ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS OF ANY ORDER

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Introduction

For some parabolic differential equations it is known that any solution in a cylindrical domain with axis $t > 0$, tends to a limit as $t \to \infty$ provided the boundary values and the coefficients of the equation tend to a limit as $t \to \infty$. Furthermore, the limit of the solution is known to be the solution of the limit equation. For second order parabolic equations, this has been proved by the author [5] for the first mixed boundary value problem, that is, when the solution u is prescribed on the lateral boundary of the cylinder. Extension to equations with a nonhomogeneous term which is "slightly" nonlinear in u, is also given in [5]. In [6] it was proved that if both the coefficients of the parabolic equation and the boundary values admit an asymptotic expansion in t^{-1} $(t \to \infty)$, then the same is true of the solution. Asymptotic convergence for solutions of second order parabolic equations satisfying a nonlinear boundary condition (generalized Newton's law of cooling) was established by the author in [7].

The present paper consists of two parts. In Part I we consider second order parabolic equations and establish the asymptotic behavior of solutions, both for the first and the second (and even more general) mixed boundary value problems. The nonhomogeneous term is a nonlinear perturbation. The domains are "almost cylindrical," i.e., the cross sections $t = \text{const.}$ tend to a limit as $t \to \infty$. For the first mixed boundary value problem, the present treatment is not only an improvement of the analogous results of [5], but it is also a much more simplified treatment. Thus for instance, we do not make here any use of existence theorems for parabolic equations. We

(1) Part I was prepared under Contract Nonr-222 (62) (NR 041 214) with the University of California in Berkeley. Part II was prepared under Contract Nonr-710 (16) (NR 044 004) between the Office of Naval Research and the University of Minnesota.

use however the Schauder existence theory for elliptic equations [17] and, for the second mixed boundary problem, recent results of Agmon, Douglis and Nirenberg [1].

In Part II we consider general nonhomogeneous parabolic equations of any order in an "almost cylindrical" domain, and solutions having prescribed Dirichlet data on the lateral boundary. We first prove that if both the coefficients of the equation and the boundary values tend to a limit as $t \to \infty$, then the solution $u(x, t)$ converges in the L_2 norm to a solution of the limit elliptic equation. The special case of homogeneous equations in a cylindrical domain with zero boundary values was proved by Vishik [20]. In our derivation of the L_2 convergence, we make essential use of some results of the paper of Agmon, et al. [1], already mentioned above. Having derived the L_2 convergence, we use it to get a *uniform* convergence. Here we make use of the fundamental solutions for parabolic equations [4] [19] and also (for cylindrical domain—where stronger results are derived) of Green's function considered by P. Rosenbloom [16]. Finally, we derive asymptotic expansions in t^{-1} for the solutions.

Part I. Second order parabolic equations

In this part we consider the asymptotic behavior of solutions of second order parabolic equations satisfying either the first or the second (and even more general) boundary conditions. In § 1 we state the main results about uniform convergence (as $t \to \infty$) of solutions of the second mixed boundary value problems (Theorems 1, 2). Theorem 1 is proved in § 2 and Theorem 2 is proved in §§ 3, 4. In § 5 we discuss the asymptotic expansion in t^{-1} of solutions, as $t \to \infty$. The results of §§ 1–5 are extended in § 6 to solutions of the first mixed boundary value problem. Finally, in § 7 we consider the behavior of solutions satisfying a *generalized* second boundary value condition.

1. Statement of results for the second boundary value problem

Let D be a domain in the $(n+1)$-dimensional space of real variables $(x, t) = (x_1, \ldots, x_n, t)$ bounded by a bounded domain B on $t=0$ and a surface S in the half space $t>0$. We denote by B_τ the intersection $D \cap \{t=\tau\}$ and assume that for every $\tau > 0$ B_τ is bounded and nonempty. We further denote by D_τ $(D_\infty = D)$ the domain $D \cap \{0<t<\tau\}$ and by S_τ the set $S \cap \{0<t<\tau\}$. The boundary of a domain G is denoted by ∂G, the closure of a set G is denoted by $\bar{G}$, and the complement in a set G_2 of a set G_1 is denoted by $G_2 - G_1$. Later on we shall assume that there exists a bounded domain C in the x-space such that, as $t \to \infty$, $B_t \to C$ in a certain sense. For simplicity we assume throughout this paper that C and S are each composed of one surface, but all the results can easily be extended to the case that C and S are each composed of a finite number of surfaces.

DEFINITION. We say that $w(y,t) \to z(x)$ uniformly in $(y,t) \in D$, $x \in C$ as $y \to x$, $t \to \infty$ and also write

$$\lim_{\substack{y \to x \\ t \to \infty}} w(y,t) = z(x),$$

if for any $\varepsilon > 0$ there exist $\delta > 0$, $t_0 > 0$ depending on ε such that $|w(y,t) - z(x)| < \varepsilon$ whenever $(y,t) \in D$, $x \in C$, $|y-x| < \delta$, $t > t_0$. A similar definition can be given for functions defined only on S.

Consider the equations

$$L u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t) u - \frac{\partial u}{\partial t} = f(x,t) + k(x,t,u) \quad \text{for } (x,t) \in D, \tag{1.1}$$

$$L_0 v \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial v}{\partial x_i} + c(x) v = f(x) + k(x,v) \quad \text{for } x \in C, \tag{1.2}$$

where $u = u(x,t)$, $v = v(x)$, and the boundary conditions

$$\frac{\partial u(x,t)}{\partial T} + g(x,t,u(x,t)) = h(x,t) \qquad \text{for } (x,t) \in S, \tag{1.3}$$

$$\frac{d v(x)}{d T} + g(x) v(x) = h(x) \qquad \text{for } x \in \partial C. \tag{1.4}$$

Here
$$\frac{\partial u(x,t)}{\partial T} = \lim_{\substack{y \to x \\ y \in \gamma}} \sum_{i,j=1}^{n} a_{ij}(x,t) \cos\,[\nu(x,t), x_j] \frac{\partial u(y,t)}{\partial y_i} \tag{1.5}$$

for all rays γ issuing from (x, t) and pointing into the interior of B_t. We call $\partial u/\partial T$ the *transversal* (or *conormal*) derivative of u. In (1.5), $\nu(x,t)$ is the outwardly directed normal to ∂B_t at the point (x,t). Similarly we define

$$\frac{d v(x)}{d T} = \lim_{\substack{y \to x \\ y \in \gamma}} \sum_{i,j=1}^{n} a_{ij}(x) \cos\,[\nu(x), x_j] \frac{\partial v(y)}{\partial x_i} \tag{1.6}$$

as the transversal derivative of $v(x)$, where the rays γ start at x and point into the interior of C. In order to avoid confusion later on, we have denoted the transversal derivative on S by $\partial/\partial T$ and on ∂C by d/dT.

DEFINITION. Given a bounded domain G in the x-space, its boundary ∂G is said to be of class $C^{m+\beta}$ (m integer, $0<\beta<1$) if to each point y of G there corresponds a sphere V (in the x-space) having y for its center and such that $V \cap \partial G$ can be represented, for some i, in the form

$$x_i = \psi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{1.7}$$

where ψ possesses m Hölder continuous (exponent β) x-derivatives. If the functions ψ are only assumed to be m times continuously differentiable, then ∂G is said to belong to the class C^m.

For any function $w = w(x)$ in G we introduce the norms:

$$|w|_0^G = \underset{x \in G}{\text{l.u.b.}}\, |w(x)|, \quad |w|_\alpha^G = |w|_0^G + H_\alpha^G(w),$$

$$|w|_{k+\alpha}^G = \sum_{|i| \leqslant k} \left| \frac{\partial^i}{\partial x^i} w \right|_\alpha^G,$$

where $\partial/\partial x$ denotes any partial derivative with respect to the x_j and $0 \leqslant \alpha < 1$, and

$$H_\alpha^G(w) = \underset{x, y \in G}{\text{l.u.b}} \frac{|w(x) - w(y)|}{|x - y|^\alpha}.$$

When there is no confusion, we omit the superscript G from the norm sign.

When we write, for functions $z(x)$ defined on ∂G, the norm $|z|_d^{\partial G}$, we mean the following. A finite covering of ∂G is given and, hence, in each such portion z becomes a function of $n-1$ variables. We then take $|z|_d^{\partial G}$ to be the sum of the d-norms of z in these portions. We shall clearly assume then that ∂G is of class C^e with $e \geqslant d$. Let the above finite covering be composed of portions ∂G_j of ∂G and let $\psi = \psi^j$ be the representation (1.7) for ∂G_j. We then define

$$|\partial G|_{m+\beta} = \sum_j |\psi^j|_{m+\beta}.$$

Finally, we denote by $|G|$ the diameter of G.

We shall need, later on, various assumptions on L, L_0, f, k, g, h and D. For the sake of clarity we list most of them now.

(A) The coefficients of L are continuous in $\bar{D}$ and are bounded by a positive constant M, and L is uniformly parabolic in $\bar{D}$, that is, there exists a positive constant M' such that, for all (x, t) in $\bar{D}$ and for all real vectors $\xi = (\xi_1, \ldots, \xi_n)$,

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\,\xi_j \geqslant M' \sum_{i=1}^{n} \xi_i^2. \tag{1.8}$$

(A_0) The following limits exist, uniformly in $(x,t) \in \bar{D}$ and $y \in \bar{C}$:

$$\lim_{\substack{x\to y\\ t\to\infty}} a_{ij}(x,t) = a_{ij}(y), \qquad \lim_{\substack{x\to y\\ t\to\infty}} b_i(x,t) = b_i(y), \qquad \lim_{\substack{x\to y\\ t\to\infty}} c(x,t) = c(y).$$

The functions $a_{ij}(y)$, $b_i(y)$, $c(y)$ are Hölder continuous (exponent α) in $\bar{C}$.

(B) $f(x,t)$ is a continuous function in $\bar{D}$.

(B_0) As $x\to y,\ t\to\infty,\ f(x,t)\to f(y)$ uniformly with respect to $(x,t)\in\bar{D}$ and $y\in\bar{C}$, and $f(y)$ is Hölder continuous (exponent α) in $\bar{C}$.

(C) $k(x,t,u)$ is continuous for $(x,t,)\in\bar{D}$, $-\infty<u<\infty$, and

$$|k(x,t,u)| \leqslant \mu_0\,|u|, \tag{1.9}$$

where μ_0 is a sufficiently small constant (depending on M, M', the α-norms of the coefficients of L and $\underset{t}{\text{l.u.b.}}\ (|B_t| + |\partial B_t|_1)$; see (D)).

(C_0) As $x\to y, t\to\infty$, $k(x,t,u)\to k(y,u)$ uniformly with respect to $(x,t)\in\bar{D}$, $y\in\bar{C}$ and u in bounded intervals. The function $k(x,u)$ is Hölder continuous in (x,u) for $x\in\bar{C}$ and u in bounded intervals, and $\partial k(x,u)/\partial u$ is continuous for $x\in\bar{C}$ and u in bounded interval, and

$$\left|\frac{\partial k(x,u)}{\partial u}\right| \leqslant \mu_0, \tag{1.10}$$

where μ_0 is a sufficiently small constant (depending on the same quantities as the μ_0 in (1.9) and, in addition, on bounds on $|f|$, g, $|h|$ and $|\partial C|_{2+\alpha}$).

(D) For every $t>0$, ∂B_t is of class C^1 and $\underset{t}{\text{l.u.b.}}\ (|B_t| + |\partial B_t|_1) < \infty$.

(D_0) ∂C is of class $C^{2+\alpha}$ and to every point x on ∂C there corresponds one and only one point (x_t, t) on each $B_t\ (t>0)$ such that (i) $x_t\to x$ as $t\to\infty$, uniformly with respect to x on ∂C, and (ii) as $t\to\infty$, the direction cosines of the normal $\nu(x_t,t)$ to ∂B_t tend to the direction cosines of the normal $\nu(x)$ to ∂C at x, uniformly with respect to x on ∂C.

Remarks. (a) By (A_0), (D_0) it follows that if $z(x)$ is continuously differentiable in a neighborhood of ∂C, then as $t\to\infty$ $\partial z(x_t)/\partial T \to dz(x)/dT$ uniformly with respect to x on ∂C. (b) If D is a cylindrical domain, then (D_0) reduces to the assumption that $\partial B_t \equiv \partial C$ is of class $C^{2+\alpha}$.

(E) $h(x,t)$ is a continuous function for (x,t) on S.

(E_0) As $t \to \infty$, $h(x_t, t) \to h(x)$ uniformly in $x \in \partial C$.

(F) $g(x, t, u)$ is continuous for $(x, t) \in \bar{S}$, $-\infty < u < \infty$, and there exists a positive constant μ_1 such that

$$\frac{g(x,t,u)}{u} \geqslant \mu_1 \quad \text{for } (x,t) \in \bar{S}, \ -\infty < u < \infty, \ u \neq 0. \tag{1.11}$$

(Note, by taking $u \gtrless 0, u \to 0$ that $g(x, t, 0) \equiv 0$.)

(F_0) As $t \to \infty$, $g(x_t, t, u) \to g(x)u$ uniformly with respect to x on ∂C and u in bounded intervals. (Note, by (F), that $g(x) \geqslant \mu_1 > 0$.)

(G_1) $a_{ij}(x)$ belong to $C^{1+\alpha}$ in some outside neighborhood of ∂C.

(G_2) ∂C is of class $C^{3+\alpha}$.

Definition. We say that $u(x, t)$ is a solution in D of the system (1.1), (1.3) if (i) u is continuous in $\bar{D}$, (ii) the derivatives $\partial u / \partial x_i$, $\partial^2 u / \partial x_i \partial x_j$, $\partial u / \partial t$ are continuous in D and (1.1), (1.3) are satisfied. We say that $v(x)$ is a solution in C of the system (1.2), (1.4) if (i) v is continuous in $\bar{C}$, (ii) the derivatives $\partial v / \partial x_i$, $\partial^2 v / \partial x_i \partial x_j$ are continuous in C and (1.2), (1.4) are satisfied.

We can now state the main results on the uniform convergence of solutions of (1.1), (1.3) as $t \to \infty$.

Theorem 1. *Assume that* (A)–(F) *hold and, in addition, that*

$$\lim_{t \to \infty} h(x, t) = 0, \quad \lim_{t \to \infty} f(x, t) = 0, \quad \lim_{t \to \infty} \sup c(x, t) \leqslant 0 \tag{1.12}$$

uniformly with respect to $(x, t) \in S$, $(x, t) \in \bar{D}$ *and* $(x, t) \in \bar{D}$ *respectively. If* $u(x, t)$ *is a solution in* D *of the system* (1.1), (1.3), *then, uniformly in* $(x, t) \in \bar{D}$,

$$\lim_{t \to \infty} u(x, t) = 0. \tag{1.13}$$

Theorem 2. *Assume that* (A)–(F), (A_0)–(F_0) (G_1), (G_2) *hold and that* $c(x) \leqslant 0$. *If* $u(x, t)$ *is a solution in* D *of the system* (1.1), (1.3), *then*

$$\lim_{\substack{x \to y \\ t \to \infty}} u(x, t) = v(y) \tag{1.14}$$

uniformly with respect to (x, t) *in* $\bar{D}$ *and* y *in* $\bar{C}$, *and* $v(y)$ *is the unique solution in* C *of the system* (1.2), (1.4).

In a preliminary report [9] we have proved Theorem 1 as stated above, and Theorem 2 without assuming (G_1), (G_2).

In the proof of Theorem 2 there appears a decisive lemma (Lemma 3, below) whose proof involved tedious potential theoretic calculations. The present proof avoids these calculations by simply using a recent result of [1]. However, we have to assume (G_1), (G_2)

In the course of the proof of Theorem 2 it will be shown that if $h(x)$, $g(x)$ belong to $C^{1+\alpha}$ then $v(x)$ belongs to $C^{2+\alpha}$ in $\bar{C}$, and thus it satisfies (1.4) in the classical sense.

From the proofs of Theorems 1, 2 it will become clear that they remain true if $\partial/\partial T$ is replaced by any other oblique derivative $\partial/\partial \tilde{T}$ provided, as $t\to\infty$, $\partial/\partial \tilde{T}\to d/dT$ (at the corresponding points).

2. Proof of Theorem 1

We introduce the function

$$\varphi(x)=e^{\lambda R}-e^{\lambda x_1}, \tag{2.1}$$

where R is a positive number satisfying $2x_1 \leqslant R$ for all $(x,t)=(x_1,\ldots,x_n,t)$ in $\bar{D}$, and λ is a positive constant. λ and R will be determined later. $\varphi(x)$ satisfies

$$(L\varphi)(x,t)=-a_{11}(x,t)\lambda^2 e^{\lambda x_1}-b_1(x,t)\lambda e^{\lambda x_1}+c(x,t)(e^{\lambda R}-e^{\lambda x_1}) \qquad \text{for } (x,t)\in D,$$

$$\frac{\partial\varphi(x)}{\partial T}+g(x,t,\varphi(x))\geqslant -\lambda e^{\lambda x_1}\frac{\partial x_1}{\partial T}+\mu_1(e^{\lambda R}-e^{\lambda x_1}) \qquad \text{for } (x,t)\in S.$$

Using (A), we may choose λ sufficiently large such that

$$(L\varphi)(x,t)<-2e^{\lambda x_1}+c(x,t)(e^{\lambda R}-e^{\lambda x_1}) \qquad \text{for } (x,t)\in D. \tag{2.2}$$

Having fixed λ, we choose R so large that

$$\frac{\partial\varphi(x)}{\partial T}+g(x,t,\varphi(x))\geqslant\mu_2>0 \qquad \text{for } (x,t)\in S. \tag{2.3}$$

Note that the constants λ, R, μ_2 are independent of (x,t). By (1.12) it follows that there exists a sufficiently large number $\bar{\sigma}$ such that

$$c(x,t)(e^{\lambda R}-e^{\lambda x_1})<e^{\lambda x_1} \qquad \text{for all } (x,t)\in\bar{D}-D_{\bar{\sigma}}. \tag{2.4}$$

Substituting (2.4) into (2.2), we get

$$(L\varphi)(x,t)<-2\delta \qquad \text{for } (x,t)\in\bar{D}-D_{\bar{\sigma}}, \tag{2.5}$$

where $2\,\delta = \underset{(x,t)\in D}{\text{g.l.b.}}\; e^{\lambda x_1} > 0$. For later purposes we define

$$\delta_0 = \underset{(x,t)\in D}{\text{g.l.b.}}\; \varphi(x), \quad \delta_1 = \underset{(x,t)\in D}{\text{l.u.b.}}\; \varphi(x). \tag{2.6}$$

The function $\varphi(x)$ will now be used to construct a comparison function which will majorize the solution $u(x,t)$. Consider the function

$$\psi(x,t) = 2\,\varepsilon\,\frac{\varphi(x)}{2\,\delta} + \varepsilon\,\frac{\varphi(x)}{\mu_2} + \frac{A\,\varphi(x)}{\delta_0}\, e^{-\gamma(t-\sigma)}, \tag{2.7}$$

where ε, A, λ are any positive numbers, and $\sigma \geqslant \bar{\sigma}$. Using the properties of $\varphi(x)$ derived above, we get

$$\psi(x,t) \leqslant \left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\mu_2} + \frac{A}{\delta_0}\, e^{-\gamma(t-\sigma)}\right)\delta_1, \tag{2.8}$$

$$L\,\psi(x,t) < -2\,\varepsilon - 2\,\delta\,\frac{\varepsilon}{\mu_2} - 2\,\delta\,\frac{A}{\delta_0}\, e^{-\gamma(t-\sigma)} + \gamma\,\frac{A\,\delta_1}{\delta_0}\, e^{-\gamma(t-\sigma)}. \tag{2.9}$$

Defining

$$\gamma = \delta/\delta_1, \quad \delta_2 = \delta/\delta_1 \tag{2.10}$$

and using (2.8), we obtain from (2.9)

$$L\,\psi(x,t) < -\varepsilon - \delta_2\,\psi(x,t) \qquad \text{for } (x,t) \in D - D_\sigma. \tag{2.11}$$

Using (1.11) and the choice of R, we also get

$$\frac{\partial\,\psi(x,t)}{\partial\,T} + g(x,t,\psi(x,t)) > \varepsilon \qquad \text{for } (x,t) \in S. \tag{2.12}$$

The function $\psi(x,t)$ will now be used to estimate $u(x,t)$.

Let ε be an arbitrary positive number. If we prove that for a sufficiently large number $\varrho = \varrho(\varepsilon)$

$$|u(x,t)| \leqslant A_0\,\varepsilon \qquad \text{for } (x,t) \in D - D_\varrho, \tag{2.13}$$

where A_0 is a constant independent of ε, ϱ, then the proof of Theorem 1 is completed. Now, by (1.12) there exists $\sigma = \sigma(\varepsilon) > 0$ such that

$$|h(x,t)| < \varepsilon \qquad \text{for } (x,t) \in S - S_\sigma, \tag{2.14}$$

$$|f(x,t)| < \varepsilon \qquad \text{for } (x,t) \in D - D_\sigma. \tag{2.15}$$

We take σ such that also $\sigma > \bar{\sigma}$ (and then (2.11), (2.12) hold). We next take in the definition of ψ above the numbers σ, ε to be the same numbers as the present ones, and

$$A = \operatorname*{l.u.b.}_{x \in B_\sigma} |u(x, \sigma)|. \tag{2.16}$$

We shall prove that

$$u(x, t) < \psi(x, t) \quad \text{in } D - D_\sigma. \tag{2.17}$$

The proof is based on an argument similar to that appearing in [21]. We first note, by (2.14), (2.15), (2.16) and (1.9), that

$$Lu > -\varepsilon - \mu_0 |u| \quad \text{for } (x, t) \in D - D_\sigma, \tag{2.18}$$

$$\partial u / \partial T + g(x, t, u) < \varepsilon \quad \text{for } (x, t) \in S - S_\sigma, \tag{2.19}$$

$$u(x, \sigma) < \psi(x, \sigma) \quad \text{for } x \in B_\sigma. \tag{2.20}$$

Consider now the set Σ of points $t \geqslant \sigma$ such that $\psi > u$ in $\bar{D}_t - D_\sigma$. By (2.20), Σ is nonempty. It is clearly an open set. If we prove that Σ is closed, then the proof of (2.17) is completed. Suppose then that t is such that $\psi(x, \tau) > u(x, \tau)$ in $D_t - D_\sigma$, and we have to prove that $\psi(x, t) > u(x, t)$ for $x \in B_t$. If this is not the case, then the function $\tilde{u}(x, t) \equiv \psi(x, t) - u(x, t)$ obtains its minimum zero in the set $\bar{D}_t - D_\sigma$ at a point (x^0, t) on $\bar{B}_t$. We shall derive a contradiction by proving that (x^0, t) can belong neither to ∂B_t nor to B_t.

If $(x^0, t) \in \partial B_t$ then, noting that $\partial / \partial T$ is a derivative along an outward direction to ∂B_t, we get

$$0 \geqslant \frac{\tilde{u}(x^0, t) - \tilde{u}(y, t)}{|x^0 - y|} = \frac{\partial}{\partial T} \tilde{u}(\tilde{y}, t) \to \frac{\partial \tilde{u}(x^0, t)}{\partial T},$$

as $y \to x^0$ along the transversal ray issuing at the point (x^0, t). Since also $g[x, t, u(x, t)] = g[x, t, \psi(x, t)]$ at $x = x^0$, we get

$$\frac{\partial \psi(x^0, t)}{\partial T} + g[x^0, t, \psi(x^0, t)] \leqslant \frac{\partial u(x^0, t)}{\partial T} + g[x^0, t, u(x^0, t)],$$

which contradicts (2.12), (2.19) combined.

If $(x^0, t) \in B_t$ then, at that point,

$$u = \psi, \quad |u| = \psi, \quad \partial u / \partial x_i = \partial \psi / \partial x_i, \quad \partial u / \partial t \geqslant \partial \psi / \partial t. \tag{2.21}$$

Also, since (a_{ij}) is a positive matrix,

$$\sum a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \geqslant \sum a_{ij} \frac{\partial^2 u}{\partial x_i x_j} \quad \text{at } (x^0, t). \tag{2.22}$$

Combining (2.21), (2.22) we conclude that, at the point (x^0, t),

$$L u + \varepsilon + \mu_0 |u| \leqslant L \psi + \varepsilon + \mu_0 \psi.$$

The last inequality however, contradicts (2.11), (2.18) combined, provided

$$\mu_0 \leqslant \delta_2. \tag{2.23}$$

Hence, assuming μ_0 to be sufficiently small so that (2.23) is satisfied, we conclude that (x^0, t) cannot belong to B_t. This completes the proof of (2.17).

In a similar way, replacing (2.11), (2.12) by

$$L \tilde{\psi} > \varepsilon + \delta_2 |\tilde{\psi}|, \quad \frac{\partial \tilde{\psi}}{\partial T} + g(x, t, \tilde{\psi}) < -\varepsilon, \tag{2.24}$$

where $\tilde{\psi} = -\psi$ and replacing (2.18), (2.19) by

$$L u < \varepsilon + \mu_0 |u|, \quad \frac{\partial u}{\partial T} + g(x, t, u) > -\varepsilon, \tag{2.25}$$

we can prove that $u > \tilde{\psi}$ in $D - D_\sigma$. Combining this ineqality with (2.17), and recalling the definition of ψ in (2.7), we have

$$|u(x, t)| \leqslant \frac{\varepsilon}{\delta} \varphi(x) + \frac{\varepsilon}{\mu_2} \varphi(x) + \frac{A}{\delta_0} \varphi(x) e^{-\gamma(t-\sigma)} \quad \text{for } (x, t) \in D - D_\sigma. \tag{2.26}$$

Taking ϱ sufficiently large such that $A \delta_1 e^{-\gamma(\varrho-\sigma)}/\delta_0 \leqslant \varepsilon$ the proof of (2.13) is completed.

From the above proof the following corollary follows.

COROLLARY 1. *If the assumption* (1.12) *in Theorem* 1 *is replaced by*

$$\limsup_{t\to\infty} |h(x, t)| \leqslant \varepsilon, \quad \limsup_{t\to\infty} |f(x, t)| \leqslant \varepsilon, \quad \limsup_{t\to\infty} c(x, t) \leqslant 0 \tag{2.27}$$

uniformly in $(x, t) \in \bar{D}$, *and if the other assumptions of Theorem* 1 *remain unchanged, then*

$$\limsup_{t\to\infty} |u(x, t)| \leqslant A_1 \varepsilon \tag{2.28}$$

uniformly in $(x, t) \in \bar{D}$, *where* A_1 *is a constant independent of* ε.

3. Proof of Theorem 2 for smooth h, g

In this paragraph we prove Theorem 2 under the additional assumption that h, g are $C^{1+\alpha}$ in some outside neighborhood of ∂C. This assumption will be removed in § 4. We need a few preliminary results.

We recall that ∂C is of class $C^{3+\alpha}$. Now at every point x^0 of ∂C we draw an outwardly directed normal $\nu(x^0)$ to ∂C and denote by $\tilde{\nu}(x^0)$ the segment on $\nu(x^0)$ of length $\delta'(\delta'>0)$ and initial point x^0. We obtain a family N of straight segments. It is elementary to see that every point x outside C and sufficiently close to ∂C lies on one and only one normal segment $\tilde{\nu}(x^0)$ provided δ' is sufficiently small, say $\delta' \leqslant \bar{\delta}$. In what follows we take $\delta' = \bar{\delta}$.

We now measure any fixed distance δ, $0<\delta\leqslant\bar{\delta}$ on each $\tilde{\nu}(x), x \in \partial C$ and denote the set of the end points by ∂C_δ. The following lemma is well known.

LEMMA 1. *Each ∂C_δ is a surface orthogonal to the family N, and* $\operatorname*{l.u.b.}_{\delta} |\partial C_\delta|_{2+\alpha} \leqslant$ *const.* $<\infty$.

Using local coordinates $x_i = f_i(s)$ $(1 \leqslant i \leqslant n, s=(s_1, \ldots, s_{n-1}))$ for ∂C, we can represent ∂C_δ locally in the form

$$x_i = f_i(s) + g_i(s)\, t, \tag{3.1}$$

where $g_i(s)$ is $(-1)^{i-1}$ times the determinant of the matrix obtained from the matrix $(\partial f_i/\partial s_j)$ (i indicates rows, j indicates columns) by erasing the ith row. t is defined by

$$t = \delta/g(s), \tag{3.2}$$

where

$$g(s) = \left[\sum_{i=1}^{n} (g_i(s))^2\right]^{\frac{1}{2}}. \tag{3.3}$$

We next need a recent result of Agmon *et al.* [1, Chapter II]:

LEMMA 2. *Consider the system* (1.2), (1.4) *with* $k \equiv 0$ *and assume that* ∂C *is of class* $C^{2+\alpha}$, *that* f *and the coefficients of* L_0 *are* $C^\alpha(\bar{C})$, *that* a_{ij} *are* $C^{1+\alpha}(\partial C)$, *and that* h, g *are* $C^{1+\alpha}(\partial C)$. *If* $c(x)\leqslant 0$, $g(x)>0$, *then there exists a unique solution of* (1.2), (1.4) *which is of class* $C^{2+\alpha}(\bar{C})$, *and*

$$|v|_{2+\alpha}^{C} \leqslant K\left(|h|_{1+\alpha}^{\partial C} + |f|_{\alpha}^{C}\right), \tag{3.4}$$

where K depends only on bounds on the quantities

$$M', \ |a_{ij}|_{1+\alpha}^{\partial C}, \ |a_{ij}|_{\alpha}^{C}, \ |b_i|_{\alpha}^{C}, \ |c|_{\alpha}^{C}, \ |g|_{1+\alpha}^{\partial C}, \ |1/g|_{0}^{\partial C}, \ |C|, \ |\partial C|_{2+\alpha}.$$

We are now going to consider differential systems analogous to (1.2), (1.4) in each C_δ, C_δ being the interior of ∂C_δ. The solution $v^\delta(x)$ will be "close" to both $u(x,t)$ $(t\to\infty)$ and $v(x)$ appearing in the formulation of Theorem 2. We put $C' = C_{\bar{\delta}}$, where $\bar{\delta}$ appears in Lemma 1, and write $\bar{C}'$ for the closure of C'. We may assume that the $a_{ij}(x)$ are $C^{1+\alpha}$ in $\bar{C}' - C$, as follows by assumption (G_1).

Every function $p(x)$ defined in $\bar{C}$ or on ∂C can be extended to $\bar{C}' - \bar{C}$ as follows. Let $x \in \bar{C}' - \bar{C}$ and let x^0 be the point on ∂C such that x lies on $\tilde{\nu}(x^0)$. We then define $p(x)=p(x^0)$. We extend in this manner the functions $b_i(x)$, $c(x)$, $f(x)$, $k(x,u)$ (u fixed). It is clear that the extended functions have the same Hölder-continuity properties (in x) as the original functions. We denote the transversal derivatives at the point x on $\bar{\nu}(x^0)$ by d/dT^δ.

Consider the system

$$L_0 v^\delta(x) = f(x) + k(x,v) \text{ in } C_\delta \tag{3.5}$$

$$\frac{d v^\delta(x)}{d T^\delta} + g(x) v^\delta(x) = h(x) \text{ on } \partial C_\delta. \tag{3.6}$$

We shall prove:

LEMMA 3. *The system* (3.5), (3.6) *has (for $0 \leqslant \delta \leqslant \bar{\delta}$) a unique solution $v^\delta(x)$ and as $\delta \to 0$,*

$$|v - v^\delta|_0^C \to 0, \quad \underset{x \in \partial C}{\text{l.u.b.}} \left\{ |v^\delta(x) - v^\delta(x')| + \left| \frac{d v^\delta(x)}{dT} - \frac{d v^\delta(x')}{dT^\delta} \right| \right\} \to 0, \tag{3.7}$$

where $v \equiv v^0$ and x' is the point on ∂C_δ which lies on $\bar{\nu}(x)$.

Proof. Using the maximum principle [11] and (1.9) we easily conclude that if a solution v^δ exists, it must be bounded independently of δ, the bound being dependent only on the given functions of the system and on $|C|$. Hence, without loss of generality we may assume that $k(x,u)$, for $|u|$ larger than a certain a priori determined constant, satisfies the regularity assumptions in (C), (C_0) with constants independent of u.

We next consider the set Z_N of functions w defined in C_δ which satisfy $|w|_\alpha \leqslant N$. We define a transformation Tw as follows. Replace in (3.5) $k(x,v)$ by $k(x,w)$. Tw is the solution of the modified system (3.5), (3.6). By Lemma 2 it exists and (using Lemma 1)

$$|Tw|_{2+\alpha}^{C_\delta} \leqslant K(|h|_{1+\alpha} + |f|_\alpha + |k|_\alpha),$$

where K is independent of δ. Noting that $|k|_\alpha \leqslant K_1 + \mu_0 K_2 N$, where K_1, K_2 are constants independent of N and δ, we conclude, upon taking $N = K(|h|_{1+\alpha} + |f|_\alpha + K_1) + 1$ and assuming μ_0 to be sufficiently small, $|Tw|_{2+\alpha} \leqslant N$. Hence, Tw maps Z_N into a compact subset.

T is also a continuous transformation on Z_N. Indeed, if we write the differential systems for Tw_1, Tw_2 and subtract one from the other, we find, using Lemma 2, that

$$|Tw_1 - Tw_2|_{2+\alpha} \leqslant K_3 |k(x,w_1) - k(x,w_2)|_\alpha \leqslant K_4 |w_1 - w_2|_\alpha.$$

Having proved that T is a continuous transformation of a convex and bounded subset Z_N

of a Banach space Z_∞ into a compact subset, we can apply Schauder's fixed point theorem [18] and conclude that there exists a fixed point $v = T v$.

To complete the proof of Lemma 3 we have to prove (3.7). The second statement of (3.7) follows from the inequality $|v^\delta|_{2+\alpha}^{C_\delta} \leqslant N$, which, in particular, guarantees the equi-continuity of $\{v^\delta\}$ and of their first derivatives in their respective domains C_δ. The first statement follows by either an appropriate use of the maximum principle, or by the comparison argument of § 2.

COROLLARY. *From the above proof if follows that the convergence in* (3.7) *is uniform with respect to f, h, provided $|f|_\alpha$, $|h|_{1+\alpha}$ are bounded by a fixed constant.*

Proof of Theorem 2. Given any positive number ε, we shall prove that there exists $\beta > 0$, $\varrho > 0$ depending on ε such that

$$|u(x,t) - v(y)| \leqslant A\,\varepsilon \quad \text{for } (x,t) \in D - D_\varrho,\ y \in \bar{C},\ |x - y| \leqslant \beta. \tag{3.8}$$

Here and in the following, A is used to denote any constant independent of ε. In [5] we simply defined $w = u - v$ and applied Theorem 1 to w. This method, however, fails in the present case, mainly since D is not necessarily a cylindrical domain. To overcome this difficulty, we shall not try to estimate $u - v$ directly. Instead, we shall approximate v by a family of functions $v_*^\delta\,(\delta \to 0)$ and estimate $u - v_*^\delta$.

We introduce the functions

$$h_*(x) = h(x) - \varepsilon \quad \text{for } x \in \partial C, \qquad f_*(x) = f(x) + \varepsilon \quad \text{for } x \in \bar{C} \tag{3.9}$$

and apply Lemma 3 with f, h replaced by f_*, h_*. We conclude that for every $0 \leqslant \delta \leqslant \bar{\delta}$ there exists a unique solution $v_*^\delta(x)$ of the system

$$L_0 v_*^\delta(x) = f_*(x) + k(x, v_*^\delta) \qquad \text{for } x \in C_\delta, \tag{3.10}$$

$$\frac{d v_*^\delta(x)}{d T^\delta} + g(x)\, v_*^\delta(x) = h_*(x) \qquad \text{for } x \in \partial C_\delta. \tag{3.11}$$

We define $v_*(x) = v_*^0(x)$. By Lemma 3 and its corollary we also conclude that there exists a *fixed* $\delta > 0$ depending on ε, such that

$$\operatorname*{l.u.b.}_{x \in C} |v_*^\delta(x) - v_*(x)| \leqslant \varepsilon, \tag{3.12}$$

$$\operatorname*{l.u.b.}_{x \in \partial C} \left| \frac{d v_*^\delta(x)}{d T} - \frac{d v_*^\delta(x')}{d T^\delta} \right| \leqslant \frac{\varepsilon}{4}, \tag{3.13}$$

$$\underset{x \in \partial C}{\text{l.u.b.}} \; |g(x)\, v_*^\delta(x) - g(x')\, v_*^\delta(x')| \leqslant \frac{\varepsilon}{4}, \tag{3.14}$$

$$\underset{x \in \partial C}{\text{l.u.b.}} \; |h(x) - h(x')| \leqslant \frac{\varepsilon}{4}. \tag{3.15}$$

Consider the function

$$w(x, t) = u(x, t) - v_*^\delta(x) \qquad \text{for } (x, t) \in D - D_\sigma.$$

Here σ is a sufficiently large number such that all the domains B_t^0 (which are the projections of B_t on $t = 0$) lie in a fixed closed set contained in the interior of C_δ, provided $t \geqslant \sigma$ (recall that δ is a fixed number). The function $w(x, t)$ is thus defined in $D - D_\sigma$, and it satisfies the differential equation

$$\begin{aligned} L\, w &= L\, u - (L - L_0)\, v_*^\delta - L_0\, v_*^\delta \\ &= [f(x, t) - f_*(x)] + [k(x, t, u) - k(x, u)] + [k(x, u) - k(x, v_*^\delta)] + (L - L_0)\, v_*^\delta \\ &\equiv F(x, t). \end{aligned} \tag{3.16}$$

By the corollary at the end of § 2 we obtain

$$|u(x, t)| \leqslant A \qquad \text{for all } (x, t) \in D - D_\sigma, \tag{3.17}$$

provided σ is sufficiently large. Hence, $k(x, t, u) \to k(x, u)$ as $t \to \infty$, uniformly in $(x, t) \in D$. We also have

$$\underset{t > \sigma}{\text{l.u.b.}} \; |v_*^\delta|_2^{B_t} \leqslant A, \tag{3.18}$$

since $\bigcup_{t > \sigma} B_t$ is contained in a closed set interior to C_δ.

Combining these remarks and using (3.9) we obtain

$$L\, w < \tilde{k}(x, t)\, w, \tag{3.19}$$

where $|\tilde{k}(x, t)| \leqslant \mu_0$ $((x, t) \in D - D_\sigma)$, provided σ is sufficiently large. We turn to the boundary condition. By (D_0),

$$x_t \to x, \text{ direction of } \nu(x_t, t) \to \text{ direction of } \nu(x) \tag{3.20}$$

as $t \to \infty$, uniformly in $x \in \partial C$. Using the definitions (1.5), (1.6) and Remark (a) in § 1 (following the assumption (D_0)), we get

$$g(x_t)\, v_*^\delta(x_t) - g(x)\, v_*^\delta(x) \to 0 \text{ as } t \to \infty, \tag{3.21}$$

$$\frac{\partial v_*^\delta(x_t)}{\partial T} - \frac{d v_*^\delta(x)}{d T} \to 0 \text{ as } t \to \infty \tag{3.22}$$

uniformly with respect to x on ∂C. Now, on ∂B_t we have

$$\begin{aligned}\frac{\partial w(x_t, t)}{\partial T} + g(x_t)\, w(x_t, t) &= [g(x_t)\, u - g(x_t, t, u)] + h(x_t, t) - \frac{\partial v_*^\delta(x_t)}{\partial T} - g(x_t)\, v_*^\delta(x_t) \\ &= [g(x_t)\, u - g(x_t, t, u)] + [h(x_t, t) - h_*(x')] + \left[\frac{d v_*^\delta(x')}{d T^\delta} - \frac{\partial v_*^\delta(x_t)}{\partial T}\right] \\ &\quad + [g(x')\, v_*^\delta(x') - g(x^t)\, v_*^\delta(x_t)] \equiv I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.23}$$

As $t \to \infty$, $I_1 \to 0$ by (F_0) and (3.17); I_2 becomes larger than $\frac{2}{3}\varepsilon$, by (E_0), (3.9) and (3.15); I_3 becomes smaller than $\frac{1}{3}\varepsilon$ by (3.13), (3.22), and I_4 becomes smaller than $\frac{1}{3}\varepsilon$ by (3.14), (3.21). The above statements hold uniformly with respect to $x \in \partial C$. We conclude that

$$\frac{\partial w(x,t)}{\partial T} + g(x)\, w(x,t) > 0 \qquad \text{for } (x,t) \in S - S_\sigma, \tag{3.24}$$

provided σ is sufficiently large.

With the aid of (3.19), (3.24) we proceed to estimate w. σ is now a fixed number. Consider the function

$$\theta(x,t) = -A_0\, \varphi(x)\, e^{-\gamma(t-\sigma)}, \tag{3.25}$$

where $\varphi(x)$ and γ are defined in § 2. Using the properties of $\varphi(x)$ derived in § 2 we conclude that

$$L\theta > \tilde{k}(x,t)\, |\theta| \qquad \text{for } (x,t) \in D - D_\sigma, \tag{3.26}$$

$$\frac{\partial \theta(x,t)}{\partial T} + g(x,t,\theta) < 0 \qquad \text{for } (x,t) \in S - S_\sigma, \tag{3.27}$$

provided $\sigma \geqslant \bar{\sigma}$, which we may assume. Taking

$$A_0 = \delta_0^{-1} \operatorname*{l.u.b.}_{x \in B_\sigma} |w(x,\sigma)| + 1,$$

we can use the comparison argument of § 2 to conclude that

$$w(x,t) > \theta(x,t) \qquad \text{for } (x,t) \in D - D_\sigma. \tag{3.28}$$

Taking $\varrho \geqslant \sigma$ such that $A_0\, \varphi(x)\, e^{-\gamma(\varrho-\sigma)} \leqslant \varepsilon$ and using (3.25), (3.28) we get, using the definition of w,

$$u(x,t) > v_*^\delta(x) - \varepsilon \qquad \text{for } (x,t) \in D - D_\varrho. \tag{3.29}$$

Since $v_*^\delta(x)$ is a continuous function in C_δ, there exists $\beta > 0$ such that

$$|v_*^\delta(x) - v_*^\delta(y)| < \varepsilon \quad \text{if } |x-y| \leqslant \beta,\; y \in \bar{C},\; x \in C_\delta. \tag{3.30}$$

Combining (3.30) with (3.29), (3.12) we get

$$u(x,t) > v_*(y) - 3\varepsilon \quad \text{if } y \in \bar{C},\; (x,t) \in D - D_\varrho,\; |x-y| \leqslant \beta. \tag{3.31}$$

Consider now the function

$$\tilde{v}(x) = v(x) - v_*(x). \tag{3.32}$$

It satisfies the system of differential inequalities

$$L_0 \tilde{v} > -\varepsilon - \mu_0 |\tilde{v}| \quad \text{for } x \in C, \tag{3.33}$$

$$\frac{d\,\tilde{v}(x)}{d\,T} + g(x)\,v \leqslant \varepsilon \quad \text{for } x \in \partial C. \tag{3.34}$$

Using the comparison argument of § 2 (considering $L_0 v$ as $(L_0 - \partial/\partial t)\,v$) we easily obtain,

$$\tilde{v}(x) \leqslant A\,\varepsilon \quad \text{for } x \in C. \tag{3.35}$$

Combining (3.35), (3.32) with (3.31) we get

$$u(x,t) > v(y) - A\,\varepsilon, \quad \text{if } (x,t) \in D - D_\varrho,\; y \in \bar{C},\; |x-y| \leqslant \beta. \tag{3.36}$$

In a similar way, by defining $h^*(x) = h(x) + \varepsilon$, $f^*(x) = f(x) - \varepsilon$ we can prove that

$$u(x,t) < v(y) + A\,\varepsilon, \quad \text{if } (x,t) \in D - D_\varrho,\; y \in \bar{C},\; |x-y| \leqslant \beta. \tag{3.37}$$

Combining (3.37) with (3.36), the proof of (3.8) is completed.

From the above proof we easily derive:

COROLLARY 2. *If the assumptions:* $f(x,t) \to f(x)$, $h(x,t) \to h(x)$, $g(x,t,u) \to g(x,t)\,u$ *are replaced by*

$$\left.\begin{aligned} &\limsup_{\substack{x \to y \\ t \to \infty}} |f(x,t) - f(y)| \leqslant \varepsilon, \qquad \limsup_{\substack{x \to y \\ t \to \infty}} |h(x,t) - h(x)| \leqslant \varepsilon, \\ &\limsup_{t \to \infty} |g(x,t,u) - g(x)\,u| \leqslant \varepsilon\,|u| \quad (\varepsilon > 0), \end{aligned}\right\} \tag{3.38}$$

uniformly with respect to $(x,t) \in \bar{D}$, $y \in \bar{C}$; $(x,t) \in S$, $y \in \partial C$ *and* $(x,t) \in S$, $y \in \partial C$, u *in bounded intervals, respectively, and if the assumptions of Theorem* 2 *are otherwise the same, and if* $g(x)$, $h(x)$ *are* $C^{1+\alpha}$ *on* ∂C, *then*

$$\limsup_{\substack{x\to y\\ t\to\infty}} |u(x,t)-v(y)|\leqslant A\,\varepsilon \tag{3.39}$$

uniformly in $(x,t)\in\bar{D}$, $y\in\bar{C}$, *where* A *is a constant independent of* ε. *If* $h(x)$, $g(x)$, $1/g(x)$ *and* $f(x)$ *are bounded by* K_0, *then* A *depends on* K_0 *but not on* $h(x)$, $g(x)$, $f(x)$.

This corollary will be used in the following section.

4. Proof of Theorem 2 (for general h, g)

It remains to prove Theorem 2 in case h, g are only assumed to be continuous. The essential point is the proof of the existence of a solution $v(x)$ of (1.2), (1.4). Once this is proved, the proof of Theorem 2 can be completed as follows.

We have to prove (3.8) for every $\varepsilon>0$. We construct $C^{1+\alpha}$ functions $\tilde{g}$, $\tilde{h}$ in a neighborhood of ∂C which satisfy

$$\underset{x\in\partial C}{\text{l.u.b.}}\,[|g(x)-\tilde{g}(x)|+|h(x)-\tilde{h}(x)|]\leqslant\varepsilon. \tag{4.1}$$

Let $\tilde{v}(x)$ be the solution of (1.2), (1.4) with g, h replaced by $\tilde{g}, \tilde{h}$. By the results of § 3, $\tilde{v}(x)$ exists and, by Corollary 2,

$$|u(x,t)-\tilde{v}(y)|\leqslant A\,\varepsilon \qquad \text{if } (x,t)\in\bar{D}-D_\varrho,\ y\in\bar{C},\ |x-y|\leqslant\beta \tag{4.2}$$

for ϱ sufficiently large.

Next, the function $w(x)=v(x)-\tilde{v}(x)$ satisfies:

$$L_0\,w(x)=\hat{k}(x)\,w \quad (|\hat{k}|\leqslant\mu_0) \qquad \text{for } x\in C, \tag{4.3}$$

$$d\,w/d\,T+g(x)\,w=[h(x)-\tilde{h}(x)]+[g(x)-\tilde{g}(x)]\,\tilde{v}\equiv H(x) \qquad \text{for } x\in\partial C. \tag{4.4}$$

By the maximum principle we find that $\tilde{v}$ is bounded independently of ε (provided we take, as we certainly may, $\tilde{h}$, $\tilde{g}$ and $1/\tilde{g}$ to be bounded independently of ε). It thus follows that $|H(x)|\leqslant A_1\,\varepsilon$, Where A_1 is independent of ε.

We can now apply to the system (4.3), (4.4) either the maximum principle, or the comparison argument of § 2 (writing $L_0\,w=(L_0-\partial/\partial t)\,w$). We conclude that

$$|w(x)|=|v(x)-\tilde{v}(x)|\leqslant A_2\,\varepsilon \qquad \text{for all } x\in\bar{C}, \tag{4.5}$$

where A_2 is independent of ε. Combining (4.5) with (4.2), the proof of (3.8) is completed. It thus remains to prove the existence of $v(x)$.

EXISTENCE OF v. We recall that the coefficients of L_0 have been extended to $\bar{C}'$. We now need to introduce a *principal fundamental solution* of L_0. This is a fundamental solution $\Gamma(x,t)$ in the whole n-dimensional space E_n of an elliptic equation which coincides with $L_0 u = 0$ on C'. Furthermore, it satisfies (uniformly in ξ in bounded sets)

$$\Gamma(x,\xi)\to 0, \quad \frac{\partial \Gamma(x,\xi)}{\partial x_i}\to 0 \text{ as } |x|\to\infty. \tag{4.6}$$

In the general case that L_0 is already defined in the whole space E_n, the construction of Γ is fairly complicated. It was given by Giraud [10]; see also [13, § 20]. In our present case, the construction can be simplified and we proceed to describe it.

We first extend the coefficients of L_0 into the whole space E_n in such a manner that for some $R>0$

$$a_{ij}(x)=\delta_{ij}, \quad b_i(x)=0, \quad c(x)=-k^2<0 \qquad \text{if } |x|>R$$

(k constant) and such that all the coefficients are again Hölder-continuous (exponent α) in E_n and $c(x)\leqslant 0$ in E_n. In what follows we shall consider only the case that $n>2$. In the case $n=2$ some of the formulas take a different form, but the methods and results are the same.

Let $J(t)$ be the Bessel function which solves the equation

$$\frac{d^2 J}{dt^2}+\frac{n-1}{t}\frac{dJ}{dt}-J=0$$

and which, for $t\to 0$, satisfies

$$J(t)=K\,t^{2-n}(1+O(t)), \quad J'(t)=(2-n)\,K\,t^{1-n}(1+O(t)), \tag{4.7}$$

where K is a positive constant. Furthermore,

$$J(t)=O(e^{-mt}), \quad J'(t)=O(e^{-mt}) \text{ as } t\to\infty, \tag{4.8}$$

where m is some positive constant. Following the parametric method we proceed to construct a fundamental solution $\Gamma_1(x,\xi)$ in E_n for the elliptic operator

$$L_1=[L_0-c(x)]-k^2, \tag{4.9}$$

which has for its essential singularity the kernel

$$\Gamma_0(x,\xi)=\frac{k^{n-2}}{|\det(a^{ij}(\xi))|^{\frac{1}{2}}}\,J\left[k\left(\sum a^{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)\right)^{\frac{1}{2}}\right]. \tag{4.10}$$

Here (a^{ij}) is the matrix inverse to (a_{ij}).

We write Γ_1 in the form (compare [13, p. 55])

$$\Gamma_1(x,\xi)=\Gamma_0(x,\xi)+\int_{E_n}\Gamma_1(x,\eta)\,K(\eta,\xi)\,d\eta. \tag{4.11}$$

Noting that $\sum a_{ij}(\xi)\,\partial^2\Gamma_0/\partial x_i\,\partial x_j=k^2\Gamma_0$, and assuming that the second term on the right side of (4.11) is of smaller order of singularity compared with the first term (this can very easily be verified a posteriori), the equation $L_1\Gamma_1=0$ implies that

$$K(x,\xi)=\sum\left[a_{ij}(x)-a_{ij}(\xi)\right]\frac{\partial^2\Gamma_0}{\partial x_i\,\partial x_j}+\sum b_i(x)\,\frac{\partial\Gamma_0}{\partial x_i}. \tag{4.12}$$

Note that

$$|K(x,\xi)|\leqslant\frac{A_0}{|x-\xi|^{n-\alpha}}\exp\{-m\,|x-\xi|\},\quad m=m(k), \tag{4.13}$$

and A_0 is independent of k. We next observe that if we prove that

$$\int_{E_n}|K(x,\xi)|\,dx\leqslant\varrho<1\quad(\varrho\text{ constant}), \tag{4.14}$$

then the solution of (4.11) is given by iteration, that is,

$$\Gamma_1(x,\xi)=\Gamma_0(x,\xi)+\sum_{m=0}^{\infty}\int_{E_n}\Gamma_0(x,\eta)\,K^{(m)}(\eta,\xi)\,d\eta, \tag{4.15}$$

where $K^0=K$. Indeed, using (4.14) and the elementary inequality

$$\int_{E_n}\frac{1}{|x-\eta|^{n-\beta}}\,\frac{1}{|\eta-\xi|^{n-\gamma}}\,d\eta\leqslant\frac{\text{const.}}{|x-\xi|^{n-\beta-\gamma}}, \tag{4.16}$$

provided $0<\beta<n$, $0<\gamma<n$, $\beta+\gamma<n$, one can prove, by induction, that for $j>n$ and for all $x\in E_n$, $\xi\in E_n$

$$|K^{(j)}(x,\xi)|+\int_{E_n}|K^{(j)}(\eta,\xi)|\,d\eta\leqslant\text{const. }\varrho^j, \tag{4.17}$$

where the constant is independent of j. Furthermore, noting by (4.8), (4.10), (4.12), that (4.6) is satisfied for Γ replaced by Γ_0 and for Γ replaced by each term

$$\int_{E_n}\Gamma_0(x,\eta)\,K^{(m)}(\eta,\xi)\,d\eta,$$

we conclude, upon using (4.17), that (4.6) is satisfied also with Γ replaced by Γ_1. It remains to prove (4.14).

Noting that in (4.13) $m(k) \to \infty$ as $k \to \infty$, it follows that if k is sufficiently large then (4.14) is satisfied.

We proceed to construct Γ. We write it in the form

$$\Gamma(x, \xi) = \Gamma_1(x, \xi) + \int_{|\eta|<R} \Gamma(x, \eta) \gamma(\eta) \Gamma_1(\eta, \xi) \, d\eta, \tag{4.18}$$

where
$$\gamma(\eta) = \begin{cases} c(\eta) + k^2 & \text{if } |\eta| \leqslant R \\ 0 & \text{if } |\eta| > R \end{cases}. \tag{4.19}$$

In the bounded domain (ξ, η), $|\xi| \leqslant R$, $|\eta| \leqslant R$ we can apply the Fredholm theory. It follows that if (for any fixed x) a unique solution $\Gamma(x, \xi)$ of (4.18) does not exist, then there exists a nontrivial function $w(\xi)$ which satisfies the equation

$$w(\xi) = \gamma(\xi) \int_{|\eta|<R} \Gamma_1(\xi, \eta) w(\eta) \, d\eta \equiv \gamma(\xi) \tilde{w}(\xi) \tag{4.20}$$

for $|\xi| \leqslant R$, where $\tilde{w}(\xi)$ is an abbreviated notation for the integral. However, we can then define $\tilde{w}(\xi)$ for all $\xi \in E_n$ (in terms of the integral) and it satisfies the equation

$$L_0 \tilde{w} = (L_1 + \gamma) \tilde{w} = 0 \quad \text{in } E_n.$$

By (4.6) with Γ replaced by Γ_1, $\tilde{w}(\xi) \to 0$ as $|\xi| \to \infty$. Applying the maximum principle [11] we conclude that $\tilde{w} \equiv 0$, which is a contradiction.

We have thus proved that for every $x \in E_n$ there exists a unique solution $\Gamma(x, \xi)$ of (4.18) for $|\xi| \leqslant R$. We can now use the right side of (4.18) to define $\Gamma(x, \xi)$ also for $|\xi| > R$.

In order to study the behavior of $\Gamma(x, \xi)$ as $x \to \xi$ and as $|x| \to \infty$ we first multiply both sides of (4.18) by $\Gamma_1(x', x)$ and integrate with respect to x, $|x| \leqslant R$. Next we multiply the resulting equation by $\Gamma_1(x'', x')$ and integrate with respect to x'. Proceeding in this manner $n-2$ additional times, we obtain $n+1$ integral equations: the first one determines Γ, the second equation determines $\int \Gamma_1 \Gamma$, etc. The last equation (with variables $x^{(n)}, \xi$) determines $\int \dots \int \Gamma_1 \dots \Gamma_1 \Gamma$ (n integrations) and the nonhomogeneous term is continuous in $x^{(n)}$, and tends to zero as $|x^{(n)}| \to \infty$. Therefore, the same can be proved for the solution $\int \dots \int \Gamma_1 \dots \Gamma_1 \Gamma$ (n integrations). We now turn to the $(n-1)$th equation, $(n-2)$th equation, etc. In this manner we conclude that

$$\Gamma(x, \xi) = \Gamma_0(x, \xi) + \Gamma'(x, \xi), \tag{4.21}$$

where Γ' satisfies (4.6) with Γ replaced by Γ', and Γ' has a smaller order of singularity than Γ_0. Thus,

$$|\Gamma'| \leqslant \frac{A}{|x-\xi|^{n-2-\alpha}}, \quad \left|\frac{\partial}{\partial x}\Gamma'\right| \leqslant \frac{A}{|x-\xi|^{n-1-\alpha}}, \quad \left|\frac{\partial^2}{\partial x^2}\Gamma'\right| \leqslant \frac{A}{|x-\xi|^{n-\alpha}}, \tag{4.22}$$

where A is a constant. We have thus completed the construction of the principal fundamental solution Γ.

We now return to the proof of the existence of $v(x)$. We consider the space Z_N of functions $w(x)$ on $\bar{C}$ with norm $|w|_\varepsilon \leqslant N$ for some $\varepsilon > 0$. We define $\tilde{w} = Tw$ as follows:

$$\tilde{w}(x) = \int_{\partial C} \Gamma(x,\xi)\,\mu(\xi)\,d\Sigma - \int_C \Gamma(x,\eta)\,[f(\eta) + k(\eta, w(\eta))]\,d\eta, \tag{4.23}$$

where $\mu(x)$ is defined for $x \in \partial C$ as the solution of

$$\tfrac{1}{2}\mu(x) + \int_{\partial C}\left[\frac{d\,\Gamma(x,\xi)}{d\,T_x} + g(x)\,\Gamma(x,\xi)\right]\mu(\xi)\,d\Sigma$$

$$= h(x) + \int_C \left[\frac{d\,\Gamma(x,\eta)}{d\,T_x} + g(x)\,\Gamma(x,\eta)\right][f(\eta) + k(\eta, w(\eta))]\,d\eta \equiv \tilde{h}(x). \tag{4.24}$$

Here $d\Sigma$ is the surface area element on ∂C. By the properties of Γ [15] [13, 28–30] it follows that if $\mu(x)$ is continuous on ∂C then $\tilde{w}(x)$ is a solution of the system

$$L_0\tilde{w} = f(x) + k(x, w) \quad \text{in } C \tag{4.25}$$

$$\frac{d\,\tilde{w}(x)}{d\,T} + g(x)\,w(x) = h(x) \quad \text{on } \partial C \tag{4.26}$$

in the sense defined in § 1. Hence it remains to prove the following two statements:

(a) $\mu(x)$ exists as a unique solution of (4.24),

(b) $\tilde{w} = Tw$ has a fixed point.

Proof of (a). Since the kernel of (4.24) is integrable, it is sufficient (by Fredholm's theory) to show that if

$$\tfrac{1}{2}\mu(x) + \int_{\partial C}\left[\frac{d\,\Gamma(x,\xi)}{d\,T_x} + g(x)\,\Gamma(x,\xi)\right]\mu(\xi)\,d\Sigma = 0, \tag{4.27}$$

then $\mu \equiv 0$. Consider the function

$$z(x) = \int_{\partial C} \Gamma(x,\xi)\,\mu(\xi)\,d\Sigma. \tag{4.28}$$

By (4.27), $d z(x)/d T + g(x) z(x) = 0$ on ∂C. Using the maximum principle and the positivity of $g(x)$ we easily conclude that $z \equiv 0$ in $\bar{C}$. We next consider $z(x)$ in $E_n - \bar{C}$. In this domain it satisfies $L_0 z = 0$ and it vanishes on ∂C. Since by (4.6) it also tends to zero as $|x| \to \infty$, the maximum principle yields $z \equiv 0$ in $E_n - C$. Applying the jump relation for the transversal derivatives of simple layers (a simple layer is a function of the form (4.28) with any function μ) we get $\mu(x) \equiv 0$.

Proof of (b). By a comparison argument similar to that given in § 2, we find that $v(x)$, if existing, is a priori bounded. Hence we may change the definition of $k(x, u)$ for large u without restricting the generality of the proof. We thus may assume that

$$|k(x, u)| \leqslant K_1, \left|\frac{\partial k(x, u)}{\partial u}\right| \leqslant K_1 \quad \text{for all } x \in \bar{C}, \ -\infty < u < \infty, \tag{4.29}$$

where K_1 is a constant. Solving (4.24) we then find that

$$\operatorname*{l.u.b.}_{x \in \partial C} |\mu(x)| \leqslant K_2 \operatorname*{l.u.b.}_{x \in \partial C} |\tilde{h}(x)|, \tag{4.30}$$

where K_2 is independent of $\tilde{h}$. Using (4.29) and the definition of $\tilde{h}$ we conclude that

$$\operatorname*{l.u.b.}_{x \in \partial C} |\mu(x)| \leqslant K_3, \tag{4.31}$$

where K_3 is independent of both N and the particular w of Z_N.

Using [15, Theorem 8] we further get $|\tilde{w}|_\varepsilon \leqslant K_4$, where K_4 is also independent of both N and w in Z_N. Hence, if we take $N = K_4$, then T maps Z_N into itself.

$T(Z_N)$ is compact, since by [15, Theorem 8] we have $|\tilde{w}|_\beta \leqslant K_5$ for any $\beta < 1$, and it is enough to take $\beta > \varepsilon$.

The continuity of T on Z_N is easily proved using (4.24) and (4.23). We can thus apply Schauder's fixed point theorem [18] and conclude the existence of a fixed point for T. Having completed the proof of (b), the proof of Theorem 2 is completed.

Remark 1. The above proof of the existence of $v(x)$ does not make use of the assumptions (G_1), (G_2). Furthermore, ∂C need only to be $C^{1+\alpha}$.

Remark 2. Corollary 2 at the end of § 3 holds also under the weaker assumption that $g(x)$ and $h(x)$ are only continuous on ∂C.

Remark 3. If $g(x, t, u)$ is monotone decreasing in u, then existence of a solution for the system (1.1), (1.3) was proved in [7].

Remark 4. If $a_{ij} \in C^{2+\alpha}(\bar{C})$, $b_i \in C^{1+\alpha}(\bar{C})$, then we can write $L_0 u$ in a variational form and use the $(1+\alpha)$ estimates of Agmon *et al.* [1] instead of the $(2+\alpha)$ estimates. It is then sufficient to assume in the above proof of Theorem 2 that ∂C belongs to $C^{2+\alpha}$.

5. Asymptotic expansion of solutions

We shall need the following assumptions:

(D*) D is a cylinder and ∂B (or ∂C) is of class $C^{2+\alpha}$.

(C*) $k(x, t, u) \equiv 0$.

(A_m) For (x, t) in $\bar{D}$,

$$a_{ij}(x, t) = \sum_{\lambda=0}^{m} a_{ij}^{\lambda}(x)\, t^{-\lambda} + t^{-m} o(1),$$

$$b_i(x, t) = \sum_{\lambda=0}^{m} b_i^{\lambda}(x)\, t^{-\lambda} + t^{-m} o(1),$$

$$c(x, t) = \sum_{\lambda=0}^{m} c^{\lambda}(x)\, t^{-\lambda} + t^{-m} o(1),$$

where $o(1) \to 0$ as $t \to \infty$, uniformly in $x \in \bar{B}$; the functions a_{ij}^{λ}, b_i^{λ}, c^{λ} belong to $C^{\alpha}(\bar{B})$ and a_{ij}^{0} also belong to $C^{1+\alpha}(\partial B)$.

(B_m) For (x, t) in $\bar{D}$,

$$f(x, t) = \sum_{\lambda=0}^{m} f^{\lambda}(x)\, t^{-\lambda} + t^{-m} o(1),$$

where $o(1) \to 0$ as $t \to \infty$, uniformly with respect to $x \in \bar{B}$, and the f^{λ} belong to $C^{\alpha}(\bar{B})$.

(E_m) For $(x, t) \in \partial B$,

$$h(x, t) = \sum_{\lambda=0}^{m} h^{\lambda}(x)\, t^{-\lambda} + t^{-m} o(1),$$

where $o(1) \to 0$ as $t \to \infty$, uniformly in $x \in \partial B$, and the h^{λ} belong to $C^{1+\alpha}(\partial B)$.

(F_m) $g(x, t, u) \equiv g(x, t)\, u$ and for $x \in \partial B$,

$$g(x, t) = \sum_{\lambda=0}^{m} g^{\lambda}(x)\, t^{-\lambda} + t^{-m} o(1),$$

where $o(1) \to 0$ as $t \to \infty$ uniformly in $x \in \partial B$, and the g^{λ} belong to $C^{1+\alpha}(\partial B)$.

We introduce the operators

$$L_\lambda v \equiv \sum_{i,j=1}^{n} a_{ij}^{\lambda}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{\lambda}(x) \frac{\partial v}{\partial x_i} + c^{\lambda}(x) v, \tag{5.1}$$

$$\frac{d v(x)}{d T^{\lambda}} = \sum_{i,j=1}^{n} a_{ij}^{\lambda}(x) \cos(\nu(x), x_j) \frac{\partial v(x)}{\partial x_i}. \tag{5.2}$$

We can now state:

THEOREM 3. *Let the assumptions* (A), (B), (C*), (D*), (E), (F) *and* (A_m), (B_m), (E_m), (F_m) *be satisfied for some non-negative integer* m *and let* $c^0(x) \leqslant 0$. *If* $u(x, t)$ *is a solution of the system* (1.1), (1.3) *then*

$$u(x, t) = \sum_{\lambda=0}^{m} u^{\lambda}(x) t^{-\lambda} + t^{-m} o(1), \tag{5.3}$$

where $o(1) \to 0$ *as* $t \to \infty$, *uniformly in* $x \in \bar{B}$, *and the* $u^{\lambda}(x)$ *are determined successively by the following system:*

$$L_0 u^{\lambda}(x) = f^{\lambda}(x) - (\lambda - 1) u^{\lambda-1}(x) - \sum_{\mu=1}^{\lambda} L_\mu u^{\lambda-\mu}(x) \quad (x \in B) \tag{5.4}$$

$$\frac{d u^{\lambda}(x)}{d T} + g^0(x) u^{\lambda}(x) = h^{\lambda}(x) - \sum_{\mu=1}^{\lambda} g^{\mu}(x) u^{\lambda-\mu}(x) - \sum_{\mu=1}^{\lambda} \frac{d}{d T^{\mu}} u^{\lambda-\mu}(x) \quad (x \in \partial B). \tag{5.5}$$

It is understood that for $\lambda = 0$ *the right sides of* (5.4), (5.5) *are replaced by* $f^0(x)$ *and* $h^0(x)$ *respectively.*

6. The first mixed boundary value problem

In this chapter we shall prove analogs of Theorem 1.2 to the case of the first mixed boundary value problem. The boundary conditions (1.3), (1.4) are replaced by

$$u(x, t) = h(x, t) \quad \text{for } (x, t) \in S, \tag{6.1}$$

$$v(x) = h(x) \quad \text{for } x \in \partial C. \tag{6.2}$$

The assumptions (D), (D_0) are replaced by the weaker assumptions:

(D′) $\operatorname{l.u.b.}_{t} |B_t| < \infty$,

(D_0') ∂C is of class $C^{2+\alpha}$ and to every x on ∂C there corresponds one and only one point (x_t, t) on each ∂B_t such that $x_t \to x$ as $t \to \infty$, uniformly in $x \in \partial C$.

THEOREM 4. *Let the assumptions* (A)–(C), (D′), (E) *be satisfied and assume that*

$$\lim_{t\to\infty} h(x, t) = 0, \quad \lim_{t\to\infty} f(x, t) = 0, \quad \limsup_{t\to\infty} c(x, t) \leqslant 0 \tag{6.3}$$

uniformly with respect to $(x, t) \in S$, $(x, t) \in \bar{D}$ *and* $(x, t) \in \bar{D}$ *respectively. If* $u(x, t)$ *is a solution in* D *of the system* (1.1), (6.1), *then* $u(x, t) \to 0$ *as* $t \to \infty$, *uniformly with respect to* (x, t) *in* $\bar{D}$.

The proof is similar to that of Theorem 1, and employs the same function $\varphi(x)$ and a comparison argument similar to that used in § 2. Details are omitted.

THEOREM 5. *Let the assumptions* (A)–(C), (D′), (E) *and* (A_0)–(C_0), (D_0'), (E_0) *be satisfied and let* $c(x) \leq 0$. *If* $u(x, t)$ *is a solution in* D *of the system* (1.1), (6.1) *then*

$$\lim_{\substack{x \to y \\ t \to \infty}} u(x, t) = v(y) \tag{6.4}$$

uniformly with respect to $(x, t) \in \bar{D}$, $y \in \bar{C}$, *and* $v(y)$ *is the unique solution in* C *of the system* (1.2), (6.2).

Proof. We first prove the theorem in the case that $h(x)$ is a polynomial. The proof is then similar to the proof in § 3, except that instead of using Lemma 2 we use Schauder's $(2+\alpha)$ estimates [17] (see also [3], [13]). The existence of $v(x)$ follows by using these estimates and Schauder's fixed point theorem, as in § 3. The family v^δ of approximating functions is constructed as follows:

Let C_δ be a sequence of domains which tend to C (as $\delta \to 0$) from the outside, and which satisfy:

$$\underset{\delta}{\text{l.u.b.}} \, |\partial C_\delta|^{2+\alpha} < \infty. \tag{6.5}$$

We can construct the C_δ in such a manner that there exists a one-to-one correspondence $x \leftrightarrow x^\delta$ from ∂C onto ∂C_δ such that $x^\delta \to x$ as $\delta \to 0$, uniformly in $x \in \partial C$.

We next take C' to be any fixed domain containing $\bar{C}$, and extend the coefficients of the system (1.2), (6.2) to C' in such a manner that they remain Hölder-continuous (exponent α). This can be done even with preserving the Hölder coefficients (see [12]).

In each C_δ we solve the problem

$$L_0 v^\delta = f(x) + k(x, v^\delta) \quad \text{in } C_\delta, \tag{6.6}$$

$$v^\delta(x) = h(x) \quad \text{on } \partial C_\delta. \tag{6.7}$$

By the Schauder estimates (and on using (6.5)) we get

$$|v^\delta|_{2+\alpha}^{C_\delta} \leq \text{const. independent of } \delta. \tag{6.8}$$

From this inequality we get a lemma analogous to Lemma 3, and we then complete the proof by the method of § 3. Furthermore, Corollary 2 can also be extended to the present case.

In the general case that $h(x)$ is not a polynomial, but only a continuous function, we construct, for any given $\varepsilon > 0$, a polynomial $\tilde{h}(x)$ such that

$$|\tilde{h} - h|_0^{\partial C} \leqslant \varepsilon. \tag{6.9}$$

The existence of $v(x)$ is proved by approximating h by smooth functions h_m and finding, by using interior $(2+\alpha)$ estimates [17, 13, 3], that the corresponding solutions v_m converge to a solution in the interior of C, whereas, by using the maximum principle, we find that the convergence is uniform in $\bar{C}$. Hence $\lim v_m$ is the desired solution v.

By the maximum principle we have

$$|\tilde{v} - v|_0^C \leqslant A\,\varepsilon, \tag{6.10}$$

where A is independent of ε, and $\tilde{v}$ is the solution of (1.2), (6.2) when h is replaced by $\tilde{h}$.

The proof of Theorem 5 can now be completed (similarly to § 4) by applying to $\tilde{v}$, u a corollary analogous to Corollary 2, and by using (6.10).

Remark 1. If $a_{ij} \in C^{2+\alpha}(\bar{C})$, $b_i \in C^{1+\alpha}(\bar{C})$, then we can write L_0 is a variational form and use the α-estimates of Agmon *et al.* [1] instead of the $(2+\alpha)$ estimates. It is then sufficient to assume that ∂C in Theorem 5, is only C^α.

Remark 2. In [6] we have proved an analogue of Theorem 3 for the first mixed boundary value problem.

7. Generalized second boundary value problem

In this section we discuss the extension of Theorems 1–3 to the case where instead of (1.3) we have

$$\frac{\partial u(x,t)}{\partial \tau} + g(x,t,u) = h(x,t) \qquad \text{on } S, \tag{7.1}$$

where $\partial u/\partial \tau = \beta(x,t)\,\partial u/\partial t + \partial u/\partial T$. It will be assumed that

(G) $\beta(x,t)$ is continuous on S and $0 \leqslant \beta(x,t) \leqslant$ const. $< \infty$.

Theorem 1 *remains true if we replace* (1.3) *by* (7.1) *and assume that* (G) *holds.*

To prove this statement we proceed along the proof of § 3 with appropriate modifications. Thus, in the definition of $\psi(x, t)$ we take γ smaller than that in (2.10), depending on l.u.b. β. We thus derive (2.11) and

$$\frac{\partial \psi(x, t)}{\partial \tau} + g(x, t, \psi(x, t)) > \varepsilon. \tag{7.2}$$

If we prove that the function $w(x, t) = \psi(x, t) - u(x, t)$ is positive in $D - D_\delta$, then the proof is easily completed.

The proof can be given similarly to that of § 2, noting that $\partial/\partial\tau$ is a derivative in an outward-upward direction.

We note that the uniqueness of u, for more general quasi-linear equations and with h in (7.1) being a nonlinear function of u, $\partial u/\partial x_i$, was proved in [8].

Theorem 2 can also be extended to the present problem, and also Theorem 3 with the $u^\lambda(x)$ depending also on the coefficients in the expansion of $\beta(x, t)$.

Part II. Higher order parabolic equations

In this part we prove that if the boundary values and the coefficients of a parabolic equation of any order tend to a limit as $t \to \infty$, then the solution also tends to a limit which will be the solution of the limit elliptic equation. The convergence is first proved in the L_2 sense and then it is extended to a uniform convergence. Naturally, since an appropriate maximum principle for higher order equations is not known, the regularity assumptions on the differential system will be stronger than in the case of second order equations. The methods are also quite different.

In § 1 we state some results of Agmon *et al.* [1], part of which overlap with results announced by Browder [2]. These are used very substantially in the following. In § 2 we formulate the main result on L_2 convergence. (The domain is not necessarily cylindrical.) The proof is given in § 3. Using the L_2 convergence we proceed in § 4 to establish uniform convergence. We finally discuss in § 5 the question of asymptotic expansion of solutions.

In what follows, the notation introduced in Part I, § 1 will be used freely. All the functions are real.

1. Auxiliary theorems on elliptic equations

Let G be an n-dimensional bounded domain and denote

$$x = (x_1, \ldots, x_n), \quad i = (i_1 \ldots, i_n),$$

$$|i| = i_1 + \cdots + i_n, \quad x^i = x_1^{i_1} \ldots x_1^{i_n}, \quad D^i = D_1^{i_1} \ldots D_1^{i_n},$$

where $D_k = \partial / \partial x_k$. Consider in G the differential equation of order $2m$

$$L_0 u \equiv \sum_{|i| \leqslant 2m} a_i(x) D^i u(x) = f(x). \tag{1.1}$$

L_0 is said to be uniformly elliptic in G if for any $x \in G$ and any real vector ξ,

$$A_0 |\xi|^{2m} \leqslant (-1)^m \sum_{|i|=2m} a_i(x) \xi^i \leqslant A_1 |\xi|^{2m} \qquad (A_0 > 0, A_1 > 0).$$

Together with (1.1) we consider the boundary conditions, on ∂G,

$$\frac{\partial^j u}{\partial \nu^j} = \varphi_j(x), \qquad 0 \leqslant j \leqslant m-1, \tag{1.2}$$

where ν is the outwardly directed normal to ∂G. We state the following results of Agmon *et al.* [1, Chapter IV] as a lemma.

LEMMA 4. *Let L_0 be uniformly elliptic in G, and assume that ∂G is $C^{2m+k+\alpha}$ for some non-negative integer k, that $f(x)$ and $a_i(x)$ are $C^{k+\alpha}(\bar{G})$ and that the φ_j belong to $C^{2m+k-j+\alpha}(\partial G)$. If the system* (1.1), (1.2) *cannot have more than one solution, then there exists a unique solution $u(x)$ of* (1.1), (1.2) *and it satisfies*

$$|u|_{2m+k+\alpha}^{G} \leqslant K \left(|f|_{k+\alpha} + \sum_{j=0}^{m-1} |\varphi_j|_{2m+k-j+\alpha} \right), \tag{1.3}$$

where K is a constant depending only on A_0, $|\partial G|_{2m+k+\alpha}$, and on the $(k+\alpha)$ norms of a_i in $\bar{B}$.

For elliptic equations in variational form Agmon *et al.* derived in [1, Chapter IV], existence and a priori estimates for $|u|_{m-1+k+\alpha}$ $(k \geqslant 0)$. We formulate this result for the equation (1.1):

LEMMA 5. *Let L_0 be uniformly elliptic in G, and assume that ∂G is $C^{m-1+k+\alpha}$ for some non-negative integer $k < m+1$, that $f(x)$ is $C^{\alpha}(\bar{G})$, that φ_j is $C^{m-1+k-j+\alpha}(\partial G)$, that $a_i(x)$ is $C^{\alpha}(\bar{G})$ and that $a_i(x)$ is $C^{|i|-m+1+\alpha}(\bar{G})$ if $|i| \geqslant m$. If the system* (1.1), (1.2) *cannot have more than one solution, then there exists a unique $u(x)$ of* (1.1), (1.2) *and it satisfies:*

$$|u|_{m-1+k+\alpha}^{G} \leqslant K \left(|f|_{\alpha} + \sum_{j=0}^{m-1} |\varphi_j|_{m-1+k-j+\alpha} \right), \tag{1.4}$$

where K is a constant depending on A_0, $|\partial G|_{m-1+k+\alpha}$, on the α-norms of the a_i and on the $(|i|-m+1+\alpha)$ norms of the a_i with $|i| \geqslant m$.

2. Statement of the main result on L_2 stability

We shall consider the parabolic equation $(u=u(x, t))$

$$\frac{\partial u}{\partial t}+Lu\equiv\frac{\partial u}{\partial t}+\sum_{|i|\leqslant 2m} a_i(x, t) D^i u=f(x, t) \quad \text{in } D \tag{2.1}$$

and the Dirichlet boundary conditions

$$\frac{\partial^j u}{\partial \nu_t^j}=\varphi_j(x, t) \quad (0\leqslant j\leqslant m-1) \quad \text{on } \partial B_t,\ 0<t<\infty, \tag{2.2}$$

where ν_t is the outwardly directed normal to ∂B_t. For clarity we first state the assumptions needed later. In what follows, A will denote any constant independent of t, h.

The assumptions on D will look somewhat complicated. Roughly speaking, it will be assumed that S is smooth and the B_t tend regularly (or smoothly) and sufficiently fast to their limit C.

Assumptions on D

(A_1) $|\partial B_t|_{2m+\alpha}\leqslant A$.

(A_2) There exists a one-to-one transformation $x_t\leftrightarrow x_\tau$ from ∂B_t onto ∂B_τ, for any t, τ, such that if $x_{t+h}-x_t=\varepsilon_{t,h}(x_t)$, then

(i) for $|h|\leqslant 1$, $\frac{1}{|h|}|\varepsilon_{t,h}|_{m-1+\alpha}^{\partial B_t}\leqslant A$,

(ii) as $h\to 0$, $\frac{1}{h}\varepsilon_{t,h}(x_t)\to\frac{\bar{d} x_t}{\bar{d} t}$ uniformly in $x_t\in\partial B_t$, and $\left|\frac{\bar{d} x_t}{\bar{d} t}\right|_{m-1+\alpha}=o(1)$ as $t\to\infty$.

(A_3) The function $N_{t,h}(x_t)=\frac{1}{h}\cos\{\nu_t(x_t),\ \nu_{t+h}(x_{t+h})\}$ satisfies

(i) for $|h|\leqslant 1$, $|N_{t,h}|_{m-1+\alpha}^{\partial B_t}\leqslant A$,

(ii) as $h\to 0$, $N_{t,h}(x_t)\to\frac{\bar{d}\nu_t(x_t)}{\bar{d} t}$ uniformly in $x_t\in\partial B_t$, and $\left|\frac{\bar{d}\nu_t}{\bar{d} t}\right|_{m-1+\alpha}=o(1)$ as $t\to\infty$.

(A_4) There exists a bounded domain $C=B_\infty$ such that there is a one-to-one correspondence $x_t\leftrightarrow x_\infty$ between ∂B_t and ∂B_∞, $|\partial B_\infty|_{m+\alpha}<\infty$ and the function $\varepsilon_t(x_t)=x_t-x_\infty$ satisfies

$$|\varepsilon_t|_{m-1+\alpha}^{\partial B_t}=o(1) \quad \text{as } t\to\infty.$$

(A_5) The function $N_t(x_t)=\cos[\nu_t(x_t),\ \nu_\infty(x_\infty)]$ satisfies

$$|N_t|_{m-1+\alpha}^{\partial B_t}=o(1) \quad \text{as } t\to\infty.$$

Assumption on the boundary values

Roughly speaking the assumptions are that the φ_j are sufficiently smooth and they converge sufficiently fast to a limit as $t \to \infty$. More precisely:

(B$_1$) $\sum_j |\varphi_j(\cdot, t)|^{\partial B_t}_{2m-j+\alpha} \leqslant A$

(B$_2$) The functions $R^j_{t,h}(x_t) = \frac{1}{h}[\varphi_j(x_{t+h}, t+h) - \varphi_j(x_t, t)]$ satisfy:

(i) for $|h| \leqslant 1$, $\sum_{j=0}^{m-1} |R^j_{t,h}|^{\partial B_t}_{m-1-j+\alpha} \leqslant A$.

(ii) as $h \to 0$, $R^j_{t,h}(x_t) \to \frac{\bar{d}\varphi_j(x_t)}{\bar{d}t}$ uniformly in $x_t \in \partial B_t$, and $\sum_{j=0}^{m-1} \left| \frac{\bar{d}\varphi_j}{\bar{d}t} \right|^{\partial B_t}_{m-1-j+\alpha} = o(1)$ as $t \to \infty$.

(B$_3$) There exist functions $\varphi_j(x_\infty)$ of class $C^{m-j+\alpha}$ on ∂B_∞ such that the functions $S^j_t(x_t) = \varphi_j(x_t, t) - \varphi_j(x_\infty)$ satisfy

$$|S^j_t|^{\partial B_t}_{m-1-j+\alpha} = o(1) \text{ as } t \to \infty. \tag{2.3}$$

Assumptions L, L_0, f

(C$_1$) L is uniformly parabolic, that is, for every $(x, t) \in D$ and any real vector ξ,

$$A_1 |\xi|^{2m} \leqslant (-1)^m \sum_{|i|=2m} a_i(x, t) \xi^i \leqslant A_2 |\xi|^{2m} \quad (A_1 > 0,\ A_2 > 0).$$

(C$_2$) L is *positive* in the L_2-norm, that is, there exists $\gamma > 0$ such that for every $t > 0$ and for every function $\mathring{v}$ of class $C^{2m}(B_t)$, $C^{m-1}(\bar{B}_t)$ which vanishes on ∂B_t together with its first $(m-1)$ normal derivatives

$$\int_{B_t} \mathring{v}(x) L \mathring{v}(x)\, dx \geqslant \gamma \int_{B_t} (v(x))^2\, dx.$$

(C$_3$) There exist functions $a_i(x)$, $f(x)$ defined in the closure of the domain $B_* = \bigcup_{t>0} B_t$, and satisfying: f and a_i belong to $C^\alpha(\bar{B}_*)$ for $|i| < m$, and the a_i belong to $C^{|i|-m+1+\alpha}(\bar{B}_*)$ if $2m \geqslant |i| \geqslant m$.

(C$_4$) As $t \to \infty$

$$\|f(\cdot, t) - f(\cdot)\|^{B_t} \to 0, \quad \sum_{|i| \leqslant 2m} \|a_i(\cdot, t) - a(\cdot)\|^{B_t} \to 0.$$

In (C$_4$) the following notation has been employed:

$$\|g\|^G = \left(\int_G (g(x))^2\, dx\right)^{\frac{1}{2}}, \quad \|g(\cdot, t)\|^G = \left(\int_G (g(x, t))^2\, dx\right)^{\frac{1}{2}}.$$

Remark 1. If in (C_2) we make a stronger assumption about the vanishing of $\mathring{v}$ on ∂B_t, namely, if we assume $\mathring{v}$ to have compact support in B_t, then we obtain a new assumption, say, (C_2'). It can be shown that (C_2') is equivalent to (C_2).

Remark 2. The assumptions (C_2), (C_4) combined imply (using Remark 1) that L_0 defined by (1.1) is a positive operator in B_∞. Hence the existence theorems of Lemmas 4, 5 can be applied.

Before stating the result of the L_2 convergence we have to introduce one more notation. We denote by $\partial B_{t,\sigma}$ $(\sigma > 0)$ the surface obtained from ∂B_t by shifting each point of ∂B_t a distance σ along the inner normal. By $B_{t,\sigma}$ we denote the interior of $\partial B_{t,\sigma}$. It is well known that $\partial B_{t,\sigma}$, for small σ, is orthogonal to the family of the normals issuing from ∂B_t.

THEOREM 6. *Let the assumptions* (A_1)–(A_5), (B_1)–(B_3), (C_1)–(C_4) *be satisfied. If* $u(x, t)$ *is a solution of* (2.1), (2.2) *in* D, *then*

$$\| u(\cdot, t) - v(\cdot) \|^{B_{t,\sigma}} \to 0 \text{ as } t \to \infty, \tag{2.4}$$

where $\sigma \equiv \underset{x_t \in \partial B_t}{\text{l.u.b.}} |x_t - x_\infty| \to 0$ *as* $t \to \infty$ (*and hence* $B_{t,\sigma} \to B_\infty$ *in a uniform manner*), *and* $v(x)$ *is the unique solution in* B_∞ *of the system* (1.1), (2.2), *where* $\varphi_j(x) = \varphi_j(x_\infty)$.

The assumptions of Theorem 6, with the exception of (2.3), seem to be quite natural. It would be desirable to assume $\alpha = 0$ in (2.3). For the case of two space dimensions this can be done (see the end of § 3).

3. Proof of Theorem 6

Let $v(x, t)$ be a solution of the Dirichlet problem

$$L_0 v(x, t) = f(x) \text{ in } B_t \tag{3.1}$$

$$\frac{\partial^j v(x, t)}{\partial \nu_t^j} = \varphi_j(x, t) \text{ on } \partial B_t. \tag{3.2}$$

Be Lemma 4 and our assumptions, v exists and satisfies

$$|v(\cdot, t)|_{2m+\alpha}^{B_t} \leqslant H, \tag{3.3}$$

where H, here and in the following, is used to denote any constant independent of t, h. We shall first estimate the $L_2(B_t)$ norm of the function

$$z(x, t) \equiv u(x, t) - v(x, t). \tag{3.4}$$

z satisfies the system

$$\frac{\partial z}{\partial t}+\sum_{|i|\leqslant 2m} a_i(x, t) D^i z=\tilde{f}(x, t) \quad \text{in } D, \tag{3.5}$$

where $\tilde{f}\equiv[f(x, t)-f(x)]-\sum_{|i|\leqslant 2m}[a_i(x, t)-a_i(x)] D^i v-\frac{\partial v}{\partial t}$,

$$\frac{\partial^j z}{\partial \nu_t^j}=0 \quad \text{on } \partial B_t, \quad 0<t<\infty, \tag{3.6}$$

In writing (3.5) we have assumed however that $\partial v/\partial t$ exists. We now proceed to prove the existence of $\partial v/\partial t$ and to estimate it.

Consider the function $v_h(x, t)=[v(x, t+h)-v(x, t)]/h$. It is defined in $B_t \cap B_{t+h}$ (here we imagine, for simplicity, that the B_σ, $0<\sigma<\infty$, lie on the hyperplane $t=0$). For small $\sigma>0$, the points $x_{t,\sigma}$ on $\partial B_{t,\sigma}$ are in one-to-one correspondence with the points x_t of ∂B_t, and the transformation $x_t \rightarrow x_{t,\sigma}$ is of class $C^{2m-1+\alpha}$. Hence, using (A_2) there is a one-to-one transformation $x_{t,\sigma} \leftrightarrow x_{t+h}$ from $\partial B_{t,\sigma}$ onto ∂B_{t+h} which is of class $C^{m-1+\alpha}$. We take

$$\sigma=\underset{x_t \in \partial B_t}{\text{l.u.b.}}\, |x_t-x_{t+h}| \tag{3.7}$$

and then $v_h(x, t)$ is defined in $B_{t,\sigma}$.

v_h satisfies the differential equation

$$L_0 v_h(x, t)=0 \quad \text{in } B_{t,\sigma}, \tag{3.8}$$

$$\frac{\partial^j}{\partial \nu_{t,\sigma}^j} v_h=\varphi_{jh}(x_{t,\sigma}) \quad \text{on } \partial B_{t,\sigma}, \tag{3.9}$$

where $\nu_{t,\sigma}$ is the outward normal to $\partial B_{t,\sigma}$ (and hence $\partial/\partial \nu_{t,\sigma}=\partial/\partial \nu_t$), and where

$$\begin{aligned}
\varphi_{jh}(x_{t,\sigma}) &= \frac{1}{h}\left[\frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma}, t+h)-\frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma}, t)\right] \\
&= \frac{1}{h}\left[\frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma}, t+h)-\frac{\partial^j}{\partial \nu_{t+h}^j} v(x_{t+h}, t+h)\right] \\
&\quad -\frac{1}{h}\left[\frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma}, t)-\frac{\partial^j}{\partial \nu_t^j} v(x_t, t)\right]+\frac{1}{h}[\varphi_j(x_{t+h}, t+h)-\varphi_j(x_t, t)] \\
&\equiv \Phi_1^h+\Phi_2^h+\Phi_3^h.
\end{aligned} \tag{3.10}$$

By assumption (B_2),

$$|\Phi_3^h|_{m-1-j+\alpha}^{\partial B_t} \leqslant A, \tag{3.11}$$

provided $|h| \leqslant 1$, which we may assume.

Next writing $$-\Phi_2^h = \frac{1}{h}\int_0^1 \frac{d}{d\lambda}\frac{\partial^j}{\partial \nu_t^j} v(\lambda x_{t,\sigma} + (1-\lambda)x_t, t)\,d\lambda$$

and using (3.3) and the differentiability assumptions on ∂B_t, we obtain

$$|\Phi_2^h|_{m-1-j+\alpha}^{\partial B_t} \leqslant H\frac{\sigma}{h}, \tag{3.12}$$

where for simplicity, we take $h > 0$, here and in the following.

To estimate Φ_1^h, we write it in the form

$$\Phi_1^h = \frac{1}{h}\left[\frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma}, t+h) - \frac{\partial^j}{\partial \nu_t^j} v(x_{t+h}, t+h)\right]$$
$$+\frac{1}{h}\left[\frac{\partial^j}{\partial \nu_t^j} v(x_{t+h}, t+h) - \frac{\partial^j}{\partial \nu_{t+h}^j} v(x_{t+h}, t+h)\right] \equiv \Phi_{11}^h + \Phi_{12}^h. \tag{3.13}$$

Φ_{11}^h can be estimated similarly to Φ_2^h. We thus get (using (A_2), (i))

$$|\Phi_{11}^h|_{m-1-j+\alpha}^{\partial B_{t+h}} \leqslant H\frac{\sigma}{h} + H. \tag{3.14}$$

Using (3.3) we obtain

$$|\Phi_{12}^h|_{m-1-j+\alpha}^{\partial B_{t+h}} \leqslant \frac{H}{h}|\cos[\nu_t(x_t), \nu_{t+h}(x_{t+h})]|_{m-1+\alpha}^{\partial B_t}. \tag{3.15}$$

Combining (3.11)–(3.15) and recalling that, by assumption, $\sigma/h \leqslant A$, we obtain, using (A_3),

$$|\varphi_{jh}|_{m-1-j+\alpha}^{\partial B_{t,\sigma}} \leqslant H. \tag{3.16}$$

Applying Lemma 5 to the system (3.8), (3.9) and using (3.16), we get

$$|v_h|_{m-1+\alpha}^{B_{t,\sigma}} \leqslant H. \tag{3.17}$$

Also, by the interior estimates of [3] we have, for any compact subset E of $B_{t,\sigma}$

$$|v_h|_{2m+\alpha}^{E} \leqslant \text{const.}\ |v_h|_0^{B_{t,\sigma}} \leqslant H', \tag{3.18}$$

where H' depends on E but not of h, if h is sufficiently small.

Using the assumptions (A_1)–(A_3) it is seen that as $h \to 0$, $\lim \varphi_{jh}(x_{t,\sigma})$ exists. Denoting it by $\tilde{\varphi}_j(x_t)$, we have

$$|\tilde{\varphi}_j|_{m-1-j+\alpha}^{\partial B_t} \leqslant H\left(\sum_{j=0}^{m-1}\left|\frac{\bar{d}\varphi_j}{\bar{d}t}\right|_{m-1-j+\alpha} + \left|\frac{\bar{d}v}{\bar{d}t}\right|_{m-1+\alpha} + \left|\frac{\bar{d}x_t}{\bar{d}t}\right|_{m-1+\alpha}\right) = o(1), \tag{3.19}$$

where $o(1) \to 0$ as $t \to \infty$.

The system

$$L_0 \tilde{v} = 0 \text{ in } B_t, \tag{3.20}$$

$$\frac{\partial^j \tilde{v}}{\partial \nu_t^j} = \tilde{\varphi}_j \text{ on } \partial B_t, \tag{3.21}$$

has, by Lemma 5, a unique solution $\tilde{v}(x, t)$ and, by (3.19),

$$|\tilde{v}|_{m-1+\alpha} = o(1). \tag{3.22}$$

We claim that $\partial v/\partial t$ exists and is equal to $\tilde{v}$. Indeed, by (3.17), (3.18) it follows that any sequence $\{h_k\}$ $(h_k \to 0)$ has a subsequence $\{h_k'\}$ such that the corresponding v_h converge in the interior of B_t to a solution v' of $L_0 v' = 0$, and v' has a finite $(m-1+\alpha)$ norm in B_t and satisfies (3.21). Hence, the limit v' coincides with $\tilde{v}$. Since $\tilde{v}$ is uniquely determined, it follows that as $h \to 0$, v_h converges to $\tilde{v}$, uniformly in every compact subset of B_t. Hence $\partial v/\partial t$ exists and is equal to $\tilde{v}$. By (3.22) we also have

$$\left|\frac{\partial v}{\partial t}\right|_{m-1+\alpha}^{B_t} = o(1), \text{ as } t \to \infty. \tag{3.23}$$

It is now easy to complete the estimation of z. By assumption (C_4),

$$\|f(\cdot, t) - f(\cdot)\|^{B_t} \to 0 \text{ as } t \to \infty. \tag{3.24}$$

By (3.3) and by assumption (C_4),

$$\sum_{|i| \leqslant 2m} \|[a_i(\cdot, t) - a_i(\cdot) D^i v\|^{B_t} \to 0 \text{ as } t \to \infty. \tag{3.25}$$

Combining (3.25), (3.24), (3.23) we conclude that, in (3.5),

$$\|\tilde{f}(\cdot, t)\|^{B_t} \equiv \varepsilon(t) \to 0 \text{ as } t \to \infty. \tag{3.26}$$

Multiplying the equation in (3.5) by $z(x, t)$ and integrating over B_t, we obtain, upon making use of the boundary conditions (3.6) and the positivity of L,

$$\psi'(t) + 2\gamma\psi(t) \leqslant 2\int_{B_t} \tilde{f}(x, t)\, z(x, t)\, dx, \tag{3.27}$$

where $\psi(t) = \int_{B_t} (z(x, t))^2\, dx$. Using Schwarz's and the inequality $\beta\gamma \leqslant \frac{1}{2}(\varepsilon\beta^2 + \gamma^2/\varepsilon)$ $(\varepsilon > 0, \beta > 0, \gamma > 0)$, we get

$$\psi'(t) + \gamma\,\psi(t) \leqslant \frac{1}{\gamma}\,\varepsilon^2(t). \tag{3.28}$$

We claim that $\psi(t) \to 0$ as $t \to \infty$. Indeed, for any given $\delta > 0$, we choose t_0 such that $\varepsilon^2(t) < \delta\gamma^2/2$ if $t > t_0$. Integrating (3.28), we obtain

$$\psi(t) \leqslant e^{-\gamma t}\left[\int_{t_0}^{t} \frac{1}{\gamma}\,\varepsilon^2(\tau)\,e^{\gamma\tau}\,d\tau + e^{\gamma t_0}\,\psi(t_0)\right] \leqslant \frac{\delta}{2} + e^{-\gamma(t-t_0)}\,\psi(t_0) < \delta$$

if t is sufficiently large. We have thus proved that

$$\|u(\cdot, t) - v(\cdot, t)\|^{B_t} \equiv \|z(\cdot, t)\|^{B_t} \to 0 \text{ as } t \to \infty. \tag{3.29}$$

We next consider the function

$$w(x, t) = v(x, t) - v(x), \tag{3.30}$$

where $v(x)$ is the solution of

$$L_0 v = f(x) \text{ in } B_\infty, \tag{3.31}$$

$$\frac{\partial^j v}{\partial \nu_\infty^j} = \varphi_j(x_\infty)\ (0 \leqslant j \leqslant m-1) \text{ on } \partial B_\infty. \tag{3.32}$$

w satisfies the system

$$L_0 w = 0 \text{ in } B_{t,\sigma}, \tag{3.33}$$

$$\frac{\partial^j w}{\partial \nu_t^j} = \varphi_j^0(x_{t,\sigma})\ (0 \leqslant j \leqslant m-1) \text{ on } \partial B_{t,\sigma}. \tag{3.34}$$

where

$$\sigma = \operatorname*{l.u.b.}_{x_t \in \partial B_t} |x_t - x_\infty| \tag{3.35}$$

and where

$$\begin{aligned}\varphi_j^0(x_{t,\sigma}) = {} & \left[\frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma}, t) - \frac{\partial^j}{\partial \nu_t^j} v(x_t, t)\right] + \left[\frac{\partial^j}{\partial \nu_\infty^j} v(x_\infty) - \frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma})\right] \\ & + [\varphi_j(x_t, t) - \varphi_j(x_\infty)] \equiv \Psi_1^t + \Psi_2^t + \Psi_3^t.\end{aligned} \tag{3.36}$$

Using (3.3) and (A_4) we get

$$|\Psi_1^t|_{m-1-j+\alpha}^{\partial B_t} \leqslant H\,\sigma. \tag{3.37}$$

Next, by assumption (B_3),

$$|\Psi_3^t|_{m-1-j+\alpha}^{\partial B_t} = o(1), \text{ as } t \to \infty. \tag{3.38}$$

To estimate Ψ_2^t, we write it in the form

$$\Psi_2^t = \left[\frac{\partial^j}{\partial \nu_\infty^j} v(x_\infty) - \frac{\partial^j}{\partial \nu_\infty^j} v(x_{t,\sigma})\right] + \left[\frac{\partial^j}{\partial \nu_\infty^j} v(x_{t,\sigma}) - \frac{\partial^j}{\partial \nu_t^j} v(x_{t,\sigma})\right] \equiv \Psi_{21}^t + \Psi_{22}^t. \tag{3.39}$$

Since, by Lemma 5 with $k=1$, we have

$$|v(\cdot)|_{m+\alpha}^{B\infty} \leqslant H, \tag{3.40}$$

we easily get, using (A_4),

$$|\Psi_{21}^t|_{m-1-j+\alpha}^{\partial B_t} \leqslant H\,|x_t - x_\infty|_{m-1+\alpha}^{\partial B_t} = o(1) \text{ as } t \to \infty. \tag{3.41}$$

Finally, using (A_5), (3.40),

$$|\Psi_{22}^t|_{m-1-j+\alpha}^{\partial B_t} \leqslant H\,|\cos[\nu_t(x_t), \nu_\infty(x_\infty)]|_{m-1+\alpha}^{\partial B_t} = o(1) \text{ as } t \to \infty. \tag{3.42}$$

Combining (3.36)–(3.39), (3.41), (3.42) we easily get

$$|\varphi_j^0|_{m-1-j+\alpha}^{\partial B_t,\sigma} = o(1), \text{ as } t \to \infty. \tag{3.43}$$

Using Lemma 5 with $k=0$ we obtain,

$$|v(\cdot, t) - v(\cdot)|_{m-1+\alpha}^{B_t,\sigma} \equiv |w(\cdot, t)|_{m-1+\alpha}^{B_t,\sigma} \to 0 \quad \text{as } t \to \infty. \tag{3.44}$$

Combining (3.44) with (3.29), the proof of Theorem 6 is completed.

Remark. From the above proof we see that the assumption (2.3) was needed in making use of Lemma 5 with $f \equiv 0$, $k=0$. Hence if Lemma 5, for $f \equiv 0$, $k=0$, holds with $\alpha=0$ in (1.4), then it is enough to assume, in (B_3), that (2.3) is satisfied for $\alpha=0$. Also, it is enough to assume that (A_2)–(A_5) and (B_2), hold with $\alpha=0$. The desired a priori inequality (that is, (1.4) with $k=\alpha=0$, $f \equiv 0$) can be viewed as a generalization of the maximum principle to higher order equations. It was recently proved by Miranda [14] for $n=2$, provided L_0 is positive in the sense that

$$\int_{B\infty} \mathring{v}(x)\, L_0 v(x)\, dx \geqslant \gamma_0 \sum_{|i| \leqslant m} \int_{B\infty} (D^i \mathring{v}(x))^2\, dx \quad (\gamma_0 > 0)$$

for any $\mathring{v} \in C^{2m}(B_\infty)$, $\mathring{v} \in C^{m-1}(\bar{B}_\infty)$, and v having zero Dirichlet data on ∂B_∞.

Added in proof: Extending Miranda's results S. Agmon (in *Bull. Amer. Math. Soc.* 66 (1960), 77–80) has very recently proved maximum principles and, in particular, Lemma 5 for $f \equiv 0$, $k=0$, $\alpha=0$, provided the $a_i(x)$ belong to $C^{|i|}(\bar{G})$ and ∂G is of class C^{2m}. Hence, if $a_i(x) \in C^{|i|}(\bar{B}_\infty)$, then Theorem 6 holds when the assumptions (A_2)–(A_5), (B_2), (B_3) are weakened by taking $\alpha=0$. A similar improvement holds also for Theorem 7–9 below.

4. Uniform convergence

Having proved the L_2 convergence of $u(x, t)$ to $v(x)$, we proceed in this section to derive, under stronger assumptions, uniform convergence. The first result is about convergence for x in compact subsets of B_∞. The second result is about convergence in the whole domain D, provided D is a cylinder. Finally we mention a few additional results that can be derived by some modifications of the methods.

4.1. Convergence in compact subsets

We need the following additional assumptions:

(C_4') As $t \to \infty$

$$|f(\cdot, t) - f(\cdot)|_0^{B_t} \to 0, \quad \sum_{|i| \leqslant 2m} |a_i(\cdot, t) - a_i(\cdot)|_0^{B_t} \to 0.$$

(C_5') The coefficients $a_i(x, t)$ of L have $|i|$ continuous derivatives in $\bar{D}$ which are bounded (in $\bar{D}$) by a constant A_3.

THEOREM 7. *Let the assumptions* (A_1)–(A_5), (B_1)–(B_3), (C_1)–(C_3) *and* (C_4'), (C_5') *be satisfied. If* $u(x, t)$ *is a solution of* (2.1), (2.2) *then for every compact subset* G *of* B_∞,

$$|u(\cdot, t) - v(\cdot)|_0^G \to 0, \text{ as } t \to \infty, \tag{4.1}$$

where $v(x)$ *is the solution of* (1.1), (1.2) *with* $\varphi_j(x) = \varphi_j(x_\infty)$.

Note that (4.1) is equivalent to the statement $u(x, t) \to v(y)$ as $x \to y$, $t \to \infty$ uniformly in $x \in G$, $y \in G$.

Proof. In the proof of Theorem 6 we introduced the functions $z(x, t) = u(x, t) - v(x, t)$ and $w(x, t) = v(x, t) - v(x)$. For the second function we derived a uniform convergence to zero (see (3.44)). For $z(x, t)$, however, we derived only L_2 convergence to zero (see (3.29)). It thus remains to prove uniform convergence for z. By the estimates in § 3 and by (C_4') we already know that z satisfies (3.5), (3.6) and

$$|\tilde{f}(\cdot, t)|_0^{B_t} \to 0, \text{ as } t \to \infty, \tag{4.2}$$

$$\|z(\cdot, t)\|_0^{B_t} \to 0, \text{ as } t \to \infty. \tag{4.3}$$

Let E be a domain which satisfies $G \subset E \subset \bar{E} \subset B_\infty$. Consider the cylinder Q with base E and $0 < t < \infty$. If t is sufficiently large, say $t \geqslant \varrho$, then $Q - Q_\varrho$ is contained in $D - D_\varrho$. Let $K(x, t; \xi, \tau)$ $(t > \tau)$ be a fundamental solution of $L^* - \partial u / \partial \tau$ (the adjoint of $L + \partial u / \partial t$) as a function of (ξ, τ), with singularity at (x, t), in the cylinder $Q - Q_\varrho$. Under the assumption (C_5'), its existence was proved by Slobodetski [19] (and, under slightly stronger assumptions, earlier by Eidelman [4]) and certain smothness and boundedness properties have been derived. In particular,

$$\int_E |K(x, t; \xi, \tau)| \, d\xi \leqslant H_0 \quad (H_0 \text{ const.}), \tag{4.4}$$

provided $\tau > \varrho$, $0 < t - \tau \leqslant 1$, and

$$\int_E [K(x, t; \xi, \tau)]^2 \, d\xi \leqslant H_1 \quad (H_1 \text{ const.}), \tag{4.5}$$

provided $\tau > \varrho$, $t - \tau = 1$.

We introduce a function $\psi(\xi)$ which is 1 in some neighborhood of G, zero outside $\bar{E}$ and which is defined and of class C^{2m} for all ξ. Writing down Green's identity for the operator $L + \partial/\partial\tau$ with the functions $z(\xi, \tau)$, $\psi(\xi) K(x, t; \xi, \tau)$ and integrating over the domain $\xi \in E$, $t-1 < \tau < t$ we find, for any fixed $x \in G$, $t-1 \geqslant \varrho$

$$\begin{aligned} z(x, t) = & \int_{t-1}^{t} \int_E \hat{f}(\xi, \tau) \psi(\xi) K(x, t; \xi, \tau) \, d\xi \, d\tau \\ & - \int_{t-1}^{t} \int_E z(\xi, t) \left(L^* - \frac{\partial}{\partial t} \right) [\psi(\xi) K(x, t; \xi, \tau)] \, d\xi \, d\tau \\ & + \int_E z(\xi, t-1) \psi(\xi) K(x, t; \xi, t-1) \, d\xi \equiv T_1 + T_2 + T_3. \end{aligned} \tag{4.6}$$

By (4.2), (4.4) and by (4.3), (4.5) we get

$$|T_1|_0^E \to 0 \text{ as } t \to \infty, \quad |T_3|_0^E \to 0 \text{ as } t \to \infty. \tag{4.7}$$

Next, since $\psi = 1$ in some neighborhood of G, $(L^* - \partial/\partial\tau)(\psi K)$ must vanish for ξ in this neighborhood. Since $x \in G$, it follows that if $(L^* - \partial/\partial\tau)[\psi(\xi) K(x, t; \xi, \tau)] \neq 0$ then $|x - \xi| \geqslant \beta > 0$ for some constant β independent of t, τ. Hence (by results of [19], [4])

$$\left| \left(L^* - \frac{\partial}{\partial t} \right) [\psi(\xi) K(x, t; \xi, \tau)] \right| \leqslant H_2, \tag{4.8}$$

where H_2 is independent of x, t, ξ, τ, provided $x \in G$.

Combining (4.8) with (4.3) we get

$$|T_2|_0^E \to 0 \text{ as } t \to \infty, \tag{4.9}$$

which, combined with (4.7), (4.6), completes the proof of the theorem.

4.2. Convergence in the whole domain D

We shall prove convergence in the whole domain D, for cylindrical domains. For such domains the assumptions (A_1)–(A_5), (B_1)–(B_3) of Theorem 6 take a much simpler form and we therefore reformulate them.

(A) D is a cylinder and ∂B is of class $C^{2m+\alpha}$.

(B) $\sum_j |\varphi_j(\cdot, t)|^{\partial B}_{2m-j+\alpha} \leq A, \quad \sum_j |\varphi_j(\cdot)|^{\partial B}_{m-j+\alpha} \leq A$

$$\sum_j \left|\frac{\partial}{\partial t}\varphi_j(\cdot, t)\right|^{\partial B}_{m-1-j+\alpha} = o(1), \quad \sum_j |\varphi_j(\cdot, t) - \varphi_j(\cdot)|^{\partial B}_{m-1-j+\alpha} = o(1) \quad (t \to \infty).$$

We shall need the following new assumption:

(C_2') L is *strongly* positive in D, that is, for any $w(x, t)$ which is of class $C^{2m}(D)$ and with compact support in D,

$$\int_D \mathring{w}\, L\, \mathring{w}\, dx\, dt \geq \gamma \sum_{|i| \leq m} \int_D (D_x^i \mathring{w})^2\, dx\, dt \quad (\gamma > 0).$$

THEOREM 8. *Let the assumptions* (A), (B), (C_1), (C_2'), (C_3), (C_4'), (C_5') *be satisfied and let* $2m > \frac{1}{2}n$. *If* $u(x, t)$ *is a solution of* (2.1), (2.2) *then*

$$|u(\cdot, t) - v(\cdot)|_0^B \to 0, \text{ as } t \to \infty, \tag{4.10}$$

where $v(x)$ *is the solution of* (1.1), (1.2).

Proof. As in the proof of Theorem 7, we only have to consider $z(x, t)$. More specifically, we have to prove that

$$\underset{x \in B}{\text{l.u.b.}}\ |z(x, t)| \to 0, \text{ as } t \to \infty, \tag{4.11}$$

where z is the solution of (3.5), (3.6) and (4.2), (4.3) hold.

We shall make use of Green's function $G(x, t; \xi, \tau)$ constructed by Rosenbloom [16, 122–123] for all $(x, t) \in \bar{D}$, $(\xi, \tau) \in \bar{D}$, $(x, t) \neq (\xi, \tau)$, $t > \tau$. By our regularity assumptions on the coefficients of L it follows that z can be represented in the form

$$z(x, t) = \int_{t-1}^{t} \int_B G(x, t; \xi, \tau)\, \hat{f}(\xi, \tau)\, d\xi\, d\tau + \int_B G(x, t; \xi, t-1)\, z(\xi, t-1)\, d\xi. \tag{4.12}$$

Furthermore, we have [16]

$$\int_B [G(x, t; \xi, \tau)]^2\, d\xi \leq \frac{H_3}{(t-\tau)^{n/2m}} \quad (H_3 \text{ const.}), \tag{4.13}$$

provided $0 < t - \tau \leq 1$. Using Schwarz's inequality in (4.12) and making use of (4.13), (4.2), (4.3), the proof of (4.11) immediately follows.

Remark 1. The assumption $2m > \frac{1}{2}n$ was used only in concluding via (4.13), that

$$\int_{t-1}^{t} \int_{B} |G(x, t; \xi, \tau)|\, d\xi\, d\tau \leqslant \text{const. independent of } t. \tag{4.14}$$

If one could establish (4.14) for any m, n then Theorem 8 would follow for any m, n.

Remark 2. The assumption (C_2') may become too restrictive in some applications. In some cases this assumption may be replaced by the assumption (C_2). We give one example:

Suppose (A), (B), (C_1)–(C_4) *are satisfied, and suppose that*

$$a_i(x, t) \equiv a_i(x), \quad \left\| \frac{\partial f(\cdot, t)}{\partial t} \right\|^B \to 0, \quad \left| \frac{\partial}{\partial t} \varphi_j(\cdot, t) \right|_{2m-j+\alpha}^{\partial B} \to 0,$$

$$\left| \frac{\partial^2}{\partial t^2} \varphi_j(\cdot, t) \right|_{m-1-j+\alpha}^{\partial B} \to 0, \ \textit{as} \ t \to \infty,$$

If $2m > \frac{1}{2}n$, *then* (4.10) *holds.*

Indeed, by the method of § 3 we can prove that

$$\left| \frac{\partial}{\partial t} v(\cdot, t) \right|_{2m+\alpha}^{B} = o(1), \ \text{as} \ t \to \infty,$$

$$\left| \frac{\partial^2}{\partial t^2} v(\cdot, t) \right|_{m-1+\alpha}^{B} = o(1), \ \text{as} \ t \to \infty.$$

We now differentiate (3.5), (3.6) with respect to t and apply to $\partial z / \partial t$ the argument applied in § 3 to z. We get

$$\left\| \frac{\partial z(\cdot, t)}{\partial t} \right\|^B \to 0, \ \text{as} \ t \to \infty. \tag{4.15}$$

Using L_2 estimates for elliptic equations (for instance [1, Chapter IV]) we obtain, using (4.15) in (3.5), (3.6) (for each fixed t),

$$\sum_{|i| \leqslant 2m} \| D^i z(\cdot, t) \|^B \to 0 \ \text{as} \ t \to \infty. \tag{4.16}$$

Since $2m > \frac{1}{2}n$, we conclude from (4.16), upon using Sobolev's lemma, that (4.11) holds.

The above method can be used even in case $2m \leqslant \frac{1}{2}n$. We then apply it several times (estimating successive t-derivatives of v, z). Naturally we then have to make further assumptions on the rate of convergence of φ_j and f as $t \to \infty$. Note, finally, that if it is a priori known that $\sum_{|i| \leqslant 2m} |D^i u(\cdot, t)|_0^B \leqslant H_4$ (H_4 independent of t) then we may take in the above proof $a_i(x, t)$ depending also on t, provided

$$\left\| \frac{\partial}{\partial t} a_i(\cdot, t) \right\|^B \to 0, \ \text{as} \ t \to \infty,$$

Remark 3. In the proof of Theorems 6–8 we could define $v(x, t)$ in a different manner, namely, v is the solution of

$$L v = f(x) \text{ in } B_t$$

$$\frac{\partial^j v}{\partial \nu_t^j} = \varphi_j(x_t, t) \text{ on } \partial B_t.$$

In the case $f \equiv 0$, $\varphi_j(x_\infty) \equiv 0$ this gives a new result. Indeed, we obtain the conclusion of Theorem 6 under somewhat different assumptions on the rate of convergence of the coefficients. The method is the same as in § 3.

Remark 4. For second order parabolic equations it is seen from the proofs of Theorems 1, 2, 4, 5 that if both the coefficients and the nonhomogeneous terms tend to their limits faster than $\varepsilon(t)$, then the same is true of the solution. Here $\varepsilon(t)$ is any monotone function which decreases to zero as $t \to \infty$ (for instance, $\varepsilon(t) = t^\lambda$, $\lambda < 0$). This result can easily be proved also for higher order equations, by following carefully the proofs of Theorems 6–8.

5. Asymptotic expansion of solutions

We shall need the following assumptions:

(B^k) For every j, $o \leqslant j \leqslant m-1$,

$$\varphi_j(x, t) = \sum_{\lambda=0}^{k} \varphi_j^\lambda(x) t^{-\lambda} + \tilde{\varphi}_j(x, t) t^{-k}$$

and
$$\sum_{\lambda=0}^{k} |\varphi_j^\lambda|_{2m-j+\alpha}^{\partial B} \leqslant A, \quad |\tilde{\varphi}_j(\cdot, t)|_{2m-j+\alpha}^{\partial B} \leqslant A,$$

$$\left|\frac{d}{dt} \tilde{\varphi}_j(\cdot, t)\right|_{m-1-j+\alpha}^{\partial B} \to 0, \quad |\tilde{\varphi}_j(\cdot, t)|_{m-1-j+\alpha}^{\partial B} \to 0, \text{ as } t \to \infty.$$

(C^k) For every j, $0 \leqslant |j| \leqslant 2m$

$$a_j(x, t) = \sum_{\lambda=0}^{k} a_j(x) t^{-\lambda} + \tilde{a}_j(x, t) t^{-k}$$

$$f(x, t) = \sum_{\lambda=0}^{k} f^\lambda(x) t^{-\lambda} + \tilde{f}(x, t) t^{-k}$$

and
$$\sum_{\lambda=0}^{k} |a_j^\lambda|_\alpha^B \leqslant A, \quad \sum_{\lambda=0}^{k} |f^\lambda|_\alpha^B \leqslant A, \quad |a_j^0|_{|j|-m+1+\alpha}^B \leqslant A, \text{ if } |j| \geqslant m,$$

$$\|\tilde{a}_j(\cdot, t)\|^B \to 0, \quad \|\tilde{f}(\cdot, t)\|^B \to 0, \text{ as } t \to \infty.$$

We can now prove the following theorem.

THEOREM 9. *Let the assumptions* (A), (B^k), (C_1), (C_2), (C^k) *be satisfied. If* $u(x, t)$ *is a solution of* (2.1), (2.2) *then*

$$u(x, t) = \sum_{\lambda=0}^{k} u^\lambda(x)\, t^{-\lambda} + \tilde{u}(x, t)\, t^{-k}, \tag{5.1}$$

where $\|\tilde{u}(\cdot, t)\|^B \to 0$ *as* $t \to \infty$, *and the* u^λ *satisfy the equations*

$$\frac{\partial u^\lambda}{\partial t} + L_0 u^\lambda = f^\lambda(x) - (\lambda - 1) u^{\lambda-1}(x) - \sum_{\mu=1}^{\lambda} L_\mu u^{\lambda-\mu}(x), \tag{5.2}$$

$$\frac{\partial^j}{\partial \nu^j} u^\lambda(x) = \varphi_j^\lambda(x) \quad (0 \leqslant j \leqslant m-1) \text{ on } \partial B, \tag{5.3}$$

where $L_\lambda \equiv \sum_{|i| \leqslant 2m} a_i^\lambda(x) D^i$, *and if* $\lambda = 0$ *it is understood that the right-hand side of* (5.2) *is replaced by* $f^0(x)$.

The proof can be given by induction on k. The case $k = 0$ is a consequence of Theorem 6. The passage from k to $k+1$ is performed similarly to the case of second order equations in [6] and, therefore, we omit further details.

In view of Theorems 7, 8, we can state a theorem similar to Theorem 9 which is concerned with uniform convergence.

References

[1]. S. AGMON, A. DOUGLIS & L. NIRENBERG, Estimates near the boundary of solutions of elliptic partial differential equations satisfying general boundary conditions, I. *Comm. Pure Appl. Math.*, 12 (1959), 623–727.

[2]. F. E. BROWDER, Estimates and existence theorems for elliptic boundary value problems. *Proc. Nat. Acad. Sci. U. S. A.*, 45 (1959), 365–372.

[3]. A. DOUGLIS & L. NIRENBERG, Interior estimates for elliptic systems of partial differential equations. *Comm. Pure Appl. Math.*, 8 (1955), 503–538.

[4]. S. D. EIDELMAN, On the fundamental solutions of parabolic systems. *Mat. Sb.*, 38 (80) (1956), 51–92.

[5]. A. FRIEDMAN, Convergence of solutions of parabolic equations to a steady state. *J. Math. Mech.*, 8 (1959), 57–76.

[6]. ——, Asymptotic behavior of solutions of parabolic equations. *J. Math. Mech.*, 8 (1959), 372–392.

[7]. ——, Generalized heat transfer between solids and gases under non-linear boundary conditions. *J. Math. Mech.*, 8 (1959), 161–183.

[8]. ——, Remarks on the maximum principle for parabolic equations and its applications. *Pacific J. Math.*, 8 (1958), 201–211.

[9]. ——, Asymptotic behavior of solutions of parabolic equations under various boundary values. *Technical Report No.* 16, *Univ. of Calif. in Berkeley*, April, 1959.

[10]. G. GIRAUD, Généralization des problèmes sur les opérations du type elliptic. *Bull. Sci. Math.*, 56 (1932), 248–272, 281–312, 316–352.

[11]. E. Hopf, Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. *Sitz. Preuss. Akad. Wiss.* 9 (1927), 147–152.

[12]. E. J. McShane, Extension of the range of functions. *Bull. Amer. Math. Soc.*, 40 (1934), 837–842.

[13]. C. Miranda, *Equazioni alle derivate parziali di tipo ellittico.* Springer Verlag, 1955.

[14]. ——, Teorema del massimo modulo e teorema di esistenza e di unicità par il problema di Dirichlet relativ alle equazioni ellittiche in due variabili. *Ann. Mat. Pura Appl.*, 46 (ser. 4) (1958), 265–311.

[15]. W. Pogorzelski, Etude de la solution fondamentale de l'équation elliptique et des problèmes aux limites. *Ann. Polon. Math.*, 3 (1957), 247–284.

[16]. P. C. Rosenbloom & G. E. Forsythe, *Numeral Analysis and Partial Differential Equations.* Surveys in Applied Mathematics, John Wiley & Sons, 1958.

[17]. J. Schauder, Über lineare elliptische Differentialgleichungen zweiter Ordnung. *Math. Z.*, 38 (1933–34), 257–282.

[18]. ——, Der Fixpunktsatz in Funktionalraumen. *Studia Math.*, 2 (1930), 171–180.

[19]. L. N. Slobodetski, Fundamental solutions and the Cauchy problem for parabolic systems. *Math. Sb.*, 46 (88) (1958), 229–258.

[20]. I. M. Višik, The problem of Cauchy with operators as coefficients, the mixed boundary problem for systems of differential equations and an approximation method of their solution. *Mat. Sb.*, 39 (81) (1956), 51–148.

[21]. H. Westphal, Zur Abschätzung der Lösungen nichtlinearer parabolischer Differentialgleichungen. *Math. Z.*, 51 (1949), 690–695.

Received May 25, 1959, in revised form March 28, 1960