x, y, y')\}/h$$

when h is any real number $\neq 0$, and $\mathcal{M}_h f = 0$ when $h = 0$. This ratio $\mathcal{M}_h f$, which vanishes identically in the case of a problem independent of y, will be restricted by conditions enabling us to generalize the results of § 4.

We shall suppose the problem to be such that *the ratio $\mathcal{M}_h f$ remains bounded when x, y, y', h range in bounded sets.* This is always assumed in the investigations of Tonelli and classical writers.[1]

We denote by (x_0, y_0) and (x_1, y_1) two fixed finite points of the plane such that $x_0 < x_1$. We call *ordinary admissible curve*, a pair of functions $y(x)$, $y'(x)$ such that

$$(7.2) \qquad y(x) = y_0 + \int_{x_0}^{x} y'(t)\,dt = y_1 - \int_{x}^{x_1} y'(t)\,dt$$

and for which the integral

$$(7.3) \qquad \int_{x_0}^{x_1} f(x, y(x), y'(x))\,dx$$

exists (finite or infinite), each of these integrals being understood in the general Denjoy sense.

For brevity we write $f(t)$ for $f(t, y(t), y'(t))$ and also $\mathcal{M}_h f(t)$ for $\mathcal{M}_h f(t, y(t), y'(t))$, etc. We say, further, that a curve $y(x)$, $y'(x)$ fulfils *the condition* (D) if the function

$$\mathcal{M}_{\eta(x)} f(x, y(x), y'(x))$$

is Denjoy integrable whenever $\eta(x)$ has bounded derivates. We shall confine ourselves to the study of »minimizing» curves which fulfil the condition (D). This restriction is also assumed, actually in a much more stringent form, in the investigations of Tonelli [14] and Mc Shane [9].

(7.4) **Theorem.** *Let $y(x)$, $y'(x)$ be an ordinary admissible curve which fulfils the condition (D) and for which (7.3) assumes a finite minimum; and let $\{h_n\}$ be a sequence of numbers tending to zero which contains an infinity of terms of both signs. Then there exists a constant c, finite or infinite, such that, for almost every x, we have*

---

[1] Here, an even weaker (*fractional* Lipschitz) condition would suffice.

$$\{f(x, y(x), y'(x) + \zeta) - f(x, y(x), y'(x))\}/\zeta \geq c + \liminf_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\, dt$$

whenever $\zeta > 0$, *and*

$$\{f(x, y(x), y'(x) + \zeta) - f(x, y(x), y'(x))\}/\zeta \leq c + \limsup_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\, dt$$

whenever $\zeta < 0$.

In the case of an infinite constant c, these inequalities are interpreted to mean that, for almost every x,

$$\limsup_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\, dt = +\infty \text{ if } c = -\infty; \text{ and}$$

$$\liminf_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\, dt = -\infty, \text{ if } c = +\infty.$$

Proof. We may suppose that neither of the extreme limits as $n \to \infty$ of the expression

$$\int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\, dt$$

is infinite for almost all x, since there is then nothing to prove. We may suppose further that $y'(x)$ is everywhere finite and measurable (B). The functions

$$\Phi_n(x, \zeta) = -\{f(x, y(x), y'(x) + \zeta) - f(x) - \zeta \cdot \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\, dt\}$$

and

$$\Phi(x, \zeta) = \liminf_n \Phi_n(x, \zeta)$$

are then also measurable (B), and moreover $\Phi(x, 1)$ and $\Phi(x, -1)$ are finite below in sets of positive measure. If the assertion of our theorem is false, the function $\Phi(x, \zeta)$ therefore fulfils the conditions of Theorem (4. 1); we shall show that this leads to a contradiction.

By Theorem (4. 1), there exist constants c_0 and $\gamma_0 > 0$, and a bounded measurable function $\zeta_0(x)$, such that both the subsets of (x_0, x_1)

$$E_1 = \mathop{\mathrm{E}}_{x} [\zeta_0(x) > 0] \quad \text{and} \quad E_2 = \mathop{\mathrm{E}}_{x} [\zeta_0(x) < 0]$$

have positive measure and that in $E_1 + E_2$ we have

$$\Phi(x, \zeta_0(x)) + c_0 \zeta_0(x) > \gamma_0.$$

By replacing $\zeta_0(x)$ by 0 at certain points, if necessary, we may suppose that E_1 and E_2 are *perfect sets of positive measure on which each of the functions*

$$\zeta_0(x),\ 1/\zeta_0(x),\ y'(x),\ f(x),\ f(x, y(x), y'(x) + \zeta_0(x))$$

is bounded and continuous, while on each of the sets E_1 and E_2 each of the functions of x

$$\liminf_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt \quad \textit{and} \quad \limsup_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt$$

is either an infinite constant or else a bounded and continuous function. Further, except in the »trivial» case in which, on the whole of E_1,

$$\liminf_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt = +\infty,$$

the function

$$\alpha_n(x) = \operatorname{Min}\left\{0, \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt - \liminf_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt\right\}$$

is finite on E_1 and tends to zero as $n \to \infty$. Hence, replacing E_1 by a perfect subset of positive measure if necessary, we may (in view of Egoroff's theorem) suppose that the sequence $\{\alpha_n(x)\}$ converges uniformly to zero in E_1. Denoting by K_0 the upper bound of $\zeta_0(x)$, we have in E_1,

$$\Phi_n(x, \zeta_0(x)) - \Phi(x, \zeta_0(x)) \geq K_0\, \alpha_n(x),$$

and it follows that, for all n exceeding a certain constant n_0,

$$\Phi_n(x, \zeta_0(x)) + c_0 \zeta_0(x) > \frac{1}{2}\gamma_0.$$

Similarly this extends to the »trivial» case and to the set E_2. Finally, by continuity, we can determine $\delta_0 > 0$ so that *if x and ξ are any two points of $E_1 + E_2$ such that $|x - \xi| < \delta_0$ and if n is any integer exceeding n_0, then*

$$(7.5) \qquad f(x, y(x), y'(x) + \zeta_0(x)) - f(x) - \zeta_0(x) \cdot \left\{c_0 + \int_{x_0}^{\xi} \mathcal{M}_{h_n} f(t)\,dt\right\} < -\frac{1}{4}\gamma_0.$$

This being so, let ξ_1 and ξ_2 denote, respectively, a point of density of E_1 and one of E_2, and suppose for simplicity that $\xi_1 < \xi_2$. (The proof proceeds similarly if $\xi_1 > \xi_2$.) We denote by $\{h'_n\}$ the subsequence of $\{h_n\}$ which consists of the positive terms not exceeding the numbers $K'_0 \,.\, (\xi_2 - \xi_1)$ and $K'_0 \,.\, (\xi_1 - x_0)$ where K'_0 is the upper bound of $1/\zeta_0(x)$ in $E_1 + E_2$. For each n we can now determine in (x_0, x_1) two intervals I_n and J_n, necessarily non-overlapping, which have the points ξ_1 and ξ_2, respectively, as their right-hand endpoints, such that

$$\int\limits_{I_n \,.\, E_1} \zeta_0(x)\, dx = - \int\limits_{J_n \,.\, E_2} \zeta_0(x)\, dx = h'_n.$$

We now define

$$\eta'_n(x) = \zeta_0(x) \;\; \textit{when} \;\; x < (E_1 I_n + E_2 J_n),$$

$$= 0 \qquad \textit{otherwise};$$

$$\eta_n(x) = \int\limits_{x_0}^{x} \eta'_n(t)\, dt = - \int\limits_{x}^{x_1} \eta'_n(t)\, dt.$$

We write for brevity $\eta(x), \eta'(x)$ instead of $\eta_1(x), \eta'_1(x)$. Evidently the function $\eta_n(x)$ which may be written Min $(\eta(x), h'_n)$ is monotone increasing in I_n, monotone decreasing in J_n and constant in each of the three remaining parts of (x_0, x_1).

We shall now show that *for sufficiently large n the integral*

$$\mathcal{J}_n = \int\limits_{x_0}^{x_1} f(x, y(x) + \eta_n(x), y'(x) + \eta'_n(x))\, dx$$

exists and has a value which is smaller than that of

$$\mathcal{J} = \int\limits_{x_0}^{x_1} f(x, y(x), y'(x))\, dx.$$

This will evidently contradict our hypotheses, since $y(x) + \eta_n(x), y'(x) + \eta'_n(x)$ is an admissible curve; and our theorem will then be established.

For brevity, we write $f_n(x)$, $\mathcal{M}_h f_n(x)$ in place of

$$f(x, y(x), y'(x) + \eta'_n(x)), \quad \mathcal{M}_h f(x, y(x), y'(x) + \eta'_n(x)).$$

Denoting by c_n any constant, we have then

$$\mathcal{J}_n - \mathcal{J} = \int_{x_0}^{x_1} \{f_n(x) - f(x) - c_n \eta'_n(x) + \eta_n(x) \mathcal{M}_{\eta_n(x)} f_n(x)\}\, dx,$$

each of the terms under the integral sign, except perhaps the last, being integrable for trivial reasons. Now, by integration by parts [lemma (6. 3)], we have

$$\int_{x_0}^{x_1} \eta_n(x) \mathcal{M}_{\eta_n(x)} f_n(x)\, dx = -\int_{x_0}^{x_1} \eta'_n(x) \left\{ \int_{x_0}^{x} \mathcal{M}_{\eta_n(t)} f_n(t)\, dt \right\} dx,$$

and here the inner integrand on the right-hand side reduces to

$$\mathcal{M}_{\eta_n(t)} f(t)$$

except when $t < (E_1 I_n + E_2 J_n)$ in which case both are bounded, since $y'(t)$ and $\eta'_n(t)$ are then bounded. Thus

$$\int_{x_0}^{x_1} \eta_n(x) \mathcal{M}_{\eta_n(x)} f(x)\, dx = -\int_{x_0}^{x_1} \eta'_n(x) \left\{ \int_{x_0}^{x} \mathcal{M}_{\eta_n(t)} f(t)\, dt + O(h'_n) \right\} dx.$$

Now if $x < (E_1 I_n + E_2 J_n)$, we can determine ξ in this set so that $|x - \xi| < O(h'_n)$ and[1] that

$$\int_{x_0}^{x} \mathcal{M}_{\eta_n(t)} f(t)\, dt = \int_{x_0}^{\xi} \mathcal{M}_{h'_n} f(t)\, dt - \int_{x_0}^{\xi_1} \mathcal{M}_{h'_n} f(t)\, dt + \int_{I_n \cdot (x_0, x)} \mathcal{M}_{\eta(t)} f(t)\, dt + \int_{J_n \cdot (x_0, x)} \mathcal{M}_{\eta(t)} f(t)\, dt.$$

The last two integrals are $o(1)$ as $n \to \infty$, by continuity of the Denjoy integral of the function $\mathcal{M}_{\eta(t)} f(t)$ which is independent of n. Hence, choosing

$$c_n = c_0 + \int_{x_0}^{\xi_1} \mathcal{M}_{h'_n} f(t)\, dt,$$

we obtain

$$\mathcal{J}_n - \mathcal{J} = \int_{x_0}^{x_1} \left(f_n(x) - f(x) - \eta'_n(x) \cdot \left\{ c_0 + \int_{x_0}^{\xi} \mathcal{M}_{h'_n} f(t)\, dt + o(1) \right\} \right) dx$$

and by (7. 5), for all large n,

---

[1] Remembering that $\mathcal{M}_h f = 0$ for $h = 0$, by definition.

$$\mathcal{J}_n - \mathcal{J} < \int\limits_{E_1 I_n + E_2 J_n} - \left(\frac{1}{4}\gamma_0 + o(1)\right) dx < 0.$$

This completes the proof.

§ 8. **Generalized curves.** By a *generalized admissible curve*, we shall mean two finite functions measurable (B), $y(x)_*$, $y'(x, \alpha)_*$, the latter defined for $0 \leq \alpha \leq 1$ as well as for $x_0 \leq x \leq x_1$, such that

$$(8.1) \qquad y(x)_* = y_0 + \int\limits_{x_0}^{x} dt \int\limits_{0}^{1} d\alpha\, y'(t, \alpha)_* = y_1 - \int\limits_{x}^{x_1} dt \int\limits_{0}^{1} d\alpha\, y'(t, \alpha)_*,$$

and for which the integral

$$(8.2) \qquad \int\limits_{x_0}^{x_1} dx \int\limits_{0}^{1} d\alpha f(x, y(x)_*, y'(x, \alpha)_*)$$

exists, each of these integrals being interpreted as a repeated integral in the general Denjoy sense.

For a given function $f(x, y, y')$ and fixed end points (x_0, y_0), (x_1, y_1), the problem of the minimum of (8.2) in the class of pairs of functions $y(x)_*$, $y'(x, \alpha)_*$ subject to (8.1) will be called the *generalized minimum problem.* It is closely connected with the ordinary minimum problem for the simple integral (7.3) subject to (7.2).

It is possible to show, by quite elementary methods, that when the functions concerned are sufficiently smooth, the double integral (8.2) for a pair of functions $y(x)_*$, $y'(x, \alpha)_*$ subject to (8.1) can always be approximated, as closely as we please, by a simple integral of the type (7.3) corresponding to a pair of functions $y(x)$, $y'(x)$ subject to (7.2). When this is the case, *the value of the minimum is clearly the same for the generalized problem as for the ordinary problem.*

To obtain such an approximation, we divide the range of x into parts A_i ($i = 1, 2, \ldots, N$), and, in each of these, we divide the range of α into parts $B_{i,k}$ ($k = 1, 2, \ldots, N_i$). Further, we divide A_i into parts $A_{i,k}$ such that the length of $A_{i,k}$ is the area in (x, α) of $[A_i \times B_{i,k}]$, and we choose for $y'(x)$, when $x < A_{i,k}$, the constant value

$$\frac{1}{|A_{i,k}|} \int\limits_{A_i} dx \int\limits_{B_{i,k}} d\alpha\, y'(x, \alpha)_*,$$

and define $y(x) = y_0 + \int_{x_0}^{x} y'(t)dt$. If the subdivision is suitably chosen in correspondence with a positive number ε, arbitrarily small, we shall have, when the functions concerned are sufficiently smooth,

$$|y(x) - y(x)_*| < \varepsilon \text{ for all } x, \ y(x_0) = y(x_0)_* = y_0, \ y(x_1) = y(x_1)_* = y_1,$$

and

$$\left| \int_{x_0}^{x_1} dx \int_0^1 d\alpha f(x, y(x)_*, y'(x, \alpha)_*) - \int_{x_0}^{x_1} dx f(x, y(x), y'(x)) \right| \leq$$
$$\leq \sum_{i,k} |A_{i,k}| \cdot |\mathrm{M}_{A_{i,k}} f(x, y(x), y'(x)) - \mathrm{M}_{A_i, B_{i,k}} f(x, y(x)_*, y'(x, \alpha)_*)|$$

where $\mathrm{M}_A f(x)$ denotes the *mean value* of a function $f(x)$ over a set A, and where $\mathrm{M}_{A,B} f(x, \alpha)$ denotes that of a function $f(x, \alpha)$ over the plane set $[A \times B]$ of values of (x, α).

If the functions are substantially constant in the divisions chosen, the two mean values occuring in each term of the sum on the right-hand side of our last inequality will differ by as little as we please, and the sum itself will be arbitrarily small.

The importance of our generalization of the problem, first given in a slightly different form in the paper [16], lies in the fact that *a minimum which is not attained in the ordinary problem may be, and frequently is, attained in the generalized problem.*

This is so, for instance, in the simple example, due to Caratheodory [3],

$$f(x, y, y') = \frac{(1 + y^2)(1 + y'^2)}{\{2(1 + y'^2)\}^{1/2} - 1}; \quad x_0 = y_0 = y_1 = 0, x_1 = 1.$$

In this example, we have $f \geq 2$, with equality only if $y = 0$ and $y' = \pm 1$. Clearly, for an ordinary curve $y(x)$, $y'(x)$ subject to (7.3), the conditions $y(x) = 0$, $y'(x) = \pm 1$ at almost all points x, are not compatible; but they become compatible as soon as we replace the former by $|y(x)| < \varepsilon$ where $\varepsilon > 0$. In fact, denoting by n any positive even integer exceeding $1/\varepsilon$, we need only choose for our curve, the zig-zag defined in each of the n intervals $(k-1)/n < x \leq k/n$ $(k = 1, 2, \ldots, n)$ by

$$y'(x) = (-1)^k, \ y(x) = \int_0^x y'(t)dt.$$

It follows that for the ordinary problem the minimum is unattained and has the value 2. The minimum for the generalized problem clearly has the same value, but this value is attained when we choose, for all x,

$$y(x)_* = 0, \ y'(x, \alpha)_* = -1 \ (\alpha < {}^1/_2), \ y'(x, \alpha) = 1 \ (\alpha \geq {}^1/_2).$$

§ 9. **The Condition (W. E.)$_*$ and Classical Non-regular Problems.** For brevity, we write

$$f(x)_* = \int_0^1 f(x, y(x)_*, y'(x, \alpha)_*)\, d\alpha, \quad \bar{f}_y(x)_* = \int_0^1 f_y(x, y(x)_*, y'(x, \alpha)_*) d\alpha,$$

$$\bar{y}'(x)_* = \int_0^1 y'(x, \alpha)_* d\alpha, \text{ etc.}$$

In the next §§, we shall establish the necessity of a condition which extends to generalized curves the results so far established for ordinary curves. In its simplest form, the condition may be written

$$f(x, y(x)_*, \bar{y}'(x)_* + \zeta) - f(x)_* - \zeta \cdot \left\{ c + \int_{x_0}^{x} \bar{f}_y(t)_* dt \right\} \geq 0,$$

and is a direct extension of the condition (W. E.) in classical form (§ 3). We shall call it *the condition* (*W. E.*)$_*$.

It is convenient to express this condition in geometrical language. We require a few preliminary definitions.

A function $F(\zeta)$ will be termed *convex at* ζ_0, if we have

$$-\infty < F(\zeta_0) \leq aF(\zeta_1) + bF(\zeta_2) \ \textit{whenever} \ a \geq 0, \ b \geq 0, \ a + b = 1, \ a\zeta_1 + b\zeta_2 = \zeta_0,$$

provided only that ζ_1 and ζ_2 belong to the interval of definition of $F(\zeta)$. [Geometrically, this implies that at ζ_0 the graph of $F(\zeta)$ does not lie vertically above any chord joining two points of the graph. Convexity at ζ_0, thus defined, is not a purely »local» property, since it depends on the values of $F(\zeta)$ in the whole interval of definition. It must not, therefore, be confused with »local convexity» such as occurs when, for instance, $F(\zeta)$ has a positive second differential coefficient $F''(\zeta)$ in the neighbourhood of a point ζ_0.] We shall understand convexity at ζ_0 to imply that $F(\zeta)$ is never $-\infty$, except in the case in which ζ_0 is an endpoint of the interval of definition of $F(\zeta)$; this is implied in the definition when a suitable convention is made regarding indeterminate forms $\infty - \infty$. With these conventions, Minkowski's theorem (§ 3) asserts that *a function* $F(\zeta)$ *defined for all* ζ *is convex at* ζ_0 *if and only if there exists a linear function* $L(\zeta)$ *such that*

$$L(\zeta) \leq F(\zeta) \text{ for all } \zeta, \text{ and } L(\zeta_0) = F(\zeta_0). \tag{9.1}$$

We say that the function $F(\zeta)$ has at ζ_0 the *flat support* $L(\zeta)$, if $L(\zeta)$ is a linear function of ζ which fulfils the conditions (9.1). Minkowski's theorem asserts that a flat support exists at ζ_0 if and only if $F(\zeta)$ is convex at ζ_0. The condition (W. E.) in its classical form requires that the function

$$F(\zeta) = f(x, y(x), y'(x) + \zeta)$$

shall have at $\zeta = 0$ the flat support

$$L(\zeta) = f(x) + \zeta \cdot \left\{ c + \int_{x_0}^{x} f_y(t)\, dt \right\}.$$

This condition can, therefore, only be fulfilled when $F(\zeta)$ is a *convex* function of ζ at $\zeta = 0$. *It is the condition* $(W.E.)_*$ *which provides the generalization applicable to the non-convex case.*

We call *generalized flat support* of $F(\zeta)$ at ζ_0, a linear function $L(\zeta)$ such that

$$L(\zeta) \leq F(\zeta) \text{ for all } \zeta$$

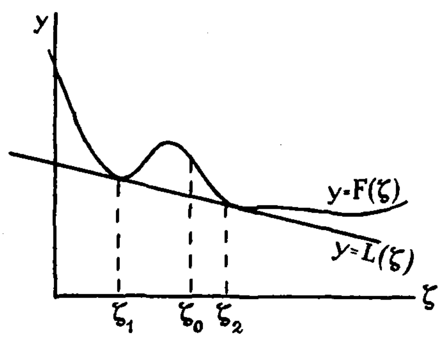
and that $L(\zeta_*) = F(\zeta_*)$ for certain values $\zeta_* = \zeta_*(\alpha)$ with the average

$$\int_0^1 \zeta_*(\alpha)\, d\alpha = \zeta_0.$$

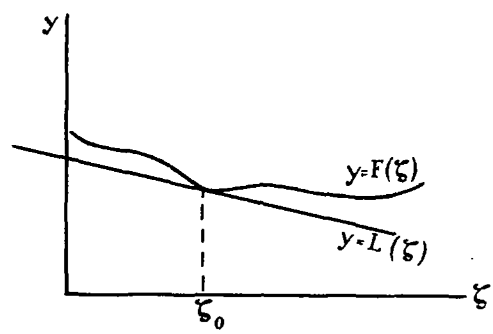
As is easily seen, in order that a linear function $L(\zeta)$ which nowhere exceeds $F(\zeta)$ be a generalized flat support of $F(\zeta)$ at ζ_0, it is necessary and sufficient that there exist *two* values ζ_1 and ζ_2 such that

$$\zeta_1 \leq \zeta_0 \leq \zeta_2 \text{ and } L(\zeta_1) = F(\zeta_1),\ L(\zeta_2) = F(\zeta_2).$$

The condition is clearly necessary; and, when it is fulfilled, we can choose for $\zeta_*(\alpha)$ a stepfunction which assumes at most the two values ζ_1 and ζ_2 and which has the average ζ_0. The case $\zeta_1 = \zeta_2 = \zeta_0$ is that of a flat support in the ordinary sense.



generalized flat support at ζ_0.



ordinary flat support at ζ_0.

With these conventions, we see that *the condition* $(W.E.)_*$ *requires that the function*

$$F(\zeta) = f(x,\ y(x)_*,\ \bar{y}'(x)_* + \zeta)$$

shall have

$$L(\zeta) = \bar{f}(x)_* + \zeta \cdot \left\{ c + \int_{x_0}^{x} \bar{f}_y(t)_* dt \right\}$$

as a generalized flat support at $\zeta = 0$.

It is clear geometrically that the condition $(\mathrm{W.E.})_*$, as distinct from the condition (W.E.), can only arise in the case of a function $f(x, y, y')$ which is not convex in y', i.e. in the case of a variational problem which is not »*regular*». In order to illustrate the meaning of the condition from the classical point of view, let us now consider a case in which $f(x, y, y')$ is differentiable as often as required but not convex in y'. We shall suppose, with C. Caratheodory [3, 4], that there exist functions $p(x, y, 0)$ and $p(x, y, 1)$, differentiable as often as necessary, such that $p(x, y, 0) < p(x, y, 1)$ in the region under consideration and that, as function of y', f is convex at every point y' outside the interval $p(x, y, 0) \leq \leq y' \leq p(x, y, 1)$ and not convex at any point y' inside. If y' lies in this interval, it is easily seen that the function

$$F(\zeta) = f(x, y, y' + \zeta)$$

has for $\zeta = 0$ exactly one generalized flat support, namely

$$P(x, y) + (y' + \zeta)\, Q(x, y)$$

where $P(x, y) + y'\, Q(x, y)$ is the linear interpolation of f, as function of y', between $y' = p(x, y, 0)$ and $y' = p(x, y, 1)$, and may therefore[1] be written

$$f(x, y, p) + (y' - p) f_p(x, y, p)$$

where p may be either $p(x, y, 0)$ or $p(x, y, 1)$. Hence, for each of these two values of p, a simple calculation shows that we have identically in x, y

$$(9.2) \qquad P_y - Q_x = f_y - p\, Q_y - Q_x.$$

This being so, let $y(x)_*$, $y'(x, \alpha)_*$ be a generalized curve along which f fulfils the condition $(\mathrm{W.E.})_*$ for almost all x, and let x be almost any point for which $y'(x, \alpha)_*$ does not reduce for almost all α to $\bar{y}'(x)_*$. Then clearly, in view of the condition $(\mathrm{W.E.})_*$,

$$p(x, y(x)_*, 0) \leq \bar{y}'(x)_* \leq p(x, y(x)_*, 1),$$

[1] It is tangent to f at the two extreme values of y'.

so that the only values that can be assumed by $y'(x, \alpha)$ in positive measure of α, are $p(x, y(x)_*, 0)$ and $p(x, y(x)_*, 1)$. For this x and for $y = y(x)_*$, we therefore have, by (9.2), neglecting a possible set of measure zero in α,

$$P_y - Q_x = f_y(x, y(x)_*, y'(x, \alpha)_*) - Q_x - y'(x, \alpha)_* Q_y.$$

Moreover, by definition

$$Q(x, y(x)_*) = f_{y'}(x, y(x)_*, y'(x, \alpha)_*)$$

independently of α. When we integrate both these relations with respect to α, the former gives, for almost any x,

$$P_y - Q_x = \bar{f}_y(x)_* - Q_x - \bar{y}'(x)_* Q_y = \bar{f}_y(x)_* - \frac{d}{dx} Q(x, y(x)_*),$$

and hence, substituting for Q,

$$P_y - Q_x = \bar{f}_y(x)_* - \frac{d}{dx} \bar{f}_{y'}(x)_*. \tag{9.3}$$

On the other hand, in view of the condition (W. E.)$_*$, we have, writing $\eta_\alpha = \zeta + y'(x, \alpha)_* - \bar{y}'(x)_*$,

$$\int_0^1 f(x, y(x)_*, y'(x, \alpha)_* + \zeta) d\alpha \geq \int_0^1 L(\eta_\alpha) d\alpha = L(\zeta), \tag{9.4}$$

where $L(\zeta) = \bar{f}(x)_* + \zeta \cdot \left\{ c + \int_{x_0}^{x} \bar{f}_y(t)_* \, dt \right\}$.

The function of ζ on the left of the inequality (9.4) thus has $L(\zeta)$ for an ordinary flat support at $\zeta = 0$, and, by the definition of tangent, we deduce at once that

$$\bar{f}_{y'}(x)_* = c + \int_{x_0}^{x} f_y(t)_* \, dt.$$

Hence by (9.3), we must have when $y = y(x)_*$,

$$P_y - Q_x = 0$$

at almost all the points x such that $y'(x, \alpha)$ does not reduce to $\bar{y}'(x)$ almost everywhere in α. In particular, if the function

$$\Omega(x, y) = P_y - Q_x$$

is different from zero throughout the region under consideration, $y'(x, \alpha)_*$ must reduce to $\bar{y}'(x)_*$ almost everywhere, and our minimum cannot be attained along a generalized curve $y(x)_*$, $y'(x, \alpha)_*$ other than a trivial variant of an ordinary minimizing curve.

We shall not dwell on the important part played by the restriction $\Omega(x, y) \neq 0$ in the classical theory of the non-regular problem as developed by Caratheodory (cf. also TONELLI [13], vol. II, p. 193—200). The object of these remarks has been to show the connection between our condition (W. E.)$_*$ and these classical ideas.

§ 10. **General Form of the Condition (W. E.)$_*$.** We shall now consider a more general form of the condition (W. E.)$_*$ which corresponds exactly to that of the condition (W. E.) in § 7, and our main theorem will be the exact analogue of Theorem (7.4). The necessary condition established in Theorem (7.1) reduces to the classical form (3.3) of the condition (W. E.), whenever there exists a sequence of numbers $\{h_n\}$ tending to zero and containing an infinity of terms of both signs, such that, for almost every x,

$$\limsup_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt = \liminf_n \int_{x_0}^{x} \mathcal{M}_{h_n} f(t)\,dt = \int_{x_0}^{x} f_y(t)\,dt.$$

In exactly the same way, the necessary condition which will be established in Theorem (10.1) below, reduces to the form of the condition (W. E.)$_*$ stated in § 9, whenever there exists a sequence of numbers $\{h_n\}$ tending to zero and containing an infinity of terms of both signs, such that, for almost every x,

$$\limsup_n \int_{x_0}^{x} \bar{\mathcal{M}}_{h_n} f(t)_*\,dt = \liminf_n \int_{x_0}^{x} \bar{\mathcal{M}}_{h_n} f(t)_*\,dt = \int_{x_0}^{x} \bar{f}_y(t)_*\,dt,$$

where, in accordance with our previous notation, $\bar{\mathcal{M}}_h f(t)_*$ stands for

$$\int_0^1 \{f(t, y(t)_* + h, y'(t, \alpha)_*) - f(t, y(t)_*, y'(t, \alpha)_*)\}\,d\alpha/h,$$

when $h \neq 0$, and for 0 when $h = 0$.

In our main theorem, the appropriate modifications, applicable to the generalized problem, require hardly any new idea. The principal restriction $(D)_*$

will be the natural generalization of our condition (D), but we shall make one further restriction $(G)_*$ which will be seen in § 11 to exclude only certain quite unimportant generalized curves, and whose effect is, in particular, to ensure that, in certain of our repeated integrals, the inner integral exists in the Lebesgue sense. (Elsewhere, as usual, integrals will be in the general Denjoy sense.)

We shall again suppose that the ratio $\mathcal{M}_h f$ remains bounded when x, y, y', h range in bounded sets. Further, we shall say that a generalized curve $y(x)_*$, $y'(x, \alpha)_*$ fulfils the condition $(G)_*$ if there exists a measurable function $\varphi(x)$, finite almost everywhere in (x_0, x_1), such that

$$|y'(x, \alpha)_*| < \varphi(x)$$

for almost all α at each point x. Finally, we shall say that a generalized curve $y(x)_*$, $y'(x, \alpha)_*$ fulfils the condition $(D)_*$, if the function

$$\mathcal{M}_{\eta(x)} f(x, y(x)_*, y'(x, \alpha)_*)$$

has for almost all x an integral in α which is Denjoy integrable in x, whenever $\eta(x)$ has bounded derivates.

(10. 1) **Theorem.** *Let $y(x)_*$, $y'(x, \alpha)_*$ be a generalized admissible curve which fulfils the conditions $(G)_*$ and $(D)_*$, and for which the integral* (8. 2) *assumes a finite minimum; and let $\{h_n\}$ be a sequence of numbers tending to zero which contains an infinity of terms of both signs. Then there exists a constant c, finite or infinite, such that, for almost every x, we have*

$$\{f(x, y(x)_*, \bar{y}'(x)_* + \zeta) - f(x, y(x)_*, \bar{y}'(x)_*)\}/\zeta \geq$$

$$\geq c + \liminf_n \int_{x_0}^{x} dt \int_0^1 d\alpha\, \mathcal{M}_{h_n} f(t, y(t)_*, y'(t, \alpha)_*)$$

whenever $\zeta > 0$, and

$$\{f(x, y(x)_*, \bar{y}'(x)_* + \zeta) - f(x, y(x)_*, \bar{y}'(x)_*)\}/\zeta \leq$$

$$\leq c + \limsup_n \int_{x_0}^{x} dt \int_0^1 d\alpha\, \mathcal{M}_{h_n} f(t, y(t)_*, y'(t, \alpha)_*)$$

whenever $\zeta < 0$. In the case of an infinite constant c, these inequalities are interpreted to mean that, for almost every x,

$$\limsup_n \int_{x_0}^{x} \bar{\mathfrak{M}}_{h_n} f(t)_* \, dt = +\infty \text{ if } c = -\infty; \text{ and } \liminf_n \int_{x_0}^{x} \bar{\mathfrak{M}}_{h_n} f(t)\, dt = -\infty \text{ if } c = +\infty.$$

Proof. We may suppose that neither of the limits as $n \to \infty$ of the expression

$$\int_{x_0}^{x} \bar{\mathfrak{M}}_{h_n} f(t)_* \, dt$$

is infinite for almost all x, and moreover that $\bar{y}'(x)$ is everywhere finite and measurable (B).

The functions

$$\Phi_n(x, \zeta) = -\left\{ f(x, y(x)_*, \bar{y}'(x)_* + \zeta) - \bar{f}(x)_* - \zeta \cdot \int_{x_0}^{x} \bar{\mathfrak{M}}_{h_n} f(t)\, dt \right\}$$

and

$$\Phi(x, \zeta) = \liminf_n \Phi_n(x, \zeta)$$

are then also measurable (B) and moreover $\Phi(x, 1)$ and $\Phi(x, -1)$ are finite below in sets of positive measure. If the assertion of our theorem is false, the function $\Phi(x, \zeta)$ therefore fulfils the conditions of Theorem (4. 1). We shall show that this leads to a contradiction.

As in the proof of Theorem (7. 4), we see that there exist constants c_0 and $\gamma_0 > 0$, and a bounded measurable function $\zeta_0(x)$, such that both the subsets of (x_0, x_1)

$$E_1 = \mathop{\mathrm{E}}_x [\zeta_0(x) > 0] \quad \text{and} \quad E_2 = \mathop{\mathrm{E}}_x [\zeta_0(x) < 0]$$

have positive measure and that in $E_1 + E_2$ we have

$$\Phi(x, \zeta_0(x)) + c_0 \zeta_0(x) > \gamma_0.$$

Moreover we may suppose further that E_1 and E_2 are perfect sets on which each of the functions

$$\zeta_0(x),\ 1/\zeta_0(x),\ \varphi(x),\ \bar{y}'(x),\ f(x)_*,\ f(x, y(x)_*, \bar{y}'(x)_* + \zeta_0(x))$$

is bounded and continuous, and that, on each of the sets E_1 and E_2, each of the functions of x

$$\liminf_n \int_{x_0}^{x} \bar{\mathfrak{M}}_{h_n} f(t)_* \, dt \quad \text{and} \quad \limsup_n \int_{x_0}^{x} \bar{\mathfrak{M}}_{h_n} f(t)\, dt$$

is either an infinite constant, or else a bounded and continuous function. And finally, that whenever x and ξ are points of $E_1 + E_2$ distant less than a certain positive number δ_0, we have for all n exceeding a certain integer n_0,

$$(10.2)\quad f(x, y(x)_*, \bar{y}'(x)_* + \zeta_0(x)) - \bar{f}(x) - \zeta_0(x) \cdot \left\{c_0 + \int_{x_0}^{\xi} \mathcal{M}_{h_n} f(t)\, dt\right\} < -\frac{1}{4}\gamma_0.$$

This being so, let ξ_1 and ξ_2 denote, respectively, a point of density of E_1 and one of E_2, and suppose for simplicity that $\xi_1 < \xi_2$. (The proof proceeds similarly if $\xi_1 > \xi_2$, and we may clearly exclude the case $\xi_1 = \xi_2$.) We denote by $\{h'_n\}$ the subsequence of $\{h_n\}$ which consists of the positive terms not exceeding the numbers $K' \cdot (\xi_2 - \xi_1)$ and $K' \cdot (\xi_1 - x_0)$ where K' is the upper bound of $1/\zeta_0(x)$ in $E_1 + E_2$. For each n we can now determine in (x_0, x_1) two intervals I_n and J_n, necessarily non-overlapping, which have the points ξ_1 and ξ_2, respectively, as their right-hand endpoints, such that

$$\int_{I_n \cdot E_1} \zeta_0(x)\, dx = -\int_{J_n \cdot E_2} \zeta_0(x)\, dx = h'_n.$$

We now define $\eta'(x)$ to be $\zeta_0(x)$ when $x < E_1 I_1 + E_2 J_1$, and 0 otherwise, and we write

$$\eta(x) = \int_{x_0}^{x} \eta'(t)\, dt = -\int_{x}^{x_1} \eta'(t)\, dt; \quad \eta_n(x) = \operatorname{Min}(\eta(x), h'_n).$$

We denote by $\eta'_n(x)$ the derivative of $\eta_n(x)$ and we consider the generalized curve $y_n(x)_*$, $y'_n(x, \alpha)_*$ obtained by writing

$$y_n(x)_* = y(x)_* + \eta_n(x); \quad y'_n(x, \alpha)_* = \begin{cases} \bar{y}'(x)_* + \zeta_0(x) & \textit{when } x < E_1 I_n + E_2 J_n, \\ \bar{y}'(x, \alpha)_* & \textit{otherwise}. \end{cases}$$

We shall show that *for sufficiently large n the integral*

$$\mathcal{J}_n = \int_{x_0}^{x_1} dx \int_0^1 d\alpha\, f(x, y_n(x)_*, y'_n(x, \alpha)_*)$$

exists and has a smaller value than the integral

$$\mathcal{J} = \int_{x_0}^{x_1} dx \int_0^1 d\alpha\, f(x, y(x)_*, y'(x, \alpha)_*).$$

This will contradict our hypotheses and so establish the theorem.

For brevity, we write $\bar{f}_n(x)_*$, $\bar{\mathcal{M}}_h f_n(x)_*$ instead of

$$\int_0^1 f(x, y(x)_*, y'_n(x, \alpha)_*)\, d\alpha, \quad \int_0^1 \mathcal{M}_h f(x, y(x)_*, y'_n(x, \alpha)_*)\, d\alpha.$$

Denoting by c_n any constant, we have then

$$\mathcal{J}_n - \mathcal{J} = \int_{x_0}^{x_1} \{\bar{f}_n(x)_* - \bar{f}(x)_* - c_n \eta'_n(x) + \eta_n(x)\, \bar{\mathcal{M}}_{\eta_n(x)} f_n(x)_*\}\, dx,$$

each of the terms under the integral sign, except perhaps the last, being integrable for trivial reasons. Now by integration by parts [lemma (6. 3)], we have

$$\int_{x_0}^{x_1} \eta_n(x)\, \bar{\mathcal{M}}_{\eta_n(x)} f_n(x)_*\, dx = -\int_{x_0}^{x_1} \eta'_n(x) \left\{ \int_{x_0}^{x} \bar{\mathcal{M}}_{\eta_n(t)} f_n(t)_*\, dt \right\} dx,$$

and here the inner integrand on the right-hand side reduces to

$$\bar{\mathcal{M}}_{\eta_n(t)}\, f(t)_*$$

except when $t < (E_1 I_n + E_2 J_n)$ in which case it reduces to

$$\mathcal{M}_{\eta_n(t)}\, f(t, y(t)_*, \bar{y}'(t)_* + \eta'_n(t))$$

which is bounded since $|y'(t, \alpha)_*| < \varphi(t)$ and $\varphi(t)$ is bounded in the set $E_1 + E_2$. Thus

$$\int_{x_0}^{x_1} \eta_n(x)\, \bar{\mathcal{M}}_{\eta_n(x)} f_n(x)_*\, dx = -\int_{x_0}^{x_1} \eta'_n(x) \left\{ \int_{x_0}^{x} \bar{\mathcal{M}}_{\eta_n(t)}\, f(t)_*\, dt + O(h'_n) \right\} dx.$$

Now if $x < (E_1 I_n + E_2 J_n)$, we can determine ξ in this set so that $|x - \xi| < O(h'_n)$ and that

$$\int_{x_0}^{x} \bar{\mathcal{M}}_{\eta_n(t)}\, f(t)_*\, dt = \int_{x_0}^{\xi} \bar{\mathcal{M}}_{h'_n}\, f(t)_*\, dt - \int_{x_0}^{\xi_1} \bar{\mathcal{M}}_{h'_n}\, f(t)_*\, dt + \int_{(I_n + J_n)\cdot(x_0, x)} \bar{\mathcal{M}}_{\eta(t)}\, f(t)_*\, dt.$$

The last integral is $o(1)$ as $n \to \infty$, by continuity of the Denjoy integral of the function $\bar{\mathcal{M}}_{\eta(t)} f(t)_*$ which is independent of n. Hence, choosing

$$c_n = c_0 + \int_{x_0}^{\xi} \bar{\mathcal{M}}_{h'_n} f(t)_* \, dt,$$

we obtain from (10.2) [exactly as in the proof of Theorem (7.4)] that, for all large n,

$$\mathcal{J}_n - \mathcal{J} < 0.$$

This completes the proof.

§ 11. **Additional Remarks on Generalized Curves.** We shall now show that the generalized minimum and the existence of a minimizing generalized curve are unaffected by the restriction to generalized curves which fulfil the condition $(G)_*$. In fact, given any generalized admissible curve, we can determine another which fulfils this condition and for which the value of the variational integral is not increased.

The generalized curve $y(x)_*$, $y'(x, \alpha)_*$ will be termed *bifurcating curve* if there exist two finite measurable functions $\zeta_1(x)$ and $\zeta_2(x)$, where $\zeta_1(x) \leq \zeta_2(x)$, such that for almost all x the set

$$\underset{\alpha}{\mathrm{E}} \, [y'(x, \alpha)_* \neq \zeta_1(x), \; y'(x, \alpha)_* \neq \zeta_2(x)]$$

is of measure zero in α. We shall only deal with the case in which there exists a measurable function $a(x)$ such that $0 \leq a(x) \leq 1$ and

$$y'(x, \alpha)_* = \zeta_1(x) \quad \textit{when} \quad 0 \leq \alpha \leq a(x),$$
$$y'(x, \alpha)_* = \zeta_2(x) \quad \textit{when} \quad a(x) \leq \alpha \leq 1.$$

Actually this is no real loss of generality, since our integrals are unaffected by a *rearrangement* of $y'(x, \alpha)_*$ as function of α. (For the definition of rearrangement, cf Hardy, Littlewood and Polya [5, p. 276].) Evidently any bifurcating curve fulfils the condition $(G)_*$. We therefore need only prove

(11.1) **Theorem.** *Given any admissible generalized curve $y(x)_*$, $y'(x, \alpha)_*$ there exists a bifurcating curve with the same $y(x)_*$, such that the value of our variational integral is not increased when we replace the given generalized curve by this bifurcating curve.*

To obtain such a bifurcating curve it is evidently sufficient to establish the existence of measurable functions $a(x)$, $\zeta_1(x)$, $\zeta_2(x)$, such that, for almost all x,

$$0 \leq a(x) \leq 1, \; \zeta_1(x) \leq \zeta_2(x), \; a(x)\zeta_1(x) + (1 - a(x))\zeta_2(x) = \bar{y}'(x),$$
$$a(x) f(x, y(x)_*, \zeta_1(x)) + (1 - a(x)) f(x, y(x)_*, \zeta_2(x)) \leq K(x),$$

where $K(x)$ is a finite function measurable (B) nowhere less than $\bar{f}(x)_*$ and with the same integral. Such a function $K(x)$ certainly exists provided that $\bar{f}(x)_*$ is finite above, and we may clearly suppose this to be the case. We may suppose further that $\bar{y}'(x)_* = 0$.

Writing $\Phi(x, \zeta) = K(x) - f(x, y(x)_*, \zeta)$, we have by hypothesis

$$(11.2) \qquad \int_0^1 \Phi(x, y'(x, \alpha)_*) d\alpha \geq 0, \quad \int_0^1 y'(x, \alpha)_* d\alpha = 0,$$

and we wish to show that there exist measurable functions $a(x)$, $\zeta_1(x)$, $\zeta_2(x)$, such that, for almost all x,

$$0 \leq a(x) \leq 1, \; \zeta_1(x) \leq \zeta_2(x), \; a(x)\zeta_1(x) + (1 - a(x))\zeta_2(x) = 0,$$
$$a(x)\Phi(x, \zeta_1(x)) + (1 - a(x))\Phi(x, \zeta_2(x)) \geq 0.$$

For this purpose, let Q be the set of four-dimensional points (a, ζ_1, ζ_2, x) such that

$$(11.3) \quad 0 \leq a \leq 1, \; \zeta_1 \leq \zeta_2, \; a\zeta_1 + (1 - a)\zeta_2 = 0, \; a\Phi(x, \zeta_1) + (1 - a)\Phi(x, \zeta_2) \geq 0.$$

Clearly Q is a Borel set. We shall see that its projection on the x-axis includes every point x of (x_0, x_1).

We denote by E_1 the set of x for which $\Phi(x, 0) \geq 0$, and by E_2 the set of the points x, not belonging to E_1, for each of which there exists a value c such that $\Phi(x, \zeta) + c\zeta \leq 0$ *for all* ζ. Finally we denote by E_3 the set of x not belonging to $E_1 + E_2$.

Let now x be any point of (x_0, x_1). If $x < E_1$, the relations (11.3) are fulfilled when $a = 0$, $\zeta_1 = \zeta_2 = 0$. If $x < E_2$, the relations (11.2) require that the non-positive function of α

$$\Phi(x, y'(x, \alpha)_*) + cy'(x, \alpha)_*$$

have a non-negative integral in α; this function then vanishes almost everywhere in α, and in particular, by the second of the relations (11.2), at two values of α for which $y'(x, \alpha)_*$ assumes values ζ_1 and ζ_2 where $\zeta_1 \leq 0$ and $\zeta_2 \geq 0$. Since $\Phi(x, 0) < 0$, this implies $\zeta_1 < 0 < \zeta_2$. Choosing a so that $a\zeta_1 + (1 - a)\zeta_2 = 0$, the relations (11.3) are fulfilled by the system (a, ζ_1, ζ_2, x). Finally if $x < E_3$, we may apply Theorem (4.1) (in the particular form relating to functions independent of x), and we find that there exist numbers $\zeta_1, \zeta_2, c_0, \gamma_0$ such that $\zeta_1 < 0 < \zeta_2, \gamma_0 > 0$, and

$$\Phi(x, \zeta_1) + c_0\zeta_1 > \gamma_0, \; \Phi(x, \zeta_2) + c_0\zeta_2 > \gamma_0.$$

Choosing a as before, the relations (11.3) are fulfilled, the last of these in the stronger form

$$a\Phi(x, \zeta_1) + (1 - a)\Phi(x, \zeta_2) > \gamma_0.$$

Thus every x of (x_0, x_1) belongs to the projection of the Borel set Q. By a simple extension of lemma (5. 2), the set Q therefore contains the graph of a vector function whose components $a(x)$, $\zeta_1(x)$, $\zeta_2(x)$ are measurable. This completes the proof.

References.

BANACH S.

[1] *Theorie des Operations Linéaires*, Monografje Matematyczne I (Warszawa 1932).

CARATHEODORY C.

[2] *Inaugural Dissertation* (Göttingen 1904).

[3] *Über starke Maxima und Minima bei einfachen Integralen*, Math. Ann. 62 (1906) 449—503.

[4] *Sur les points singuliers du problème du Calcul des Variations dans le plan*, Annali di Mat. Pura e Applicata (3) 21 (1913) 153—171.

HARDY G H., LITTLEWOOD J. E., and POLYA.

[5] *Inequalities* (Cambridge 1934).

LAVRENTIEFF M.

[6] *Sur quelques problèmes du Calcul des Variations*, Annali di Mat. (4) 4 (1927) 7—28.

LUSIN N.

[7] *Leçon sur les ensembles analytiques*, Collection Borel (Paris 1930).

MC SHANE E. J.

[8] *Über die Unlösbarkeit eines einfachen Problems der Variationsrechnung*, Nachr. d. Gesell. d. Wiss. z. Göttingen (1933) 358—364.

[9] *The Du Bois Reymond relation in the Calculus of Variations*, Math. Ann. 109 (1934) 746—755.

MENGER K.

[10] *Courbes minimisantes non rectifiables et champs généraux de courbes admissibles dans le calcul des variations*, C. R. Acad. d. Sc. Paris *(11 mai 1936)*.

[11] *Calcul des Variations dans les espaces distanciés généraux*, C. R. Acad. d. Sc. Paris *(16 mars 1936)*.

SAKS S.

[12] *Theory of the integral*, Monografje Matematyczne VII (Warszawa 1937).

TONELLI L.

[13] *Fondamenti di Calcolo delle Variazioni*, (Bologna 1921).

TONELLI L.

[14] *Sulle proprieta delle estremanti,* Annali d. R. Scuola Normale Sup. d. Pisa (2) 3 (1934) 213—237.

[15] *Sur une question du Calcul des Variations,* Rec. Math. Moscou 33 (1926) 87—98.

YOUNG L. C.

[16] *Approximation by polygons in the Calculus of Variations,* Proc. Roy. Soc. A 141 (1933) 325—341.