# ON THE COHOMOLOGY OF TWO-STAGE POSTNIKOV SYSTEMS

BY

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## 1. Introduction

In the recent development of algebraic topology, cohomology operations have proved to be of vital importance. Several examples of such operations (primary and higher orders) have been constructed by Adem, Massey, Pontrjagin, Steenrod, Thomas, and others. A cohomology operation (primary) relative to dimensions $q$, $q+i$ and coefficient groups $G_1$, $G_2$ is a natural transformation (see Eilenberg–MacLane [5]) between the cohomology functors $H^q(\ ,G_1)$ and $H^{q+i}(\ ,G_2)$ defined on the category of topological spaces:

$$\theta : H^q(\ ,G_1) \to H^{q+i}(\ ,G_2).$$

Primary cohomology operations are closely connected with the cohomology of Eilenberg–MacLane spaces (see Serre [11]). The secondary operations are in a similar way connected with the cohomology of spaces with two non-vanishing homotopy groups and of spaces closely connected with these. This has been shown in works by Adams [1] and Peterson–Stein [10].

In this paper we shall compute the cohomology of certain spaces $P_{n,h}$ (see below) with two non-vanishing homotopy groups. This computation is carried out by means of a spectral sequence argument.

The spectral sequence argument giving the cohomology of $K(\pi, n)$'s (see Serre [11]) relies heavily on the fact the transgression commutes with Steenrod operations. In the computation of $H^*(P_{n,h})$ this however does not suffice. We need to have some information about the differentials of $\mathrm{Sq}^1\alpha$, where $\alpha \in H^*(F)$, $F$ the fibre in a fibration $E \to B$, even if $\alpha$ is not transgressive, provided the differentials on $\alpha$ are known. Sections 3–8 in this paper are devoted to the study of this and of related problems.

Using the method developed by N. E. Steenrod we construct and study mappings $\varphi$: $W \otimes C \to C^{(n)}$ preserving certain filtrations. These mappings are used in the study of the problem mentioned above. The mapping $\varphi$ is also used to define some spectral operations (natural transformations between spectral sequence functors $E_r^{p,q}$ and $E_s^{a,b}$, $r, s \geqslant 2$, defined on the category of fibre spaces). Certain spectral operations have earlier been constructed by R. Vazquez [13] and by Araki [2]. The operations constructed in this paper are related to or coincide with the operations introduced in these papers.

Sections 9–12 contain a computation of the ring structure of $H^*(P_{n,h}, Z_2)$ where $P_{n,h} = P(Z_2, n; Z_2, 2^h n-1, \varepsilon_n^{2^h})$, $n \geqslant 2, h \geqslant 1$. Some information about the action of the Steenrod algebra $A^*$ on $H^*(P_{n,h})$ is contained in the Theorems 11.1 and 12.1. It is of some interest to get the complete action of $A^*$ on $H^*(P_{n,h})$. Apart from Theorems 11.1 and 12.1, however, we have at the present only scattered information about this action of $A^*$. Because of incompleteness, this is not included in this paper. By xtending the methods used in this paper further computations can be carried out. This will be done in a subsequent paper.

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## 2. Preparations

In this section we are going to review a few quite well known things needed in the following.

Let us consider graded filtered differential modules with decreasing filtratration and differential of order $+1$. Although the modules are graded we shall in the following often suppress the grading to simplify the notation. A mapping $f$: $A \to B$ between two such modules must satisfy $df = fd$ and $f(F^p A) \subseteq F^p B$. The d's and the F's denote the differential operators and the filtrations on $A$ and $B$. Such a map induces the homomorphisms

$$f^*: H(A) \to H(B), \quad f_r^*: E_r(A) \to E_r(B) \quad (r = 0, 1, \ldots, \infty). \tag{1}$$

with the property
$$d_r f_r^* = f_r^* d_r, \tag{2}$$

where $d_r$ is the differential in the $r$th term $E_r$ of the spectral sequence $\{E_r\}$ of the filtered differential modules. A homotopy $s$: $f \simeq g$ of degree $\leqslant k$ between two maps $f$, $g$: $A \to B$ is a module homomorphism $s$: $A \to B$ satisfying

$$s(F^p A) \subseteq F^{p-k} B \quad \text{and} \quad ds + sd = g - f \quad \text{for all } p. \tag{3}$$

**Lemma** 2.1. *If* $s\colon f \simeq g$ *is a homotopy of degree* $\leqslant k$, *then*

$$f_r^* = g_r^* \colon E_r(A) \to E_r(B) \quad \textit{for} \quad r > k.$$

*Proof.* By definition we have

$$E_r^p = Z_r^p / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}),$$

where $Z_r^p$ as usual denotes the module

$$Z_r^p = \{x \mid x \in F^p,\ dx \in F^{p+r}\}.$$

Let $e \in Z_r^p(A)$ then $$g(a) - f(a) = dsa + sda,$$

where $g(a) - f(a) \in Z_r^p(B)$, $da \in F^{p+r}(A)$, $sda \in P^{p+r-k}(B) \subseteq F^{p+1}(B)$ (since $r - k \geqslant 1$), and $sa \in F^{p-k}(B)$. Hence we have

$$dsa \in Z_{r-1}^{p-r+1}(B) \subseteq Z_{r-1}^{p+1}(B).$$

Since $d(sda) = d(g(a) - f(a)) \in F^{p+r}(B)$ we have

$$s\,da \in Z_{r-1}^{p+1}(B).$$

This means that $g(a) - f(a)$ determines zero in $E_r^p$, and hence that

$$f_r^* = g_r^* \quad \text{for} \quad r = k+1,\ k+2,\ \ldots,\ \infty,$$

which was to be proved.

In a later section the following algebraic lemma will be needed. Let $T_h$, $h = 0, 1, \ldots$, be a vector space over $Z_2$ (the integers modulo 2) generated by $1$, $\alpha_h$, $\beta_h$, and $\gamma_h$ and let $T_h$ be mapped into $Z_2[x, y]$, the polynomial algebra generated by $x$ and $y$, by a vector space mapping

$$f_h\colon T_h \to Z_2[x, y], \tag{4}$$

defined by $f_h(1) = 1$, $f_h(\alpha_h) = x^{2^h}$, $f_h(\beta_h) = (x^2 + y)^{2^h}$, and $f_h(\gamma_h) = (xy)^{2^h}$. By tensoring we get the mapping

$$F\colon \bigotimes_h T_h \xrightarrow{\otimes f_h} \bigotimes_h Z_2[x, y] \to Z_2[x, y], \tag{5}$$

where the last mapping is the multiplication mapping. Let $V_h$ be a vector space generated by $1$ and $\alpha_h$ and let $g_h\colon V_h \to Z_2[x]$ be defined by $g_h(1) = 1$ and $g_h(\alpha_h) = x^{2^h}$. As before we get a mapping

$$G\colon \bigotimes_h V_h \xrightarrow{\otimes g_h} \bigotimes_h Z_2[x] \to Z_2[x]. \tag{6}$$

LEMMA 2.2 *The mappings $F$ and $G$ are isomorphisms.*

*Proof.* Let us consider the systems

$$\{a_0 \otimes a_1 \otimes \ \dots \ \otimes a_n \mid n \geqslant 0;\ a_h = 1,\ \alpha_h,\ \beta_h,\ \text{or}\ \gamma_h\} \tag{7}$$

and

$$\{x^s y^t \mid s, t \geqslant 0\} \tag{8}$$

of (vector space) generators of $\underset{h}{\otimes} T_h$ and $Z_2[x, y]$. Let us define a grading in $\underset{h}{\otimes} T_h$ and $Z_2[x, y]$ by dim $\alpha_h = 2^h$, dim $\beta_h = 2 \cdot 2^h$, dim $\gamma_h = 3 \cdot 2^h$, dim $x = 1$, and dim $y = 2$. The mapping $F$ is then easily seen to be homogeneous. Let $\eta(1) = 0$, $\eta(\alpha_h) = 2^h$, $\eta(\beta_h) = 2^h i$, and $\eta(\gamma_h) = 2^h + 2^h i$. There is then to each generator $a = a_0 \otimes \dots \otimes a_n$ of (7) associated a gaussian integer with non-negative components

$$\eta(a_0 \otimes \ \dots \ \otimes a_n) = \eta(a_0) + \dots + \eta(a_n). \tag{9}$$

This correspondence is obviously $(1-1)$ and onto. If we write $F(a)$ as a sum of generators (8), then it is easily seen that the term with a maximal number of $y$'s in $F(a)$ is $x^s y^t$ with $s + ti = \eta(a)$. This shows us that in each dimension $\underset{h}{\otimes} T_h$ and $Z_2[x, y]$ have the same (finite) number of generators. Furthermore $x^s y^t$ is in the image under $F$ of the subspace generated by $\{a \mid I(\eta(a)) \leqslant t\}$ with $a$ as in (7), and $I(c)$ the imaginary component of the complex number $c$. This follows by a trivial induction on $t$ and shows that $F$ is onto and hence an isomorphism. That $G$ is an isomorphism is trivial. This proves the lemma.

In section 11 we shall need

LEMMA 2.3. *Let $X$ be a topological space and let $x$ be a homogeneous element of $H^*(X, Z_2)$. Then*

$$x^{2^h} = 0 \Rightarrow (\mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \ \dots \ \mathrm{Sq}^{i_r} x)^{2^h} = 0 \quad (r, h = 1, 2, \dots).$$

*Proof.* The lemma for $r > 1$ follows by a trivial induction from the case $r = 1$. Now let $r = 1$ and $h = 1$. Since by the Cartan formula

$$(\mathrm{Sq}^i x)^2 = \mathrm{Sq}^{2i}(x^2) = 0,$$

the theorem is true in this case. Also

$$(\mathrm{Sq}^i x)^{2^h} = (\mathrm{Sq}^{2i}(x^2))^{2^{h-1}}$$

and the lemma follows for $r = 1$ and $h$ arbitrary by induction with respect to $h$.

### 3. The Eilenberg–Zilber theorem

In this section we shall prove a strengthened form of the Eilenberg–Zilber theorem (see [6]). In the formulation of the Eilenberg–Zilber theorem we shall follow Dold [3].

Let $\mathcal{K}_n$ be the category of $n$-tuples $(K_0, K_1, \ldots, K_{n-1})$ of *css*-complexes. Let $C_*$ denote the functor taking any *css*-complex $K$ into its (non-normalized) chain complex $C_*(K)$ with coefficients in the integers $Z$. Let $A$ and $B$ denote the functors defined by

$$\left.\begin{aligned} A(K_0, K_1, \ldots, K_{n-1}) &= C_*(K_0 \times K_1 \times \ldots \times K_{n-1}) \\ B(K_0, K_1, \ldots, K_{n-1}) &= C_*(K_0) \otimes C_*(K_1) \otimes \ldots \otimes C_*(K_{n-1}). \end{aligned}\right\} \tag{1}$$

Both $A$ and $B$ have values in the category of chain complexes. For any *css*-complex $K$ we can in $C_*(K)$ not only define a grading and a differential operator but also a filtration. Let namely $\sigma_q$ denote a $q$-simplex in $K$. We can then in a unique way write $\sigma_q$ in the form

$$\sigma_q = s_{i_1} s_{i_2} \ldots s_{i_{q-p}} \sigma_p \quad (0 \leqslant i_{q-p} < \ldots < i_1 < q) \tag{2}$$

where $\sigma_p$ is a non-degenerate $p$-simplex in $K$, and $s_i$ denotes a degeneracy operator in $K$. The generator $\sigma_q \in C_q(K)$ is then said to be of filtration $p$,

$$\sigma_q \in F_p C_*(K) \tag{3}$$

This defines a filtration in $C_*(K)$.

Defining the filtration in a tensor product of filtered modules $D_i$ by the formula

$$F_p(D_0 \otimes \ldots \otimes D_{n-1}) = \sum_{i_0 + \ldots + i_{n-1} = p} F_{i_0}(D_0) \otimes \ldots \otimes F_{i_{n-1}}(D_{n-1}), \tag{4}$$

the equations (1) show that $A$ and $B$ are filtered chain complexes.

We define a complex $\mathrm{Hom}(A,B)$ as follows. An element $f \in \mathrm{Hom}(A,B)_r$, $r \geqslant 0$, is a natural transformation $f\colon A \to B$ increasing grading by $r$ and filtration at most by $r$ (of degree $\leqslant r$ with respect to filtration)

$$f(A_m) \subseteq B_{m+r}, \quad f(F_p A) \subseteq F_{p+r} B, \tag{5}$$

such that
$$d(f) = df + (-1)^{r+1} fd \in \mathrm{Hom}(A, B)_{r-1} \quad (\mathrm{Hom}(A,B)_{-1} = 0). \tag{6}$$

It is easily seen that $d(d(f)) = 0$ so that the requirement (6) only means, that $d(f)$ must increase filtration by at most $r-1$. Equation (6) defines a differential in $\mathrm{Hom}(A,B)$ which is hence a chain complex (functor taking $\mathcal{K}_n \times \mathcal{K}_n$ into the category of chain complexes). The chain complex $\operatorname{Hom}(A, B)$ is augmented. If $f \in \operatorname{Hom}(A, B)_0$ then the restriction

$$f \mid A_0 \colon A_0 \to B_0$$

is multiplication by an integer $k$. Putting $\varepsilon(f) = k$ and $\varepsilon(g) = 0$ for $g \in \operatorname{Hom}(A, B)_r$, $r > 0$, it is easily seen, that we get an augmentation.

THEOREM 3.1 *The complex* $\operatorname{Hom}(A, B)$ *is acyclic,*

$$H_r \operatorname{Hom}(A, B) = 0 \quad \textit{for} \quad r > 0,$$

$$\varepsilon_* \colon H_0 \operatorname{Hom}(A, B) \to Z \quad \textit{is an isomorphism.}$$

*Proof.* The proof is by the method of acyclic models, as it is developed in Eilenberg–MacLane [4] and Gugenheim–Moore [7] (cf. also Moore [8]).

Let $\sigma_q$ be as in (2), then there is a unique mapping

$$u = u(\sigma_q) \colon \Delta_p \to K, \tag{7}$$

such that $u$ takes the basic simplex $\eta_p$ of the standard simplex $\Delta_p$ (*css*-complex) into $\sigma_p$ and hence $s_{i_1} s_{i_2} \dots s_{i_{q-p}} \eta_p$ into $\sigma_q$. Similarly, if $a_0 \times \dots \times a_{n-1}$ is a $q$-simplex in $K_0 \times \dots \times K_{n-1}$, it can in a unique way be written as

$$a_0 \times \dots \times a_{n-1} = s_{j_1} s_{j_2} \dots s_{j_{q-p}} (b_0 \times \dots \times b_{n-1}) \quad (0 \leqslant j_{q-p} < \dots < j_1 < q), \tag{8}$$

with $b_0 \times \dots \times b_{n-1}$ a non-degenerate $p$-simplex in $K_0 \times \dots \times K_{n-1}$. Again there are unique mappings

$$u_i \colon \Delta_p \to K_i, \tag{9}$$

such that $u_i(\eta_p) = b_i$. We put

$$u = u(a_0 \times \dots \times a_{n-1}) = B(u_0, \dots, u_{n-1}) : B(\Delta_p, \dots, \Delta_p) \to B(K_0, \dots, K_{n-1}), \tag{10}$$

then

$$\left.\begin{aligned} u(a_0 \times \dots \times a_{n-1})(\eta_p \otimes \dots \otimes \eta_p) &= b_0 \otimes \dots \otimes b_{n-1}, \\ u(a_0 \times \dots \times a_{n-1})(S\eta_p \otimes \dots \otimes S\eta_p) &= a_0 \otimes \dots \otimes a_{n-1}, \end{aligned}\right\} \tag{11}$$

where $S = s_{j_1} \dots s_{j_{q-p}}$ in (8)

The *css*-complexes $\Delta_p$, $p = 0, 1, \dots$, are acyclic. A contracting homotopy

$$\bar{U} = \bar{U}(\Delta_p) : C_*(\Delta_p) \to C_*(\Delta_p) \tag{12}$$

is defined by

$$\bar{U}(m_0, \dots, m_r) = (0, m_0, \dots, m_r), \tag{13}$$

where $$(m_0, \ldots, m_r) \in \Delta_p \quad (0 \leqslant m_0 \leqslant m_1 \leqslant \ldots \leqslant m_r \leqslant p), \tag{14}$$

is an r-simplex in $\Delta_p$.

Let $\varepsilon$: $C_*(\Delta_p) \to H_0(C^*(\Delta_p))$ denote the augmentation mapping. Since $H_0(C_*(\Delta_p)) \cong Z$ and since the class of (0) is a generator of this group, we can define a mapping

$$\eta\colon H_0(C_*(\Delta_p)) \to C_0(\Delta_p) \tag{15}$$

by $$\eta((0)) = (0). \tag{16}$$

Since degenerate mappings $\Delta_p \to \Delta_{p-r}$ map zero into zero, it is easily seen that $\bar{U}$ and $\eta$ are natural with respect to degenerate mappings. It is easy to see that

$$d\bar{U} + \bar{U}d = 1 - \eta\varepsilon \tag{17}$$

so that $\bar{U}$ is a contracting homotopy.

A contracting homotopy

$$U\colon B(\Delta_p, \ldots, \Delta_p) \to B(\Delta_p, \ldots, \Delta_p) \tag{18}$$

is defined by $U(\sigma_{p_0}, \sigma_{p_1}, \ldots, \sigma_{p_{n-1}}) =$

$$\sum_{i=0}^{n-1} \eta\varepsilon(\sigma_{p_0}) \otimes \eta\varepsilon(\sigma_{p_1}) \otimes \ldots \otimes \eta\varepsilon(\sigma_{p_{i-1}}) \otimes \bar{U}(\sigma_{p_i}) \otimes \sigma_{p_{i+1}} \otimes \ldots \otimes \sigma_{p_{n-1}}. \tag{19}$$

Then $$dU + Ud = 1 - \eta\varepsilon, \tag{20}$$

where $\varepsilon : B \to H_0(B)$ is the augmentation and $\eta : B(\Delta_p, \ldots, \Delta_p) \to H_0 B(\Delta_p, \ldots, \Delta_p)$ is defined by $\eta((0) \otimes \ldots \otimes (0)) = (0) \otimes \ldots \otimes (0)$ (cf. (15)).

A simplex $\sigma_r = (m_0, m_1, \ldots, m_r) \in \Delta_p$ is said to contain zero if $m_0 = 0$. A simplex $\sigma_r \times \sigma_r \times \ldots \times \sigma_r \in \Delta_p \times \Delta_p \times \ldots \times \Delta_p$ is said to contain zero if $\sigma_r$ contains zero. All elements in the subgroup of $C_*(\Delta_p)$ ($C_*(\Delta_p \times \ldots \times \Delta_p)$) generated by such simplexes are said to contain zero. A generator $\sigma_{p_0} \otimes \ldots \otimes_{p_{n-1}}$ ($\dim \sigma_{p_i} = p_i$) in $C_*(\Delta_p) \otimes \ldots \otimes C_*(\Delta_p)$ is said to contain zero if and only if $p_j = 0$ for $j < i$ implies that $\sigma_{p_i}$ contains zero. As before all elements in the subgroup generated by elements of this sort are said to contain zero.

To prove Theorem 3.1 we must show two things:

(i) If $f$, $g \in \operatorname{Hom}(A, B)_r$, and if in case $f_* = g_*$ $(= T$, say$)$: $H_0(A(\Delta_p, \ldots, \Delta_p)) \to H_0(B(\Delta_p, \ldots, \Delta_p))$, $p = 0, 1, \ldots$, then $f \simeq g$ by a homotopy $h$ of degree $\leqslant r+1$.

This shows that $H_r \operatorname{Hom}(A, B) = 0$ for $r > 0$ and that $\varepsilon_*$ is monic.

(ii) For any $k \in Z$ there is a mapping $f$: $A \to B$ preserving grading and filtration such that $\varepsilon(f) = k$.

This shows that $\varepsilon_*$ is onto.

*Proof. of* (i) *and* (ii). For $a_0 \times \ldots \times a_{n-1}$ a 0-simplex in $K_0 \times \ldots \times K_{n-1}$ we define

$$h_0(a_0 \times \ldots \times a_{n-1}) = u(a_0 \times \ldots \times a_{n-1})\, U_0(f_0 - g_0)(\eta_0 \times \ldots \times \eta_0), \tag{21}$$

where the different symbols are defined in (8), (10), (12), and (19). The general definition of $h$ is by induction. If $a_0 \times \ldots \times a_{n-1}$ is a $q$-simplex in $K_0 \times \ldots \times K_{n-1}$, then

$$h_q(a_0 \times \ldots \times a_{n-1}) = u(a_0 \times \ldots \times a_{n-1})\, U_q(f_q - g_q - h_{q-1} d)\,(S\eta_p \times \ldots \times S\eta_p). \tag{22}$$

A standard computation shows that $h$ is natural and that $dh + hd = f - g$. We therefore only need to show that $h$ is of degree $\leqslant r+1$.

By (13) and (19) we see that $U$ increases the filtration by at most one, and that $U$ preserves the filtration of elements containing zero.

If $c \in C_*(\Delta_p \times \ldots \times \Delta_p)$ contains zero, then $h(c)$ contains zero. This follows from (21) and (22) by trivial induction using the special form of $U$ given in (13) and (19).

In (22) $d(S\eta_p \times \ldots \times S\eta_p)$ is a sum of simplices all except possibly one containing zero. If there is a simplex not containing zero, this one will be of filtration $p-1$. By induction (21) and (22) now show that $h$ is of degree $\leqslant r+1$.

Proposition (ii) is proved in a similar way. The function $f$ is defined by

$$f_0(a_0 \times \ldots \times a_{n-1}) = k \cdot u(a_0 \times \ldots \times a_{n-1})(\eta_0 \times \ldots \times \eta_0), \tag{23}$$

$$f_q(a_0 \times \ldots \times a_{n-1}) = u(a_0 \times \ldots \times a_{n-1})\, U f_{q-1} d\,(S\eta_p \times \ldots \times S\eta_p), \tag{24}$$

with the same notation as above. That $f$ preserves filtration follows by induction on $q$, first noting that $f$ maps elements containing zero into elements containing zero.

## 4. The Steenrod construction

This section follows the paper [3] by A. Dold, and for further details we refer to this paper.

Let $\pi$ be a permutation group on $n$ letters $(0, 1, \ldots, n-1)$, then $\pi$ operates in $\mathcal{K}_n$ by permutation of the factors. Let $T \in \pi$, then

$$T(K_0, K_1, \ldots, K_{n-1}) = (K_{T(0)}, K_{T(1)}, \ldots, K_{T(n-1)}). \tag{1}$$

Associated with $T$ there are chain mappings

$$T_* = T\colon A(\bar{K}) \to A(T\bar{K}), \quad \bar{K} = (K_0, K_1 \ldots, K_{n-1}), \tag{2}$$

$$T_* = T\colon B(\bar{K}) \to B(T\bar{K}), \tag{3}$$

defined by

$$T_*(a_0 \times a_1 \times \ldots \times a_{n-1}) = a_{T(0)} \times a_{T(1)} \times \ldots \times a_{T(n-1)}, \tag{4}$$

$$T_*(a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}) = (-1)^* \, a_{T(0)} \otimes a_{T(1)} \otimes \ldots \otimes a_{T(n-1)}, \tag{5}$$

where the sign $(-1)^*$ is given by the usual sign convention. It is clear that in both cases

$$T_* T'_* = (TT')_* \quad (T, T' \in \pi). \tag{6}$$

Because of (6) the group $\pi$ acts on $\mathrm{Hom}(A, B)$ by

$$Tf = T_* f T_*^{-1} \quad \text{for} \quad f \in \mathrm{Hom}(A, B). \tag{7}$$

By Theorem 3.1 $\mathrm{Hom}(A,B)$ is hence an acyclic $\pi$-complex. Let $V$ be an arbitrary $\pi$-free $\pi$-complex over $Z$, then by a fundamental theorem in homological algebra we can construct a $\pi$-mapping

$$\varphi'' \colon V \to \mathrm{Hom}(A, B) \tag{8}$$

preserving augmentation. Since

$$\mathrm{Hom}(V, \mathrm{Hom}(A, B)) \cong \mathrm{Hom}(V \otimes A, B), \tag{9}$$

we get

THEOREM 4.1. *There exists a natural transformation*

$$\varphi' \colon V \otimes A \to B$$

*satisfying*

$$\varphi'(v \otimes (a_0 \times a_1 \times \ldots \times a_{n-1})) = \varepsilon(v)\, a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \quad (v \in V,\ a_i \in K_i),$$

*whenever* $\dim v = \dim a_i = 0$, *and such that*

$$\begin{array}{ccc} V \otimes A(\bar{K}) & \xrightarrow{\varphi'} & B(\bar{K}) \\ \downarrow{\scriptstyle T \otimes T_*} & & \downarrow{\scriptstyle T_*} \\ V \otimes A(T\bar{K}) & \xrightarrow{\varphi'} & B(T\bar{K}) \end{array} \qquad \bar{K} = (K_0, K_1 \ldots, K_{n-1}) \tag{10}$$

*is commutative for all* $T \in \pi$. *Also*

$$\varphi'(v \otimes \eta) \in F_{i+p} B(\bar{K}), \tag{11}$$

*if* $\dim v = i$ *and* $\eta \in F_p(A(\bar{K}))$.

If $\bar{\varphi}' \colon V \otimes A \to B$ is another transformation satisfying the above conditions, then $\varphi'$ and $\bar{\varphi}'$ are homotopic by a natural homotopy $H$. The diagram obtained by replacing $\varphi'$ by $H$ in (10) is commutative and

$$H(v\otimes\eta)\in F_{p+i+1},$$

if $\dim v=i$ and $\eta\in F_p$.

Putting $K_0=K_1=\ldots=K_{n-1}=K$ then $A(K,K,\ldots,K)$ and $B(K,K,\ldots,K)$ can be considered as functors of one variable. The diagonal map

$$\Delta\colon K\to K\times K\times\ldots\times K \tag{12}$$

defined by $\Delta(x)=(x,x,\ldots,x)$ for $x\in K$ induces a natural transformation

$$\Delta\colon C_*(K)\to A(K,K,\ldots,K) \tag{13}$$

preserving filtration. The following composition is also denoted by $\varphi'$

$$V\otimes C_*(K)\xrightarrow{1\otimes\Delta} V\otimes A(\bar{K})\xrightarrow{\varphi'} B(\bar{K}),\quad \bar{K}=(K,K,\ldots,K). \tag{14}$$

By (2) and (3) $A(K,\ldots,K)$ and $B(K,\ldots,K)$ are $\pi$-complexes. The complex $C_*(K)$ is a $\pi$-complex letting $\pi$ operate trivially and so are $V\otimes C_*(K)$ and $V\otimes A(\bar{K})$ letting $\pi$ operate diagonally. The mappings in the composition (14) are easily seen to be $\pi$-homomorphisms.

Let $f\colon E\to B$ be an arbitrary *css*-mapping. Let the $n$-skeleton of $B$ be the subcomplex generated by all $n$-simplices in $B$. Then the inverse images of skeletons in $B$ define an increasing collection of subcomplexes in $B$ which in turn gives rise to an increasing filtation on $C^*(E)$. The formula (4) in section 3 then defines a filtration on $C_*(E)^{(n)}$, the $n$-fold tensor product of $C_*(E)$. Since $\varphi'$ is natural we have a commutative diagram

$$\begin{array}{ccc} V\otimes C_*(E) & \xrightarrow{\varphi'} & C_*(E)^{(n)} \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle f^{(n)}} \\ V\otimes C_*(B) & \xrightarrow{\varphi'} & C_*(B)^{(n)}. \end{array} \tag{15}$$

For $v\in V$ and $c$ a $(p+q)$-simplex in $E$ belonging to $F_pC_*(E)$, this diagram shows that

$$\varphi'(v\otimes c)\in F_{p+i}(C_*(E)^{(n)}),\ (C_*(E)^{(n)}=B(E,\ldots,E)). \tag{16}$$

Since $c\in F_pC_*(E)$, $f(c)$ can be written $f(c)=s_{i_1}s_{i_2}\ldots s_{i_q}a$, $0\leqslant i_q<i_{q-1}<\ldots<i_1<p+q$, where $a$ is a $p$-simplex in $B$.

There is a unique map $u\colon \Delta_p\to B$ such that the basic simplex $\sigma_p$ in $\Delta_p$ is mapped into $a$. Hence $u(s_{i_1}\ldots s_{i_q}\sigma_p)=f(c)$. By naturality the diagram

$$\begin{array}{ccc} V\otimes C_*(\Delta_p) & \xrightarrow{\varphi'} & C_*(\Delta_p)^{(n)} \\ \downarrow{\scriptstyle 1\otimes u_*} & & \downarrow{\scriptstyle u_*^{(n)}} \\ V\otimes C_*(B) & \xrightarrow{\varphi'} & C_*(B)^{(n)} \end{array} \tag{17}$$

is commutative. Since $F_{np}(C_*(\Delta_p)^{(n)}) = F_{np+1}(C_*(\Delta_p)^{(n)}) = \dots = C_*(\Delta_p)^{(n)}$, it follows that if in (16) $i \geqslant (n-1)p$, then

$$\varphi'(v \otimes c) \in F_{np}(C_*(E)^{(n)}). \tag{18}$$

We can therefore define a filtration in $V \otimes C_*(E)$ by

$$v \otimes c \in F_{\min(p+i,\, np)}(V \otimes C_*(E)), \quad \dim v = i, \quad c \in F_p\,(C_*(E)). \tag{19}$$

With this definition, (16) and (18) show that $\varphi'$ preserves filtration ($C_*(E)^{(n)}$ filtered as usual).

Thus far we have only considered the non-normalized chain complex of *css*-complexes. However, since the mapping $\varphi'$ preserves filtration, it follows that $\varphi'(v \otimes c)$ is degenerate in $C_*(E)^{(n)}$ whenever $c$ is. We can therefore factor out the degeneracies, and we get a mapping

$$\varphi' \colon V \otimes C_{*N} \to C_{*N}^{(n)},$$

where $C_{*N} = C_{*N}(E)$ denotes the normalized chain complex of $E$. The filtration considered above induces filtrations in $V \otimes C_{*N}$ and $C_{*N}^{(n)}$. The new mapping $\varphi'$ preserves these filtrations. In the following we shall only consider the normalized chain complex $C_{*N}$ of *css*-complexes. For convenience of notatation we shall therefore drop the $N$, so that in the following $C_* = C_{*N}$ denotes the normalized chain functor.

As suggested by (19), we shall define a filtration on the tensor product $A_* \otimes B_*$ of two filtered chain complexes $A_*$ and $B_*$ by

$$\text{Type } 1_n. \qquad F_p(A_* \otimes B_*) = A_* \otimes F_{[p/n]}(B_*) + \sum_{q+i=p} F_i(A_*) \otimes F_q\,(B_*), \tag{20}$$

where $n$ is a fixed integer $\geqslant 1$ and $[c]$ the greatest integer $\leqslant c$. This filtration is easily seen to have the property (cf. (19))

$$a \in F_i(A_*),\ b \in F_q\,(B_*) \Rightarrow a \otimes b \in F_{\min(nq,\, q+i)}(A_* \otimes B_*). \tag{21}$$

In the following we shall also consider tensor products of chain complexes with cochain complexes. By the tensor product of a graded chain complex $A_*$ and a graded cochain complex $B^*$ we mean the tensor product of the two cochain complexes $A^*$ and $B^*$, where $(A^*)^n = (A_*)_{-n}$. The grading of $A_* \otimes B^*$ is therefore defined by

$$(A_* \otimes B^*)^n = \bigoplus_i (A_*)_i \otimes (B^*)^{(n+i)}. \tag{22}$$

If the complexes are filtered (chain complexes increasingly ($F_j$, $j = 0, 1, \dots$), cochain complexes decreasingly ($F^j$, $j=0, 1, \ldots$)), then the tensor product $A_*\otimes B^*$ is usually given the filtration

$$F^k(A_*\otimes B^*) = \sum_{j-i=k} F_i(A_*)\otimes F^j(B^*). \tag{23}$$

A different filtration (type $2_n$) corresponding to the one defined in (20) is given as follows

$$\text{Type } 2_n. \quad F^p(A_*\otimes B^*) = A_*\otimes F^{np-(n-1)}(B^*) + \sum_{q-i=p} F_i(A_*)\otimes F^q(B^*). \tag{24}$$

The filtrations of type $1_n$ and $2_n$ are clearly compatible with the differential.

The filtration of type $2_n$ we shall use in the case $B^* = B_0^*\otimes B_1^*\otimes\ldots\otimes B_{n-1}^*$, where $B_j^*$ is a filtered cochain complex and the filtration of the tensor product is the usual one. The filtration of $A_*\otimes(B_0^*\otimes\ldots\otimes B_{n-1}^*)$ is easily seen to have the following property:

$$\left.\begin{array}{ll}\text{If} & a\in F_i(A_*),\ b_j\in F^{p_j}(B_j)\\ \text{then} & a\otimes b_0\otimes b_1\otimes\ldots\otimes b_{n-1}\in F^p(A_*\otimes(B_0^*\otimes\ldots\otimes B_{n-1}^*)),\end{array}\right\} \tag{25}$$

for $p\leqslant$ l.i.g. (max $((1/n)\sum_j p_j,\ \sum_j p_j - i)$), where l.i.g. ($\alpha$) denotes the least integer greater than or equal to $\alpha$.

Just as the chain complex $C_*(E)$ the cochain complex $C(E)$ (dual of $C_*(E)$) is a filtered complex. The filtration on $C(E)$ is defined by

$$u\in F^pC(E)\Leftrightarrow\langle u, c\rangle = 0 \quad \text{for all} \quad c\in F_{p-1}(C_*(E)).$$

A dual
$$\varphi\colon V\otimes C^{(n)}\to C \tag{26}$$

of the mapping $\varphi'$ is defined by the formula

$$\langle\varphi(v\otimes u_0\otimes\ldots\otimes u_{n-1}), c\rangle = (-1)^{\frac{1}{2}i(i-1)}\langle u_0\otimes\ldots\otimes u_{n-1}, \varphi'(v\otimes c)\rangle, \tag{27}$$

where $\dim v = i$ and $u_j\in C = C(E)$. The sign on the right hand side makes $d\varphi=\varphi d$ hold true.

If we give $V\otimes C^{(n)}$ the filtration of type $2_n$ (cf. (24) and (25)), then $\varphi$ is filtration preserving. If namely $c\in F_pC_*$ and $p<\max\ ((1/n)\sum_j p_j,\ \sum_j p_j - i)$, then $\min(np, p+i)<\sum_j p_j$, and the right hand side of (27) is zero, as an elementary argument shows using the fact $\varphi'(e_i\otimes c)\in F_{\min(np,p+i)}(C_*^{(n)})$.

The mapping $\varphi$ (26) is a $\pi$-homomorphism. We can therefore factor out with the action of $\pi$, and we get

$$\varphi\colon V\otimes_\pi C^{(n)}\to C. \tag{28}$$

Since the action of $\pi$ is filtration preserving, the filtration of $V\otimes C^{(n)}$ (type $2_n$) induces a filtration in $V\otimes_\pi C^{(n)}$ also named type $2_n$. With this filtration on $V\otimes_\pi C^{(n)}$ the mapping $\varphi$ is filtration preserving.

Thus far we have been working over the integers. We could, however, have chosen any ring as ground ring. In the following we shall be working over a ground field $K$.

Summarizing we have the theorems:

THEOREM 4.2. *Let $\pi$ be a permutation group on $n$ letters. Let $V$ be a $\pi$-free complex over $K$. Let $f: E\to B$ be a mapping of css-complexes and let $C_*=C_*(E)$ be filtered by inverse images of skeletons in $B$. Let $\pi$ act in $C_*^{(n)}$ by permutation, trivially in $C_*$ and diagonally in $V\otimes C_*$. There then exists a $\pi$-equivariant filtration preserving transformation*

$$\varphi': V\otimes C_*\to C_*^{(n)},$$

*natural with respect to mappings $g: E\to E_1$, $\bar{g}: B\to B_1$ with $\bar{g}f=f_1 g$ and satisfying*

$$\varphi'(v\otimes a)=\varepsilon(v)\cdot(a)^{(n)} \quad \textit{for} \quad v\in V,\ a\in C_*,$$

*whenever* $\dim v=\dim a=0$. *The filtration in $V\otimes C_*$ is of type $1_n$, and the filtration in $C_*^{(n)}$ the usual one. If $\bar{\varphi}': V\otimes C_*\to C_*^{(n)}$ is another such transformation, then $\varphi'$ and $\bar{\varphi}'$ are homotopic by a $\pi$-equivariant natural homotopy $H$ of degree $\leqslant 1$.*

THEOREM 4.3. *The duals of the transformations in Theorem 4.2 give rise to filtration preserving natural transformations*

$$\varphi: V\otimes_\pi C^{(n)}\to C.$$

*The filtration on $V\otimes_\pi C^{(n)}$ is of type $2_n$. Any two mappings $\varphi$ and $\bar{\varphi}$ are homotopic by a homotopy of degree $\leqslant 1$.*

By Lemma 2.1 we get

THEOREM 4.4. *Any mapping $\varphi$ as in Theorem 4.3 induces a mapping*

$$\varphi_*: E_t(V\otimes_\pi C^{(n)})\to E_t(C).$$

*For $t\geqslant 2$, this mapping is independent of the choice of the mapping $\varphi$.*

By the naturality of $\varphi$ we get

THEOREM 4.5. *Let*

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "f_1"'] & E \arrow[d, "f"] \\ B_1 \arrow[r, "\bar{g}"] & B \end{tikzcd}$$

*be commutative and let $g^*$ be the induced mapping $g^*: E_t(f) \to E_t(f_1)$. Then for $t \geqslant 0$ the diagram*

$$\begin{array}{ccc} E_t(V \otimes_\pi C(E)^{(n)}) & \xrightarrow{\varphi_*} & E_t(C(E)) \\ \Big\downarrow {\scriptstyle (1\otimes g^{(n)})_*} & & \Big\downarrow {\scriptstyle g^*} \\ E_t(V \otimes_\pi C(E_1)^{(n)}) & \xrightarrow{\varphi_*} & E_t(C(E_1)) \end{array}$$

*is commutative.*

## 5. Spectral operations

Let $f: E \to B$ be a mapping of *css*-complexes, and let $\bar{u} \in E_r^p(f)$ be represented by a cochain $x$

$$\bar{u} = \{x\} \in E_r^p = Z_r^p / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}). \tag{1}$$

We shall use the mapping $\varphi$ constructed in section 4 (with respect to a permutation group $\pi$, an augmented $\pi$-free $\pi$-complex $V$ and $n$) to define operations in the spectral sequence $\{E_r, d_r\}$, $r \geqslant 2$. Expressions like e.g. $e_i \otimes x \otimes \ldots \otimes x + e_{i+1} \otimes x \otimes \ldots \otimes x \otimes dx$ in $x$, $dx$, and in elements $e_i$ from $V$, belonging to $V \otimes_\pi C^{(n)}$ determine elements in $E_t^{p,q}(V \otimes_\pi C^{(n)})$ for certain $t$, $p$ and $q$. If for a certain $t$, the element in $E_t(V \otimes_\pi C^{(n)})$ determined by such an expression is independent of the choice of representative $x$ of $\bar{u}$, then this expression will determine a spectral operation, i.e. a transformation

$$\theta: E_r^{p,q}(C) \to E_t^{a,b}(C),$$

$p, q, a, b, r, t$ fixed, $r, t \geqslant 2$, natural with respect to mappings $g: E \to E_1$ and $\bar{g}: B \to B_1$ such that $f_1 g = \bar{g} f$, where $f_1: E_1 \to B_1$. The image of $\bar{u}$ under this spectral operation is the image in $E_t(C)$ of the element in $E_t(V \otimes_\pi C^{(n)})$ determined by the expression under the mapping

$$\varphi^*: E_t(V \otimes_\pi C^{(n)}) \to E_t(C). \tag{2}$$

We can express this in a slightly different way. Let $M = M_r^{p,q}$ be a filtered, graded, differential module on two generators $u$ and $v$, $du = v$, where $u$ and $v$ have dimension (grading) $p+q$ and $p+q+1$ respectively. The filtration is as follows

$$M = F^0 M = \cdots = F^p M \supseteq F^{p+1} M = \ldots = F^{p+r} M \supseteq F^{p+r+1} M = \ldots,$$

where $u \in F^p M$, $u \notin F^{p+1} M$, $v \in F^{p+r} M$, and $F^{p+r+1} M = 0$. If $\bar{u} \in E_r^{p,q}$, then $x \in Z_r^p$, which means that $x \in F^p$ and $dx \in F^{p+r}$. We can therefore define a mapping $g: M_r^{p,q} \to C = C(E)$ of filtered, graded, differential modules by setting $g(u) = x$ and $g(v) = dx$. This defines a map

$$\psi(x) = 1 \otimes g^{(n)} : V \otimes M^{(n)} \to V \otimes C^{(n)},$$

where the filtration of $V\otimes M^{(n)}$ (and of $V\otimes C^{(n)}$) is of type $2_n$. This mapping clearly preserves filtration, commutes with the differential, and is $\pi$-equivariant. It hence gives rise to a map

$$\psi=\psi(x): V\otimes_\pi M^{(n)}\to V\otimes_\pi C^{(n)}, \tag{3}$$

also denoted by $\psi$. Letting the functor $E_t$ act we get the induced map

$$\psi^*=\psi(x)^*: E_t(V\otimes_\pi M^{(n)})\to E_t(V\otimes_\pi C^{(n)}). \tag{4}$$

An expression in $x$, $dx$, etc. as considered above now corresponds to an element $\vartheta$ in $E_t^{a,b}(V\otimes_\pi M^{(n)})$. If for all $\bar{u}\in E_r^{p,q}$ the image of $\vartheta$ under the mapping $\psi(x)^*$ (see (4)) does not depend upon the choice of representative $x$ of $\bar{u}$, then we define an operation $\theta=\theta(\vartheta)$

$$\theta: E_r^{p,q}\to E_t^{a,b}, \tag{5}$$

by the formula
$$\theta(\bar{u})=\varphi^*\psi^*(\vartheta). \tag{6}$$

We shall now prove a few general theorems about operations of that sort. We remark, that for any operation $\theta=\theta(\vartheta)$ (see (5)) considered in this section we assume $\psi^*(\vartheta)$ to be independent of choice of representative $x$ of $\bar{u}$. In later sections we shall consider specific operations; we will then have to prove that $\psi^*(\vartheta)$ does not depend upon choice of representative $x$ of $\bar{u}$. The image of $\vartheta$ under $\psi^*$ will of course usually depend upon $\bar{u}$.

Let two permutation groups $\pi$ and $\pi_1$ ($n$ letters) be given, and let $\sigma:\pi\to\pi_1$ be a permutation group mapping. Let $V$ and $V_1$ be $\pi$- and $\pi_1$-free resolutions of $K$. There then exists a $\sigma$-equivariant map $h: V\to V_1$, which gives us the filtration preserving map

$$h\otimes 1^{(n)}: V\otimes_\pi M^{(n)}\to V_1\otimes_{\pi_1} M^{(n)}.$$

In the spectral sequence this induces the mapping

$$E_t^{a,b}(V\otimes_\pi M^{(n)})\to E_t^{a,b}(V_1\otimes_{\pi_1} M^{(n)}). \tag{7}$$

Let $\vartheta$ be an element from the left of (7) determining a spectral operation $\theta$, and let $\vartheta_1$ be the image of $\vartheta$ on the right. The element $\vartheta_1$ will then clearly determine an operation $\theta_1$.

THEOREM 5.1. *For any* $\bar{u}\in E_r^{p,q}(f)$ *we have*

$$\theta(\bar{u})=\theta_1(\bar{u})\in E_t^{a,b}(f).$$

*Proof.* Let $\varphi_1$ be a mapping as constructed in section 4 (cf. Theorem 4.2). The composition $\varphi_1(h\otimes 1)$ will then have the properties stated in Theorem 4.2 with respect to $\pi$ and $V$. We now get the commutative diagram

$$\begin{array}{ccccc} V\otimes_\pi M^{(n)} & \xrightarrow{\varphi} & V\otimes_\pi C^{(n)} & \xrightarrow{\varphi} & C \\ \downarrow{\scriptstyle h\otimes 1^{(n)}} & & \downarrow{\scriptstyle h\otimes 1^{(n)}} & & \| \\ V_1\otimes_{\pi_1} M^{(n)} & \xrightarrow{\varphi_1} & V_1\otimes_{\pi_1} C^{(n)} & \xrightarrow{\varphi_1} & C. \end{array}$$

By applying $E_t$ to this diagram we obtain the result (cf. Theorem 4.3).

Let us remark, that if we have two equivariant maps

$$h,\ h_1 : V \to V_1,$$

then there exists an equivariant homotopy $s: h \simeq h_1$. Such a homotopy gives a homotopy $s\otimes 1^{(n)} : V\otimes_\pi M^{(n)} \to V_1\otimes_{\pi_1} M^{(n)}$ of degree $\leqslant 1$. Using this remark the above theorem applied to $\sigma=$ identity immediately yields.

THEOREM 5.2. *The spectral operations associated with a permutation group $\pi$ are independent of the choice of the free resolution $V$ used in their construction.*

The following theorem is immediate.

THEOREM 5.3. *Let $\vartheta \in E_t^{a,b}(V\otimes_\pi M^{(n)})$ determine a spectral operation $\theta$, and let $d_t\vartheta=0$. The class $\bar{\vartheta}=\{\vartheta\}\in E_{t+1}^{a,b}$ will then determine a spectral operation $\bar{\theta}$. For any $\bar{u}\in E_r(C)$ we have $d_t\theta(\bar{u})=0$ and*

$$\{\theta(\bar{u})\} = \bar{\theta}(\bar{u}) \in E_t^{a,b}(C).$$

In the definition of the concept "spectral operation" we required naturality. We shall see, that the operations constructed here are natural (and hence spectral operations).

THEOREM 5.4. *Let*

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ B_1 & \xrightarrow{g_1} & B \end{array}$$

*be commutative. Let $g^*$ be the induced map $g^*: E_t(f)\to E_t(f_1)$. Let $\vartheta\in E_t^{a,b}(V\otimes_\pi M^{(n)})$ determine an operation (see (6)). Then for any $\bar{u}\in E_r^{p,q}(f)$ we have*

$$g^*\theta(\bar{u}) = \theta(g^*(\bar{u})) \in E_t^{a,b}(f_1).$$

*The operation $\theta$ is hence a spectral operation.*

*Proof.* Follows immediately from Theorem 4.5.

THEOREM 5.5. *Let $\{x\} = \bar{u} \in E_r^{p,q}(f)$ and let $\eta \in V \otimes_\pi M^{(n)}$. Let the total dimension $p+q+\alpha+\beta$ of $\eta$ be less than the total dimension of $\bar{u}$, then $\varphi\psi(\eta) = 0$, where $\psi = \psi(x)$.*

*Proof.* Let $\eta = \sum v_i \otimes s_1 \otimes \dots \otimes s_n$. Then since $\dim(s_1 \otimes \dots \otimes s_n) = p+q+\alpha+\beta+i$, $\dim(u \otimes \dots \otimes u) = n(p+q)$, $\dim(s_1 \otimes \dots \otimes s_n) \geqslant \dim(u \otimes \dots \otimes u)$, and $\alpha+\beta < 0$ we get

$$p+q+\alpha+\beta+i \geqslant n(p+q) > n(p+q+\alpha+\beta). \tag{8}$$

Let $\sum v_i \otimes \bar{s}_1 \otimes \dots \otimes \bar{s}_n$ denote the image of $\eta$ under $\psi$. We then get (see (27) in section 4)

$$\langle \varphi\psi(\eta), c \rangle = \sum \pm \langle \bar{s}_1 \otimes \dots \otimes \bar{s}_n, \varphi'(v_i \otimes c) \rangle, \tag{9}$$

where $c$ is a $(p+q+\alpha+\beta)$-simplex. Let $c$ be the image of the basic simplex $\eta_{p+q+\alpha+\beta}$ in $\Delta_{p+q+\alpha+\beta}$ under a mapping $g: \Delta_{p+q+\alpha+\beta} \to E$. Since $\varphi'(v_i \otimes c)$ is in the image of $(C(\Delta_{p+q+\alpha+\beta}))^{(n)}$ under $g^{(n)}$, whose top dimension $n(p+q+\alpha+\beta)$ by (8) is less than the dimension $p+q+\alpha+\beta+i$ of $v_i \otimes c$, we get $\varphi'(v_i \otimes c) = 0$. Equation (9) now shows that $\varphi\psi(\eta) = 0$, which proves the theorem.

COROLLARY 5.6. *Let $\vartheta \in E_t^{a,b}(V \otimes_\pi M^{(n)})$ determine a spectral operation*

$$\theta : E_r^{p,q} \to E_t^{a,b}.$$

*If $a+b < p+q$, then $\theta(\bar{u}) = 0$ for any $\bar{u} \in E_r^{p,q}(f)$.*

## 6. Cyclic reduced powers

In the following sections we shall look at the operations obtained from cyclic groups. These operations we shall for an obvious reason call cyclic reduced powers.

Let $n$ denote a prime number. Let $\pi$ be a cyclic group with $n$ elements operating on $n$ letters $(0, 1, \dots, n-1)$ by cyclic permutation. Let $T$ be the generator defined by the equation

$$T(i) = i+1 \pmod{n}.$$

Let there be given, as before, an element $\bar{u} \in E_r^{p,q}(C)$. The cyclic reduced powers of $\bar{u}$ are then determined by the map

$$\varphi: E_t(V \otimes_\pi C^{(n)}) \to E_t(C).$$

By Theorem 5.2 the reduced powers are independent of the choice of the resolution $V$. Let us therefore choose $V = W$, where $W$ is the standard resolution

$$K \xleftarrow{\varepsilon} K\pi \xleftarrow{\Delta} K\pi \xleftarrow{\Sigma} K\pi \xleftarrow{\Delta} \dots, \tag{1}$$

where $K\pi$ denotes the group ring of $\pi$ over $K$. The maps in this resolution are multiplications by the elements

$$\Delta = T - 1 \text{ and } \sum = 1 + T + T^2 + \dots + T^{n-1}. \tag{2}$$

Let $e_k$ denote the generator of $W$ in dimension $k$, then

$$\partial e_{2k} = \sum e_{2k-1},\ \partial e_{2k+1} = \Delta e_{2k}. \tag{3}$$

## 7. The mod 2 case

In this section let $K$ denote the field of residue classes of integers modulo 2, $K = \{0, 1\}$, and let $n = 2$.

For any cochain $x$ in $F^pC^{p+q} = F^pC \cap C^{p+q}$ we shall consider an expression of the form

$$\alpha^i(x) = e_{p+q-i} \otimes x^2 + e_{p+q-i+1} \otimes x\, dx \in W \otimes_\pi C^{(2)}, \quad \text{all } i, \tag{1}$$

with the agreement that $e_j = 0$ for $j < 0$.

By the definition of the filtration of type $2_2$ (section 4) we see, that the term $e_{p+q-1} \otimes x^2$ has filtration as follows

$$e_{p+q-i} \otimes x^2 \in F^m(W \otimes_\pi C^{(2)}), \quad \text{with} \quad m = \max(p, p+i-q). \tag{2}$$

Since $e_j \otimes x^2 = Te_j \otimes x^2 \in W \otimes_\pi C^{(2)}$ we get, by taking the differential of (1),

$$d(e_{p+q-i} \otimes x^2 = e_{p+q-i+1} \otimes x\, dx) = e_{p+q-i+1} \otimes (dx)^2.$$

Hence
$$d(\alpha^i(x)) = \alpha^i(dx). \tag{3}$$

Let us now choose a function $\varphi: W \otimes_\pi C^{(2)} \to C$ as constructed in section 4 and keep it fixed in the following. The image under $\varphi$ of the cochain $\alpha^i(x)$ we shall denote by $\mathrm{sq}^i x$,

$$\varphi(\alpha^i(x)) = \varphi(e_{p+q-i} \otimes x^2 + e_{p+q-i+1} \otimes x\, dx) = \mathrm{sq}^i x \in C. \tag{4}$$

By Theorem 5.5 we have $\mathrm{sq}^i x = 0$ for $i < 0$. It is clear that $\mathrm{sq}^i x = 0$ for $i \geqslant p + q + 2$. If $dx \in F^{p+1}$, then by (2) we have

$$\mathrm{sq}^i x \in F^m(C), \tag{5}$$

with $m = \max(p, p+i-q)$. Since $ddx = 0$ equation (3) gives

$$d\, \mathrm{sq}^i x = \mathrm{sq}^i dx. \tag{6}$$

Now let $\{x\}=\bar{u}\in E_r^{p,q}(C)$, $r\geqslant 2$. As mentioned in section 5, to decide if the expression (1) determines a spectral operation we must examine if for some $t$ expression (1) determines a class in $E_t(W\otimes_\pi C^{(2)})$, independent of the choice of representative $x$ of $\bar{u}$. To this end let $x$ and $y$ both represent $\bar{u}$. Then

$$x-y=da+b, \tag{7}$$

with $a\in F^{p-r+1}$ and $b\in F^{p+1}$ (cf. (1) section 5). By (1), (3), and (7) we now get

$$\begin{aligned}
&d(\alpha^i(a)+e_{p+q-i+1}\otimes xy)\\
&\quad=\alpha^i(da)+e_{p+q-i}\otimes(xy+yx)+e_{p+q-i+1}\otimes(x\,dy+dx\cdot y)\\
&\quad=e_{p+q-i}\otimes((x-y-b)^2+xy+yx)+e_{p+q-i+1}\otimes(x\,dy+dx\cdot y)\\
&\quad=e_{p+q-i}\otimes(x^2+y^2+(x-y)\,b+b(x-y)+b^2)+e_{p+q-i+1}\otimes(x\,dy+dx\cdot y)\\
&\quad=\alpha^i(x)+\alpha^i(y)+e_{p+q-i}\otimes((x-y)\,b+b(x-y)+b^2)\\
&\qquad\quad+e_{p+q-i+1}\otimes(x\,dy+dx\cdot y+x\,dx+y\,dy).
\end{aligned}\tag{8}$$

By (2) and (3) we see (cf. also Fig. 1)

$$\left.\begin{aligned}
&\alpha^i(x)\in F^p,\ d\alpha^i(x)\in F^{\max(p+r,\,p+i-q+2r-1)} &&\text{for}\quad i\leqslant q,\\
&\alpha^i(x)\in F^{p+i-q},\ d\alpha^i(x)\in F^{p+i-q+2r-1} &&\text{for}\quad i\geqslant q,
\end{aligned}\right\}\tag{9}$$

and also
$$\alpha^i(a)+e_{p+q-i+1}\otimes xy\in F^{\max(p-r+1,\,p+i-q-2r+3)}. \tag{10}$$

Since in the last line of (8) the terms $\alpha^i(x)$ and $\alpha^i(y)$ have smaller filtration than all other terms occurring, it follows from (8), (9), and (10), that

$$\left.\begin{aligned}
&\alpha^i(x)\in Z_r^p, &&\text{for}\quad 0\leqslant i\leqslant q-r+1,\\
&\alpha^i(x)\in Z_{2r-1+i-q}^p, &&\text{for}\quad q-r+1\leqslant i\leqslant q,\\
&\alpha^i(x)\in Z_{2r-1}^{p+i-q}, &&\text{for}\quad q\leqslant i\leqslant p+q,
\end{aligned}\right]\tag{11}$$

and also
$$\left.\begin{aligned}
&\{\alpha^i(x)\}\in E_r^{p,q+i}, &&\text{for}\quad 0\leqslant i\leqslant q,\\
&\{\alpha^i(x)\}\in E_{r+\min(i-q,\,r-2)}^{p+i-q,\,2q}, &&\text{for}\quad q\leqslant i\leqslant p+q,
\end{aligned}\right\}\tag{12}$$

are independent of the choice of representative $x\in\bar{u}\in E_r^{p,q}$. For $i=p+q$, $\{\alpha^i(x)\}$ is well determined in $E_r$ since $\alpha^{p+q}(a)=e_0\otimes ada$. This, however, shall not concern us.

We are now in a position where we can define the reduced powers in spectral sequences. By (4) and by (12), namely, we known when $\mathrm{sq}^i x$ is independent of the representative $x$ of $\bar{u}$. We therefore make the definition for $\{x\}=\bar{u}\in E_r^{p,q}$

$$\left.\begin{aligned}
&\mathrm{Sq}^i\,\bar{u}=\{\mathrm{sq}^i x\}\in E_r^{p,q+i}, &&\text{for}\quad 0\leqslant i\leqslant q,\\
&\mathrm{Sq}^i\,\bar{u}=\{\mathrm{sq}^i x\}\in E_{r+\min(i-q,\,r-2)}^{p+i-q,\,2q}, &&\text{for}\quad q\leqslant i\leqslant p+q.
\end{aligned}\right\}\tag{13}$$

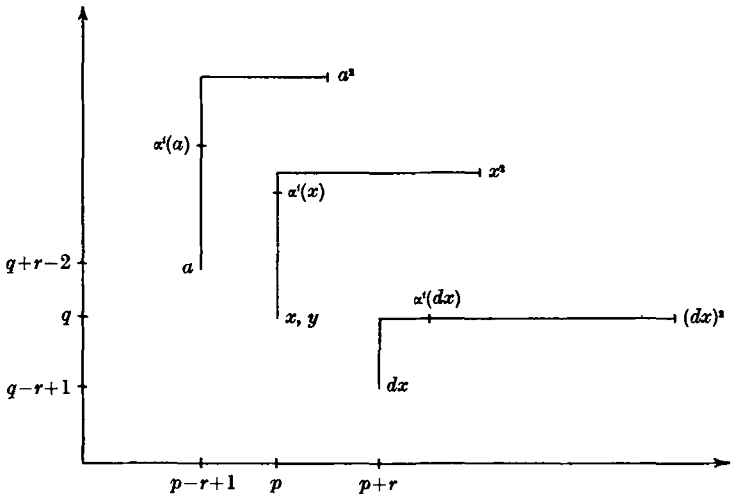


Fig. 1.

By (11) we see, that $\mathrm{sq}^i x$ for some i determine elements in later stages of the spectral sequence than the ones given in (13). It will be convenient in the following to denote these classes by $\mathrm{Sq}^i \bar{u}$ also. Explicitly we have:

for $0 \leqslant i \leqslant q-r+i$

$$\mathrm{Sq}^i \bar{u} = \{\mathrm{sq}^i x\} \in E_r^{p,q+i}, \tag{14}$$

for $q-r+1 \leqslant i \leqslant q$

$$\mathrm{Sq}^i \bar{u} = \{\mathrm{sq}^i x\} \in E_{r+j}^{p,q+i} \quad \text{for any } j, \quad 0 \leqslant j \leqslant i-q+r-1, \tag{15}$$

for $q \leqslant i \leqslant p+q$

$$\mathrm{Sq}^i \bar{u} = \{\mathrm{sq}^i x\} \in E_{r+j}^{p+i-q,2q} \quad \text{for any } j, \quad \min(i-q,\, r-2) \leqslant j \leqslant r-1. \tag{16}$$

If we disregard the stage of the spectral sequence, the formulas show that $\mathrm{Sq}^i$ increases the total dimension by $i$, and that furthermore $\mathrm{Sq}^i \bar{u}$ belongs to a group situated on the angle line from $(p, q)$ vertical to $(p, 2q)$ thence horizontally to $(2p, 2q)$ as displayed in Fig. 2. Formula (6) shows, that spectral operations defined here commute with the differentials in the spectral sequence (see Fig. 2).

THEOREM 7.1. *Let* $\bar{u} \in E_r^{p,q}(C)$ *and let* $d_r \bar{u} = \bar{v} \in E_r^{p+r,q-r+1}(C)$. *Then*

$$d_t \mathrm{Sq}^i \bar{u} = \mathrm{Sq}^i d_r \bar{u} = \mathrm{Sq}^i \bar{v},$$

*where* $t = \min(\max(r,\, i-q+2r-1),\, 2r-1)$. (See Fig. 2.)

Thus $t$ has the property that $d_t$ goes from the angle line beginning in $\bar{u}$ to the one beginning in $\bar{v}$.

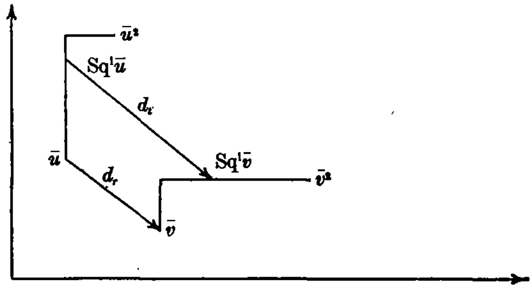


Fig. 2.

We remark that we are only interested in $E_r^{p,q}$ for $p \geqslant 0$, $q \geqslant 0$, and $r \geqslant 2$. In Theorem 7.1, therefore, we assume without mentioning it that the fibre degree $q-r+1$ of $\bar{v}$ is greater than or equal to zero. This same assumption we have made in e.g. (14).

For $i<0$ it follows from Corollary 5.6, that $\mathrm{Sq}^i\bar{u}=0$. For $i>p+q$ let us define $\mathrm{Sq}^i\bar{u}=0 \in E_{2r-2}^{p+i-q,2q}$. With this extension the $\mathrm{Sq}^i$'s still commute with the differentials. This is clear except for the case

$$d_{2r-1}\{\mathrm{Sq}^{p+q+1}\bar{u}\}(=d_{2r-1}\{0\}=0)=\{\mathrm{Sq}^{p+q+1}\bar{v}\}=\{\bar{v}^2\}.$$

Since, however, $d_r(\bar{u}\cdot\bar{v})=\bar{v}^2$, we have $\{\bar{v}^2\}=0 \in E_{2r-1}$, which is what we wanted to show.

The $\mathrm{Sq}^i$'s are additive. To show this we make the following computation. Let $x$ and $y$ belong to $Z_r^{p,q}$, then

$$\begin{aligned}\mathrm{sq}^i(x+y) &= \varphi(e_{p+q-i}\otimes(x+y)^2+e_{p+q-i+1}\otimes(x+y)(dx+dy))\\ &= \varphi(e_{p+q-i}\otimes(x^2+y^2+xy+yx)+e_{p+q-i+1}\otimes(x\,dx+y\,dy+x\,dy+y\,dx))\\ &= \mathrm{sq}^i x+\mathrm{sq}^i y+d\varphi(e_{p+q-i+1}\otimes xy+e_{p+q-i+2}\otimes dx\cdot y)\\ &\quad +\varphi(e_{p+q-i+2}\otimes dx\,dy).\end{aligned} \tag{17}$$

Now let $\bar{u}_1, \bar{u}_2 \in E_r^{p,q}$ be represented by $x$ and $y$ respectively. Since in (17) the last two terms in the last line determine zero in the group to which $\mathrm{Sq}^i(\bar{u}_1+\bar{u}_2)$ belongs (see (13)), we have proved

THEOREM 7.2. *For* $\bar{u}_1, \bar{u}_2 \in E_r^{p,q}$

$$\mathrm{Sq}^i(\bar{u}_1+\bar{u}_2)=\mathrm{Sq}^i(\bar{u}_1)+\mathrm{Sq}^i(\bar{u}_2).$$

Let
$$\mu_*: C_* \to C_* \otimes C_*, \quad \mu: C\otimes C \to C$$

be dual filtration preserving diagonal approximations (filtration on $C_*\otimes C_*$ and $C\otimes C$ the usual ones). The product operation in $E_t(C)$ is then induced by $\mu$,

$$\mu: E_r^{p_1, q_1}(C)\otimes E_r^{p_2, q_2}(C)\to E_r^{p_1+p_2, q_1+q_2}(C).$$

We remark that $E_t(C\otimes C)=E_t(C)\otimes E_t(C)$ (usual filtration on $C\otimes C$). In the following we shall choose $\mu_*$ and $\mu$ to be defined respectively by $\mu_*(c)=\varphi'(e_0\otimes c)$ and $\mu(x\otimes y)=\varphi(e_0\otimes xy)$. Furthermore we shall write $\mu(x\otimes y)=xy$. By the proof of Theorem 4.1 we can obviously choose the mapping $\varphi$ so that the multiplication becomes associative. We shall assume in the following, that the mapping $\varphi$ we are working with has this property. We remark that the usual (explicit) Eilenberg–Zilber mapping gives rise to an associative multiplication.

To derive the Cartan formula in the case of spectral operations let us consider the diagram

$$\begin{tikzcd}
W\otimes_\pi (C\otimes C)^{(2)} \arrow[r, "1\otimes\mu\otimes\mu"] \arrow[d, "\Delta\otimes(1\otimes 1)^{(2)}"] & W\otimes_\pi C^{(2)} \arrow[r, "\varphi"] & C \arrow[d, "\mu"] \\
(W\otimes W)\otimes_\pi(C\otimes C)^{(2)} \arrow[r, "\sigma"] & (W\otimes_\pi C^{(2)})\otimes(W\otimes_\pi C^{(2)}) \arrow[r, "\varphi\otimes\varphi"] & C\otimes C.
\end{tikzcd} \tag{18}$$

The mapping $\Delta$ is the equivariant diagonal map $\Delta: W\to W\otimes W$ given by $\Delta e_k=\sum_{i+j=k} e_i\otimes T^i e_j$ ($\pi$ operates diagonally in $W\otimes W$); $\sigma$ is a permutation of the factors, $\sigma: (W\otimes W)\otimes_\pi (C_1\otimes C_2)^{(2)}\to (W\otimes_\pi C_1^{(2)})\otimes(W\otimes_\pi C_2^{(2)})$.

The diagram (18) is commutative up to a homotopy $H$ of degree $\leqslant 1$. This is seen by considering the diagram

$$\begin{tikzcd}
W\otimes C_* \arrow[r, "\varphi'"] \arrow[d, "\Delta\otimes\mu_*"] & C_*^{(2)} \arrow[r, "\mu_*\otimes\mu_*"] & C_*^{(4)} \arrow[d, "P"] \\
W\otimes W\otimes C_*\otimes C_* \arrow[r] & (W\otimes C)_*\otimes(W\otimes C^*) \arrow[r, "\varphi'\otimes\varphi'"] & C_*^{(2)}\otimes C_*^{(2)},
\end{tikzcd} \tag{19}$$

which is dual to (18). The mapping $P$ is the permutation of the factors given by $(1, 2, 3, 4)\to(1, 3, 2, 4)$. The element $T\in\pi$ operates in $C^{(4)}$ by the permutation $(1, 2, 3, 4)\to(3, 4, 1, 2)$ of the factors.

Theorem 4.1 shows, that the two compositions in (19) mapping $W\otimes C_*$ into $C_*^{(4)}$ are homotopic by a homotopy of degree $\leqslant 1$. The diagram (18) is therefore commutative up to a homotopy $H$ of degree $\leqslant 1$.

Let $x\in F^pC^{p+q}$, $y\in F^sC^{s+t}$, take $k\geqslant 0$, and let $\eta=e_{p+q+s+t-k}\otimes(x\otimes y)^2+e_{p+q+s+t-k+1}\otimes(x\otimes y)\otimes d(x\otimes y)$. Then applying the maps in (18) to $\eta$ we get

$$\varphi(1\otimes\mu\otimes\mu)(\eta)=\mathrm{sq}^k(xy),$$

$$\begin{aligned}\mu(\varphi\otimes\varphi)\,\sigma(\Delta\otimes(1\otimes 1)^{(2)})(\eta) &= \mu(\varphi\otimes\varphi)\,\sigma\Big(\sum_{i+j=k} e_{p+q-i}\otimes T^{p+q-i}e_{s+t-j}\otimes(x\otimes y)^2 \\ &\quad+\sum_{i+j=k} e_{p+q-i+1}\otimes T^{p+q-i+1}e_{s+t-j}\otimes xy\,dx\cdot y \\ &\quad+\sum_{i+j=k} e_{p+q-i}\otimes T^{p+q-i}e_{s+t-j+1}\otimes xyx\,dy\Big) \\ &= \mu(\varphi\otimes\varphi)\Big(\sum_{i+j=k}\alpha^i(x)\otimes\alpha^j(y)+d\Big(\sum_{\substack{i+j=k\\ p+q-i\text{ even}}} e_{p+q-i+1}x^2\otimes e_{s+t-j+1}\,y\,dy\Big) \\ &\quad+\sum_{\substack{i+j=k\\ p+q-i\text{ even}}} e_{p+q-i+1}x^2\otimes e_{s+t-j+1}(dy)^2 \\ &\quad+\sum_{i+j=k} e_{p+q-i+1}T^{p+q-i+1}x\,dx\otimes e_{s+t-j+1}\,y\,dy\Big) \\ &= \sum_{i+j=k}\mathrm{sq}^i x\cdot\mathrm{sq}^j y+db+c,\end{aligned} \tag{20}$$

where

$$\begin{aligned} b &= \sum_{\substack{i+j=k\\ p+q-i\text{ even}}}\varphi(e_{p+q-i+1}x^2)\cdot\varphi(e_{s+t-j+1}\,y\,dy), \\ c &= \sum_{\substack{i+j=k\\ p+q-i\text{ even}}}\varphi(e_{p+q-i+1}x^2)\cdot\varphi(e_{s+t-j+1}(dy)^2) \\ &\quad+\sum_{i+j=k}\varphi(e_{p+q-i+1}T^{p+q-i+1}x\,dx)\cdot\varphi(e_{s+t-j+1}y\,dy).\end{aligned} \tag{21}$$

We remark that $b$ and $c$ are zero if $y$ is a cocycle. Since (18) is commutative up to a homotopy $H$, we get

$$\mathrm{sq}^k(xy)=\sum_{i+j=k}\mathrm{sq}^i x\cdot\mathrm{sq}^j y+db+c+dH(\eta)+H(d\eta), \tag{22}$$

with $$\eta=e_{p+q+s+t-k}\otimes(x\otimes y)^2+e_{p+q+s+t-k+1}\otimes(x\otimes y)\otimes d(x\otimes y).$$

This equation implies (proof below) the Cartan formulas.

THEOREM 7.3. *For any two classes $\bar u_1\in E_r^{p,q}$ and $\bar u_2\in E_r^{s,t}$ the following formulas hold true,*

$$\mathrm{Sq}^k(\bar u_1\bar u_2)=\sum_{\substack{i+j=k\\ 0\le i\le q,\ 0\le j\le t}}\mathrm{Sq}^i\bar u_1\cdot\mathrm{Sq}^j\bar u_2\in E_r^{p+s,q+t+k}\quad(0\le k\le q+t)$$

$$\mathrm{Sq}^k(\bar u_1\bar u_2)=\sum_{\substack{i+j=k\\ q\le i\le p+q,\ t\le j\le s+t}}\mathrm{Sq}^i\bar u_1\cdot\mathrm{Sq}^j\bar u_2\in E_{r+\min(k-q-t,\,r-2)}^{p+s+k-q-t,\,2(q+t)}\quad(q+t\le k\le p+q+s+t).$$

*Proof.* We first remark that in the second equation we are considering $\mathrm{Sq}^i\bar u_1$ and $\mathrm{Sq}^j\bar u_2$ as belonging to $E_{r+\min(k-q-t,\,r-2)}$. In (16) we saw that this is legitimate. The proof of the Cartan formulas follows from (22). The only terms on the right hand side of (22) that will make any contribution are the ones of least filtration. Since $H$ is of degree $\leqslant 1$, these are in the two cases $\sum_{i+j=k} \mathrm{sq}^i x\, \mathrm{sq}^j y$ with respectively $0 \leqslant i \leqslant q$, $0 \leqslant j \leqslant t$ and $q \leqslant i \leqslant p+q$, $t \leqslant j \leqslant s+t$. This proves the theorem.

If $A$ is an algebra over a field of characteristic $n$, then the algebra homomorphism $a \to a^n$ is denoted by $\zeta$. The iterations of $\zeta$ are denoted $\zeta^s$ ($\zeta^s = \zeta \cdot \zeta^{s-1}$, $\zeta^1 = \zeta$).

THEOREM 7.4. *Let $\bar{u} \in E_2^{p,q}$ and let $\bar{u} = \sum_\alpha a_\alpha \cdot b_\alpha$, where $a_\alpha \in E_2^{p,0}$ and $b_\alpha \in E_2^{0,q}$. Then*

$$\mathrm{Sq}^i \bar{u} = \sum_\alpha a_\alpha \cdot \mathrm{Sq}^i b_\alpha \in E_2^{p,q+i} \quad \textit{for} \quad 0 \leqslant i \leqslant q,$$

$$\mathrm{Sq}^i \bar{u} = \sum_\alpha \mathrm{Sq}^{i-q} a_\alpha \cdot b_\alpha^2 \in E_2^{p+i-q,2q} \quad \textit{for} \quad q \leqslant i \leqslant p+q.$$

*This means, that if $E_2^{*,0} \otimes E_2^{0,*} \to E_2^{*,*}$ is an isomorphism, then*

$$\mathrm{Sq}^i = 1 \otimes \mathrm{Sq}^i : E_2^{p,q} \to E_2^{p,q+i} \quad \textit{for} \quad 0 \leqslant i \leqslant q,$$

$$\mathrm{Sq}^i = \mathrm{Sq}^{i-q} \otimes \zeta : E_2^{p,q} \to E_2^{p+i-q,2q} \quad \textit{for} \quad q \leqslant i \leqslant p+q.$$

*Proof*. Since the $\mathrm{Sq}^i$'s are additive, we only need to show that for $a$ in the base and $b$ in the fibre $\mathrm{Sq}^i(a \cdot b) = a \cdot \mathrm{Sq}^i b$ for $i \leqslant q$ and $\mathrm{Sq}^i(a \cdot b) = \mathrm{Sq}^{i-q} a \cdot b^2$ for $q \leqslant i$. This, however, is a trivial consequence of the Cartan formula.

THEOREM 7.5. *Let $\bar{u} \in E_r^{p,q}$, let $d_r \bar{u} = 0$, and let $\bar{u}$ determine the class $\{\bar{u}\} \in E_{r+1}^{p,q}$. Then*

$$\{\mathrm{Sq}^i \bar{u}\} = \mathrm{Sq}^i \{\bar{u}\} \in \begin{cases} E_{r+1}^{p,q+i} & \textit{for} \quad 0 \leqslant i \leqslant q \\ E_{r+1+\min(i-q,r-1)}^{p+i-q,2q} & \textit{for} \quad q \leqslant i \leqslant p+q. \end{cases}$$

*Proof*. Since $d_r \bar{u} = 0$ it follows that there is a cochain $x \in Z_{r+1}^{p,q}$ representing $\bar{u}$; $x$ will then clearly also represent $\{\bar{u}\}$. By the definition (13) of $\mathrm{Sq}^i$ we see, that $\mathrm{sq}^i x$ represents both $\mathrm{Sq}^i \bar{u}$ and $\mathrm{Sq}^i \{\bar{u}\}$. This implies the theorem.

By the Theorems 7.4 and 7.5 we immediately get

THEOREM 7.6. *Let $\bar{u} = \{\sum_\alpha a_\alpha \cdot b_\alpha\} \in E_r^{p,q}$, where $\sum_\alpha a_\alpha \cdot b_\alpha \in E_2^{p,q}$ with $a_\alpha \in E_2^{p,0}$ and $b_\alpha \in E_2^{0,q}$. Then*

$$\mathrm{Sq}^i \bar{u} = \{\sum_\alpha a_\alpha \cdot \mathrm{Sq}^i b_\alpha\} \in E_r^{p,q+i} \quad \textit{for} \quad 0 \leqslant i \leqslant q,$$

$$\mathrm{Sq}^i \bar{u} = \{\sum_\alpha \mathrm{Sq}^{i-q} a_\alpha \cdot b_\alpha^2\} \in E_{r+\min(i-q,r-2)}^{p+i-q,2q} \quad \textit{for} \quad q \leqslant i \leqslant p+q.$$

As before let $f: E \to B$ be a mapping of css-complexes. Let $b_0$ (vertex) be a base point in $B$. The inverse of $b_0$, $F = f^{-1}(b_0)$, is a subcomplex of $E$. We therefore get the commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{\text{incl.}} & E & \xrightarrow{f} & B \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1_B} \\ b_0 & \xrightarrow{\text{incl.}} & B & \xrightarrow{1_B} & B \end{array}, \tag{23}$$

where $b_0$ also denotes the subcomplex of $B$ generated by $b_0$. The pairs (incl., incl.) and $(f, 1_B)$ of horizontal mappings will be denoted by $\alpha$ and $\gamma$. These induce mappings

$$E_r(1_B) \xrightarrow{\gamma^*} E_r(f) \xrightarrow{\alpha^*} E_r(k). \tag{24}$$

We have for all $r \geqslant 2$

$$\left.\begin{array}{l} E_r^{0,q}(k) \cong H^q(F), \; E_r^{p,q}(k) = 0 \quad \text{for} \quad p \geqslant 1, \\ E_r^{p,0}(1_B) \cong H^p(B), \; E_r^{p,q}(1_B) = 0 \quad \text{for} \quad q \geqslant 1. \end{array}\right\} \tag{25}$$

If $\bar{u}$ is an element in the fibre or in the base of the spectral sequences (25) then by $\bar{u}'$ we shall denote the corresponding cohomology class under the isomorphisms given in (25).

THEOREM 7.7. *Let $\bar{u}_1 \in E_r^{0,q}(k)$ and let $\bar{u}_2 \in E_r^{p,0}$. Then*

$$(\mathrm{Sq}^i \bar{u}_1)' = \mathrm{Sq}^i(\bar{u}_1') \in H^{q+i}(F),$$

$$(\mathrm{Sq}^i \bar{u}_2)' = \mathrm{Sq}^i(\bar{u}_2') \in H^{p+i}(B).$$

*Proof.* By comparison of the construction of Steenrod powers in the spectral sequence and in cohomology the proof follows trivially.

From the naturality (Theorem 5.4) of the $\mathrm{Sq}^i$'s we get from (24)

THEOREM 7.8. *Let $\bar{u} \in E_2^{p,0}(1_B)$ and let $\bar{v} \in E_2^{0,q}(f)$. Then*

$$\gamma^*(\mathrm{Sq}^i \bar{u}) = \mathrm{Sq}^i(\gamma^* \bar{u}),$$

$$\alpha^*(\mathrm{Sq}^i \bar{v}) = \mathrm{Sq}^i(\alpha^* \bar{v}).$$

The infinity term, $E_\infty = E_\infty(f)$, of our spectral sequence is isomorphic to the graded module associated with the filtered module $H^* = H^*(E)$,

$$0 = F^{p-1}H^p \subseteq F^pH^p \subseteq \ldots \subseteq F^iH^p \subseteq \ldots \subseteq F^1H^p \subseteq F^0H^p = H^p(E),$$

$$F^iH^p / F^{i+1}H^p \cong E_\infty^{i,p-i}.$$

If we disregarded the filtration, the mapping $\varphi$: $W \otimes_\pi C(E)^{(2)} \to C(E)$ used to define the spectral operations can also be used to determine the Steenrod operations in $H^*(E)$. The proof of the following theorem is trivial.

THEOREM 7.9. *Let* $\bar{u} \in F^p H^{p+q}(E)$ *determine* $\{\bar{u}\}$ *in* $E_\infty^{p,q}(f)$. *Then we have*

$$\mathrm{Sq}^i \bar{u} \in F^{p+j} H^{p+q+i}(E)$$

*and*

$$\{\mathrm{Sq}^i \bar{u}\} = \mathrm{Sq}^i \{\bar{u}\} \in E^{p+j,\, q+i-j},$$

*where* $j = \max(0, i-q)$. (*Here* $\mathrm{Sq}^i$ *denotes a cohomology operation when operating on the cohomology class* $\bar{u}$ *and a spectral operation when operating on* $\{\bar{u}\}$).

Exactly as in the case of cohomology operations we can consider iterated operations and ask for relations between them. In the case $E_2^{*,*} \cong H^*(B) \otimes H^*(F)$, certain relations are, however, easily derived from the Adem relations by means of Theorems 7.4 and 7.6. Since such relations are not used in this paper, we shall not write them down.

## 8. Some lemmas

In certain computations coming up in later sections we meet the following situation: In the spectral sequence $\{E_r, d_r\}$ of a certain mapping $f\colon E \to B$ we know that three elements $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, and $\gamma \in E_n^{0,2(n-1)}$ $(n \geq 2)$ have the properties

$$d_n \alpha = \beta, \quad d_n \gamma = \alpha \cdot \beta,$$

$$E_2^{p,q} = 0 \quad \text{for} \quad 1 \leq q < n-1, \quad \text{any } p.$$

We are then interested in determining the differentials of $\mathrm{Sq}^i \gamma$ and of other elements in the fibre. The lemmas proved in this section treat this and a similar situation. First let us make the following

REMARK 8.1. *Let* $f\colon E \to B$ *be a map of css-complexes and* $\{E_r, d_r\}$ *the corresponding spectral sequence. Let* $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, *and* $\gamma \in E_n^{0,2(n-1)}$ $(n \geq 2)$ *with* $d_n \alpha = \beta$, $d_n \gamma = \alpha\beta$. *Let* $E_j^{2n-j,j-1} = 0$, $t = 2, 3, \ldots, n-1$. *Then there exist cochain representatives* $u$, $v$, *and* $x$ *of* $\alpha$, $\beta$, *and* $\gamma$ *respectively with the property*

$$dx = uv + a \tag{1}$$

*with* $a \in F^{2n-1}(C^{2n-1}(E))$.

*Proof*. The cochain $a$ we shall say is "in the base". In general we shall say that any cochain belonging to $\sum_j F^j C^j$ is in the base. Let $u$ be a representative of $\alpha$. The cochain $v = du$ is then in the base and represents $\beta$. Let $y$ be an arbitrary representative of $\gamma$. Then, since $d_n \gamma = \alpha\beta$, the cochain $dy$ must represent $\alpha\beta$, which is also represented by $uv$. By (1) in Section 5 we therefore get

$$dy = uv + db + c$$

for some $b \in F^1C^{2n-2}$ and $c \in Z_{n-1}^{n+1,n-2}$. The cochain $c$ determines a class in $E_{n-1}^{n+1,n-2}$. Since this group by assumption is zero, we get

$$c = db_1 + c_1$$

with $b_1 \in F^3$ and $c_1 \in Z_{n-2}^{n+2,n-3}$. Since $c_1$ determines a class in $E_{n-2}^{n+2,n-3} = 0$, we can iterate this process. We therefore get

$$dy = uv + d(b + b_1 + \dots + b_{n-2}) + c_{n-2}$$

with $c_{n-2} = a$ in the base and $b + b_1 + \dots + b_{n-2} \in F^1$. Since $d(b + b_1 + \dots + b_{n-2}) \in F^n$, the element $x = y - (b + b_1 + \dots + b_{n-2})$ represents $\gamma$ and we get (1).

LEMMA 8.2. *Let $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, and $\gamma \in E_n^{0,2(n-1)}$ be elements in the spectral sequence $\{E_r, d_r\}$ associated with a css-map $f\colon E \to B$. Let $u$, $v$, and $x$ be cochains representing $\alpha$, $\beta$, and $\gamma$ respectively with the properties $du = v$, $dx = uv + a$, where $a$ is in the base. Then*

$$\tau^{(2k+1)} = \mathrm{Sq}^{2k+1}\gamma + \sum_{\sigma=0}^{k} \mathrm{Sq}^\sigma \alpha \cdot \mathrm{Sq}^{2k+1-\sigma}\alpha \quad (0 \leqslant k < n-1),$$

*is transgressive, while*

$$\tau^{(2k)} = \mathrm{Sq}^{2k}\gamma + \sum_{\sigma=0}^{k-1} \mathrm{Sq}^\sigma \alpha \cdot \mathrm{Sq}^{2k-\sigma}\alpha \quad (0 < k \leqslant n-1),$$

*persists to $E_{n+k}$ and has*

$$d_{n+k}\{\tau^{(2k)}\} = \{\mathrm{Sq}^k \alpha \cdot \mathrm{Sq}^k \beta\}. \tag{2}$$

*Furthermore there are cochains $u_1$, $v_1$, and $x_1$ representing $\mathrm{Sq}^k\alpha$, $\mathrm{Sq}^k\beta$, and $\tau^{(2k)}$ respectively such that*

$$du_1 = v_1 \quad \textit{and} \quad dx_1 = u_1 v_1 + a_1,$$

*where $a_1$ is in the base. (The existence of $u_1$, $v_1$, $x_1$, and $a_1$ with this property clearly implies* (2).)

*Proof*. Since $dx = uv + a$ and $ddx = 0$ we get

$$d(uv) + da = v^2 + da = 0.$$

By (17) and (22) of section 7 we get for $\varepsilon = 0, 1$

$$\begin{aligned} d\,\mathrm{sq}^{2k+\varepsilon}(x) &= \mathrm{sq}^{2k+\varepsilon}(dx) = \mathrm{sq}^{2k+\varepsilon}(uv + a) = \mathrm{sq}^{2k+\varepsilon}(uv) + \mathrm{sq}^{2k+\varepsilon}(a) \\ &\quad + d\varphi(e_{2n-2k-\varepsilon} \otimes (uv)\,a + e_{2n-2k-\varepsilon+1} \otimes v^2 a) + \varphi(e_{2n-2k-\varepsilon+1} \otimes (v^2)^2) \\ &= \mathrm{sq}^{2k+\varepsilon}(uv) + \mathrm{sq}^{2k+\varepsilon}(a) + d(\varphi(e_{2n-2k-\varepsilon} \otimes (uv)\,a + e_{2n-2k-\varepsilon+1} \otimes v^2 a) + \mathrm{sq}^{2k+\varepsilon-1}(a)) \\ &= \sum_{i+j=2k+\varepsilon} \mathrm{sq}^i u \cdot \mathrm{sq}^j v + H(e_{2n-2k-\varepsilon} \otimes v^4) + \mathrm{sq}^{2k+\varepsilon}(a) \\ &\quad + d(H(\eta) + \varphi(e_{2n-2k-\varepsilon} \otimes (uv)\,a + e_{2n-2k-\varepsilon+1} \otimes v^2 a) + \mathrm{sq}^{2k+\varepsilon-1}(a)) \end{aligned} \tag{3}$$

with $\eta = e_{2n-1-2k-\varepsilon}\otimes uvuv + e_{2n-2k-\varepsilon}\otimes uvvv$ (see (22) in section 7; since $v$ is a cocycle, $b$ and $c$ there are zero). By the definition of the filtration of type $2_4$ we see that the filtration of $\eta$ is greater than or equal to $2n-(2n-1-2k-\varepsilon) = 2k+\varepsilon+1$, which is $\geqslant 2$ since $2k+\varepsilon \geqslant 1$. Since $H$ is of degree $\leqslant 1$, we see that all terms in

$$b = H(\eta) + \varphi(e_{2n-2k-\varepsilon}\otimes(uv)\,a + e_{2n-2k-\varepsilon+1}\otimes v^2 a) + \mathrm{sq}^{2k+\varepsilon-1}(a) \tag{4}$$

are of filtration $\geqslant 1$. The sum

$$\sum_{\sigma=0}^{k+\varepsilon-1} \mathrm{Sq}^{\sigma}\alpha \cdot \mathrm{Sq}^{2k+\varepsilon-\sigma}\alpha$$

is represented by

$$Q = \sum_{\sigma=0}^{k+\varepsilon-1} \mathrm{sq}^{\sigma} u \cdot \mathrm{sq}^{2k+\varepsilon-\sigma} u.$$

Applying the coboundary operator to $Q$ we get

$$\begin{aligned} dQ &= \sum_{\sigma=0}^{k+\varepsilon-1} (\mathrm{sq}^{\sigma} v \cdot \mathrm{sq}^{2k+\varepsilon-\sigma} u + \mathrm{sq}^{\sigma} u \cdot \mathrm{sq}^{2k+\varepsilon-\sigma} v) \\ &= \sum_{\sigma=0}^{k+\varepsilon-1} (\mathrm{sq}^{2k+\varepsilon+\sigma} u \cdot \mathrm{sq}^{\sigma} v + \mathrm{sq}^{\sigma} u \cdot \mathrm{sq}^{2k+\varepsilon-\sigma} v) + d\left(\sum_{\sigma=0}^{k+\varepsilon-1} \varphi(e_1\otimes \mathrm{sq}^{\sigma} v\, \mathrm{sq}^{2k+\varepsilon-\sigma} u)\right) \\ &\quad + \sum_{\sigma=0}^{k+\varepsilon-1} \varphi(e_1\otimes \mathrm{sq}^{\sigma} v\, \mathrm{sq}^{2k+\varepsilon-\sigma} v). \end{aligned} \tag{5}$$

The term

$$c = \sum_{\sigma=0}^{k+\varepsilon-1} \varphi(e_1\otimes \mathrm{sq}^{\sigma} v\, \mathrm{sq}^{2k+\varepsilon-\sigma} u) \tag{6}$$

we note has filtration $\geqslant 1$.

If $\varepsilon = 1$, we get from (3) and (5)

$$d(\mathrm{sq}^{2k+1} x + Q + b + c) = H(e_{2n-2k-1}\otimes v^4) + \sum_{\sigma=0}^{k} \varphi(e_1\otimes \mathrm{sq}^{\sigma} v\, \mathrm{sq}^{2k+1-\sigma} v) + \mathrm{sq}^{2k+1}(a). \tag{7}$$

Since $H$ is of degree $\leqslant 1$, it follows that the right hand side is in the base. Also since, as we observed above, $b+c$ is of filtration $\geqslant 1$, it follows that $\mathrm{sq}^{2k+1}x + Q + b + c$ represents $\tau^{(2k+1)}$. The equation (7) hence shows that $\tau^{(2k+1)}$ is transgressive.

If $\varepsilon = 0$, we get from (3) and (5)

$$d(\mathrm{sq}^{2k} x + Q + b + c) = \mathrm{sq}^{k} u \cdot \mathrm{sq}^{k} v + H(e_{2n-2k}\otimes v^4) + \sum_{\sigma=0}^{k-1} \varphi(e_1\otimes \mathrm{sq}^{\sigma} v\, \mathrm{sq}^{2k-\sigma} v) + \mathrm{sq}^{2k}(a). \tag{8}$$

As before

$$x_1 = \mathrm{sq}^{2k} x + Q + b + c \tag{9}$$

represents $\tau^{(2k)}$, and

$$a_1 = H(e_{2n-2k}\otimes v^4) + \sum_{\sigma=0}^{k-1} \varphi(e_1\otimes \mathrm{sq}^{\sigma} v\, \mathrm{sq}^{2k-\sigma} v) + \mathrm{sq}^{2k}(a) \tag{10}$$

is in the base. Putting

$$u_1 = \mathrm{sq}^k u, \quad v_1 = \mathrm{sq}^k v$$

we get from (8) the second statement of the lemma. This completes the proof.

LEMMA 8.3. *With the same assumptions as in Lemma* 8.2

$$\gamma \cdot d_n(\gamma) = \gamma\alpha\beta \in E_n^{n,3(n-1)}$$

*is transgressive, i.e. persists till* $E_{3n-2}$.

*Proof.* The element $\gamma\alpha\beta$ is represented by $x \cdot dx$. Taking coboundary we get by (17) and (22) of section 7

$$\begin{aligned} d(x \cdot dx) &= (dx)^2 \\ &= \mathrm{sq}^{2n-1}(uv + a) = \mathrm{sq}^{2n-1}(uv) + \mathrm{sq}^{2n-1} a + d\varphi(e_1 \otimes (uv)\, a + e_2 \otimes v^2 a) + \varphi(e_2 \otimes (v^2)^2) \\ &= \mathrm{sq}^{n-1} u \cdot \mathrm{sq}^n v + \mathrm{sq}^n u \cdot \mathrm{sq}^{n-1} v + \mathrm{sq}^{2n-1} a + dH(\eta) + H(e_1 \otimes v^4) + db \end{aligned} \tag{11}$$

with $\eta = e_0 \otimes (u \otimes v)^2 + e_1 \otimes (u \otimes v \otimes v \otimes v)$ and $b = \varphi(e_1 \otimes (uv)\, a + e_2 \otimes v^2 a) + \mathrm{sq}^{2n-2} a$. Since

$$\mathrm{sq}^{n-1} u \cdot \mathrm{sq}^n v + \mathrm{sq}^n u \cdot \mathrm{sq}^{n-1} v = \mathrm{sq}^{n-1} u \cdot v^2 + uv \cdot \mathrm{sq}^{n-1} v = db_1 + \varphi(e_1 \otimes v^2\, \mathrm{sq}^{n-1} v) \tag{12}$$

with

$$b_1 = \mathrm{sq}^{n-1} u \cdot a + x \cdot \mathrm{sq}^{n-1} v + \varphi(e_1 \otimes a\, \mathrm{sq}^{n-1} v), \tag{13}$$

we get from (11)

$$d(xdx + H(\eta) + b + b_1) = H(e_1 \otimes v^4) + \varphi(e_1 \otimes v^2\, \mathrm{sq}^{n-1} v) + \mathrm{sq}^{2n-1} a. \tag{14}$$

Since $\eta$ is of filtration $2n$, $H(\eta)$ is of filtration $2n - 1 \geqslant n + 1$. It follows that $xdx + H(\eta) + b + b_1$ is a cochain representative for $\gamma\alpha\beta$. Since the right hand side of (14) is in the base, this equation gives us the conclusion. This completes the proof.

We shall now consider analogs of the above lemmas. As before we make the following

REMARK 8.4. *Let $f$: $E \to B$ be a mapping of css-complexes and let $\{E_r, d_r\}$ be the corresponding spectral sequence. Let $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, and $\gamma \in E_n^{0,2^h n-2}$ ($n \geqslant 2$, $h \geqslant 2$) with $d_n \alpha = \beta$ and $d_{(2^h n-n)}(\gamma) = \alpha\beta^{2^h-1}$. Let $E_j^{2^h n-j,j-1} = 0$ for $j = 2, 3, \ldots, n-1$. There then exist cochains $u$, $v$, and $x$ representing $\alpha$, $\beta$, and $\gamma$ respectively with the property*

$$dx = uv^{2^h-1} + a$$

*with $a$ in the base.*

LEMMA 8.5. *Let* $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, *and* $\gamma \in E_n^{0,2^h n-2}$ $(n \geqslant 2, h \geqslant 2)$ *be elements in the spectral sequence* $\{E_r, d_r\}$ *associated with a css-map* $f\colon E \to B$. *Let* $u$, $v$, *and* $x$ *be cochains representing* $\alpha$, $\beta$, *and* $\gamma$ *respectively with the properties* $du = v$, $dx = uv^{2^h} + a$ *where* $a$ *is in the base. Then*

$$\mathrm{Sq}^k \gamma, \ k \leqslant 2^h n - 2$$

*is transgressive if* $n$ *is not divisible by* $2^h$. *If* $k = s \cdot 2^h$, *then*

$$\mathrm{Sq}^k \gamma = \mathrm{Sq}^{s \cdot 2^h} \gamma$$

*persists to* $E_{(2^h-1)(n+s)}$ *and has*

$$d_{(2^h-1)(n+s)}\{\mathrm{Sq}^{s \cdot 2^h}\gamma\} = \{\mathrm{Sq}^s \alpha \cdot (\mathrm{Sq}^s \beta)^{2^h-1}\}.$$

*Furthermore there are cochains* $u_1$, $v_1$, *and* $x_1$ *representing* $\mathrm{Sq}^s \alpha$, $\mathrm{Sq}^s \beta$, *and* $\mathrm{Sq}^{s \cdot 2^h}\gamma$ *respectively such that*

$$du_1 = v_1, \ dx_1 = u_1 {v_1}^{2^h-1} + a_1$$

*with* $a_1$ *in the base.*

*Proof*. From the equation $dx = uv^{2^h-1} + a$ we get

$$d(uv^{2^h-1}) + da = v^{2^h} + da = 0. \tag{15}$$

Putting $\sum_i \mathrm{sq}^i y = \mathrm{sq}\, y$ for any cochain $y$ we get from (17) of section 7

$$d\,\mathrm{sq}(x) = \mathrm{sq}(dx) = \mathrm{sq}(uv^{2^h-1} + a) = \mathrm{sq}(uv^{2^h-1}) + \mathrm{sq}(a) + db, \tag{16}$$

where $$b = \sum_i (\varphi(e_{2^h n - i} \otimes (uv^{2^h-1})\, a + e_{2^h n - i + 1} \otimes v^{2^h} a) + \mathrm{sq}^{i-1} a).$$

By an obvious generalization of (22) of section 7 we get from (16)

$$d\,\mathrm{sq}(x) = \mathrm{sq}(u) \cdot (\mathrm{sq}(v))^{2^h-1} + db_1 + c_1 + \mathrm{sq}(a) + db, \tag{17}$$

where $$b_1 = \sum_i \overline{H}(\bar{\eta}_i) \tag{18}$$

$$c_1 = \sum_i \overline{H}(e_{2^h n - i} \otimes (v \otimes v \otimes \ldots \otimes v)^2). \tag{19}$$

The homotopy in the generalized diagram (18) of Section 7 is here denoted by $\overline{H}$: $W \otimes (C^{(2^h)})^{(2)} \to C$; $\bar{\eta}_i$ denotes the element $e_{2^h n - 1 - i} \otimes (uv^{2^h-1})^2 + e_{2^h n - i} \otimes (uv^{2^h-1})(v^{2^h})$.

Putting

$$Q = \sum_{\substack{0 \leqslant t \leqslant h-1 \\ i < j}} (\mathrm{sq}^i u (\mathrm{sq}^i v)^{2^t-1})(\mathrm{sq}^j u (\mathrm{sq}^j v)^{2^t-1})(\mathrm{sq}\, v)^{2^h - 2^{t+1}} \tag{20}$$

we get

$$dQ = \sum_{\substack{0 \leqslant t \leqslant h-1 \\ i<j}} ((\mathrm{sq}^i v(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j u(\mathrm{sq}^j v)^{2^j-1})$$

$$+ (\mathrm{sq}^i u(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j v(\mathrm{sq}^j v)^{2^j-1}))(\mathrm{sq}\, v)^{2^h-2^{t+1}}$$

$$= \sum_{\substack{0 \leqslant t \leqslant h-1 \\ i\neq j}} (\mathrm{sq}^i u(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j v)^{2^j}(\mathrm{sq}\, v)^{2^h-2^{t+1}} + db_2 + c_2 \tag{21}$$

with

$$b_2 = \sum_{\substack{0 \leqslant t \leqslant h-1 \\ i<j}} \varphi(e_1 \otimes (\mathrm{sq}^i v(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j u(\mathrm{sq}^j v)^{2^j-1}))(\mathrm{sq}\, v)^{2^h-2^{t+1}}, \tag{22}$$

$$c_2 = \sum_{\substack{0 \leqslant t \leqslant h-1 \\ i<j}} \varphi(e_1 \otimes ((\mathrm{sq}^i v)^{2^i})((\mathrm{sq}^j v)^{2^j}))(\mathrm{sq}\, v)^{2^h-2^{t+1}}. \tag{23}$$

If for a moment $a$ and $b$ are arbitrary cocycles in the base and $c$ is a cochain with the property that $c$ and $dc$ are in the base, then an easy computation shows

$$(a+b+dc)^2 = a^2 + b^2 + d\bar{c}$$

with $\bar{c}$ and $d\bar{c}$ in the base. Using this an inductive argument shows

$$(\mathrm{sq}\, v)^{2^t} = \sum_i (\mathrm{sq}^i v)^{2^t} + dc(t) \tag{24}$$

with $c(t)$ and $dc(t)$ in the base. Furthermore the components of $c(t)$ are all of dimension $>n$ ($=$dimension of $v$). By (21) and (24) we now get

$$dQ + db_2 + c_2 + \mathrm{sq}(u)(\mathrm{sq}\, v)^{2^h-1}$$

$$= \sum_{\substack{0 \leqslant t \leqslant h-1 \\ i\neq j}} (\mathrm{sq}^i u(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j v)^{2^j}(\mathrm{sq}\, v)^{2^h-2^{t+1}} + \mathrm{sq}\, u(\mathrm{sq}\, v)^{2^h-1}$$

$$= \sum_{\substack{1 \leqslant t \leqslant h-1 \\ i\neq j}} (\mathrm{sq}^i u(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j v)^{2^j}(\mathrm{sq}\, v)^{2^h-2^{t+1}} + \sum_s \mathrm{sq}^s u \cdot \mathrm{sq}^s v(\mathrm{sq}\, v)^{2^h-2}$$

$$= \sum_{\substack{2 \leqslant t \leqslant h-1 \\ i\neq j}} (\mathrm{sq}^i u(\mathrm{sq}^i v)^{2^i-1})(\mathrm{sq}^j v)^{2^j}(\mathrm{sq}\, v)^{2^h-2^{t+1}} + \sum_s \mathrm{sq}^s u(\mathrm{sq}^s v)^3(\mathrm{sq}\, v)^{2^h-4} + db_3 + c_3, \tag{25}$$

where $b_3 = \sum_s \mathrm{sq}^s u \cdot \mathrm{sq}^s v \cdot c(1) \cdot (\mathrm{sq}\, v)^{2^h-4}$ and $c_3 = \sum_s (\mathrm{sq}^s v)^2 \cdot c(1) \cdot (\mathrm{sq}\, v)^{2^h-4}$. We note that $b_3$ has positive filtration, and that $c_3$ is in the base. Iterating the process from (25) we get

$$dQ + db_2 + c_2 + \mathrm{sq}\, u(\mathrm{sq}\, v)^{2^h-1} = \sum_s \mathrm{sq}^s u(\mathrm{sq}^s v)^{2^h-1} + d(b_3 + \ldots + b_{h+1}) + c_3 + \ldots + c_{h+1}, \tag{26}$$

where $b_3 + \ldots + b_{h+1}$ is of positive filtration and $c_3 + \ldots + c_{h+1}$ is in the base.

Using (17) and (26) we now get

$$d(\mathrm{sq}(x) + b + b_1 + b_2 + \ldots + b_{h+1} + Q) = \sum_s \mathrm{sq}^s u(\mathrm{sq}^s v)^{2^h-1} + \mathrm{sq}\, a + c_1 + c_2 + \ldots + c_{h+1}. \tag{27}$$

Since $\operatorname{sq} x+b+b_1+\ldots+b_{h+1}+Q$ is a representative of $\operatorname{Sq}\gamma=\sum_i \operatorname{Sq}^i\gamma$ and since $\operatorname{sq} a + c_1+c_2+\ldots+c_{h+1}$ is in the base the conclusion of the lemma follows. This completes the proof.

LEMMA 8.6. *With the same assumptions as in Lemma* 8.5

$$\gamma\cdot d_{(2^k-1)n}(\gamma)=\gamma\cdot\alpha\cdot\beta^{2^k-1}\in E_{(2^k-1)n}$$

*is transgresssive; i.e. persists till* $E_{(2^k+1)n-2}$.

*Proof*. The element $\gamma\cdot\alpha\cdot\beta^{2^k-1}$ has a cochain representative $x\cdot dx$. Taking the coboundary we get

$$\begin{aligned} d(x\cdot dx)=(dx)^2=\operatorname{sq}^{2^hn-1}(dx)&=\operatorname{sq}^{2^hn-1}(uv^{2^k-1}+a)\\ &=\operatorname{sq}^{2^hn-1}(uv^{2^k-1})+\operatorname{sq}^{2^hn-1}(a)+d(b), \end{aligned}\tag{28}$$

with

$$b=\varphi(e_1\otimes(uv^{2^k-1})a+e_2\otimes v^{2^k}a)+\operatorname{sq}^{2^hn-2}a.\tag{29}$$

Using the generalized formula (22) of section 7 (see also (17), (18), and (19) in this section) we get from (28)

$$\begin{aligned} d(x\cdot dx)=&\sum_{\substack{\Sigma i(j)=2^kn-1\\ j}}\operatorname{sq}^{i(1)}(u)\operatorname{sq}^{i(2)}(v)\ldots\operatorname{sq}^{i(2^h)}(v)\\ &+d\bar{H}(\bar{\eta}_{2^kn-1})+\bar{H}(e_1\otimes(v^{2^k})^2)+\operatorname{sq}^{2^hn-1}a+db\\ =&\operatorname{sq}^{n-1}u(\operatorname{sq}^n v)^{2^h-1}+\sum\operatorname{sq}^n u\operatorname{sq}^n v\operatorname{sq}^n v\ldots\operatorname{sq}^n v\operatorname{sq}^{n-1}v\operatorname{sq}^n v\ldots\operatorname{sq}^n v\\ &+d(\bar{H}(\bar{\eta}_{2^kn-1})+b)+\bar{H}(e_1\otimes(v^{2^h})^2)+\operatorname{sq}^{2^hn-1}a\\ =&\operatorname{sq}^{n-1}u\cdot v^{2^{k+1}-2}+\sum_{0\leqslant i\leqslant 2^k-2}uv\cdot v^{2i}\cdot\operatorname{sq}^{n-1}v\cdot v^{2^{k+1}-4-2i}\\ &+d(\bar{H}(\bar{\eta}_{2^kn-1})+b)+\bar{H}(e_1\otimes(v^{2^k})^2)+\operatorname{sq}^{2^hn-1}a. \end{aligned}\tag{30}$$

Since

$$\sum_{0\leqslant i\leqslant 2^{k-1}-2}uv\cdot v^{2i}\cdot\operatorname{sq}^{n-1}v\cdot v^{2^{k+1}-4-2i}=db_1+c_1\tag{31}$$

with

$$\left.\begin{aligned} b_1&=\sum_{0\leqslant i\leqslant 2^{k-1}-2}uv^{2i+1}\cdot\operatorname{sq}^{n-1}v\cdot v^{2^k-4-2i}\cdot a,\\ c_1&=\sum_{0\leqslant i\leqslant 2^{k-1}-2}v^{2i+2}\cdot\operatorname{sq}^{n-1}v\cdot v^{2^k-4-2i}\cdot a, \end{aligned}\right\}\tag{32}$$

$$\sum_{2^{k-1}-1\leqslant i\leqslant 2^k-1}uv^{2i+1}\operatorname{sq}^{n-1}v\cdot v^{2^{k+1}-4-2i}=db_2+c_2\tag{33}$$

with

$$\left.\begin{aligned} b_2&=\sum_{i=2^{k-1}-1}^{2^k-1}xv^{2i+2-2^k}\cdot\operatorname{sq}^{n-1}v\cdot v^{2^{k+1}-4-2i},\\ c_2&=\sum_{i=2^{k-1}-1}^{2^k-1}av^{2i+2-2^k}\cdot\operatorname{sq}^{n-1}v\cdot v^{2^{k+1}-4-2i}, \end{aligned}\right\}\tag{34}$$

and
$$\mathrm{sq}^{n-1} u \cdot v^{2^{\lambda+1}-2} = db_3 + c_3, \tag{35}$$
with
$$b_3 = \mathrm{sq}^{n-1} u \cdot v^{2^{\lambda}-2} \cdot a, \quad c_3 = \mathrm{sq}^{n-1} v \cdot v^{2^{\lambda}-2} \cdot a, \tag{36}$$
we get from (30)
$$d(xdx + \bar{H}(\bar{\eta}_{2^{\lambda}n-1}) + b + b_1 + b_2 + b_3) = \bar{H}(e_1 \otimes (v^{2^{\lambda}})^2) + \mathrm{sq}^{2^{\lambda}n-1} a + c_1 + c_2 + c_3. \tag{37}$$
Since $xdx + \bar{H}(\bar{\eta}_{2^{\lambda}n-1}) + b + b_1 + b_2 + b_3$ is a cochain representing $\gamma\alpha\beta^{2^{\lambda}-1}$ and since the right hand side of (37) is in the base we get the conclusion of the lemma. This completes the proof.

## 9. Mappings of spectral sequences

For the rest of this paper we shall be working in the category of topological spaces. It is well known that there is defined a functor $S$, the singular complex functor, taking this category into the category of *css*-complexes. We shall consider the (normalized) singular homology theory of topological spaces $X$ and use the following definitions and notation
$$C_*(X) = C_*(SX), \quad C(X) = C(SX),$$
$$H_*(X) = H_*(SX), \quad H^*(X) = H^*(SX).$$
If $f\colon E \to B$ is a mapping of topological spaces, then
$$E(f) = E(Sf),$$
where $E(Sf)$ denotes the spectral sequence of the css-mapping $Sf$. In Gugenheim–Moore [7] it was shown that if $f$ is a fibre mapping, then
$$E(f)_2^{*,*} \cong H^*(B, H^*(F)) \quad \text{(local coefficients)}.$$
If $\pi_1(B)$ operates trivially on $H^*(F)$, then
$$E(f)_2^{*,*} \cong H^*(B) \otimes H^*(F).$$
The coefficient group is here assumed to be our ground field $K$.

In this section the ground field $K$ is an arbitrary field of characteristic $n$. It is well known that the end-point projection $LX \to X$, $LX$ the space of paths over $X$ based at a point $x_0 \in X$, is a fibre map. This is an example of a fibre space with total space having trivial cohomology. The infinity term $E$ of the spectral sequence of such a fibration is trivial (except for $E_\infty^{0,0} \cong K$). In this section we shall mostly consider spectral sequences with trivial $\infty$-term. The spectral sequences considered will not necessarily be spectral sequences of a fibration.

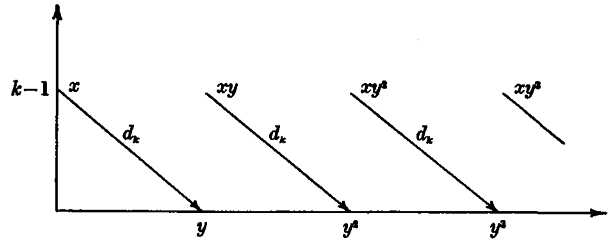


Fig. 1.

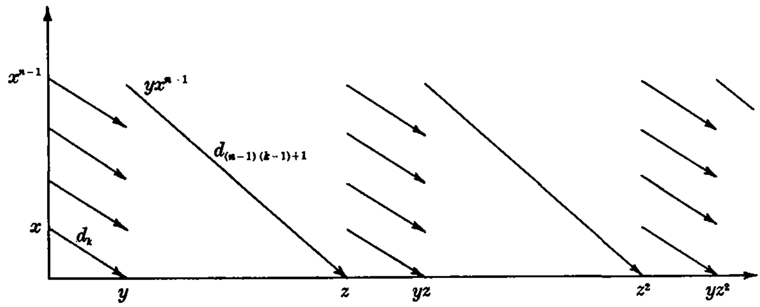


Fig. 2.

Let us define three kinds of elementary spectral sequences $A=\{A_r, d_r\}$ $B=\{B_r, d_r\}$, and $C=\{C_r, d_r\}$ $(r\geqslant 2)$. These spectral sequences are bigraded sequences of commutative algebras. The differentials are derivations and have bigradings as in the cohomology spectral sequence of fibre spaces.

The spectral sequence $A(k)=A=\{A_r, d_r\}$ (see Fig. 1) is given by

$$A_2=P(y)\otimes\Lambda(x) \quad (y\in A_2^{k,0},\ x\in A_2^{0,k-1}), \tag{1}$$

where $P(y)$ denotes the polynomial algebra generated by $y$ and $\Lambda(x)$ the exterior algebra generated by $x$, and by

$$\left.\begin{aligned} d_r=0 \quad \text{for} \quad r\neq k,\\ d_k\{x\}=\{y\}. \end{aligned}\right\} \tag{2}$$

The spectral sequence $B(k)=B=\{B_r, d_r\}$ (see Fig. 2) is given by

$$B_2=(\Lambda(y)\otimes P(z))\otimes P_n(x),\ y\in B_2^{k,0},\ z\in B_2^{n(k-1)+2,0},\ x\in B_2^{0,k-1}, \tag{3}$$

where $P_n(x)$ denotes the truncated polynomial algebra of height $n$ generated by $x$ (i.e. $x^{n-1}\neq 0$, $x^n=0$), and by

$$\left.\begin{aligned} d_r=0 \quad \text{for} \quad r\neq k \quad \text{and} \quad (n-1)(k-1)+1,\\ d_k\{x\}=\{y\},\\ d_{(n-1)(k-1)+1}\{yx^{n-1}\}=\{z\}. \end{aligned}\right\} \tag{4}$$

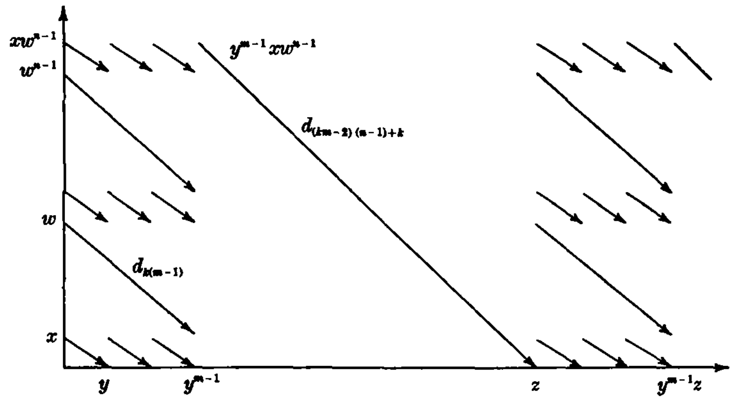


Fig. 3.

We remark that if $n \neq 2$, then $k$ must be odd. If $k$ was even then by commutativity $x^2 = 0$ which contradicts $x^{n-1} \neq 0$.

The spectral sequence $C(k, m) = C = \{C_r, d_r\}$ (see Fig. 3) is given by

$$C_2 = (P_m(y) \otimes P(z)) \otimes (\Lambda(x) \otimes P_n(w)) \tag{5}$$

where $$y \in C_2^{k,0}, \quad z \in C_2^{(km-2)(n-1)+km,0}, \quad x \in C_2^{0,k-1}, \quad w \in C_2^{0,km-2}, \tag{6}$$

and by $$d_r = 0 \quad \text{for} \quad r \neq k,\ k(m-1),\ (km-2)(n-1)+k,$$

$$\left.\begin{aligned} d_k\{x\} &= \{y\}, \\ d_{k(m-1)}\{w\} &= \{y^{m-1}x\}, \\ d_{(km-2)(n-1)+k}\{y^{m-1}xw^{n-1}\} &= \{z\}. \end{aligned}\right\} \tag{7}$$

In case $n \neq 2$, $km$ must be even. If $m > 2$, then $k$ must be even.

In all three cases the differentials of any element are easily derived using the fact that the differentials are derivations. Since we later on are going to need the explicit expression of the differentials in the third case, we shall write them down here:

$$\{y^\alpha z^\beta x w^\gamma\} \text{ persists to } C_k \quad (0 \leqslant \alpha \leqslant m-2,\ 0 \leqslant \beta < \infty,\ 0 \leqslant \gamma \leqslant n-1)$$
$$\text{(i.e. } d_\sigma\{y^\alpha z^\beta x w^\gamma\} = 0 \quad \text{for} \quad (0 \leqslant \sigma < k) \tag{8}$$

and $$d_k\{y^\alpha z^\beta x w^\gamma\} = \{y^{\alpha+1} z^\beta w^\gamma\}.$$

$$\{z^\beta w^\gamma\} \text{ persists to } C_{k(m-1)} \quad (0 \leqslant \beta < \infty,\ 0 \leqslant \gamma \leqslant n-1)$$

and $$d_{k(m-1)}\{z^\beta w^\gamma\} = \{\gamma y^{m-1} z^\beta x w^{\gamma-1}\}.$$

$$\{y^{m-1} z^\beta x w^{n-1}\} \text{ persists to } C_{(km-2)(n-1)+k} \quad (0 \leqslant \beta < \infty)$$

and $$d_{(km-2)(n-1)+k}\{y^{m-1}z^{\beta}xw^{n-1}\}=\{z^{\beta+1}\}.$$

The differentials of all other generators are zero.

The infinity terms of the three spectral sequences are quite obviously trivial.

The spaces $X$ we shall consider in the following are all assumed to have locally finite cohomology, i.e. $H^i(X,K)$ is finitely generated for all $i$. The cohomology $H^*(X,K)$ is a vector space graded by dimension.

Definition. The cohomology $H^*(X)=H^*(X,K)$ of $X$ is said to be decomposed into a tensor product if we have given graded vector spaces $T_i$ and maps

$$f_i\colon\ T_i\to H^*(X)\quad (i=1,2,\ldots,h),$$

of graded vector spaces such that the composition

$$\bigotimes_{i=1}^{h} T_i \xrightarrow{f_1\otimes\cdots\otimes f_h} \bigotimes_{i=1}^{h} H^*(X)_i \to H^*(X)\quad (H^*(X)_i=H(X))$$

is one–one and onto in dimensions $0,1,\ldots$. The map on the right of the composition is $h$-fold cup-product. If $h=\infty$ the tensor product $\bigotimes_{i=1}^{\infty}T_i$ is defined by

$$\varinjlim_{p\to\infty}\bigotimes_{i=1}^{p}T_i=\bigotimes_{i=1}^{\infty}T_i.$$

The map on the right in the composition is defined to be the direct limit of $p$-fold cup-products $\bigotimes_{i=1}^{p}H^*(X)_i\to H^*(X)$.

Lemma 9.1. *Let $f$: $E\to B$ be a fibre mapping with fibre $F$. Let $E(f)=\{E_r^{*,*},d_r\}$ denote the associated spectral sequence.*

*Let*

$$\left.\begin{aligned} f&\colon\ A_2^{0,*}\to E_2^{0,*},\\ g&\colon\ B_2^{0,*}\to E_2^{0,*},\\ h&\colon\ C_2^{0,*}\to E_2^{0,*},\end{aligned}\right\}\qquad(9)$$

*be mappings of graded vector spaces satisfying*

(a) *$f(x)$ persists to $E_k$,*
*$d_k\{f(x)\}=\{a\}$, where $a\in E_2^{k,0}$;*

(b) *$g(x^{\alpha})$ persists to $E_k$,*
*$d_k\{g(x^{\alpha})\}=\{\alpha b g(x^{\alpha-1})\}$, where $0\leqslant\alpha\leqslant n-1$, $b\in E_2^{k,0}$ (remember $g$ is not assumed to be multiplicative),*
*$bg(x^{n-1})$ persists to $E_{(n-1)(k-1)+1}$,*
*$d_{(n-1)(k-1)+1}\{bg(x^{n-1})\}=\{c\}$, where $c\in E_2^{n(k-1)+2,0}$;*

(c) $h(xw^{\alpha})$ *persists to* $E_k$,

$d_k\{h(xw^{\alpha})\}=\{\bar{a}h(w^{\alpha})\}$, *where* $0\leqslant\alpha\leqslant n-1$, $\bar{a}\in E_2^{k,0}$,

$h(w^{\alpha})$ *persists to* $E_{k(m-1)}$,

$d_{(m-1)k}\{h(w^{\alpha})\}=\{\alpha\bar{a}^{m-1}h(xw^{\alpha-1})\}$, *where* $1\leqslant\alpha\leqslant n-1$,

$\bar{a}^{m-1}h(xw^{n-1})$ *persists to* $E_{(km-2)(n-1)+k}$,

$d_{(km-2)(n-1)+k}\{\bar{a}^{m-1}h(xw^{n-1})\}=\{\bar{b}\}$, *where* $\bar{b}\in E_2^{(km-2)(n-1)+km,0}$.

*Then the (additive) spectral sequence mappings*

$$f\colon A\to E,\quad g\colon B\to E,\quad h\colon C\to E, \tag{10}$$

*defined by*

$$\left.\begin{aligned} f(\{y^{\alpha}x^{\varepsilon}\})&=\{a^{\alpha}f(x)^{\varepsilon}\}, && (0\leqslant\alpha<\infty,\ \varepsilon=0,1),\\ g(\{y^{\varepsilon}z^{\alpha}x^{\beta}\})&=\{b^{\varepsilon}c^{\alpha}g(x)^{\beta}\}, && (0\leqslant\alpha<\infty,\ \varepsilon=0,1,\ 0\leqslant\beta\leqslant n-1),\\ h(\{y^{\alpha}z^{\beta}x^{\varepsilon}w^{\gamma}\})&=\{\bar{a}^{\alpha}\bar{b}^{\beta}h(x^{\varepsilon}w^{\gamma})\}, && (0\leqslant\alpha\leqslant m-1,\ 0\leqslant\beta,\ \varepsilon=0,1,\ 0\leqslant\gamma\leqslant n-1),\end{aligned}\right\} \tag{11}$$

*are extensions of the mapings* (9). *Furthermore* $f\,|\,A_2^{*,0}$ *is an algebra homomorphism. The same is true for* $g\,|\,B_2^{*,0}$ *and* $h\,|\,C_2^{*,0}$ *provided* $b^2=0\in E_2^{2k,0}$, *and* $\bar{a}^m=0\in E_2^{mk,0}$ *respectively.*

*Proof.* Provided that we know that (11) really defines mappings of spectral sequences, then it is obvious that the mappings (10) extend the mappings (9). By the formulas (11) it is also obvious that the restrictions of (10) to the base give algebra homomorphisms under the assumptions $\bar{b}^2=0$, $\bar{a}^m=0$.

We shall restrict ourselves to show that the third equation in (11) defines a mapping $h\colon C\to E$ of spectral sequences. This is the most complicated case. A complete description of $C$ was given in (8). We see that we only need to show that

$$\begin{aligned} &h\{y^{\alpha}z^{\beta}xw^{\gamma}\}=\{\bar{a}^{\alpha}\bar{b}^{\beta}h(xw^{\gamma})\} \text{ persists to } E_k,\\ &d_k\{\bar{a}^{\alpha}\bar{b}^{\beta}h(xw^{\gamma})\}=\{\bar{a}^{\alpha+1}\bar{b}^{\beta}h(w^{\gamma})\},\\ &h\{z^{\beta}w^{\gamma}\}=\{\bar{b}^{\beta}h(w^{\gamma})\} \text{ persists to } E_{k(m-1)},\\ &d_{k(m-1)}\{\bar{b}^{\beta}h(w^{\gamma})\}=\{\gamma\bar{a}^{m-1}\bar{b}^{\beta}h(xw^{\gamma-1})\},\\ &h\{y^{m-1}z^{\beta}xw^{n-1}\}=\{\bar{a}^{m-1}\bar{b}^{\beta}h(xw^{n-1})\} \text{ persists to } E_{(km-2)(n-1)+k},\\ &d_{(km-2)(n-1)+k}\{\bar{a}^{m-1}\bar{b}^{\beta}h(xw^{n-1})\}=\{\bar{b}^{\beta+1}\}.\end{aligned}$$

Since the differentials in $E$ are derivations this follows immediately from the assumption (c).

LEMMA 9.2. *Let* $F\to E\to B$ *be a fibration. Let the total space have trivial cohomology and let* $\pi_1(B)$ *operate trivially on* $H^*(F)$ *such that in the spectral sequence* $E=\{E_r,d_r\}$ *we have* $E_2^{*,*}\cong H^*(B)\otimes H^*(F)$. *Let*

$$f_i\colon A(k_i)\to E,\quad g_j\colon B(k_j)\to E,\quad h_\sigma\colon C(k_\sigma,m_\sigma)\to E \tag{12}$$

*be (additive) mappings of spectral sequences. The indices $i$, $j$, and $\sigma$ are understood to run through given indexing sets. If the restrictions of $f_i$, $g_j$, and $h_\sigma$ to the fibres decompose $H^*(F)$ into a tensor product, then additively $H^*(B)$ is isomorphic to the tensor product of exterior algebras each having one generator $g_j(y(k_j)) \in H^*(B) \cong E_2^{*,0}$, of polynomial algebras having the generators $f_i(y(k_i))$, $g_j(z(k_j))$, $h_\sigma(z(k_\sigma, m_\sigma))$, and of truncated polynomial algebras with generators $h_\sigma(y(k_\sigma, m_\sigma))$ of height $m_\sigma$.*

$$H^*(B) \cong \Lambda(\{g_j(y)\}) \otimes K[\{f_i(y), g_j(z), h_\sigma(z)\}] \otimes \bigotimes_\sigma K[h_\sigma(y), m_\sigma]. \tag{13}$$

*If further (as in Lemma 9.1) $h_\sigma(y(k_\sigma, m_\sigma)^\beta) = h_\sigma(y(k_\sigma, m_\sigma))^\beta$, $1 \leqslant \beta \leqslant m_\sigma - 1$, then (13) is an algebra isomorphism provided in $H^*(B)$ $g_j(y(k_j))^2 = 0$ and $h_\sigma(y(k_\sigma, m_\sigma))^{m_\sigma} = 0$.*

*Proof.* The composition

$$\bigotimes_i A(k_i) \otimes \bigotimes_j B(k_j) \otimes \bigotimes_\sigma C(k_\sigma, m_\sigma) \to \bigotimes_{i,j,\sigma} E \to E \tag{14}$$

of $\bigotimes_i f_i \otimes \bigotimes_j g_j \otimes \bigotimes_\sigma h_\sigma$ and cup-product is by assumption an isomorphism on the fibre. Since the $\infty$-term of both the range and domain in (14) are trivial it follows from Moore's comparison theorem (see e.g. Zeeman [14]) that the composition (14) is an isomorphism in the base. The comparison theorem only tells us that (14) restricted to the base is an isomorphism on the additive structure, but with the final assumption in the lemma this map is an algebra homomorphism and hence an algebra isomorphism. This completes the proof.

## 10. Spaces with two non-vanishing homotopy groups

A space with only a finite number of non-vanishing homotopy groups is called a Postnikov space. An important subclass is the class of spaces with only one non-vanishing homotopy group, the Eilenberg-MacLane spaces (denoted $K(\pi, n)$, $\pi_i(K(\pi, n)) = 0$ for $i \neq n$ and $= \pi$ for $i = n$). Spaces with two non-vanishing homotopy groups $\pi$ and $\tau$ in dimensions $n$ and $m$ $(n < m)$ can be constructed from Eilenberg-MacLane spaces. Let $P$ be the total space of the fibre space induced by a mapping $K(\pi, n) \to K(\tau, m+1)$,

$$\begin{tikzcd}
 & K(\tau, m) = \Omega K(\tau, m+1) \arrow[dl] \arrow[dr] & \\
P \arrow[rr] \arrow[d] & & LK(\tau, m+1) \arrow[d] \\
K(\pi, n) \arrow[rr, "k"] & & K(\tau, m+1).
\end{tikzcd}$$

The space $P$ has the homotopy groups $\pi$ and $\tau$ in dimensions $n$ and $m$. This follows easily from the homotopy sequence of the induced fibration. The mapping $K(\pi, n) \to K(\tau, m+1)$ determines a class $(k \in H^{m+1}(K(\pi, n); \tau)$ called the $k$-invariant of $P$. The cohomology class $k$ is also the image of the basic class $\varepsilon_m \in H^m(K(\tau, m); \tau)$ under the transgression in the fibration $K(\tau, m) \to P \to K(\pi, n)$. Since the singular homotopy type of $P$ is determined by $k$ (the groups $\pi$ and $\tau$ and their dimensions), we denote $P$ by $P(\pi, n; \tau, m, k)$. The $k$-invariant was introduced by Eilenberg-MacLane. It is the first of a sequence of $k$-invariants associated with any topological space. The higher $k$-invariants were introduced by Zilber and by Postnikov. We remark that the singular homotopy type of any space with two non-vanishing homotopy groups can be obtained by the above construction.

Spaces with two non-vanishing homotopy groups (or closely related spaces) are of importance in different situations in algebraic topology. They are used in Peterson's definition of functional cohomology operations [9] and in the definition of secondary cohomology operations (see e.g. Peterson-Stein [10] or Adams [1]).

## 11. Computation of $H^*(P(Z_2, n; Z_2, 2n-1, \varepsilon_n^2), Z_2)$

The method in computing the cohomology of spaces $P = P(\pi, n; \tau, m, k)$ is simply to compute the spectral sequence of the fibration $\Omega P \to LP \to P$ by using Lemma 9.1 and 9.2. This can be done in certain cases if the cohomology of $\Omega P = P(\pi, n-1; \tau, m-1, \sigma k)$ is known. When the $k$-invariant $\sigma k$, the suspension of $k$, of the fibre is zero, this cohomology is known, for in this case $\Omega P$ is of the same homotopy type as $K(\pi, n-1) \times K(\tau, m-1)$. We remark that if in particular $k$ is decomposable, then $\sigma k = 0$, and we are in the above case. In this section let $K = Z_2$.

A sequence $I = (a_1, a_2, \ldots, a_r)$ of non-negative integers is said to be admissible if

$$a_i \geqslant 2a_{i+1} \quad (i = 1, 2, \ldots, r-1). \tag{1}$$

The degree and the excess of $I$ are defined by

$$\left.\begin{aligned} \deg I &= \sum_{i=1}^{r} a_i, \\ e(I) &= \sum_{i=1}^{r-1} (a_i - 2a_{i+1}) + a_r. \end{aligned}\right\} \tag{2}$$

We then have the relation

$$\deg I + e(I) = 2a_1. \tag{3}$$

Sequences (of non-negative integers) are multiplied by juxtaposition, and a sequence is multiplied by a (non-negative) integer by multiplying each of the components with the integer. The empty sequence is also considered a sequence in the following. Sequences of the type $(2^{h-1}d, 2^{h-2}d, \ldots, d)$ will occur frequently in the following. We shall use the short notation

$$L(d,h) = (2^{h-1}d, 2^{h-2}d, \ldots, d). \tag{4}$$

Finally let us recall that if $I = (a_1, a_2, \ldots, a_r)$, then

$$\mathrm{Sq}^I = \mathrm{Sq}^{a_1}\mathrm{Sq}^{a_2}\ldots\mathrm{Sq}^{a_r}. \tag{5}$$

THEOREM 11.1. *Let $P_n = P(Z_2, n; Z_2, 2n-1, \varepsilon_n^2)$. For each admissible sequence $J$, $e(J) \leqslant 2(n-1)$, containing odd components and each admissible sequence $N, e(N) < n-1$, there are classes $\beta(J)$ and $\gamma(2N)$ in $H^*(P_n)$ of dimensions $2n-1+\deg J$ and $2(2n-1+2\deg N)$ respectively, satisfying*

$$\beta(J) = \mathrm{Sq}^{\bar{J}}(\beta(2j+1)J_1)$$

*whenever $J = \bar{J}(2j+1)J_1$ with all components of $J_1$ even.*

*Let $\alpha$ be the non-zero class in $H^*(P_n)$, then*

$$H^*(P_n) = Z_2[\{\beta(J)\}] \otimes \Lambda(\{\mathrm{Sq}^I\alpha\}) \otimes Z_2[\{\mathrm{Sq}^{L(4(n-1+\deg N),\,h)}\gamma(2N)\}],$$

*where $h = 0, 1, \ldots$ and where $J$, $I$, and $N$ run through all admissible sequences satisfying $e(J) \leqslant 2(n-1)$, $e(I) \leqslant n-1$, and $e(N) < n-1$; further it is required that $J$ contains odd components.*

*Proof.* Since $\Omega P_n \cong K(Z_2, n-1)) \times K(Z_2, 2(n-1))$, it follows that

$$H^*(P_n) \cong H^*(K(Z_2, n-1)) \otimes H^*(Z_2, 2(n-1)) \tag{6}$$

as algebras. By Serre [11] we then get

$$H^*(\Omega P_n) \cong Z_2[\{\mathrm{Sq}^I\varepsilon, \mathrm{Sq}^J\gamma\}], \tag{7}$$

where $I$ and $J$ run through all admissible sequences of excess less than $n-1$ and $2(n-1)$ respectively, and where $\varepsilon$ and $\gamma$ are (the images of) the basic classes in $K(Z_2, n-1)$ and $K(Z_2, 2(n-1))$ respectively.

In the spectral sequence of $\Omega P_n \to LP_n \to P_n$ we have $d_r = 0$ for $2 \leqslant r < n$, $d_n\varepsilon = \alpha$, and $d_n(\alpha \otimes \varepsilon) = \alpha^2 = 0$. As mentioned above, $\alpha$ is the non-zero class in $H^n(P_n)$ $(H^i(P_n) = 0$ for $0 < i < n)$, and $\alpha^2 = 0$ follows from the special form of the (first) $k$-invariant of $P_n$.

We see now that there must be a class $x$ in $E_n^{0,2(n-1)}$ with the property $d_n(x)=\alpha\otimes\varepsilon$ since $\alpha\otimes\varepsilon$ otherwise would determine a non-zero class in $E_\infty$ contradicting the contractibility of $LP_n$. From (7) and the fact that the Steenrod squares commute with transgression it follows that $\gamma$ has this property.

$$d_n(\varepsilon)=\alpha, \quad d_n(\gamma)=\alpha\otimes\varepsilon. \tag{8}$$

The set of generators for $H^*(\Omega P_n)$ given in (7) is not so useful for our purposes. Before changing to another set of generators let us define certain elements $\tau^I$, $I$ admissible, $H^*(\Omega P_n)$. This is done inductively,

$$\left.\begin{aligned} \tau^I &= \gamma \quad \text{for } I \text{ empty},\\ \tau^{(2j)(2J)} &= \sum_{\sigma=0}^{j-1} \mathrm{Sq}^\sigma \mathrm{Sq}^J \varepsilon \cdot \mathrm{Sq}^{2j-\sigma}\mathrm{Sq}^J\varepsilon + \mathrm{Sq}^{2j}\tau^{2J},\\ \tau^{(2j+1)(2J)} &= \sum_{\sigma=0}^{j} \mathrm{Sq}^\sigma \mathrm{Sq}^J \varepsilon \cdot \mathrm{Sq}^{2j+1-\sigma}\mathrm{Sq}^J\varepsilon + \mathrm{Sq}^{2j+1}\tau^{2J}. \end{aligned}\right\} \tag{9}$$

If $I=(j)J$ and $J$ contains an odd component, then

$$\tau^I = \mathrm{Sq}^j \tau^J.$$

This defines $\tau^I$ for all admissible sequences $I$. It is easy to see that

$$\left.\begin{aligned} &\tau^J \in H^{2(n-1)+\deg J}(\Omega P_n),\\ &H^*(\Omega P_n) = Z_2[\{\mathrm{Sq}^I\varepsilon, \tau^J\}], \end{aligned}\right\} \tag{10}$$

where $I$ and $J$ run through all admissible sequences of excess less than $n-1$ and $2(n-1)$ respectively.

Next we must determine the differentials of the generators $\tau^J$ in the spectral sequence of

$$\Omega P_n \to LP_n \to P_n. \tag{11}$$

LEMMA 11.2. *If $J$ contains odd components, then $\tau^J$ is transgressive. If $I=2N$, then $\tau^I$ persists to $E_{n+\deg N}$ and*

$$d_{n+\deg N}\{\tau^I\} = \{\mathrm{Sq}^N \alpha\otimes\mathrm{Sq}^N\varepsilon\}. \tag{12}$$

*The element $\mathrm{Sq}^N\alpha\otimes\mathrm{Sq}^N\varepsilon\cdot\tau^I$ is transgressive, i.e. persists to $E_{3(n-1+\deg N)+1}$.*

*Proof.* Since $d_n\gamma=\alpha\otimes\varepsilon$ (see (8)), the assumptions in remark 8.1 are clearly satisfied for the classes $\varepsilon\in E_n^{0,n-1}$, $\alpha\in E_n^{n,0}$, and $\gamma\in E_n^{0,2(n-1)}$ ($\gamma=\tau^I$ for $I$ empty). There therefore exist cochain representatives satisfying (1) of section 8. This means that $\varepsilon$, $\alpha$, $\gamma$ satisfy the assumptions of Lemma 8.2. Suppose inductively that $\mathrm{Sq}^N\varepsilon$, $\mathrm{Sq}^N\alpha$, and $\tau^I$ satisfy the assumptions of Lemma 8.2 for all $I=2N$ of length $<r$. then Lemma 8.2 shows that if $I=2N$ is of length $r$, then (12) holds true and that $\mathrm{Sq}^N\varepsilon$, $\mathrm{Sq}^N\alpha$, $\tau^I$ satisfy the assumptions (for definition of $\tau^I$ see (9)). This proves that not only is (12) true in general but $\mathrm{Sq}^N\varepsilon$, $\mathrm{Sq}^N\alpha$, $\tau^I$ satisfy the assumptions of Lemma 8.2 for all $I=2N$. By Lemma 8.2 we therefore also get that $\tau^{(2j+1)(2N)}$ is transgressive (transgresses into $\beta((2j+1)(2N))$ say) for all $N$. Also by Lemma 8.3, for each $N, \mathrm{Sq}^N\alpha\otimes\mathrm{Sq}^N\varepsilon\cdot\tau^I$ is transgressive (and transgresses into $\gamma(I)$ say). We now only need to show that if $J=\bar{J}(2j+1)(2N)$, then $\tau^J$ is transgressive. Since by (9)

$$\tau^J=\mathrm{Sq}^{\bar{J}}(\tau^{(2j+1)(2N)}),$$

since $\tau^{(2j+1)(2N)}$ transgresses into $\beta((2j+1)(2N))$, and since $\mathrm{Sq}^{\bar{J}}$ commutes with transgression, it follows that $\tau^J$ transgresses into $\beta(J)=\mathrm{Sq}^{\bar{J}}(\beta((2j+1)(2N)))$. This completes the proof.

The proof of this lemma shows that for all $J$ and $I$ of excess $<2(n-1)$, $J$ containing odd components and all components of $I$ even, there are classes

$$\left.\begin{aligned}\beta(J)&\in H^{2n-1+\deg J}(P_n),\\ \gamma(I)&\in H^{2(2n-1+\deg I)}(P_n),\end{aligned}\right\}\tag{13}$$

such that

$$\left.\begin{aligned}d_{2n-1+\deg J}\{\tau^J\}&=\{\beta(J)\},\\ d_{3(n-1+\deg N)+1}\{\mathrm{Sq}^N\alpha\otimes\mathrm{Sq}^N\varepsilon\cdot\tau^I\}&=\{\gamma(I)\},\end{aligned}\right\}\tag{14}$$

and such that if $J=\bar{J}(2j+1)J_1$, where all components of $J_1$ are even, then

$$\beta(J)=\mathrm{Sq}^{\bar{J}}(\beta((2j+1)J_1)).\tag{15}$$

LEMMA 11.3. *If $J$ contains odd components, then $(\tau^J)^{2^h}, h=0,1,\ldots,$ is transgressive and*

$$d_t\{(\tau^J)^{2^h}\}=\{\mathrm{Sq}^{L(2(n-1)+\deg J,h)}\beta(J)\},$$

*where* $t=2^h(2(n-1)+\deg J)+1$.

*Proof*. This follows from the fact

$$(\tau^J)^{2^h}=\mathrm{Sq}^{L(2(n-1)+\deg J,h)}\tau^J.$$

For each pair $(J,h)$ let $V(J,h)$ be the fibre in the elementary spectral sequence $A(k)$ (see section 9) with $k=\dim((\tau^J)^{2^h})+1$ and let a mapping

$$f(J,h)\colon\ V(J,h)\to H^*(\Omega P_n) \tag{16}$$

be defined by $f(J,h)(x)=(\tau^J)^{2^h}$. By Lemma 2.2 we get

LEMMA 11.4. *If $J$ contains odd components, then the mapping*

$$\bigotimes_{h=0}^{\infty} f(J,h)\colon\ \bigotimes_{h=0}^{\infty} V(J,h)\to Z_2[\tau^J]\subset H^*(\Omega P_n)$$

*is an isomorphism as a mapping of graded vector spaces.*

Now let $I=2N$ be admissible and of excess $<2n-2$.

LEMMA 11.5. *The element $\mathrm{Sq}^N\varepsilon\cdot\tau^{2N}$ persists to $E_{n+\deg N}$ and*

$$d_{n+\deg N}\{\mathrm{Sq}^N\varepsilon\cdot\tau^{2N}\}=\{\mathrm{Sq}^N\alpha\otimes((\mathrm{Sq}^N\varepsilon)^2+\tau^{2N})\}.$$

*Proof.* Since $d_{n+\deg N}\{\mathrm{Sq}^N\varepsilon\}=\{\mathrm{Sq}^N\alpha\}$, the lemma follows from Lemma 11.2 using the fact that $d_{n+\deg N}$ is a derivation.

LEMMA 11.6. *The element $(\mathrm{Sq}^N\varepsilon)^2+\tau^{2N}$ persists to $E_{n+\deg N}$ and*

$$d_{n+\deg N}\{(\mathrm{Sq}^N\varepsilon)^2+\tau^{2N}\}=\{\mathrm{Sq}^N\alpha\otimes\mathrm{Sq}^N\varepsilon\}.$$

*Proof.* Since $d_{n+\deg N}$ is a differential, it follows that $d_{n+\deg N}((\mathrm{Sq}^N\varepsilon)^2)=0$. The lemma now follows from Lemma 11.2.

LEMMA 11.7. *The elements $(\mathrm{Sq}^N\varepsilon)^{2^h}$, $((\mathrm{Sq}^N\varepsilon)^2+\tau^{2N})^{2^h}$, and $(\mathrm{Sq}^N\varepsilon\cdot\tau^{2N})^{2^h}$ persist till the $(2^h(n-1+\deg N)+1)$-term in the spectral sequence and*

$$d_s\{(\mathrm{Sq}^N\varepsilon)^{2^h}\}=\{\mathrm{Sq}^{LN}\alpha\}, \tag{17}$$

$$d_s\{((\mathrm{Sq}^N\varepsilon)^2+\tau^{2N})^{2^h}\}=\{\mathrm{Sq}^{LN}\alpha\otimes(\mathrm{Sq}^N\varepsilon)^{2^h}\}, \tag{18}$$

$$d_s\{(\mathrm{Sq}^N\varepsilon\cdot\tau^{2N})^{2^h}\}=\{\mathrm{Sq}^{LN}\alpha\otimes((\mathrm{Sq}^N\varepsilon)+\tau^{2N})^{2^h}\}, \tag{19}$$

*where $s=2^h(n-1+\deg N)+1$ and $L=L(n-1+\deg N,h)$. The element*

$$\mathrm{Sq}^{LN}\alpha\otimes(\mathrm{Sq}^N\varepsilon\cdot\tau^{2N})^{2^h}$$

*persists till the $(3\cdot 2^h(n-1+\deg N)+1)$-term and*

$$d_t\{\mathrm{Sq}^{LN}\alpha\otimes(\mathrm{Sp}^N\varepsilon\cdot\tau^{2N})^{2^h}\}=\{\mathrm{Sq}^{L(4(n-1+\deg N),h)}(\gamma(2N))\}, \tag{20}$$

*where $\gamma(I)=\gamma(2N)$ is as in (12) and $t=3\cdot 2^h(n-1+\deg N)+1$.*

*Proof.* Since $(\mathrm{Sq}^N\varepsilon)^{2^h}=\mathrm{Sq}^{LN}\varepsilon$, (17) follows from (8) and the commutativity of the squaring operation with differentials. For $h=0$ (18), (19), and (20) are proved in the

Lemmas 11.5, 11.6, and 11.2. Since the differentials are derivations, we see that to prove (18) and (19) in the general case we only need to show

$$d_s\{(\tau^{2N})^{2^h}\} = \{\mathrm{Sq}^{\mathrm{LN}}\alpha\otimes(\mathrm{Sq}^{\mathrm{N}}\varepsilon)^{2^h}\}. \tag{21}$$

We do this by induction on $h$, assuming (21) is true for all powers of 2 less than $2^h$. By the commutativity of the squaring operation and the differentials (Theorem 7.1) and by Theorem 7.6 we have

$$\begin{aligned}
d_s\{(\tau^{2\mathrm{N}})^{2^h}\} &= d_s(\mathrm{Sq}^{2^h(\mathrm{n}-1+\deg \mathrm{N})}\{(\tau^{2N})^{2^{h-1}}\}) \\
&= \mathrm{Sq}^{2^{\mathrm{h}}(\mathrm{n}-1+\deg \mathrm{N})}(d_{2^{h-1}(n-1+\deg N)+1}\{(\tau^{2N})^{2^{h-1}}\}) \\
&= \mathrm{Sq}^{2^{\mathrm{h}}(\mathrm{n}-1+\deg \mathrm{N})}\{\mathrm{Sq}^{\mathrm{L}(\mathrm{n}-1+\deg \mathrm{N},\, \mathrm{h}-1)}\mathrm{Sq}^{\mathrm{N}}\alpha\otimes(\mathrm{Sq}^{\mathrm{N}}\varepsilon)^{2^{h-1}}\} \\
&= \{\mathrm{Sq}^{2^{\mathrm{h}-1}(\mathrm{n}-1+\deg \mathrm{N})}\mathrm{Sq}^{\mathrm{L}(\mathrm{n}-1+\deg \mathrm{N},\, \mathrm{h}-1)}\mathrm{Sq}^{\mathrm{N}}\alpha\otimes(\mathrm{Sq}^{\mathrm{N}}\varepsilon)^{2^h}\} \\
&= \{\mathrm{Sq}^{\mathrm{LN}}\alpha\otimes(\mathrm{Sq}^{\mathrm{N}}\varepsilon)^{2^h}\}.
\end{aligned}$$

This proves (21). To prove (20) in general we again proceed by induction on $h$. The general step is here as follows

$$\begin{aligned}
&d_t\{\mathrm{Sq}^{\mathrm{LN}}\alpha\otimes(\mathrm{Sq}^{\mathrm{N}}\varepsilon\cdot\tau^{2N})\,2^h\} \\
&\quad= d_t(\mathrm{Sq}^{2^{\mathrm{h}+1}(\mathrm{n}-1+\deg \mathrm{N})}\{\mathrm{Sq}^{\mathrm{L}(\mathrm{n}-1+\deg \mathrm{N},\, \mathrm{h}-1)}\mathrm{Sq}^{\mathrm{N}}\alpha\otimes(\mathrm{Sq}^{\mathrm{N}}\varepsilon\cdot\tau^{2N})^{2^{h-1}}\}) \\
&\quad= \mathrm{Sq}^{2^{\mathrm{h}+1}(\mathrm{n}-1+\deg \mathrm{N})}d_{3.2^{h-1}(n-1+\deg N)+1}\{\mathrm{Sq}^{\mathrm{L}(\mathrm{n}-1+\deg \mathrm{N},\, \mathrm{h}-1)\mathrm{N}}\alpha\otimes(\mathrm{Sq}^{N}\varepsilon\tau^{2N})^{2^{h-1}}\} \\
&\quad= \mathrm{Sq}^{2^{\mathrm{h}+1}(\mathrm{n}-1+\deg \mathrm{N})}\{\mathrm{Sq}^{\mathrm{L}(4(\mathrm{n}-1+\deg \mathrm{N}),\, \mathrm{h}-1)}(\gamma(2N))\} \\
&\quad= \{\mathrm{Sq}^{\mathrm{L}(4(\mathrm{n}-1+\deg \mathrm{N}),\, \mathrm{h})}(\gamma(2N))\},
\end{aligned}$$

which is what we wanted to prove.

For each pair $(N,h)$, $e(N)<n-1$, let $T(N,h)$ be the fibre in the elementary spectral sequence $C(k,2)$ (cf. section 9), with $k=\dim(\mathrm{Sq}^{\mathrm{N}}\varepsilon)^{2^h})+1$, and let a vector space mapping

$$g(N,h)\colon\ T(N,h)\to H^*(\Omega P_n) \tag{22}$$

be defined by

$$\left.\begin{aligned}
g(N,h)\,(x) &= (\mathrm{Sq}^{\mathrm{N}}\varepsilon)^{2^h}, \\
g(N,h)\,(w) &= ((\mathrm{Sq}^{\mathrm{N}}\varepsilon)^2+\tau^{2\mathrm{N}})^{2^h}, \\
(g(N,h)\,(xw) &= (\mathrm{Sq}^{\mathrm{N}}\varepsilon\cdot\tau^{2\mathrm{N}})^{2^h}.
\end{aligned}\right\} \tag{23}$$

By Lemma 2.2 we get

LEMMA 11.8. *The mapping*

$$\bigotimes_{h=0}^{\infty} g(N,h)\colon\ \bigotimes_{h=0}^{\infty} T(N,h)\to Z_2[\mathrm{Sq}^{\mathrm{N}}\varepsilon,\tau^{2\mathrm{N}}]\subset H^*(\Omega P_n) \tag{24}$$

*is an isomorphism as a mapping of graded vector spaces.*

By (10), (11), and the Lemmas 11.4 and 11.8 we get

LEMMA 11.9. *The mappings* $f(J,h)$ *and* $g(N,h)$ *defined in* (16) *and* (23) *decompose* $H^*(\Omega P_n)$ *into a tensor product, i.e.*

$$\bigotimes_{(J,h)} f(J,h) \otimes \bigotimes_{(N,h)} g(N,h)\colon \bigotimes_{(J,h)} V(J,h) \otimes \bigotimes_{(N,h)} T(N,h) \to H^*(\Omega P_n)$$

*is an isomorphism as a mapping of graded vector spaces.*

For each $(J,h)$ and $(N,h)$ the mappings $f(J,h)$ (16) and $g(N,h)$ (22) satisfy the conditions $(a)$ and $(c)$ in Lemma 9.1 respectively. These conditions are namely nothing but the statements of Lemma 11.3 and Lemma 11.7 with $a = \mathrm{Sq}^{\mathrm{L}(2(\mathrm{n}-1)+\deg \mathrm{J},\mathrm{h})}(\beta(J))$, $\bar{a} = \mathrm{Sq}^{\mathrm{LN}}\alpha$, and $\bar{b} = \mathrm{Sq}^{\mathrm{L}(4(\mathrm{n}-1+\deg \mathrm{N}),\mathrm{h})}(\gamma(2N))$. Lemma 9.1 therefore shows that $f(J,h)$ and $g(N,h)$ can be extended to mappings

$$\left.\begin{aligned} f(J,h)&\colon A(k)\to E, && k=\dim((\tau^J)^{2^h})+1,\\ g(N,h)&\colon C(k,2)\to E, && k=\dim((\mathrm{Sq}^N\varepsilon)^{2^h})+1. \end{aligned}\right\} \tag{25}$$

Since also by Lemma 2.3, $(\mathrm{Sq}^{\mathrm{LN}}\alpha)^2 = 0$ all assumptions of Lemma 9.2 are satisfied we get

$$\left.\begin{aligned} H^*(P_n) = {} & \Lambda(\{\mathrm{Sq}^{\mathrm{L}(\mathrm{n}-1+\deg \mathrm{N},\mathrm{h})\mathrm{N}}(\alpha)\})\\ & \otimes Z_2[\{\mathrm{Sq}^{\mathrm{L}(4(\mathrm{n}-1+\deg \mathrm{N}),\mathrm{h})}(\gamma(2N))\}]\\ & \otimes Z_2[\{\mathrm{Sq}^{\mathrm{L}(2(\mathrm{n}-1)+\deg \mathrm{J},\mathrm{h})}(\beta(J))\}], \end{aligned}\right\} \tag{26}$$

where as mentioned earlier $J$ and $2N$ run through all admissible sequences of excess less than $2n-2$ such that $J$ contains odd components, and $h$ runs through all non-negative integers. An easy rewriting of (26) gives Theorem 11.1.

As an example we get

$$H^*(P_2) \cong Z_2[\{\mathrm{Sq}^{\mathrm{J}}\beta\}] \otimes \Lambda(\{\mathrm{Sq}^{\mathrm{L}(1,\mathrm{h})}\alpha\}) \otimes Z_2[\{\mathrm{Sq}^{\mathrm{L}(4,\mathrm{h})}\gamma\}], \tag{27}$$

where the dimensions of $\alpha$, $\beta$, and $\gamma$ are 2, 4, and 6 respectively, $J = (j_1, j_2, \dots, j_r)$ is admissible, of excess $\leqslant 3$, and $j_r > 1$. Again $h$ runs through all non-negative integers.

We remark that since $P_n = \Omega P(Z_2, n+1;\ Z_2, 2n, \mathrm{Sq}^n \varepsilon_{n+1})$, it follows that the spectral sequence of $\Omega P_n \to LP_n \to P_n$ is a sequence of Hopf-algebras. It follows that $\varepsilon$ and $\alpha$ are primitive. Since $d_n\{\gamma\} = \alpha \otimes \varepsilon$ is not primitive, $\gamma$ cannot be primitive. The diagonal of $\gamma$ must therefore be as follows,

$$\psi(\gamma) = 1 \otimes \gamma + \varepsilon \otimes \varepsilon + \gamma \otimes 1. \tag{28}$$

This shows that (6) is false considered as a tensor product of Hopf-algebras. By (28), however, we have complete knowledge about the diagonal in $H^*(\Omega P_n)$. For example, it is not hard to show that

$$\left.\begin{aligned}\psi(\tau^{2J}) &= 1\otimes\tau^{2J} + \mathrm{Sq}^J\varepsilon\otimes\mathrm{Sq}^J\varepsilon + \tau^{2J}\otimes 1,\\ \psi(\tau^I) &= 1\otimes\tau^I + \tau^I\otimes 1,\end{aligned}\right\} \tag{29}$$

for $I$ containing odd components.

A theorem of W. Browder on spectral sequences of Hopf-algebras, as yet not published, implies that a primitive element in the fibre must be transgressive provided its dimension is $\not\equiv 2 \pmod 4$. This provides a second proof that $\tau^I$ is transgressive for $I$ containing odd components.

Let us consider the spectral sequence of the fibration

$$P_2 \to LB \to B, \quad B = P(Z_2, 3;\ Z_2, 4, \mathrm{Sq}^2\varepsilon_3). \tag{30}$$

This is also a sequence of Hopf-algebras. The basic class $\varepsilon \in H^3(B)$ has the property $\mathrm{Sq}^2\varepsilon = 0$. Hence we have

$$\varepsilon^2 = \mathrm{Sq}^3\varepsilon = \mathrm{Sq}^1\mathrm{Sq}^2\varepsilon = \mathrm{Sq}^1 0 = 0. \tag{31}$$

In the spectral sequence of (30) (Fig. 1) we therefore have

$$d_3(\varepsilon\otimes\alpha) = \varepsilon^2 = 0 \tag{32}$$

and hence

$$d_3(\beta) = \varepsilon\otimes\alpha. \tag{33}$$

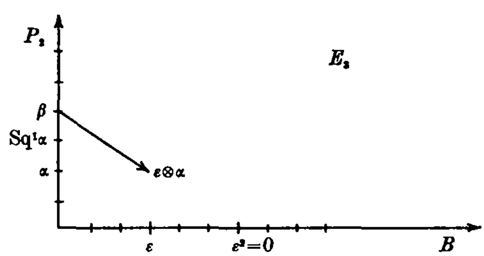


Fig. 1.

Since $\varepsilon$ and $\alpha$ are primitive, $\varepsilon\otimes\alpha$ is not primitive and (33) then shows that $\beta$ cannot be primitive. We therefore have in $H^*(P_2)$ (cf. (27))

$$\psi(\beta) = 1\otimes\beta + \alpha\otimes\alpha + \beta\otimes 1. \tag{34}$$

In the spectral sequence of

$$\Omega P_2 \to LP_2 \to P_2, \tag{35}$$

which we used in the determination of $H^*(P_2)$, we saw that $\tau^{(1)}=\varepsilon^3+\mathrm{Sq}^1\gamma$ had the property (cf. (14))

$$d_4\{\tau^{(1)}\}=d_4\{\varepsilon^3+\mathrm{Sq}^1\gamma\}=\beta \quad (=\beta(1)). \tag{36}$$

Since $\mathrm{Sq}^1$ commutes with transgression we get

$$d_5\,\mathrm{Sq}^1\{\varepsilon^3+\mathrm{Sq}^1\gamma\}=d_5\{\varepsilon^4\}=\{\mathrm{Sq}^1\beta\} \tag{37}$$

and hence in $E_5$, by (17),

$$\{\mathrm{Sq}^2\mathrm{Sq}^1\alpha\}=\{\mathrm{Sq}^1\beta\}. \tag{38}$$

In $H^*(P_2)$ this gives

$$\mathrm{Sq}^1\beta=\mathrm{Sq}^2\mathrm{Sq}^1\alpha+k\alpha\,\mathrm{Sq}^1\alpha \quad (k=0,1\in Z_2). \tag{39}$$

Applying $\psi$ to both sides we get by (34)

$$\begin{aligned}&1\otimes\mathrm{Sq}^1\beta+\alpha\otimes\mathrm{Sq}^1\alpha+\mathrm{Sq}^1\alpha\otimes\alpha+\mathrm{Sq}^1\alpha\otimes 1\\&\quad=1\otimes\mathrm{Sq}^2\mathrm{Sq}^1\alpha+\mathrm{Sq}^2\mathrm{Sq}^1\alpha\otimes 1+k(1\otimes\mathrm{Sq}^1\alpha+\alpha\otimes\mathrm{Sq}^1\alpha+\mathrm{Sq}^1\alpha\otimes\alpha+\alpha\,\mathrm{Sq}^1\alpha\otimes 1).\end{aligned} \tag{40}$$

This shows that $k$ must be one, and we have

$$\mathrm{Sq}^1\beta=\mathrm{Sq}^2\mathrm{Sq}^1\alpha+\alpha\,\mathrm{Sq}^1\alpha. \tag{41}$$

Further results on the action of Steenrod algebra in $H^*(P_n)$ can be obtained along these lines.

## 12. Computation of $H^*(P(Z_2,n;\,Z_2,2^hn-1,\varepsilon_n^{2^h}),Z_2)$

If all components in a sequence $I$ are divisible by an integer $k$ we write

$$I\equiv 0 \pmod k.$$

If this is not so we write

$$I\not\equiv 0 \pmod k.$$

THEOREM 12.1. *Let $P_{n,h}=P(Z_2,n;\,Z_2,2^hn-1,\varepsilon_n^{2^h})$ $(n\geqslant 2, h\geqslant 2)$. For each admissible sequence $J$, $e(J)\leqslant 2^hn-2$, $J\not\equiv 0 \pmod{2^h}$, and for each admissible sequence $I$, $e(I)\leqslant n-1$, there are classes $\beta(J)$ and $\gamma(I)$ in $H^*(P_{n,h})$ of dimensions $2^hn-1+\deg J$ and $2^{h+1}(n+\deg I)-2$ respectively, satisfying*

$$\beta(J)=\mathrm{Sq}^{\bar J}(\beta((j)\,J_1)$$

*whenever $J=\bar J(j)J_1$ with $j\not\equiv 0 \pmod{2^h}$ and $J_1\equiv 0 \pmod{2^h}$.*

*Let $\alpha$ be the non-zero class in $H^n(P_{n,h})$ then*

$$H^*(P_{n,h})=Z_2[\{\beta(J)\}]\otimes Z_2[\{\mathrm{Sq}^I\alpha\},2^h]\otimes Z_2[\{\gamma(I)\}],$$

*where* $Z_2[\{x_i\}, 2^h]$ *denotes the truncated polynomial algebra of height* $2^h$ *in the generators* $\{x_i\}$ $(x_i^{2^h}=0)$, *and where* $J$ *and* $I$ *run through all admissible sequences satisfying* $e(J) \leqslant 2^h n - 2$, $J \not\equiv 0 \pmod{2^h}$, *and* $e(I) \leqslant n-1$.

*Proof.* Since $\Omega P_{n,h} \simeq K(Z_2, n-1) \times K(Z_2, 2^h n - 2)$, it follows that

$$H^*(\Omega P_{n,h}) = H^*(K(Z_2, n-1)) \otimes H^*(K(Z_2, 2^h n - 2)) \tag{1}$$

as algebras. By Serre [11] a set of polynomial generators of $H^*(\Omega P_{n,h})$ is given by

$$\left.\begin{array}{l} \{\mathrm{Sq}^I \varepsilon\}, I \text{ admissible, } e(I) < n-1, \dim \varepsilon = n-1, \\ \{\mathrm{Sq}^J \gamma\}, J \text{ admissible, } e(J) < 2^h n - 2, \dim \gamma = 2^h n - 2. \end{array}\right\} \tag{2}$$

Let us note that $(\mathrm{Sq}^I \varepsilon)^{2^k} = \mathrm{Sq}^{LI} \varepsilon$ with $L = L(n-1+\deg I, k)$. We see that $e(LI) = n-1$, and also that if $N$ is an admissible sequence with $e(N) = n-1$, then in one and only one way $N$ can be written in the form $LI$ with $e(I) < n-1$. From this it follows (cf. Lemma 2.2) that as vector spaces

$$H^*(\Omega P_{n,h}) \cong \Lambda(\{\mathrm{Sq}^I \varepsilon\}) \otimes \Lambda(\{\mathrm{Sq}^J \gamma\}), \tag{3}$$

where $I$ and $J$ run through all admissible sequences with $e(I) \leqslant n-1$ and $e(J) \leqslant 2^h n - 2$.

By an argument similar to the one given in section 11 it follows that in the spectral sequence of

$$\Omega P_{n,h} \to L P_{n,h} \to P_{n,h} \tag{4}$$

we have

$$\left.\begin{array}{l} \alpha^{2^h} = 0, \\ d_r = 0 \quad \text{for} \quad 2 \leqslant r < n, \\ d_n \varepsilon = \alpha,\ d_n \varepsilon = \alpha^{2^h - 1} \otimes \varepsilon. \end{array}\right\} \tag{5}$$

We proceed to determine the differentials in the spectral sequence of (4). The following lemma follows by induction from Remark 8.4 and the Lemmas 8.5 and 8.6 in the same way Lemma 11.2 followed from Remark 8.1 and the Lemmas 8.2 and 8.3.

LEMMA 12.2. *If* $J \not\equiv 0 \pmod{2^h}$ *then* $\mathrm{Sq}^J \gamma$ *is transgressive. If* $J = 2^h I$ *then* $\mathrm{Sq}^J \gamma = \mathrm{Sq}^{2^h I} \gamma$ *persists to* $E_{(2^h-1)(n+\deg I)}$ *and*

$$d_{(2^h-1)(n+\deg I)}\{\mathrm{Sq}^{2^h I} \gamma\} = \{(\mathrm{Sq}^I \alpha)^{2^h - 1} \otimes \mathrm{Sq}^I \varepsilon\}.$$

*The element* $(\mathrm{Sq}^I \alpha)^{2^h-1} \otimes \mathrm{Sq}^I \varepsilon \cdot \mathrm{Sq}^{2^h I} \gamma$ *is transgressive (i.e. persists to* $E_{(2^h+1)(n+\deg I)-2}$).

This lemma shows that for all $J$, $J \not\equiv 0 \pmod{2^h}$, $e(J) \leqslant 2^h n - 2$, and all $I$, $e(I) \leqslant n-1$, there are classes

$$\left.\begin{array}{l}\beta(J)\in H^{2^h n-1+\deg J}(P_{n,h}),\\ \gamma(I)\in H^{2^{h+1}(n+\deg I)-2}(P_{n,h}),\end{array}\right\} \tag{6}$$

such that
$$\left.\begin{array}{l}d_{2^h n-1+\deg J}\{\mathrm{Sq}^J\gamma\}=\{\beta(J)\},\\ d_{(2^h+1)(n+\deg I)-2}\{(\mathrm{Sq}^I\alpha)^{2^h-1}\otimes\mathrm{Sq}^I\varepsilon\cdot\mathrm{Sq}^{2hI}\gamma\}=\{\gamma(I)\},\end{array}\right\} \tag{7}$$

and such that if $J=\bar{J}(j)J_1$ with $j\not\equiv 0\ (\mathrm{mod}\ 2^h)$ and $J_1\equiv 0\ (\mathrm{mod}\ 2^h)$, then

$$\beta(J)=\mathrm{Sq}^{\bar{J}}(\beta((j)J_1)). \tag{8}$$

LEMMA 12.3. *The elements* $\mathrm{Sq}^I\varepsilon$ *and* $\mathrm{Sq}^I\varepsilon\cdot\mathrm{Sq}^{2hI}\gamma$ *persist to* $E_{n+\deg I}$ *and*

$$d_{n+\deg I}\{\mathrm{Sq}^I\varepsilon\}=\{\mathrm{Sq}^I\alpha\},$$
$$d_{n+\deg I}\{\mathrm{Sq}^I\varepsilon\cdot\mathrm{Sq}^{2hI}\gamma\}=\{\mathrm{Sq}^I\alpha\otimes\mathrm{Sq}^{2hI}\gamma\}$$

*Proof.* The first equation follows from (5) and the commutativity of $\mathrm{Sq}^I$ with differentials. By Lemma 12.2 $d_{n+\deg I}\{\mathrm{Sq}^{2hI}\gamma\}=0$. The second equation is therefore a consequence of the first and the fact that differentials are derivations. This completes the proof.

For each admissible sequence $J, J\not\equiv 0\ (\mathrm{mod}\ 2^h)$ and $e(J)\leqslant 2^h n-2$, let $V(J)$ be the fibre in the elementary spectral sequence $A(k)$ (see section 9) with $k=2^h n-1+\deg J$, and let a mapping

$$f(J)\colon\ V(J)\rightarrow H^*(\Omega P_{n,h}) \tag{9}$$

be defined by $f(J)(x)=\mathrm{Sq}^J\gamma$. Since by Lemma 12.2, this mapping satisfies condition $(a)$ in Lemma 9.1 with $a=\beta(J)$ it can be extended to a mapping

$$f(J)\colon\ A(k)\rightarrow E,\quad k=2^h n-1+\deg J. \tag{10}$$

For each admissible sequence $I$ with $e(I)\leqslant n-1$ let $T(I)$ be the fibre in the elementary spectral sequence $C(k,2^h)$ with $k=n+\deg I$, and let a mapping

$$g(I)\colon\ T(I)\rightarrow H^*(\Omega P_{n,h}) \tag{11}$$

be defined by
$$\left.\begin{array}{r}g(I)(x)=\mathrm{Sq}^I\varepsilon,\\ g(I)(w)=\mathrm{Sq}^{2hI}\gamma,\\ g(I)(xw)=\mathrm{Sq}^I\varepsilon\cdot\mathrm{Sq}^{2hI}\gamma.\end{array}\right\} \tag{12}$$

Since by the Lemmas 12.2 and 12.3, this mapping satisfies condition $c$ of Lemma 9.1 with $\bar{a}=\mathrm{Sq}^I\alpha$ and $\bar{b}=\gamma(I)$ it can be extended to a mapping

$$g(I)\colon\ C(k,2^h)\rightarrow E\quad (k=n+\deg I). \tag{13}$$

LEMMA 12.4. *The mapping* $f(J)$ *and* $g(I)$ *defined in* (9) *and* (11) *decompose* $H^*(\Omega P_{n,h})$ *into a tensor product, i.e.*

$$\bigotimes_J f(J) \otimes \bigotimes_I g(I): \bigotimes_J V(J) \otimes \bigotimes_I T(I) \to H^*(\Omega P_{n,h})$$

*is an isomorphism as a mapping of graded vector spaces.*

*Proof.* When $I$ runs through all admissible sequences of excess $\leqslant n-1$, $2^h I$ will run through all sequences congruent to zero $(\bmod 2^h)$ of excess $\leqslant 2^h n - 2$. Since $J$ runs through all sequences $\not\equiv 0 \pmod{2^h}$, the lemma follows from (3).

Since by (5) and by Lemma 2.3, $(\mathrm{Sq}^I \alpha)^{2^h} = 0$, all the conditions of Lemma 9.2 are satisfied (see (10), (13), and Lemma 12.4). We therefore get

$$H^*(P_{n,h}) \cong Z_2[\{\beta(J)\}] \otimes Z_2[\{\mathrm{Sq}^I \alpha\}, 2^h] \otimes Z_2[\{\gamma(I)\}], \tag{14}$$

where as mentioned earlier $J$ and $I$ run through all admissible sequences satisfying $J \not\equiv 0 \pmod{2^h}$, $e(J) \leqslant 2^h n - 2$, and $e(I) \leqslant n-1$. This completes the proof of theorem 12.1.

We remark that contrary to the similar situation in section 11, it can be shown that (1) is an isomorphism as a mapping of Hopf-algebras. It is enough to show that $\gamma$ is primitive. By Lemma 12.2 we see that $\gamma^2 = \mathrm{Sq}^{2^h-2}\gamma$ is transgressive and hence primitive. Let

$$\psi(\gamma) = 1 \otimes \gamma + \sum_i \gamma_i' \otimes \gamma_i'' + \gamma \otimes 1 \tag{15}$$

be the diagonal of $\gamma$ with all the $\gamma_i'$'s and $\gamma_i''$'s in the vector space basis obtained by taking powers of the polynomial algebra generators (2). We then have

$$\psi(\gamma^2) = (\psi(\gamma))^2 = 1 \otimes \gamma^2 + \sum_i (\gamma_i')^2 \otimes (\gamma_i'')^2 + \gamma^2 \otimes 1. \tag{16}$$

Since $\gamma^2$ is primitive and since $(\gamma_i')^2$ and $(\gamma_i'')^2$ are in the above mentioned vector space basis of $H^*(\Omega P_{n,h})$ it follows that in (15) no $\gamma_i'$'s and $\gamma_i''$'s actually occur. Hence the primitivity of $\gamma$.

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