

ANALYTIC FUNCTIONS AND LOGMODULAR BANACH ALGEBRAS

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1. Introduction

The first part of this paper presents a generalization of a portion of the theory of analytic functions in the unit disc. The theory to be extended consists of some basic theorems related to the Hardy class H^p ($1 \leq p \leq \infty$). For example, (i) the theorem of Szegö, Kolmogoroff and Krein on mean-square approximation of 1 by polynomials which vanish at the origin, (ii) the theorems of F. and M. Riesz, on the absolute continuity of “analytic” measures, and on the integrability of $\log |f|$ for f in H^1 , (iii) Beurling’s theorem on invariant subspaces of H^2 , (iv) the factorization of H^p functions into products of “inner” and “outer” functions. The second part of the paper discusses the embedding of analytic discs in the maximal ideal space of a function algebra.

The paper was inspired by the work of Arens and Singer [3; 4], Bochner [6], Helson and Lowdenslager [14; 15], Newman [24], and Wermer [27]. Some of the proofs we employ are minor modifications of arguments due to these authors; however, the paper is self-contained and assumes only standard facts of abstract “real variable” theory, e.g., fundamental theorems on measure and integration, Banach spaces, and Hilbert spaces. In particular, very little knowledge of analytic function theory is essential for reading the paper, since the classical results which are to be generalized are special cases of the theorems here.

The Hardy class H^p ($1 \leq p < \infty$) consists of those analytic functions f in the unit disc for which the integrals

$$\int |f(re^{i\theta})|^p d\theta$$

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are bounded as r tends to 1. Since these spaces arise naturally in the study of Abel–Poisson summability of Fourier series, it was realized early in their development that many properties of H^p functions belonged in the realm of “real variable” theory. Specifically, we mean that general compactness arguments enable one to write each H^p function as the Poisson integral of an L^p function on the boundary (or, a measure in the case $p=1$); and thus, by identifying an H^p function with its boundary-values, some results can be deduced without reference to the geometry of analytic mappings. It now appears that the portion of the theory which is susceptible of a “real variable” treatment is much larger than one would have imagined.

From the point of view which we adopt, the basic vehicle for the study of H^p is the algebra of continuous functions on the unit circle whose Fourier coefficients vanish on the negative integers. The space H^p is regarded as the completion of this algebra in the Banach space L^p (of the unit circle). Thus, in our general treatment, we begin with a uniformly closed algebra A , consisting of continuous complex-valued functions on a compact Hausdorff space X . The role of the points of the unit disc is played by certain positive measures on the space X . In the case of the disc, these are the harmonic (Poisson) measures for the various points. What is most important to us is that these measures are multiplicative on the algebra:

$$\int fg dm = \int f dm \int g dm \quad (f, g \in A).$$

If we fix a measure m on X which is multiplicative on the algebra A , we can introduce $H^p(dm)$, the completion of A in the Banach space $L^p(dm)$. Of course, we cannot expect many interesting results about H^p in such extreme generality. We need some hypothesis on the underlying algebra A which (roughly) forces it to resemble an algebra of analytic functions. Our hypothesis is that A is a *logmodular algebra*, by which we mean that each real-valued continuous function on X can be uniformly approximated by functions $\log |f|$, where both f and $1/f$ belong to A . For such algebras, and any multiplicative measure m , we can study $H^p(dm)$ with considerable success. We shall concentrate on the values $p=1, 2$, and ∞ .

In their paper [14], Henry Helson and David Lowdenslager studied algebras of continuous functions on special compact abelian groups. The algebras were isomorphic to rings of analytic almost periodic functions in a half-plane, and they are described in Example 3, section 3. Arens and Singer [4] began the study of analytic function theory for these algebras, in the sense of working directly on the compact groups. Helson and Lowdenslager discovered elegant proofs for most of the theorems itemized in the first paragraph of the Introduction

(as they apply to H^p as the completion of the algebra in L^p of the compact group). After their paper appeared, Bochner [6] pointed out that several of their proofs could be applied to a general class of rings of functions. The abstract rings which Bochner described are much like certain function algebras which were being studied at the time, namely, Dirichlet algebras. (The algebra of continuous functions A is called a Dirichlet algebra provided each real-valued continuous function on X is a uniform limit of real parts of functions in A .) Indeed, a careful reading of Helson-Lowdenslager revealed that their proofs were valid for Dirichlet algebras, virtually without change. Wermer [27] made use of some of the arguments to embed analytic discs in the maximal ideal space of a Dirichlet algebra. The Helson-Lowdenslager arguments are applied to the classical case in the author's book [18]. A description of the form of some of the results for Dirichlet algebras appears there, as well as in Wermer's expository paper [28]. However, a detailed treatment of the proofs in the Dirichlet algebra context has not been available until now.

Dirichlet algebras are special cases of what we call logmodular algebras. Thus, this paper will include a detailed development of the Dirichlet algebra results which we have been discussing. But, it will go considerably beyond this, for two reasons. First, even in the Dirichlet case, we shall affect a reorganization of the order of the theorems, as well as an increase in the number of theorems which are generalized. Second, for logmodular algebras, one cannot simply repeat the Helson-Lowdenslager arguments. They can be used, with some modifications; however, one must first prove some basic theorems about a logmodular algebra. The most basic of these theorems are (i) each complex homomorphism of the algebra A has a unique (positive) representing measure on the space X , (ii) if m is such a representing measure on X , the functions in A and their complex conjugates span $L^2(dm)$. These facts are evident for a Dirichlet algebra, but far from obvious for a logmodular algebra.

It is a tribute to the clarity and elegance of the Helson-Lowdenslager arguments that they are capable of generalization in many directions. If one wants only part of the results, there are various other hypotheses which one can place on the ring of functions A ; for example, hypotheses such as (i) and (ii) of the last paragraph. We shall try to indicate some of these weakened hypotheses as we go along. It is the author's feeling that, if one wants the full strength of the results, logmodular algebras provide the natural setting. In any event, this setting does capture the full strength of the theorems; and, it allows for a considerable amount of non-trivial generality, as one can see from the examples in section 3. There are two objections to our approach which might occur to one. First, we insist that our ring A should consist of continuous functions on a compact space. Second, we treat $H^p(dm)$, where m is an arbitrary measure which is multiplicative on A . One meets situations

in which A is a ring of bounded measurable functions on a measure space (S, Σ, m) , the measure m is multiplicative on A , and the interest is in the particular spaces $H^p(dm)$. But such a ring of functions is isomorphic to a ring of continuous functions on a compact space X (the maximal ideal space of $L^\infty(dm)$), and m can easily be transferred to a measure on X .

After H^p spaces have been discussed, we turn to analytic structures on subsets of the maximal ideal space of a logmodular algebra. Given our basic knowledge of logmodular algebras, one can employ the argument which Wermer [27] gave for Dirichlet algebras, to show that each Gleason "part" of the maximal ideal space is either one point or is an "analytic disc". This extension of Wermer's result is particularly interesting, because it applies to the algebra of bounded analytic functions in the unit disc. This algebra is (isomorphic to) a logmodular algebra, and very few things about the structure of its maximal ideal space are easy to treat.

The author would like to express his appreciation to Professor Richard Arens, for many enlightening discussions during the evolution of the concept of a logmodular algebra.

2. Notation and basic definitions

Throughout this paper, X will denote a compact Hausdorff space. We denote by $C(X)$ [$C_{\mathbb{R}}(X)$] the complex [real] linear algebra of all continuous complex [real] valued functions on X . Each of these algebras is a Banach space (Banach algebra) under the sup norm

$$\|f\| = \sup_X |f|.$$

By a *measure* on X we shall understand a finite complex Baire measure on X . We shall make frequent use of the Riesz representation theorem, in this form. Every bounded (i.e., continuous) linear functional L on $C_{\mathbb{R}}(X)$ is induced by a real measure μ on X ,

$$L(f) = \int f d\mu.$$

Similarly every bounded linear functional on $C(X)$ is induced by a (complex) measure on X . The norm of the linear functional L ,

$$\|L\| = \sup_{\|f\| \leq 1} |L(f)|,$$

is precisely the total variation of the measure μ . See [9; Chap. IV, §6].

DEFINITION 2.1. A *sup norm algebra* on X is a complex linear subalgebra A of $C(X)$ which satisfies

- (i) A is uniformly closed;
- (ii) the constant functions are in A ;
- (iii) A separates the points of X , i.e., if x and y are distinct points of X , there is an f in A with $f(x) \neq f(y)$.

If A is a sup norm algebra on X , we shall have occasion to discuss other classes of functions associated with A , and we shall adopt a uniform type notation for these classes. For example, A^{-1} will denote the set of invertible elements of A , that is, the set of all functions f in A such that $f^{-1} = 1/f$ is also in A ; $\text{Re } A$ will denote the set of all real parts of functions in A ; \bar{A} will denote the set of complex conjugates of functions in A ; and $\log |A^{-1}|$ will denote the set of logarithms of moduli of invertible elements of A .

DEFINITION 2.2. Let A be a sup norm algebra on X . A *complex homomorphism* of A is an algebra homomorphism, from A onto the field of complex numbers.

Since the sup norm algebra A is uniformly closed, it is a Banach space (Banach algebra) under the sup norm. We need to know that each complex homomorphism Φ is a bounded linear functional on that Banach space, indeed that

$$|\Phi(f)| \leq \|f\| \quad (f \in A). \quad (2.11)$$

This has a simple proof. If (2.11) does not hold, there is an f in A with $\Phi(f) = 1$ but $\|f\| < 1$. Since A is a uniformly closed algebra, the series expansion $(1-f)^{-1} = 1 + f + f^2 + \dots$ shows that $(1-f)$ is invertible in A . Since Φ is not the zero homomorphism, we must have $\Phi(1) = 1$; hence $\Phi(1-f) = 0$. We have the contradiction

$$1 = \Phi(1) = \Phi(1-f)\Phi([1-f]^{-1}) = 0.$$

Of course, (2.11) together with $\Phi(1) = 1$ tells us that the norm of Φ is precisely 1.

DEFINITION 2.3. Let A be a sup norm algebra on X , and let Φ be a complex homomorphism of A . A *representing measure* for Φ is a positive measure m on X such that

$$\Phi(f) = \int f dm \quad (f \in A).$$

An *Arens-Singer measure* for Φ is a positive measure m on X such that

$$\log |\Phi(f)| = \int \log |f| dm \quad (f \in A^{-1}).$$

It is important to note that both representing measures and Arens-Singer measures are required to be positive. Since $\Phi(1) = 1$, either type of measure satisfies

$$\int dm = 1$$

and is, consequently, a probability measure (positive measure of mass 1).

THEOREM 2.1 (Arens-Singer [3]). *Let Φ be a complex homomorphism of the sup norm algebra A . There exists at least one Arens-Singer measure for Φ . Furthermore, every Arens-Singer measure for Φ is a representing measure for Φ .*

Proof. Given Φ , we define a function L on the set $\log |A^{-1}|$ by

$$L(\log |f|) = \log |\Phi(f)| \quad (f \in A^{-1}).$$

We now extend L linearly to the linear span of $\log |A^{-1}|$. This linear span is a subspace of $C_R(X)$, and we shall verify that L is a well-defined function on that subspace and that L is bounded by 1:

$$|L(u)| \leq \sup_x |u|.$$

Obviously the second condition implies the first. If L is not bounded by 1, then, since we are dealing with real-valued functions, there will exist some u in the linear span of $\log |A^{-1}|$ with

$$L(u) > \max_x u. \tag{2.12}$$

Now u has the form $u = t_1 u_1 + \dots + t_n u_n$, where each u_j is of the form $u_j = \log |f_j|$, $f_j \in A^{-1}$, and the t_j are real numbers. The number $L(u)$ is defined by $L(u) = \sum t_j \log |\Phi(f_j)|$. We may assume that each t_j is rational, since (2.12) will not be affected by a small change of any t_j . Choose a positive integer r such that every rt_j is an integer, say $rt_j = p_j$. Then (2.12) says

$$\sum_j \frac{p_j}{r} \log |\Phi(f_j)| > \max_x \sum_j \frac{p_j}{r} \log |f_j|. \tag{2.13}$$

If we let $f = f_1^{p_1} \dots f_n^{p_n}$, then $f \in A^{-1}$ and (2.13) becomes

$$\log |\Phi(f)| > \max_x \log |f|.$$

This contradicts the fact that Φ is bounded by 1.

Now we have a linear function L on a subspace of $C_R(X)$, and L is bounded by 1. The Hahn-Banach theorem tells us that we can extend L to a linear functional \tilde{L} on $C_R(X)$ which is also bounded by 1. This \tilde{L} has the form

$$\tilde{L}(u) = \int u \, dm \quad (u \in C_R(X)),$$

where m is a real measure on X of total variation at most 1. But the constant function 1 is in $\log |A^{-1}|$ and $L(1) = 1$. Thus $\int dm = 1$. Since m has integral 1 and total variation at most 1, it is clear that m is a positive measure. Hence, we have produced an Arens-Singer measure for Φ .

If m is any Arens-Singer measure for Φ , then m is a representing measure for Φ . To show this, it will suffice to prove that

$$\int \operatorname{Re} f dm = \operatorname{Re} \Phi(f) \quad (f \in A).$$

If $f \in A$, then $e^f \in A^{-1}$; consequently

$$\int \operatorname{Re} f dm = \int \log |e^f| dm = \log |\Phi(e^f)| = \log |e^{\Phi(f)}| = \operatorname{Re} \Phi(f).$$

This concludes the major part of what we need to know about sup norm algebras in general. However, for our work in Section 7, and also to understand some of the examples in the next section, we require some familiarity with the maximal ideal space of a sup norm algebra.

For the sup norm algebra A , we denote by $M(A)$ the set of all complex homomorphisms of A . With each f in A we associate a complex-valued function \hat{f} on $M(A)$ by

$$\hat{f}(\Phi) = \Phi(f) \quad (\Phi \in M(A)). \quad (2.14)$$

If we topologize $M(A)$ with the weakest topology which makes all these functions \hat{f} continuous, then $M(A)$ becomes a compact Hausdorff space. This is a consequence of the fact that the Cartesian product of compact spaces is compact [10; 21]. This space $M(A)$ is known as the *space of complex homomorphisms* of A or the *maximal ideal space* of A . The latter terminology arises from the fact that there is a one-one correspondence between complex homomorphisms Φ of A and maximal ideals M in the algebra A . It is defined by $M = \ker(\Phi) = \{f \in A; \Phi(f) = 0\}$. One may consult [10; 21] for a proof, although we shall not need this result.

Each point x in X gives rise to a complex homomorphism Φ_x of A by

$$\Phi_x(f) = f(x).$$

It is not difficult to see that $x \rightarrow \Phi_x$ is a continuous map of X into $M(A)$. Since A separates the points of X and X is compact, this map is a homeomorphism of X into $M(A)$.

We have the representation $f \rightarrow \hat{f}$ of A by an algebra \hat{A} of continuous functions on $M(A)$. This representation is not only one-one but also isometric. The inequality

$$\sup_{M(A)} |\hat{f}| \leq \|f\|$$

results from the fact that each Φ in $M(A)$ is bounded by 1. On the other hand,

$$\sup_{M(A)} |\hat{f}| \geq \sup_X |\hat{f}(\Phi_x)| = \sup_X |f(x)| = \|f\|.$$

Therefore, when it is convenient, we may employ the isometric isomorphism $f \rightarrow \hat{f}$ to regard A as a uniformly closed algebra of continuous functions on $M(A)$; and we may also employ the homeomorphism $x \rightarrow \Phi_x$ to regard X as a compact subset of $M(A)$.

3. Logmodular algebras

We now introduce the class of algebras in which we are primarily interested.

DEFINITION 3.1. Let A be a sup norm algebra on X . We say that A is a *logmodular algebra on X* if the set of functions $\log |A^{-1}|$ is uniformly dense in $C_{\mathbb{R}}(X)$.

It is important to note that, in order for A to be logmodular, we require the set $\log |A^{-1}|$ to be dense, not its linear span. As we shall see, the distinction is important.

Dirichlet algebras provide a class of examples of logmodular algebras. A *Dirichlet algebra on X* is a sup norm algebra A on X such that the space $\operatorname{Re} A$ is uniformly dense in $C_{\mathbb{R}}(X)$. Certainly such an algebra is a logmodular algebra, because $\operatorname{Re} A$ is contained in $\log |A^{-1}|$:

$$\operatorname{Re} f = \log |e^f|.$$

It is easy to see that A is a Dirichlet algebra on X if, and only if, $A + \bar{A}$ is uniformly dense in $C(X)$, or, if, and only if, there is no non-zero real measure on X which is orthogonal to A .

We shall now give some specific examples of logmodular algebras, some of which are Dirichlet algebras and some of which are not.

Example 1. Let X be the unit circle in the plane, and let A be the algebra of all continuous complex-valued functions f on X such that the negative Fourier coefficients of f are zero:

$$\int_{-\pi}^{\pi} e^{in\theta} f(\theta) d\theta = 0 \quad (n = 1, 2, 3, \dots).$$

Then A is a Dirichlet algebra on X . This is a consequence of Fejer's theorem, or of the Weierstrass approximation theorem. We shall refer to this algebra as the *standard algebra on the unit circle*. It may also be described as the uniform closure (on the circle) of the polynomials $p(z)$, or, as the algebra of boundary-values of continuous functions on the closed unit disc which are analytic in the interior. The last description arises from the fact that each f in A can be (analytically) extended to the disc by the Poisson integral formula:

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \operatorname{Re} \left[\frac{e^{i\theta} + z}{e^{i\theta} - z} \right] d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) P_z(\theta) d\theta. \quad (3.11)$$

The maximal ideal space of A is the closed unit disc in the plane. The complex homomorphism which corresponds to z , $|z| < 1$, is $\Phi_z(f) = f(z)$, where $f(z)$ is given by (3.11). The measure

$$dm_z(\theta) = \frac{1}{2\pi} P_z(\theta) d\theta$$

is the unique representing measure for the homomorphism Φ_z .

Example 2. The following is a generalization of Example 1. Let X be a compact set in the plane, with the property that each point of X is in the closure of the unbounded component of the complement of X . Let A be the algebra of continuous functions on X which can be uniformly approximated (on X) by polynomials $p(z)$. Then A is a Dirichlet algebra on X . See Wermer [28, p. 68].

Example 3. One can generalize Example 1 in another direction. Let G be a (non-trivial) subgroup of the additive group of real numbers. Regard G as a discrete topological group, and let \hat{G} be its compact character group. Let A_G be the algebra of continuous functions f on \hat{G} whose (generalized) Fourier transforms vanish on the negative part of G :

$$\int_{\hat{G}} \langle x, \alpha \rangle f(\alpha) d\alpha = 0 \quad (x < 0). \quad (3.31)$$

In (3.31), α denotes the typical element of \hat{G} , that is, a mapping of G into the unit circle such that $\alpha(x_1 + x_2) = \alpha(x_1)\alpha(x_2)$. Of course, $\langle x, \alpha \rangle$ denotes $\alpha(x)$. The measure $d\alpha$ is the Haar measure on \hat{G} , that is, the unique probability measure on \hat{G} which is translation invariant.

The algebra A_G is a Dirichlet algebra on \hat{G} ; because, the "trigonometric" polynomials

$$P(\alpha) = \sum_{n=1}^N \lambda_n \langle t_n, \alpha \rangle \quad (t_n \in G) \quad (3.32)$$

are dense in $C(\hat{G})$, and each such function has the form $f + \bar{g}$, where f and g belong to A_G . Indeed, A_G is the uniform closure of the polynomials (3.32) for which each t_n is a non-negative element of G .

Now A_G is isomorphic to an algebra of analytic almost periodic functions in the upper half-plane. If P is a function of the form (3.32), we associate with P an exponential polynomial Q on the real line, by

$$Q(x) = \sum_{n=1}^N \lambda_n e^{it_n x}. \quad (3.33)$$

The map $P \rightarrow Q$ is easily seen to be an algebra isomorphism. Furthermore, it is isometric:

$$\sup_{\alpha} |P(\alpha)| = \sup_x |Q(x)|.$$

This isometric isomorphism can therefore be extended to one between the uniform closures of the two algebras of "polynomials". The uniform closure of the functions (3.32) is $C(\hat{G})$, and the uniform closure of the functions (3.33) consists of the almost periodic functions F on the real line such that the Dirichlet series for F is supported on G :

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-itx} F(x) dx = 0 \quad (t \notin G).$$

Under the same isomorphism, the algebra A_G is carried onto the algebra consisting of the almost periodic functions F on the line which have a Dirichlet series supported on the non-negative part of G . In other words, the image algebra is the uniform closure of the exponential polynomials (3.33) for which each t_n is a non-negative element of G . Each function F in this algebra has an analytic extension to the upper half-plane. For the polynomials Q (3.33), the extension is

$$Q(z) = \sum \lambda_n e^{it_n z}.$$

When each t_n is non-negative

$$\sup_{\text{Im } z \geq 0} |Q(z)| = \sup_x |Q(x)|$$

and this permits the analytic extension of any uniform limit of such polynomials.

When G is the group of integers, \hat{G} is the unit circle and A_G is the standard algebra of Example 1. When G is the group of all real numbers, \hat{G} is the Bohr compactification of the real line, and A_G is isomorphic to the algebra of all analytic almost periodic functions in the upper half-plane which have continuous boundary-values. Another interesting case is obtained as follows. Choose an irrational number γ , and let G be the group of numbers of the form $m + n\gamma$, where m and n are integers. The group \hat{G} is the torus, and A_G consists of the continuous functions f on the torus whose Fourier coefficients

$$a_{mn} = \frac{1}{4\pi^2} \int \int e^{-im\theta} e^{in\psi} f(\theta, \psi) d\theta d\psi$$

vanish outside the half-plane of lattice points for which $m + n\gamma \geq 0$.

Algebras of analytic almost periodic functions were studied some time ago by Bohr [7] and others. A systematic study of A_G as a sup norm algebra was begun by Arens and Singer [4], and continued by Arens [1], the author [17], Helson and Lowdenslager [15], and deLeeuw and Glicksberg [8]. Arens and Singer identified the maximal ideal space of A_G , and it will be helpful for us to describe it. Topologically, it is the Cartesian product of the unit interval and \hat{G} , with all the points $(0, \alpha)$ identified to a single point.

Suppose $0 < r \leq 1$ and $\alpha \in \hat{G}$. Define a complex homomorphism on the polynomials P , of the form (3.32) with $t_n \geq 0$, by

$$\Phi_{r\alpha}(P) = \sum_n \lambda_n r^{t_n} \alpha(t_n).$$

Then $|\Phi_{r\alpha}(P)| \leq \|P\|$ and so Φ extends uniquely to a complex homomorphism of A_G . These functionals $\Phi_{r\alpha}$ exhaust the complex homomorphisms of A_G , except for the Haar homomorphism

$$\Phi_0(f) = \int_{\hat{G}} f(\alpha) d\alpha. \quad (3.34)$$

The points of \hat{G} , which define homomorphisms of A_G by point evaluation, are then embedded in the maximal ideal space of A_G as the homomorphisms Φ_{1z} .

When G is not isomorphic to the group of integers, the maximal ideal space of A_G is often called a "Big Disc". The reason for the terminology should be evident from the analogy between the discussion above and Example 1. The group \hat{G} is the "boundary" of the Big Disc, and the Haar Homomorphism (3.34) is the "origin".

Example 4. The algebra H^∞ , consisting of all bounded analytic functions in the unit disc, is (isomorphic to) a logmodular algebra. A classical theorem of Fatou states that if $f \in H^\infty$ then f is representable as the Poisson integral of a bounded Baire function F on the unit circle:

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) P_z(\theta) d\theta.$$

Furthermore, the bound of f is the (Lebesgue) essential sup norm of F :

$$\sup_{|z| < 1} |f(z)| = \|F\|_\infty = \text{ess sup } |F(\theta)|.$$

This identifies H^∞ with a closed subalgebra of L^∞ , the algebra of bounded measurable functions on the circle. The subalgebra, which we shall also call H^∞ , consists of those functions in L^∞ whose Fourier coefficients c_n vanish for $n < 0$. The algebra L^∞ is isometrically isomorphic to $C(X)$, where X is the maximal ideal space of L^∞ . This isomorphism carries H^∞ onto a sup norm algebra A on the space X . Now A is a logmodular algebra on X ; indeed, $\log |A^{-1}| = C_{\mathbb{R}}(X)$. This simply states that each real-valued function u in L^∞ is the logarithm of the modulus of an invertible H^∞ function F . The appropriate F is the boundary function for the bounded analytic function f , defined by

$$f(z) = \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(\theta) d\theta \right].$$

This logmodular algebra A is *not* a Dirichlet algebra. For a proof of this and other facts pertinent to the foregoing discussion, see [18; Chapter 10].

Example 5. Let G be a (non-trivial) subgroup of the additive group of real numbers. We refer to the algebra A_G of Example 3, and its maximal ideal space $M(A)_G$. The algebra A_G is the uniformly closed linear span of the monomials

$$M_t(\alpha) = \alpha(t) \quad (t \in G, t \geq 0)$$

that is, the characters of \hat{G} which arise from non-negative elements of G . If we neglect the "origin" of $M(A_G)$, the function M_t is extended to $M(A_G)$ by

$$\hat{M}_t(r, \alpha) = \hat{M}_t(\Phi_{r\alpha}) = r^t \alpha(t).$$

We now fix an “annulus” S in $M(A_G)$:

$$S = \{(r, \alpha); r \geq e^{-1}\}.$$

Let B_G be the uniformly closed subalgebra of $C(S)$ which is generated by the functions \hat{M}_t and their reciprocals. It is easy to see that each function in B_G attains its maximum over S on the “boundary”,

$$X = (e^{-1}\hat{G}) \cup \hat{G},$$

which consists of two disjoint copies of \hat{G} . We are going to show that B_G is a logmodular algebra on X if, and only if, G is a dense subgroup of the real numbers. Before we do this, let us note the concrete representation of the algebra B_G . If one refers to that portion of Example 3 in which we identified A_G with an algebra of analytic almost periodic functions, one sees that B_G is isomorphic to an algebra of analytic almost periodic functions in the strip $0 \leq \text{Im } z \leq 1$. The algebra B_G is the uniform completion on S of the functions

$$P(r, \alpha) = \sum \lambda_n r^{t_n} \alpha(t_n) \quad (t_n \in G).$$

From this it is easy to see that B_G is isomorphic to the uniform completion of the exponential polynomials

$$Q(z) = \sum \lambda_n e^{it_n z} \quad (t_n \in G)$$

on the strip $0 \leq \text{Im } z \leq 1$.

Note that B_G is *not* a Dirichlet algebra on X . Let μ_0 be the measure on X which is “Haar measure on \hat{G} minus Haar measure on $(e^{-1}\hat{G})$ ”:

$$\int_X f d\mu_0 = \int_{\hat{G}} f(\alpha) d\alpha - \int_{e^{-1}\hat{G}} f(e^{-1}, \alpha) d\alpha.$$

It is easy to check that μ_0 is orthogonal to B_G ; hence, B_G is not a Dirichlet algebra on X . We need to observe that any real measure on X which is orthogonal to B_G is a scalar multiple of μ_0 . Suppose μ is such a measure, and write μ as the sum of two measures, the real measure μ_1 on \hat{G} and the real measure μ_2 on $(e^{-1}\hat{G})$. Since μ is orthogonal to each monomial M_t , $t \in G$, we have

$$\int_{\hat{G}} \langle t, \alpha \rangle d\mu_1(\alpha) + e^{-t} \int_{e^{-1}\hat{G}} \langle t, \alpha \rangle d\mu_2(\alpha) = 0. \tag{3.51}$$

If we replace t by $(-t)$, i.e., replace M_t by its reciprocal, and then conjugate, we have

$$\int_{\hat{G}} \langle t, \alpha \rangle d\mu_1(\alpha) + e^t \int_{e^{-1}\hat{G}} \langle t, \alpha \rangle d\mu_2(\alpha) = 0. \tag{3.52}$$

From (3.51) and (3.52) we see that

$$\int_{\hat{G}} \langle t, \alpha \rangle d\mu_1(\alpha) = 0 \quad (t \neq 0).$$

But, since μ_1 is a real measure, we then have $d\mu_1(\alpha) = cd\alpha$, for some real scalar c . It is easy to see that $d\mu_2(\alpha) = -cd\alpha$, and hence that $\mu = c\mu_0$.

When G is a dense subgroup of the line, we wish to show that $\log |B_G^{-1}|$ is dense in $C_R(X)$. We know that $\log |B_G^{-1}|$ contains $\text{Re } B_G$, and that the annihilator of $\text{Re } B_G$ is the one dimensional space spanned by the measure μ_0 . Therefore, in order to prove that B_G is a logmodular algebra on X , it will suffice to prove that the uniform closure of $\log |B_G^{-1}|$ contains a linear subspace N of $C_R(X)$, such that N contains $\text{Re } B_G$ and one function which is not annihilated by μ_0 . Let t_0 be a fixed non-zero element of G , and let

$$u = \log |\hat{M}_{t_0}|.$$

Then $u \in \log |B_G^{-1}|$, and $\int u d\mu_0 \neq 0$ because

$$u = \begin{cases} 0, & \text{on } \hat{G} \\ -t_0, & \text{on } e^{-1}\hat{G}. \end{cases}$$

Now $\log |B_G^{-1}|$ contains

$$cu + \text{Re } f$$

provided $(ct_0) \in G$ and $f \in B_G$. When G is a dense subgroup of the line, there is a dense set of real numbers c such that ct_0 lies in G . Hence, the uniform closure of $\log |B_G^{-1}|$ contains the linear subspace spanned by u and $\text{Re } B_G$.

When G is not dense in the line, i.e., when G is isomorphic to the group of integers, the algebra B_G consists of the continuous functions on the annulus $e^{-1} \leq |z| \leq 1$, which are analytic in the interior; and X is the boundary of the annulus. In this case, B_G is not a logmodular algebra on X . Points inside the annulus do not have unique representing measures on X , whereas, we shall soon show that representing measures are unique for a logmodular algebra. It is worth noting (in the case of this annulus algebra) that the linear span of $\log |B_G^{-1}|$ is dense in $C_R(X)$, although $\log |B_G^{-1}|$ itself is not dense.

Other examples of Dirichlet algebras may be found in Wermer [27; 28]. We shall see other examples of logmodular (non-Dirichlet) algebras later. We might point out that, if A is a logmodular algebra which is not a Dirichlet algebra, then the maximal ideal space $M(A)$ cannot be simply connected, i.e., the Čech cohomology group $H^1(M(A); \mathbb{Z})$ cannot be trivial. For, if this group is trivial, a theorem of Arens and Calderon [2] states that every invertible element of A is of the form e^f , with $f \in A$.

4. Representing measures

From this point on, we shall be studying a fixed logmodular algebra A on the space X .

THEOREM 4.1. *Let Φ be a complex homomorphism of A . Then there is a unique Arens–Singer measure m for Φ , and that measure satisfies Jensen’s inequality:*

$$\log |\Phi(f)| = \log \left| \int f dm \right| \leq \int \log |f| dm \quad (f \in A).$$

Proof. Since all Arens–Singer measures for Φ agree on $\log |A^{-1}|$, which is uniformly dense in $C_{\mathbb{R}}(X)$, there is not more than one such measure. By Theorem 2.1, there exists such a measure m , and it is a representing measure for Φ . To establish the Jensen inequality, we argue as follows. Let $f \in A$ and let $\varepsilon > 0$. Then $\log(|f| + \varepsilon)$ is a continuous real-valued function on X . Hence there is a function u in $\log |A^{-1}|$ which is uniformly within ε of $\log(|f| + \varepsilon)$:

$$u - \varepsilon < \log(|f| + \varepsilon) < u + \varepsilon. \tag{4.11}$$

If $u = \log |g|$, $g \in A^{-1}$, let $h = fg^{-1}$. Then $h \in A$; and, by the right-hand inequality of (4.11), we have $|h| < e^\varepsilon$ on X . Therefore $|\Phi(h)| < e^\varepsilon$. But then

$$\left. \begin{aligned} &|\Phi(f)| |\Phi(g)|^{-1} < e^\varepsilon \\ &\log |\Phi(f)| - \log |\Phi(g)| < \varepsilon. \end{aligned} \right\} \tag{4.12}$$

Now m is an Arens–Singer measure for Φ and g is invertible. Consequently

$$\log |\Phi(g)| = \int \log |g| dm = \int u dm.$$

By the left-hand inequality of (4.11)

$$\int u dm < \varepsilon + \int \log(|f| + \varepsilon) dm.$$

If we combine this with (4.12) we obtain

$$\log |\Phi(f)| = \log \left| \int f dm \right| < 2\varepsilon + \int \log(|f| + \varepsilon) dm.$$

As ε tends monotonically to 0, we obtain the Jensen inequality.

THEOREM 4.2. *Let Φ be a complex homomorphism of A . Then Φ has a unique representing measure.*

Proof. Let m_1 and m_2 be representing measures for Φ . Let $f \in A^{-1}$. Then

$$\begin{aligned} \Phi(f) &= \int f dm_1; & |\Phi(f)| &\leq \int |f| dm_1, \\ \Phi(f^{-1}) &= \int f^{-1} dm_2; & |\Phi(f^{-1})| &\leq \int |f|^{-1} dm_2. \end{aligned}$$

But $\Phi(f)\Phi(f^{-1})=1$, hence

$$1 \leq \int |f| dm_1 \int |f|^{-1} dm_2.$$

Since this holds for every $f \in A^{-1}$ and $\log |A^{-1}|$ is dense in $C_R(X)$, we have

$$1 \leq \int e^u dm_1 \int e^{-u} dm_2 \quad (u \in C_R(X)). \quad (4.21)$$

Fix $u \in C_R(X)$ and define

$$\rho(t) = \int e^{tu} dm_1 \int e^{-tu} dm_2 \quad (-\infty < t < \infty).$$

It is clear that ρ is an analytic function on the real line. By (4.21) we have $\rho(t) \geq 1$ for all t ; because m_1 and m_2 are probability measures, $\rho(0) = 1$. Therefore $\rho'(0) = 0$. But

$$\rho'(0) = \int u dm_1 - \int u dm_2.$$

We conclude that m_1 and m_2 define the same linear functional on $C_R(X)$, and hence that $m_1 = m_2$.

Several remarks are in order. Theorem 4.1 is due to Arens and Singer [3]. Indeed, they proved this theorem under the hypothesis that the linear span of $\log |A^{-1}|$ is dense in $C_R(X)$; this can be proved by slightly modifying the proof we gave for logmodular algebras. The present proof is a minor modification of a proof for Dirichlet algebras which was shown to me by John Wermer. Theorem 4.2 is a special case of a result about the general sup norm algebra, as can be seen by examining the proof. If Φ is a complex homomorphism of the sup norm algebra A , then all representing measures for Φ agree on every $u \in C_R(X)$ which has the property that every (real) scalar multiple of u is in the weak closure of a bounded subset of $\log |A^{-1}|$.

The uniqueness Theorem (4.2) is (of course) trivial if A is a Dirichlet algebra. For the "Big Annulus" algebra of Example 5, section 3, the uniqueness of representing measures was proved by Wermer. For the algebra H^∞ of Example 4, section 3, Gleason and Whitney [12] proved that the homomorphism "evaluation at the origin" has a unique representing measure. The uniqueness of all representing measures for H^∞ was proved by the author [18; page 182]. Of course, the uniqueness of the representing measure for Φ can also be stated as follows: the linear functional Φ has a unique norm-preserving extension to a linear functional on $C(X)$.

Because of Theorem 4.2, there is really no need to speak any longer of complex homo-

morphisms of the logmodular algebra A , we may instead discuss probability measures m on X which are *multiplicative* on A :

$$\int fg \, dm = \int f \, dm \int g \, dm \quad (f, g \in A).$$

In this language, the last two theorems say the following.

COROLLARY. *Let m be any probability measure on X which is multiplicative on A . Then*

$$\log \left| \int f \, dm \right| \leq \int \log |f| \, dm \quad (f \in A).$$

In particular, if $\int f \, dm \neq 0$ then $\log |f| \in L^1(dm)$. If μ is any positive measure such that $\int f \, dm = \int f \, d\mu$ for all $f \in A$, then $\mu = m$.

For a discussion of the integrability of $\log |f|$ when $\int f \, dm = 0$, see the remarks after the corollaries to Theorem 6.3.

We shall now extend to the class of logmodular algebras a theorem of Szegő [26] and Kolmogoroff and Krein [19], concerning mean-square approximation of 1 by polynomials which vanish at the origin. The role of the polynomials will be played by the functions in A , and that of the origin will be played by a multiplicative measure on A .

DEFINITION 4.1. Let m be a probability measure on X which is multiplicative on A . We denote by A_m the set of all functions f in A such that $\int f \, dm = 0$.

Of course, A_m is a maximal ideal in the algebra A . The setting for the Szegő theorem is this. We are given m and also an arbitrary positive measure μ on X . We wish to compute the distance from the constant function 1 to the space A_m in the Hilbert space $L^2(d\mu)$. The square of this distance is

$$\inf_{f \in A_m} \int |1 - f|^2 \, d\mu$$

and the result is that the above infimum is equal to $\exp[\int \log h \, dm]$, where h is the derivative of μ with respect to m . The proof we give is a modification of a proof due to Helson and Lowdenslager [14]. Their proof is also discussed in the author's book [18; page 46].

THEOREM 4.3. *Let m be a probability measure on X which is multiplicative on A . Let μ be an arbitrary positive measure on X , and let μ_a be the absolutely continuous part of μ with respect to m . Then*

$$\inf_{f \in A_m} \int |1 - f|^2 \, d\mu = \inf_{f \in A_m} \int |1 - f|^2 \, d\mu_a.$$

In particular, if μ is mutually singular with m , then 1 lies in the closed subspace of $L^2(d\mu)$ which is spanned by A_m .

Proof. In the Hilbert space $L^2(d\mu)$, let F be the orthogonal projection of 1 into the closed subspace spanned by A_m . Then

$$\int |1 - F|^2 d\mu = \inf_{f \in A_m} \int |1 - f|^2 d\mu.$$

If $f \in A_m$ then $(1 - F)$ is orthogonal to f in $L^2(d\mu)$. Choose a sequence of functions f_n in A_m which converge to F . If $f \in A_m$ then $(1 - f_n)f$ is in A_m , because A_m is an ideal in A . Since f is bounded, $(1 - f_n)f$ converges to $(1 - F)f$, and the latter function is (therefore) in the closure of A_m . Hence $(1 - F)$ is orthogonal to $(1 - F)f$. In other words

$$\int f |1 - F|^2 d\mu = 0 \quad (f \in A_m).$$

Let

$$k = \int |1 - F|^2 d\mu.$$

Of course, we may have $k = 0$; this happens if, and only if, 1 is in the $L^2(d\mu)$ closure of A_m . If $k > 0$, the measure $d\mu_1 = k^{-1} |1 - F|^2 d\mu$ satisfies $\int f d\mu_1 = \int f dm$ for every f in A . By Theorem 4.2 we have $\mu_1 = m$. Thus, whether k is 0 or not, we have

$$|1 - F|^2 d\mu = k dm. \quad (4.31)$$

If we write $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to m and μ_s is mutually singular with m , then it is evident from (4.31) that $(1 - F)$ vanishes almost everywhere with respect to μ_s . For any f in A_m we then have

$$\int (1 - \bar{F}) f d\mu_a = \int (1 - \bar{F}) f d\mu = 0. \quad (4.32)$$

If we assume (as we may) that F is a Baire function, then F belongs to the closure of A_m in $L^2(d\mu_a)$, because

$$\int |F - f_n|^2 d\mu_a \leq \int |F - f_n|^2 d\mu.$$

If we combine this observation with (4.32), we see that F is the orthogonal projection of 1 into the $L^2(d\mu_a)$ closure of A_m . This proves the equality of the two infima in the statement of the theorem:

$$\inf_{f \in A_m} \int |1 - f|^2 d\mu = \int |1 - F|^2 d\mu = \int |1 - F|^2 d\mu_a = \inf_{f \in A_m} \int |1 - f|^2 d\mu_a.$$

In case μ is mutually singular with m we have $\mu_a = 0$, and hence 1 is in the $L^2(d\mu)$ closure of A_m , i.e., $F = 1$ almost everywhere with respect to μ .

Now we proceed to investigate the infimum for absolutely continuous measures. We shall need two lemmas concerning probability measures.

LEMMA 4.4. *Let m be a probability measure on X , and let g be any real-valued function in $L^1(dm)$. Then*

$$\int e^g dm \geq \exp \left[\int g dm \right].$$

Proof. This is a well-known theorem. It is a consequence of the familiar inequality between arithmetic and geometric means. It is included because the author assumes he was not alone in being unaware of the following elegant proof, which was shown to him by Steven Orszag. Apparently this proof is originally due to F. Riesz. It clearly suffices to prove the inequality when $\int g dm = 0$. In this case simply observe that $e^g \geq 1 + g$ and integrate.

LEMMA 4.5. *Let m be a probability measure on X and let h be a non-negative function in $L^1(dm)$. Then*

$$\exp \left[\int \log h dm \right] - \inf_g \int e^g h dm = \left(\int g dm - 0 \right), \quad (4.51)$$

where g ranges over any one of the three following spaces of functions: (a) the space of real functions in $L^1(dm)$; (b) the space of real bounded Baire functions; (c) the space $C_R(X)$.

Proof. Since $h \in L^1(dm)$ and $\log h \leq h$, we can only have $\log h$ non-integrable if $\int \log h dm = -\infty$. In this case the left-hand member of (4.51) is defined to be 0. Let g be any real function in $L^1(dm)$ such that $\int g dm = 0$. By Lemma 4.4 we have

$$\int e^g h dm \geq \exp \left[\int (g + \log h) dm \right] = \exp \left[\int \log h dm \right], \quad (4.52)$$

at least in the case when $\log h$ is integrable. If $\log h$ is not integrable the inequality is trivial. Thus

$$\exp \left[\int \log h dm \right] \leq \inf_g \int e^g h dm \quad \left(g \in L_R^1(dm), \quad \int g dm = 0 \right).$$

($L_R^1(dm)$ denotes the space of real L^1 functions.) Suppose $\log h$ is integrable. Let

$$g = -\log h + \int \log h dm$$

so that $g \in L_R^1(dm)$ and $\int g dm = 0$. For this g , equality holds in (4.52). If $\log h$ is not integrable then, for any $\varepsilon > 0$, $\log(h + \varepsilon)$ is integrable; and if we let ε tend monotonically to 0 we obtain (4.51). Thus (4.51) holds for all non-negative h in $L^1(dm)$, where g ranges over $L_R^1(dm)$.

The infimum on the right of (4.51) is unaffected if $L^1_R(dm)$ is replaced by the class of real bounded Baire functions. Given any $g \in L^1_R(dm)$ for which $\int g dm = 0$, we can select of sequence of real bounded Baire functions g_n such that $\int g_n dm \rightarrow 0$,

$$g_n \vee 0 = \max[g_n, 0]$$

increases monotonically to $g \vee 0$, and

$$g_n \wedge 0 = \min[g_n, 0]$$

decreases monotonically to $g \wedge 0$. By the monotone convergence theorem,

$$\lim_n \int e^{g_n} h dm = \int e^g h dm. \tag{4.53}$$

From this it is clear that the infima corresponding to $L^1_R(dm)$ and the space of bounded functions are equal.

If g is a real bounded Baire function with $\int g dm = 0$, we can find a sequence of functions $g_n \in C_R(X)$ which is bounded, satisfies $\int g_n dm = 0$, and converges pointwise to g almost everywhere relative to m . By the bounded convergence theorem, we have (4.53). That proves the lemma.

Now we return to our logmodular algebra A and prove the generalized Szegő theorem.

THEOREM 4.6. *Let m be a probability measure which is multiplicative on A and let h be a non-negative function in $L^1(dm)$. Then*

$$\inf_{f \in A_m} \int |1 - f|^2 h dm = \exp \left[\int \log h dm \right].$$

Proof. By the last lemma

$$\exp \left[\int \log h dm \right] = \inf_u \int e^u h dm \quad \left(u \in C_R(X), \int u dm = 0 \right).$$

Since $\log |A^{-1}|$ is uniformly dense in $C_R(X)$ we have

$$\exp \left[\int \log h dm \right] = \inf_f \int |f|^2 h dm \quad \left(f \in A^{-1}, \int f dm = 1 \right).$$

Here, we have used the fact that m is an Arens-Singer measure, so that when we approximate $u \in C_R(X)$ by $\log |f|^2$, $f \in A^{-1}$, the number $\log |\int f dm|^2 = \int \log |f|^2 dm$ will be near $\int u dm$. If $f \in A^{-1}$ and $\int f dm = 1$, then $f = 1 - g$, where $g \in A_m$. Thus

$$\exp \left[\int \log h dm \right] \geq \inf_{g \in A_m} \int |1 - g|^2 h dm. \tag{4.61}$$

On the other hand, if $g \in A_m$

$$\int \log |1-g|^2 dm \geq 2 \log \left| \int (1-g) dm \right| = 0$$

because m satisfies Jensen's inequality (Theorem 4.1). By Lemma 4.4 we then have

$$\int |1-g|^2 h dm \geq \exp \left\{ \int \left[\log |1-g|^2 + \log h \right] dm \right\} \geq \exp \left[\int \log h dm \right].$$

This establishes the reverse inequality in (4.61), and the theorem is proved.

THEOREM 4.7. *Let m be a probability measure on X which is multiplicative on A . Let μ be a positive measure on X , and let $d\mu = h dm + d\mu_s$ be the Lebesgue decomposition of μ relative to m . Then*

$$\inf_{f \in A_m} \int |1-f|^2 d\mu = \exp \left[\int \log h dm \right].$$

Proof. The absolutely continuous part of μ relative to m is $h dm$. Now apply Theorems 4.3 and 4.6.

5. The space H^2

We now begin to study the analogue (for logmodular algebras) of a segment of analytic function theory in the unit disc.

DEFINITION 5.1. Let μ be a positive measure on X . The space $H^p(d\mu)$, $1 \leq p < \infty$, is the closure of the algebra A in the Banach space $L^p(d\mu)$.

When A is the standard algebra on the unit circle and m is normalized Lebesgue measure on the circle, the spaces $H^p(dm)$ are (isomorphic to) the Hardy spaces H^p . These spaces are discussed from this point of view in [18].

In this section, we shall fix a probability measure m on X which is multiplicative on our logmodular algebra A , and concentrate on a study of the Hilbert space $H^2(dm)$.

THEOREM 5.1. *The measure m is multiplicative on $H^2(dm)$. Also, we have the Jensen inequality*

$$\log \left| \int f dm \right| \leq \int \log |f| dm \quad (f \in H^2(dm)).$$

Thus, if $f \in H^2(dm)$ and $\int f dm \neq 0$ the function $\log |f|$ is integrable with respect to m .

Proof. If f, g are in $H^2(dm)$ we choose sequences of functions f_n, g_n in A which converge (respectively) to f and g in L^2 norm. Since $\int f_n g_n = \int f_n \int g_n$, it is clear that $\int fg dm = \int f dm \int g dm$, i.e., that m is multiplicative on $H^2(dm)$.

Given $f \in H^2(dm)$ choose functions f_n in A which converge to f in $L^2(dm)$ and which satisfy $\int f_n dm = \int f dm$. Let $\varepsilon > 0$. Then $|f_n| + \varepsilon$ converges to $|f| + \varepsilon$ in $L^1(dm)$. Therefore $\log(|f_n| + \varepsilon)$ converges to $\log(|f| + \varepsilon)$ in $L^1(dm)$. For, it is easy to verify that if m is any positive measure and $\{p_n\}$ is a sequence of positive functions in $L^1(dm)$ which converges to p in $L^1(dm)$, and if $p_n \geq \varepsilon > 0$ for each n , then $\log p_n$ converges to $\log p$ in L^1 . Thus

$$\begin{aligned} \int \log(|f| + \varepsilon) dm &= \lim_n \int \log(|f_n| + \varepsilon) dm \geq \overline{\lim}_n \int \log |f_n| dm \\ &\geq \overline{\lim}_n \log \left| \int f_n dm \right| = \log \left| \int f dm \right|. \end{aligned}$$

Here we have used the Jensen inequality for functions in the algebra A . If we let ε tend monotonically to 0, we are done.

The Jensen inequality for functions in $H^2(dm)$ can also be deduced from the Szegő theorem (4.6). This proof is not only simpler, but has two other advantages. It works just as well for functions in $H^1(dm)$; and it does not depend upon the fact that A is a logmodular algebra. It assumes only that the measure m is multiplicative on the sup norm algebra A and that m satisfies the Jensen inequality on the algebra.

Now we come to one of the most basic facts about a logmodular algebra. What we want to prove is that $L^2(dm)$ is the (Hilbert space) direct sum of $H^2(dm)$ and $\overline{H}_m^2(dm)$, the space of complex conjugates of H^2 functions which are annulled by m . This amounts to showing that if $g \in L^2(dm)$ and $\int fg dm = 0$ for all $f \in A_m$, then g is actually in $H^2(dm)$. In case A is a Dirichlet algebra, this is almost evident, because the space $A + \overline{A}$ is uniformly dense in $C(X)$. For the general logmodular algebra, $A + \overline{A}$ is not dense in $C(X)$, but we shall show that it is dense in $L^2(dm)$.

THEOREM 5.2. *Let $g \in L^1(dm)$, and suppose that $\int fg dm = 0$ for all $f \in A$. Then*

$$\int \log |1 - g| dm \geq 0.$$

Proof. Let $f \in A^{-1}$. Then

$$\int f dm = \int (1 - g) f dm$$

and so

$$\left| \int f dm \right| \leq \int |f| |1 - g| dm,$$

$$\log \left| \int f dm \right| \leq \log \int |f| |1 - g| dm.$$

Since m is an Arens-Singer measure and $f \in A^{-1}$, we have

$$\int \log |f| dm \leq \log \int |f| |1-g| dm$$

or

$$\exp \left[\int \log |f| dm \right] \leq \int |f| |1-g| dm.$$

Since $\log |A^{-1}|$ is uniformly dense in $C_R(X)$, we have

$$1 \leq \int e^u |1-g| dm \quad \left(u \in C_R(X), \int u dm = 0 \right). \quad (5.21)$$

The infimum over u of the right-hand member of (5.21) is

$$\exp \left[\int \log |1-g| dm \right]$$

by Lemma 4.4. That proves the theorem.

LEMMA 5.3. *Let m be a probability measure, and let g be a real-valued function in $L^2(dm)$. Suppose that*

$$\int \log |1-tg| dm \geq 0 \quad (5.31)$$

for every real number t in some interval $|t| < \delta$. Then $g=0$ almost everywhere with respect to m .

Proof. If we employ (5.31) for t and $-t$ and then add, we obtain

$$0 \leq \int \log |1-t^2g^2| dm \quad (0 < t < \delta).$$

Let E_t be the set on which $1-t^2g^2 > 0$, and let E_t' be its complement. Then

$$0 \leq \int_{E_t} \log (1-t^2g^2) dm + \int_{E_t'} \log (t^2g^2-1) dm.$$

We apply the inequality $\log (1-x) \leq -x$ in the first integrand and the inequality $\log (x-1) \leq x$ in the second integrand. We obtain

$$0 \leq -t^2 \int_{E_t} g^2 + t^2 \int_{E_t'} g^2$$

or

$$0 \leq - \int_{E_t} g^2 + \int_{E_t'} g^2.$$

Now let t approach 0. Since $E_t = \{g^2 < 1/t^2\}$ and g^2 is integrable, $\lim_{t \rightarrow 0} m(E_t) = 1$. Thus we obtain

$$0 \leq - \int y^2 dm$$

and therefore $g = 0$.

THEOREM 5.4. *The space $(A + \bar{A})$ is dense in $L^2(dm)$. Thus*

$$L^2(dm) = H^2(dm) \oplus \bar{H}_m^2(dm),$$

where H_m^2 is the closure of A_m in $L^2(dm)$. In particular, a function g in $L^2(dm)$ belongs to $H^2(dm)$ if, and only if,

$$\int fg dm = 0 \quad (f \in A_m).$$

Proof. If $A + \bar{A}$ is not dense in $L^2(dm)$, there exists a non-zero function in $L^2(dm)$ which is orthogonal to A and \bar{A} . Thus there exists a non-zero real-valued g in $L^2(dm)$ which is orthogonal to A . Each real scalar multiple of g will also be orthogonal to A . By Theorem 5.2 this g satisfies

$$\int \log |1 - tg| dm \geq 0 \quad (-\infty < t < \infty).$$

By Lemma 5.3, this is impossible with $g \neq 0$. We conclude that $A + \bar{A}$ is dense in $L^2(dm)$. Since A and \bar{A}_m are orthogonal subspaces of $L^2(dm)$ and their sum is dense, we have $L^2 = H^2 \oplus \bar{H}_m^2$. Thus H^2 is characterized as the subspace orthogonal to \bar{A}_m , and we are done.

Needless to say, it follows from the same argument that $A + \bar{A}$ is dense in $L^p(dm)$, $1 \leq p \leq 2$. In the next section we shall prove the density for $p > 2$.

We turn now to the discussion of certain invariant subspaces of $H^2(dm)$. The result is a generalization of Beurling's theorem [5], for the case in which A is the standard algebra on the unit circle. The proof we employ is due to Helson and Lowdenslager [14]. Other relevant references are Lax [20], Masani [22], Halmos [13], and [18, Chapter 7].

THEOREM 5.5. *Let S be a closed subspace of $H^2(dm)$ which is invariant under multiplication by the functions in A . Suppose that $\int g dm \neq 0$ for at least one function g in S . Then there is a function F in $H^2(dm)$ such that*

- (i) $|F| = 1$ almost everywhere with respect to m ;
- (ii) $S = FH^2(dm)$.

The function F with these two properties is unique to within a constant factor of modulus 1.

Proof. Let G be the orthogonal projection of 1 into S . Then $(1 - G)$ is orthogonal to S . Since G belongs to S and S is invariant under multiplication by the functions in A , the function $(1 - G)$ is orthogonal to fG , $f \in A$; that is

$$0 = \int (1 - \bar{G})Gf \, dm \quad (f \in A).$$

In case $f \in A_m$ then $(fG) \in H_m^2$ and so $\int fG \, dm = 0$. Therefore, we have

$$0 = \int f|G|^2 \, dm \quad (f \in A_m).$$

By the uniqueness of m (Theorem 4.2), we must have

$$|G|^2 \, dm = k^2 \, dm$$

that is, $|G|$ is a constant k almost everywhere.

Since G belongs to S , the subspace S contains GA . But G is a bounded function, and so S contains GH^2 . We can see that $S = GH^2$ as follows. Suppose $g \in S$ and g is orthogonal to GH^2 . Then

$$0 = \int f\bar{G}g \, dm \quad (f \in A). \tag{5.51}$$

Since $g \in S$ we have $(fg) \in S$ and $(1 - g)$ is orthogonal to fg . In case $f \in A_m$ this says

$$0 = \int f\bar{G}g \, dm \quad (f \in A_m). \tag{5.52}$$

Combining (5.51) and (5.52), we see that $\bar{G}g$ is orthogonal to $(A + \bar{A})$. By Theorem 5.4 it must be that $\bar{G}g = 0$. But G is a function of constant modulus k , and that constant cannot be 0 because we have assumed that 1 is not orthogonal to S . (1 is orthogonal to S if and only if $\int h \, dm = 0$ for all $h \in S$.) Therefore $g = 0$. We conclude that $S = GH^2$. If we let $F = G/k$ then F has the properties required.

Suppose F_1 is a function in $H^2(dm)$ such that $S = F_1H^2$ and $|F_1| = 1$ almost everywhere. Let $\lambda = \int F_1 \, dm$. It is trivial to verify that $\bar{\lambda}F_1$ is the orthogonal projection of 1 into S . Hence, F_1 is a constant multiple of F .

THEOREM 5.6. *Let g be a function in $H^2(dm)$. The set of functions Ag is dense in $H^2(dm)$ if, and only if,*

$$\int \log |g| \, dm = \log \left| \int g \, dm \right| > -\infty. \tag{5.61}$$

Proof. The space Ag is dense in $H^2(dm)$ if and only if 1 belongs to the closure of Ag . In order for this density to prevail, it is clearly necessary that $\int g \, dm \neq 0$. Since (5.61) is

not affected if g is multiplied by a nonzero scalar, we may assume that $\int g dm = 1$. As we already noted, Ag is dense in H^2 if and only if there exist functions f_n in A with

$$\lim_n \int |1 - f_n g|^2 dm = 0.$$

Since $\int g dm = 1$ we may assume that $\int f_n dm = 1$, i.e., that $f_n = 1 - g_n$, with $g_n \in A_m$. Now

$$\int |1 - (1 - g_n)g|^2 dm = -1 + \int |1 - g_n|^2 |g|^2 dm$$

and so Ag is dense if, and only if, there exist functions g_n in A_m with

$$\lim_n \int |1 - g_n|^2 |g|^2 dm = 1.$$

But

$$\int |1 - f|^2 |g|^2 dm \geq \left| \int (1 - f)g dm \right|^2 = \left| \int g dm \right|^2 = 1$$

for any $f \in A_m$. Thus the density of Ag in H^2 is equivalent to

$$\inf_{f \in A_m} \int |1 - f|^2 |g|^2 dm = 1.$$

By the generalized Szegő theorem (4.6), this infimum is equal to

$$\exp \left[\int \log |g|^2 dm \right].$$

Since $\int g^2 dm = 1$ we obtain (5.61) as the necessary and sufficient condition for the density of Ag in H^2 .

DEFINITION 5.2. An *inner function* is any F in $H^2(dm)$ such that $|F| = 1$ almost everywhere relative to m . An *outer function* in H^2 is any g in $H^2(dm)$ which satisfies (5.61).

Our last two results may then be stated as follows. If S is a closed subspace of $H^2(dm)$ which is invariant under multiplication by the functions in A , and if not every function in S “vanishes at m ”, then $S = FH^2$, where F is an (essentially unique) inner function. A function g in $H^2(dm)$ is an outer function if, and only if, Ag is dense in H^2 , i.e., if, and only if, g lies in no proper “invariant” subspace of H^2 . It should be noted that when A is the standard algebra on the unit circle, Theorem 5.5 is valid if one merely assumes that S is not the zero subspace; however, the hypothesis that 1 is not orthogonal to S cannot be deleted for the general logmodular (or even Dirichlet) algebra. Consider the Dirichlet algebra A_G of Example 3, section 3. Let m be Haar measure on G , and let S be the space

H_m^2 . Then, unless G is isomorphic to the group of integers, the subspace S is not of the form $FH^2(dm)$.

THEOREM 5.7. *Let f be any function in $H^2(dm)$ for which $\int f dm \neq 0$. Then $f = Fg$, where F is an inner function and g is an outer function in $H^2(dm)$. This factorization of f is unique up to a constant of modulus 1.*

Proof. Let S be the closure of Af in $H^2(dm)$. Then S is “invariant”; and $\mathbf{1}$ is not orthogonal to S , because $\int f dm \neq 0$. Thus $S = FH^2$, where F is an inner function. In particular $f = Fg$, where $g \in H^2$. Since $Af = F \cdot (Ag)$ and $S = FH^2$, it is clear that the closure of Ag is H^2 . Thus g is an outer function in H^2 . The factorization is unique, by the uniqueness statement of Theorem 5.5.

The following theorem will be important for our work in the next section, and will help us to characterize moduli of H^2 functions. The proof is due to Helson and Lowdenslager [14]. Also see [18; p. 44].

THEOREM 5.8. *Let μ be a positive measure on X , and assume that $\mathbf{1}$ is not in the $L^2(d\mu)$ closure of A_m . Let G be the orthogonal projection of $\mathbf{1}$ into the closed subspace of $L^2(d\mu)$ which is spanned by A_m .*

- (i) $|1 - G|^2 d\mu = k dm$, where k is a positive constant.
- (ii) The function $(1 - G)^{-1}$ is an outer function in $H^2(dm)$.
- (iii) If $d\mu = h dm + d\mu_s$, where $h \in L^1(dm)$ and μ_s is mutually singular with m , then $(1 - G)h$ is in $L^2(dm)$.
- (iv) $k = \exp [\int \log h dm]$.

Proof. Statement (i) results from the definition of G and the uniqueness of m . See the proof of Theorem 4.3. We then have $|1 - G|^2 h dm = k dm$, from which it is clear that $(1 - G)^{-1}$ is in $L^2(dm)$. Let $f \in A_m$. Then

$$\int f(1 - G)^{-1} d\mu = \frac{1}{k} \int f(1 - G)^{-1} |1 - G|^2 d\mu = \frac{1}{k} \int f \cdot (1 - \bar{G}) d\mu = 0$$

because $(1 - G)$ is orthogonal to A_m in $L^2(d\mu)$. By Theorem 5.4 we see that $(1 - G)^{-1}$ is in $H^2(dm)$. Since

$$(1 - G)h dm = k \cdot (1 - G)^{-1} dm$$

and $(1 - G)^{-1}$ is in $L^2(dm)$, we see that $(1 - G)h \in L^2(dm)$. Statement (iv) is simply a repetition of the Szegő theorem (4.7). This completes the proof, except for the assertion that $(1 - G)^{-1}$ is an outer function. This we see as follows. We have

$$h = k|1 - G|^{-2}, \text{ almost everywhere } dm.$$

Thus $\log h = \log k + 2 \log |1 - G|^{-1}$.

By statement (iv), $\int \log h \, d\mu = \log k$. Therefore

$$\int \log |1 - G|^{-1} \, d\mu = 0.$$

On the other hand,

$$\begin{aligned} \int (1 - G)^{-1} \, d\mu &= \frac{1}{k} \int (1 - \bar{G}) \, d\mu = \frac{1}{k} \int (1 - \bar{G}) \, d\mu - \frac{1}{k} \int (1 - \bar{G}) G \, d\mu \\ &= \frac{1}{k} \int |1 - G|^2 \, d\mu = \int d\mu = 1. \end{aligned}$$

Here, we have used the fact that $(1 - G)$ is orthogonal to G in $L^2(d\mu)$. Therefore

$$\int \log |1 - G|^{-1} \, d\mu = \log \left| \int (1 - G)^{-1} \, d\mu \right| = 0$$

and $(1 - G)^{-1}$ is an outer function.

THEOREM 5.9. *Let h be a non-negative function in $L^1(dm)$. The following are equivalent.*

- (i) $\log h$ is integrable with respect to m .
- (ii) $h = |f|^2$, where $f \in H^2(dm)$ and $\int f \, dm \neq 0$.
- (iii) $h = |g|^2$, where g is an outer function in $H^2(dm)$.

Proof. The equivalence of (ii) and (iii) is evident from Theorem 5.7. It is also clear that (iii) implies (i). Suppose (i) holds. Let $d\mu = h \, dm$. Since $\log h$ is integrable, 1 is not in the closed subspace of $L^2(d\mu)$ which is spanned by A_m (Theorem 4.6). Let G be the orthogonal projection of 1 into that subspace of $L^2(d\mu)$. By Theorem 5.8 (and its proof), $(1 - G)^{-1}$ is an outer function in $H^2(d\mu)$ and

$$h = k |1 - G|^{-2} \quad \left(k = \exp \left[\int \log h \, d\mu \right] \right).$$

Let $g = \sqrt{k}(1 - G)^{-1}$, and then g is an outer function with $|g|^2 = h$.

6. The spaces H^1 and H^∞

We retain our fixed logmodular algebra A and the fixed probability measure m , which is multiplicative on A . In the last section, we introduced the spaces $H^p(dm)$, $1 \leq p < \infty$, and studied $H^2(dm)$ to a certain extent. Now we consider $H^1(dm)$ and its basic properties. We shall also define and discuss the space $H^\infty(dm)$. Our first problem is to characterize

H^1 as the space of all functions h in $L^1(dm)$ such that $\int fh dm = 0$ for every f in A_m . This leads to the factorization of H^1 functions. We shall also treat complex measures which are orthogonal to A_m .

THEOREM 6.1. *Let h be a function in $L^1(dm)$ such that $\int fh dm = 0$ for every f in A_m and for which $\int h dm \neq 0$. Then h is the product of two functions in $H^2(dm)$.*

Proof. Let us assume that $\int h dm = 1$. Then $h = 1 - g$, where g is a function in $L^1(dm)$ which satisfies $\int fg dm = 0$ for every f in the algebra A . According to Theorem 5.2, the function $\log |1 - g| = \log |h|$ is in $L^1(dm)$. By the general Szegő theorem (4.6), the constant function 1 does not lie in the closed subspace of $L^2(|h| dm)$ which is spanned by A_m . Now Theorem 5.8 tells us that, if G is the orthogonal projection of 1 into that closed subspace, the function $(1 - G)^{-1}$ is in $H^2(dm)$ and $(1 - G)|h|$ is in $L^2(dm)$. Therefore $(1 - G)h$ is in $L^2(dm)$. The claim is that $(1 - G)h$ is in $H^2(dm)$. By Theorem 5.4, this is equivalent to the statement that

$$\int f(1 - G)h dm = 0 \quad (f \in A_m).$$

Choose a sequence of elements f_n in A_m which converge to G in $L^2(|h| dm)$. Clearly, for any f in A_m we have

$$\int f(1 - G)h dm = \lim_n \int f(1 - f_n)h dm.$$

But each function $(1 - f_n)f$ is in A_m , and h is "orthogonal" to A_m . We conclude that $(1 - G)h$ is in $H^2(dm)$, and the factorization of h which we seek is

$$h = (1 - G)^{-1}[(1 - G)h].$$

COROLLARY. *The space $H^1(dm)$ consists of those functions h in $L^1(dm)$ such that $\int fh dm = 0$ for every $f \in A_m$.*

Proof. Obviously any H^1 function is "orthogonal" to A_m . On the other hand, if we are given an h in $L^1(dm)$ with this orthogonality property, we can choose a constant c so that $(c + h)$ is orthogonal to A_m and has mean different from 0. By the Theorem, $(c + h)$ is the product of two H^2 functions. From this it is obvious that $(c + h)$ is in the L^1 closure of A . Hence h as in the L^1 closure of A , i.e., h is in $H^1(dm)$.

We can now proceed to factor H^1 functions, just as we did H^2 functions.

DEFINITION 6.1. An *outer function* in $H^1(dm)$ is a function g in $H^1(dm)$ such that

$$\int \log |g| dm = \log \left| \int g dm \right| > -\infty.$$

There is no need for us to have another definition of inner function. Such a function F was defined as an element of $H^2(dm)$ which has modulus 1 almost everywhere. Certainly

this is the same as saying that F is an element of $H^1(dm)$ which has modulus 1 almost everywhere. One can say slightly more.

DEFINITION 6.2. The space $H^\infty(dm)$ is the set of all h in $L^\infty(dm)$ such that $\int fh dm = 0$ for every f in A_m .

It is clear that $H^\infty(dm)$ consists of the bounded functions in $H^2(dm)$; or, equivalently, it consists of the bounded functions in $H^1(dm)$. We see also that an inner function is precisely an element of $H^\infty(dm)$ which has modulus 1 almost everywhere.

THEOREM 6.2. Let f be a function in $H^1(dm)$ such that $\int f dm \neq 0$. Then $f = Fg$, where F is an inner function and g is an outer function in $H^1(dm)$.

Proof. Since $\int f dm \neq 0$, we know from Theorem 6.1 that $f = f_1 f_2$, where f_1 and f_2 are in $H^2(dm)$. Obviously $\int f_1 dm \neq 0 \neq \int f_2 dm$. By the factorization theorem (5.7) for H^2 , we have $f_j = F_j g_j$, $j = 1, 2$, where F_1 and F_2 are inner functions and g_1 and g_2 are outer functions in $H^2(dm)$. Certainly $F = F_1 F_2$ is an inner function, and it is easily checked that $g = g_1 g_2$ is an outer function in $H^1(dm)$.

Of course, we want to know that this factorization for H^1 functions is unique, just as it is for H^2 functions. To see this, it is both convenient and instructive to proceed as follows.

THEOREM 6.3. Let g be a function in $H^1(dm)$. The following are equivalent.

- (i) g is an outer function.
- (ii) The space Ag is dense in $H^1(dm)$.
- (iii) $\int g dm \neq 0$; and if h is any function in $H^1(dm)$ such that $|h| \leq |g|$, then h/g is in $H^\infty(dm)$.

Proof. Suppose g is an outer function. By Theorem 6.1, $g = g_1 g_2$, where g_1 and g_2 are in $H^2(dm)$. It is clear that g_1 and g_2 are outer functions. By Theorem 5.6, the $L^2(dm)$ closure of Ag_2 is $H^2(dm)$. Since g_1 is in $L^2(dm)$, it follows that the $L^1(dm)$ closure of $Ag = g_1(Ag_2)$ contains Ag_1 . Since g_1 is outer, 1 is in the $L^2(dm)$ closure of Ag_1 . Thus 1 is in the $L^1(dm)$ closure of Ag , from which it is apparent that Ag is dense in $H^1(dm)$. Thus (i) implies (ii). Suppose (ii) holds. Since 1 is in the $L^1(dm)$ closure of Ag , it is evident that $\int g dm \neq 0$. Let h be a function in $H^1(dm)$ such that $|h| \leq |g|$. Choose a sequence of elements f_n in A such that $f_n g$ converges to 1 in L^1 norm. For any f in A_m

$$\int f \frac{h}{g} dm = \lim_n \int f f_n g \frac{h}{g} dm$$

because the function $f[h/g]$ is bounded. Since $\int f f_n h dm = 0$ for each n , we see that h/g is bounded and "orthogonal" to A_m , i.e., h/g is in $H^\infty(dm)$. Thus (ii) implies (iii). Suppose

(iii) holds. Since $\int g \, dm \neq 0$ we have $g = Fg_1$, where F is an inner function and g_1 is an outer function. Now $|g_1| = |g|$ and so (by (iii)) the function g_1/g is in $H^\infty(dm)$. But $g_1/g = 1/F = \bar{F}$. Thus both F and \bar{F} belong to $H^\infty(dm)$. This implies that F is constant, either by the observation that any real-valued function in $H^2(dm)$ is constant, or by the following. Let $\lambda = \int F \, dm$. Then $F = \lambda + G$, where $G \in H^\infty(dm)$ and $\int G \, dm = 0$. Since $\bar{F} = \bar{\lambda} + \bar{G}$ is also in $H^\infty(dm)$

$$\begin{aligned} 1 &= \int |F|^2 \, dm = \int (\lambda + G)(\bar{\lambda} + \bar{G}) \, dm \\ &= |\lambda|^2 + \lambda \int G \, dm + \bar{\lambda} \int \bar{G} \, dm + \int G\bar{G} \, dm \\ &= |\lambda|^2 + 0 + 0 + 0 = |\lambda|^2. \end{aligned}$$

Since $|F| = 1$ and $|\int F \, dm| = 1$ it is clear that $F = \lambda$ almost everywhere. Since F is constant, $g = \lambda g_1$, an outer function.

COROLLARY. *If g and g_1 are outer functions in $H^1(dm)$ such that $|g| = |g_1|$, then $g = \lambda g_1$, where λ is a constant of modulus 1.*

COROLLARY. *The factorization in Theorem 6.2 is unique up to a constant of modulus 1.*

Proof. If $Fg = F_1g_1$, where F and F_1 are inner functions and g and g_1 are outer functions, then $|g| = |g_1|$, so that $g = \lambda g_1$. Therefore $\lambda Fg_1 = F_1g_1$. Since $\log |g_1|$ is integrable, g_1 cannot vanish on a set of positive measure; hence $\lambda F = F_1$.

COROLLARY. *Let g be an outer function in $H^1(dm)$. Then $g = h^2$, where h is an outer function in $H^2(dm)$.*

Proof. In the Hilbert space $L^2(|g| \, dm)$, the constant function 1 is not in the closed subspace spanned by A_m . (Because, $\log |g|$ is integrable (4.6).) Let G be the orthogonal projection of 1 into that subspace. Then, by Theorem 5.8, the function $(1 - G)^{-1}$ is an outer function in $H^2(dm)$, and

$$|1 - G|^2 |g| \, dm = k \, dm,$$

where $k = \exp [\int \log |g| \, dm] = |\int g \, dm|$. Thus $|g| = k |1 - G|^{-2}$, and since both g and $(1 - G)^{-2}$ are outer functions $g = k^{\frac{1}{2}} \lambda (1 - G)^{-2}$, where $|\lambda| = 1$. Thus g is the square of an outer function in $H^2(dm)$.

COROLLARY. *Let f be a function in $H^1(dm)$ such that $\int f \, dm \neq 0$. Let G be the orthogonal projection of 1 into the closed subspace of $L^2(|f| \, dm)$ which is spanned by A_m , and let*

$$c = \exp \left[\frac{1}{2} \int \log |f| \, dm \right].$$

Then $g=c(1-G)^{-2}$ is the outer part of f , that is, g is the (essentially unique) outer function occurring in the factorization of Theorem 6.2.

In the classical study of H^p spaces, one deals with the special case in which A is the standard algebra on the unit circle and m is normalized Lebesgue measure. In this case, the factorization theory is valid without the hypothesis that $\int f dm \neq 0$. One need only assume that $f \neq 0$. This results from the fact that an analytic function in the unit disc which is not identically zero can be written in the form $z^k f$, where f does not vanish at the origin. Thus, any non-zero function in H^1 has an integrable logarithm. This fails for the general logmodular algebra A , even in the case when m is non-trivial, i.e., when m is not simply a point mass on X . Indeed, there may exist functions f in the algebra A which vanish on an open subset of X with positive m measure, but which are not 0 almost everywhere relative to m . In special cases, the integrability of $\log |f|$ does go through. Consider the case in which X is the Bohr compactification of the real line and A is the algebra of analytic almost periodic functions (Example 3, Section 3). Take m to be Haar measure on X . This is a situation which closely resembles the standard one on the unit circle. Arens [1] proved that if f is a non-zero element of A , then $\log |f| \in L^1(dm)$, even if $\int f dm = 0$; however, Helson and Lowdenslager [15] showed that there exist non-zero functions in $H^\infty(dm)$ which do not have an integrable logarithm. It is interesting to note that this algebra $H^\infty(dm)$ is again a logmodular algebra. This is true for the general logmodular algebra.

THEOREM 6.4. *Let h be a non-negative function in $L^1(dm)$. Then $h = |f|$, where $f \in H^1(dm)$ and $\int f dm \neq 0$ if, and only if, $\log h$ is integrable with respect to m . A non-negative function h in $L^\infty(dm)$ has the form $h = |f|$ with $f \in H^\infty(dm)$ and $\int f dm \neq 0$ if, and only if, $\log h$ is in $L^1(dm)$.*

Proof. Since every f in $H^1(dm)$ for which $\int f dm \neq 0$ is the product of two functions in $H^2(dm)$, this is immediate from Theorem 5.9.

COROLLARY. *The algebra $H^\infty(dm)$, with the m -essential sup norm, is a logmodular algebra on the maximal ideal space of $L^\infty(dm)$.*

Proof. From the Theorem, every real-valued function u in $L^\infty(dm)$ has the form $u = \log |f|$, where f is in $H^\infty(dm)$. We can (of course) arrange that f is an outer function in $H^\infty(dm)$, just as in Theorem 5.9. This f is invertible in $H^\infty(dm)$. Simply choose an outer function g in $H^\infty(dm)$ such that $|g| = e^{-u}$. Then the outer function fg has modulus 1 and must be constant; hence, a scalar multiple of g is the inverse of f . We conclude that

$$\log |(H^\infty)^{-1}| = L_R^\infty$$

and therefore $H^\infty(dm)$ is a logmodular algebra on the maximal ideal space of $L^\infty(dm)$. See Example 4, Section 3.

One of the most beautiful theorems in the theory of analytic functions in the unit disc is due to F. and M. Riesz [25]. It provides a characterization of functions in the Hardy class H^1 . It states that if μ is a (complex) measure on the unit circle whose Fourier coefficients vanish on the negative integers, then μ is absolutely continuous with respect to Lebesgue measure. The analogue of this theorem is false for logmodular algebras; however, there is a general theorem of this sort which easily implies the classical result. The proof is due to Helson and Lowdenslager [14]. See [18, page 46] for a discussion.

THEOREM 6.5. *Let μ be a complex measure on X such that μ is orthogonal to A_m :*

$$\int f d\mu = 0 \quad (f \in A_m).$$

Let μ_a and μ_s be (respectively) the absolutely continuous and singular parts of μ with respect to m . Then μ_a and μ_s are separately orthogonal to A_m , and μ_s is also orthogonal to 1. Furthermore, $d\mu_a = h dm$, where $h \in H^1(dm)$.

Proof. Let $d\mu_a = h dm$, and let ρ be the positive measure on X defined by

$$d\rho = (1 + |h|) dm + d|\mu_s|,$$

where $|\mu_s|$ denotes the total variation (measure) of μ_s . If $f \in A_m$, then,

$$\int |1 - f|^2 d\rho \geq \int |1 - f|^2 dm \geq 1. \tag{6.51}$$

Let G be the orthogonal projection of 1 into the closed subspace of $L^2(d\rho)$ which is spanned by A_m . By (6.51)

$$\int |1 - G|^2 d\rho \geq 1.$$

By Theorem 5.8, the function $(1 - G)^{-1}$ is (an outer function) in $H^2(dm)$, and $(1 - G)(1 + |h|)$ is in $L^2(dm)$. Therefore $(1 - G)h$ is in $L^2(dm)$.

Again, let $f \in A_m$. We claim that

$$\int (1 - G) f d\mu = 0.$$

For, choose a sequence of elements f_n in A_m which converge to G in $L^2(d\rho)$. Since μ is absolutely continuous with respect to ρ and $d\mu/d\rho$ is bounded

$$\int (1 - G) f d\mu = \lim_n \int (1 - f_n) f d\mu = 0$$

because each $(1 - f_n)f$ is in A_m and μ is orthogonal to A_m . From Theorem 5.8 we know that

$(1-G)$ vanishes almost everywhere with respect to the singular part of ϱ . Thus $(1-G)$ vanishes almost everywhere relative to μ_s , and so

$$0 = \int (1-G) f d\mu_a = \int (1-G) f h dm \quad (f \in A_m). \quad (6.52)$$

Let g_n be a sequence of elements of A which converge to $(1-G)^{-1}$ in $L^2(dm)$. There is such a sequence, since $(1-G)^{-1}$ is in $H^2(dm)$. By (6.52)

$$\int g_n f (1-G) h dm = 0 \quad (6.53)$$

for each n . Now $(1-G)h$ is in $L^2(dm)$ and g_n converges to $(1-G)^{-1}$ in $L^2(dm)$. We may, therefore, pass to the limit in (6.53) and obtain

$$\int f h dm = 0 \quad (f \in A_m).$$

This proves that μ_a is orthogonal to A_m ; and hence that μ_s is also.

According to Theorem 4.3, 1 belongs to the closure of A_m in L^2 of the positive singular measure $|\mu_s|$. If we choose functions f_n in A_m which converge to 1 in $L^2(d|\mu_s|)$, we shall have

$$\int d\mu_s = \lim_n \int f_n d\mu_s.$$

But μ_s is orthogonal to A_m ; consequently $\int d\mu_s = 0$. By the Corollary to Theorem 6.1, $h \in H^1(dm)$. That completes the proof.

We should like to relate some further results concerning measures which are orthogonal to A_m (or A). We shall need the following lemma which extends Lemma 5.3. This argument is basically due to R. Arens, and we should like to thank him for allowing us to use his proof.

LEMMA 6.6. *Let m be a probability measure and let g be a real-valued function in $L^1(dm)$. Suppose that*

$$\int \log |1 - tg| dm \geq 0 \quad (6.61)$$

for every real number t in some interval $|t| < \delta$. Then g vanishes almost everywhere with respect to m .

Proof. First, let g be any function in $L^1_R(dm)$. If z is a complex number in the (open) upper half-plane, define

$$u(z) = \int \log |1 - zg| dm.$$

It is easy to see that u is a harmonic function in the upper half-plane. Furthermore,

$$\lim_{y \rightarrow 0} \frac{u(iy)}{y} = 0. \quad (6.62)$$

We see this as follows. First,

$$u(iy) = \frac{1}{2} \int \log |1 - iyg|^2 dm = \frac{1}{2} \int \log (1 + y^2 g^2) dm.$$

$$\text{Thus } \frac{u(iy)}{y} = \frac{1}{2} \int \frac{1}{y} \log (1 + y^2 g^2) dm = \frac{1}{2} \int \frac{1}{yg} \log (1 + y^2 g^2) g dm. \quad (6.63)$$

For each $y > 0$, the function $\log(1 + y^2 g^2)/yg$ is defined to be 0 on the set, where $g = 0$. As $y \rightarrow 0$, this one-parameter family of functions converges pointwise to 0; and, the convergence is bounded, since the function $\log(1 + x^2)/x$ is bounded on the real line. Since the function g is integrable, the bounded (dominated) convergence theorem tells us that we may pass to the limit in (6.63) and obtain (6.62).

Now suppose (6.61) holds for $-\delta < t < \delta$. Then the function u is non-negative in the strip $-\delta < \text{Re } z < \delta$. This follows from the observation that

$$u(x + iy) = \frac{1}{2} \int \log |1 - (x + iy)g|^2 dm = \frac{1}{2} \int \log [(1 - xg)^2 + y^2 g^2] dm \geq \int \log |1 - xg| dm.$$

The remainder of the proof consists in showing that, if u is a non-negative harmonic function in the half-strip $-\delta < \text{Re } z < \delta$, and if $u(iy)/y$ tends to 0 as $y \rightarrow 0$, then $u = 0$. Actually, u does not have to be defined in so large a region. Let

$$v(w) = u[i\delta(1 - w)] \quad (|w| < 1).$$

Then v is a non-negative harmonic function in the unit disc. Also, (6.62) becomes

$$\lim_{r \rightarrow 1} \frac{v(r)}{1 - r} = 0. \quad (6.64)$$

According to the theorem of Herglotz [16; 18; 23], v has the form

$$v(w) = \int P_w(\theta) d\mu(\theta),$$

where μ is a finite positive measure on the unit circle (and P_w is the Poisson kernel).

Now

$$\min_{\theta} P_r(\theta) = \frac{1 - r}{1 + r}$$

so that

$$v(r) \geq \frac{1-r}{1+r} \int d\mu.$$

From (6.64) we have immediately $\mu=0$. Therefore $v=0$, hence $u=0$.

$$\text{Since } u=0, \quad 0 = u(i) = \frac{1}{2} \int \log(1+g^2) dm$$

from which it is apparent that $g=0$.

THEOREM 6.7. *Let g be a function in $L^1(dm)$ which is orthogonal to $(A + \bar{A})$. Then $g=0$ almost everywhere relative to m .*

Proof. It suffices to prove that if g is a real-valued function in $L^1(dm)$ such that $\int fg dm = 0$ for every f in A , then $g=0$. Any such function g satisfies the hypotheses of Lemma 6.6, as follows from Theorem 5.2. Hence $g=0$.

COROLLARY. *If $1 \leq p < \infty$, then $(A + \bar{A})$ is dense in $L^p(dm)$. In particular, any real-valued function in $H^p(dm)$ is constant.*

COROLLARY. *Let μ be a (complex) measure on X which is orthogonal to $(A + \bar{A})$. Then μ is mutually singular with m .*

Proof. By the generalized Riesz theorem (6.5), if $d\mu = h dm + d\mu_s$, where $h \in L^1(dm)$ and μ_s is mutually singular with m , then both h and μ_s are orthogonal to $(A + \bar{A})$. By 6.7, $h=0$.

COROLLARY. *Let g be a function in $L^1(dm)$, and let L be the linear functional on the subspace $(A + \bar{A})$ of $C(X)$ which is defined by*

$$L(f) = \int fg dm \quad (f \in (A + \bar{A})).$$

Then L has a unique Hahn-Banach (norm preserving) extension to a linear functional \tilde{L} on $C(X)$. Furthermore, this extension \tilde{L} is defined by

$$\tilde{L}(f) = \int fg dm \quad (f \in C(x)).$$

In particular
$$\int |g| dm = \sup_f \left| \int fg dm \right| \quad (f \in (A + \bar{A}), \quad \|f\| \leq 1).$$

Proof. Let \tilde{L} be any linear functional on $C(X)$ such that \tilde{L} is an extension of L and

$$\|\tilde{L}\| = \|L\| = \sup_f \left| \int fg dm \right| \quad (f \in (A + \bar{A}), \quad \|f\| \leq 1).$$

Then L arises from a (complex) measure μ on X , say

$$d\mu = h dm + d\mu_s,$$

where $h \in L^1(dm)$ and μ_s is mutually singular with m . Since \tilde{L} is an extension of L , the measure

$$d\mu - g dm = (h - g) dm + d\mu_s,$$

is orthogonal to $(A + \bar{A})$. By the generalized Riesz theorem (6.5), the measures $(h - g) dm$ and $d\mu_s$ are separately orthogonal to $(A + \bar{A})$. Thus $(h - g)$ is a function in $L^1(dm)$ which is orthogonal to $(A + \bar{A})$. Hence, $h = g$. Since the norm of L is

$$\int |g| dm + \|\mu_s\|$$

and is equal to the norm of L , it is clear that $\mu_s = 0$ and

$$\|L\| = \int |g| dm.$$

COROLLARY. Let $g \in L^1(dm)$ and let L be the linear functional on A defined by

$$L(f) = \int fg dm \quad (f \in A).$$

Then L has a unique Hahn-Banach extension to a linear functional \tilde{L} on $C(X)$, and \tilde{L} has the form

$$\tilde{L}(f) = \int f\tilde{g} dm \quad (f \in C(X)),$$

where $\tilde{g} \in L^1(dm)$.

Proof. Let L be any norm-preserving extension of L to a linear functional on $C(X)$. Then

$$\tilde{L}(f) = \int f d\mu \quad (f \in C(X)),$$

where μ is a complex measure on X , and the total variation of μ equals $\|L\|$. Let

$$d\mu = \tilde{g} dm + d\mu_s,$$

where $\tilde{g} \in L^1(dm)$ and μ_s and m are mutually singular. Since \tilde{L} is an extension of L , the measure $d\mu - g dm$ is orthogonal to A . By the general F. and M. Riesz theorem, the absolutely continuous and singular parts of this measure are separately orthogonal to A . Thus, μ_s is orthogonal to A and $(\tilde{g} - g)$ is in $H_m^1(dm)$. Therefore, if $f \in A$

$$\int f d\mu = \int f\tilde{g} dm = \int fg dm.$$

Hence
$$\sup_{\substack{f \in \tilde{A} \\ \|f\| \leq 1}} \left| \int f \tilde{g} \, dm \right| = \sup_{\substack{f \in \tilde{A} \\ \|f\| \leq 1}} \left| \int fg \, dm \right| = \|L\|.$$

From this we see that $\|L\| \leq \|\tilde{g}\|_1$. But

$$\|L\| = \|\mu\| = \|\tilde{g}\|_1 + \|\mu_s\|.$$

We conclude that

$$\mu_s = 0 \quad \text{and} \quad \|\tilde{g}\|_1 = \|L\|.$$

Any other Hahn-Banach extension of L must have the form

$$\int f(\tilde{g} + h) \, dm \quad (f \in C(X)),$$

where $h \in H_m^1(dm)$ and $\|\tilde{g} + h\|_1 = \|\tilde{g}\|_1$.

Now
$$\|\tilde{g}\|_1 = \sup_{\substack{f \in \tilde{A} \\ \|f\| \leq 1}} \left| \int f \tilde{g} \, dm \right|.$$

Since the unit ball in L^∞ is weak-star compact, we can find an $f \in H^\infty(dm)$ such that $\|f\|_\infty = 1$ and

$$\int f \tilde{g} \, dm = \|g\|_1.$$

This results from the evident fact that the weak-star closure of the unit ball in A is contained in the unit ball of $H^\infty(dm)$. Since $|f| \leq 1$, it is clear that $fg = |g|$. But, note that

$$\int f(\tilde{g} + h) \, dm = \int f \tilde{g} \, dm = \|g\|_1 = \|g + h\|_1.$$

Thus $f(g + h) = |g + h|$. Since fg and $f(g + h)$ are non-negative, fh is real-valued. But $f \in H^\infty(dm)$ and $h \in H^1(dm)$. Therefore $(fh) \in H^1(dm)$, and being real-valued, is constant. Since $\int h \, dm = 0$, we have $fh = 0$. Now, from

$$fg = |g|, \quad fh = 0, \quad |g| = |g + h|$$

it is evident that $h = 0$. This proves the corollary.

7. Analytic structures in $M(A)$

We shall now regard the space X as embedded in the maximal ideal space $M(A)$, as we described in Section 2. We shall once again discuss complex homomorphisms of A . If Φ is such a homomorphism, we know that there is a unique positive measure m on X such that

$$\hat{f}(\Phi) = \int_X f \, dm \quad (f \in A)$$

We are primarily interested in Φ 's which do not lie in X , that is homomorphisms which are not simply evaluation at a point of X . It is worthwhile to ask what distinguishes X among the closed subsets of $M(A)$.

THEOREM 7.1. *The space X is the Šilov boundary for A ; that is, X is the smallest closed subset of $M(A)$ on which every function $f, f \in A$, attains its maximum modulus.*

Proof. Since A was originally defined as a sup norm algebra on X ,

$$\sup_{M(A)} |f| = \sup_X |f| = \sup_X |f|$$

for every f in A . Let $x \in X$ and let U be a relatively open subset of X which contains x . Since $\log |A^{-1}|$ is uniformly dense in $C_R(X)$, it is clear that we can find a function f in A such that $f(x) = 1$, $\|f\| < 1 + \varepsilon$, and $|f| < \varepsilon$ on $X - U$. Therefore f does not attain its maximum modulus on $X - U$.

We are interested in strengthening the analogy of logmodular algebras with algebras of analytic functions, by finding subsets of $M(A)$ which can be endowed with an analytic structure in which the functions f are analytic. Whether this is always possible or not remains an unsolved problem; however, we can show that if Φ is a point of $M(A)$ such that there is at least one other point Φ_1 "closely related" to it, then there passes through Φ an "analytic disc" in $M(A)$. The argument we present is due to Wermer [27], who proved the corresponding result for Dirichlet algebras. We shall reorganize Wermer's proof considerably, in order to gain a slightly more general theorem. We continue to work with our logmodular algebra A , and will comment on the generality later.

DEFINITION 7.1. Let Φ be a complex homomorphism of A with representing measure m . Let ψ be another complex homomorphism of A . We say that ψ is *bounded on $H^2(dm)$* if there exists a positive constant K such that

$$|\psi(f)| \leq K \cdot \left[\int |f|^2 dm \right]^{\frac{1}{2}} \quad (f \in A).$$

The reason for the terminology should be apparent. We are discussing a linear functional ψ which is bounded on the pre Hilbert space consisting of A with the (semi) norm of the space $L^2(dm)$. If ψ is so bounded, it is clear that ψ has a unique extension to a bounded linear functional on $H^2(dm)$, the completion of A in $L^2(dm)$. We shall call this extended functional ψ also. It is clearly multiplicative:

$$\psi(fg) = \psi(f)\psi(g) \quad (f, g \in H^2(dm)).$$

THEOREM 7.2. *Let Φ be a complex homomorphism of A with representing measure m . Let ψ be a complex homomorphism of A which is distinct from Φ and which is bounded on*

$H^2(dm)$. Let G be the orthogonal projection of 1 into the closed subspace of $H^2(dm)$ which is spanned by kernel $(\psi) = \{f \in A; \psi(f) = 0\}$. Then $\int G dm = k^2$, where $0 < k < 1$. Furthermore, if we define

$$Z = \frac{1}{k} \cdot \frac{k^2 - G}{1 - G} \tag{7.21}$$

we have the following.

- (i) Z is an inner function in $H^2(dm)$.
- (ii) The measure

$$d\mu = \frac{|1 - G|^2}{1 - k^2} dm = \frac{1 - k^2}{|1 - kZ|^2} dm \tag{7.22}$$

is a (the) representing measure for ψ .

- (iii) $ZH^2 = H_m^2$, the space of functions in $H^2(dm)$ such that $\Phi(f) = \int f dm = 0$.
- (iv) If $f \in H^2(dm)$ and $a_n = \int \bar{Z}^n f dm$, then

$$\psi(f) = \sum_{n=0}^{\infty} a_n [\psi(Z)]^n.$$

Proof. Let S be the closed subspace of $H^2(dm)$ which is spanned by the kernel of ψ . Then S is invariant under multiplication by the functions in A . Also, 1 is not orthogonal to S ; because, $\psi \neq \Phi$ and so there exists an f in A such that $\psi(f) = 0$ but $\Phi(f) = \int f dm \neq 0$. By Theorem 5.5, G is a function of constant modulus

$$|G| = k > 0$$

and $S = GH^2(dm)$. Since $(1 - G)$ is orthogonal to G ,

$$\int G dm = k^2.$$

Evidently $k < 1$; for $k^2 \leq \int |G| dm = k$, and if $k = 1$ then G is constant. But this means that $S = H^2$, which is impossible since ψ is a bounded and non-zero functional on $H^2(dm)$. (Its kernel cannot be dense, unless $\psi = 0$.) Now define Z by (7.21). Then

$$Z = \frac{k - \frac{1}{k}G}{1 - k\left(\frac{1}{k}G\right)}$$

and since $|k^{-1}G| = 1$, Z is a function of modulus 1. Certainly $Z \in H^2(dm)$ because Z is the sum of a uniformly convergent power series in G . That proves (i).

Define μ by (7.22). Let $f \in A$. Then

$$(1 - k^2) \int f d\mu = \int f \cdot (1 - G)(1 - \bar{G}) dm.$$

If $\psi(f) = 0$ then $(1 - G)f$ is in the subspace S ; hence $(1 - G)$ is orthogonal to $(1 - G)f$ and $\int f d\mu = 0$. Since $(1 - G)$ is orthogonal to G

$$\int |1 - G|^2 dm = \int (1 - \bar{G}) dm = 1 - k^2$$

and thus
$$\int d\mu = 1.$$

Therefore μ is a representing measure for ψ .

It is clear that $\int Z dm = 0$, since $\int G dm = k^2$. Since $|Z| = 1$ we have ZH^2 contained in H_m^2 . Let $g \in H_m^2$ and suppose that g is orthogonal to ZH^2 . Then $g \in S$, i.e., $\psi(g) = 0$. For

$$\psi(g) = \int g d\mu = \int g \cdot \frac{|1 - G|^2}{1 - k^2} dm.$$

From (7.21), G is the sum of a uniformly convergent power series in Z . So $|1 - G|^2$ is a power series in Z and \bar{Z} and we shall have $\psi(g) = 0$ provided we can show that

$$\int Z^n g dm = 0 \quad (n = 0, \pm 1, \pm 2, \dots).$$

For $n \geq 0$, this follows from the fact that $\int g dm = 0$. For $n < 0$, it follows from the fact that g is orthogonal to ZH^2 . Now, since $\psi(g) = 0$, we have $g = Gh$, where $h \in H^2$. Note that

$$\int |h|^2 dm = \frac{1}{k^2} \int Gh \bar{g} dm.$$

By (7.21), $G = k^2 + Zf$, where $f \in H^2$. Since g is orthogonal to ZH^2 ,

$$\int |h|^2 dm = \frac{1}{k^2} \int k^2 h \bar{g} dm = \int \bar{G} |h|^2 dm.$$

But $|G| = k < 1$ and so $h = 0$, i.e., $g = 0$. That proves (iii).

Let $f \in H^2$. Then $f - \Phi(f) = f - \int f dm$ is in H_m^2 . So

$$f - \Phi(f) = Zg,$$

where g is in H^2 . Obviously g is uniquely determined by f . The constant functions are orthogonal to ZH^2 and so

$$\int |f|^2 dm \geq \int |Zg|^2 dm = \int |g|^2 dm.$$

If we define a linear operator T by $Tf = g$, i.e.,

$$f = \Phi(f) + Z(Tf)$$

then the operator T is bounded by 1 on $H^2(dm)$. If we define

$$a_n = \int \bar{Z}^n f dm$$

then it is clear that

$$\begin{aligned} f &= a_0 + Z(Tf) \\ Tf &= a_1 + Z(T^2f) \text{ etc.} \end{aligned}$$

For any n we (therefore) have

$$f = a_0 + a_1 Z + \dots + a_{n-1} Z^{n-1} + Z^n (T^n f).$$

Now $\psi(Z) = k < 1$. The numbers $\psi(T^n f)$ are bounded, because ψ is a bounded functional on H^2 and $\|T\| \leq 1$. Therefore we let $n \rightarrow \infty$ and obtain

$$\psi(f) = \sum_{j=0}^{\infty} a_j [\psi(Z)]^j.$$

THEOREM 7.3. *Let Φ , m , ψ , and Z be as in Theorem 7.2. Let θ be any complex homomorphism of A which is bounded on $H^2(dm)$. Then, for every f in $H^2(dm)$*

$$\theta(f) = \sum_{n=0}^{\infty} a_n [\theta(Z)]^n.$$

Proof. If $\theta = \Phi$, the statement is evident. Suppose $\theta \neq \Phi$. We apply Theorem 7.2 with ψ replaced by θ . We obtain an inner function Z_1 associated by θ , such that $H_m^2 = Z_1 H^2$ and the corresponding series expansion for $\theta(f)$ is valid. Since

$$ZH^2 = Z_1 H^2$$

and $|Z| = |Z_1| = 1$, the function Z/Z_1 and its complex conjugate both belong to $H^2(dm)$. Since $(A + \bar{A})$ is dense in $L^2(dm)$ (Theorem 5.4), there are no non-constant real-valued functions in $H^2(dm)$. Therefore $Z_1 = \lambda Z$, where λ is a constant of modulus 1. It is easy to see that the series

$$\sum_{n=0}^{\infty} a_n [\theta(Z)]^n, \quad a_n = \int \bar{Z}^n f dm$$

is unaffected if Z is replaced by λZ , $|\lambda| = 1$. We are done.

THEOREM 7.4. *Let Φ be a complex homomorphism of A with representing measure m , and suppose there exists a complex homomorphism ψ of A which is distinct from Φ and which is bounded on $H^2(dm)$. Let Z be the function defined in Theorem 7.2. Let D be the set of all complex homomorphisms of A which are bounded on $H^2(dm)$. If $\theta \in D$ define $\hat{Z}(\theta) = \theta(Z)$. Then*

\hat{Z} is a one-one map of D onto the open unit disc in the plane. The inverse mapping τ of \hat{Z} is a continuous one-one map of the open unit disc onto D , and for every f in A the composed function $f \circ \tau$ is analytic.

Proof. Obviously \hat{Z} maps D into the open unit disc. For suppose $\theta \in D$. Associated with θ is a function Z_1 , as in the last proof; and $Z_1 = \lambda Z$, where $|\lambda| = 1$. By Theorem 7.2, $|\theta(Z_1)| < 1$, so $|Z(\theta)| < 1$. If $f \in H^2$ we have

$$\theta(f) = \sum_{n=0}^{\infty} a_n [\hat{Z}(\theta)]^n, \quad a_n = \int \bar{Z}^n f \, dm$$

for all θ in D . If $\hat{Z}(\theta_1) = \hat{Z}(\theta_2)$, then $\theta_1(f) = \theta_2(f)$ for all f in A ; hence $\theta_1 = \theta_2$. Thus \hat{Z} is a one-one map of D into the open unit disc.

To see that \hat{Z} is onto, we argue as follows. Suppose f and g belong to A . Let

$$a_n = \int \bar{Z}^n f \, dm, \quad b_n = \int \bar{Z}^n g \, dm. \quad (7.41)$$

Then

$$\begin{aligned} f &= a_0 + a_1 Z + \dots + a_n Z^n + Z^{n+1} h_1 \\ g &= b_0 + b_1 Z + \dots + b_n Z^n + Z^{n+1} h_2, \end{aligned}$$

where h_1 and h_2 are (bounded) functions in $H^2(dm)$. From this it is easy to see that if

$$c_n = \int \bar{Z}^n (fg) \, dm$$

we have

$$\left. \begin{aligned} c_0 &= a_0 b_0, \\ c_1 &= a_0 b_1 + a_1 b_0, \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0, \\ &\text{etc.} \end{aligned} \right\} \quad (7.42)$$

Now let λ be a complex number, $|\lambda| < 1$. For any f in A define

$$\theta(f) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad (7.43)$$

where the a_n are defined by (7.41). Obviously θ is a linear functional on A , and by (7.42) θ is multiplicative. Suppose $f \in A$ and

$$\int |f|^2 \, dm < \varepsilon^2.$$

Then $|a_n| < \varepsilon$ for each n and

$$|\theta(f)| < \varepsilon \cdot \frac{1}{1 - |\lambda|}.$$

(i) $\|\Phi_1 - \Phi_2\| < 2$, i.e., there exists a constant K , $0 < K < 2$, such that $|\Phi_1(f) - \Phi_2(f)| \leq K\|f\|$ for every $f \in A$.

(ii) There is a positive constant $c < 1$ such that $|\Phi_2(f)| \leq c\|f\|$ for every f in A which satisfies $\Phi_1(f) = 0$.

Proof. Suppose (ii) fails to hold. Then there is a sequence of functions f_n in A such that $\Phi_1(f_n) = 0$, $\|f_n\| \leq 1$, and $|\Phi_2(f_n)| \rightarrow 1$. If $\{\lambda_n\}$ is a sequence of points in the closed unit disc such that $|\lambda_n| \rightarrow 1$, it is easy to find a sequence of linear fractional maps L_n (of the unit disc onto itself) such that $|L_n(0) - L_n(\lambda_n)| \rightarrow 2$. Each L_n can be uniformly approximated on the unit disc by polynomials. Hence, if $f \in A$ and $\|f\| \leq 1$, then $L_n \circ f$ belongs to A . Find such a sequence of maps for $\lambda_n = \Phi_2(f_n)$. Then define $g_n = L_n \circ f_n$. We have $\|g_n\| \leq 1$ and $|\Phi_1(g_n) - \Phi_2(g_n)| \rightarrow 2$. Thus (i) does not hold, i.e., $\|\Phi_1 - \Phi_2\| = 2$.

If (i) does not hold, i.e., if $\|\Phi_1 - \Phi_2\| = 2$, there is a sequence of functions g_n in A with $\|g_n\| \leq 1$ and $|\Phi_1(g_n) - \Phi_2(g_n)| \rightarrow 2$. Let $f_n = \frac{1}{2}[g_n - \Phi_1(g_n)]$. Then $f_n \in A$, $\|f_n\| \leq 1$, $\Phi_1(f_n) = 0$, and $|\Phi_2(f_n)| \rightarrow 1$. Thus (ii) does not hold.

THEOREM 7.6. *Let A be a logmodular algebra on X , and let Φ_1 and Φ_2 be complex homomorphisms of A . The following are equivalent.*

- (i) $\|\Phi_1 - \Phi_2\| < 2$.
- (ii) If m_1 and m_2 are the (respective) representing measures for Φ_1 and Φ_2 , then m_2 is absolutely continuous with respect to m_1 , and the derivative dm_2/dm_1 is bounded.
- (iii) Φ_2 is bounded on $H^2(dm_1)$.

Proof. Let K be a positive constant, and suppose that we do not have $m_2 \leq Km_1$. Then there is a positive continuous function u on X such that

$$\int u dm_2 > K \int u dm_1. \tag{7.61}$$

Since $\log |A^{-1}|$ is uniformly dense in $C_R(X)$, we may assume that $u = -\log |f|$, $f \in A^{-1}$. Since $u > 0$ we have $\|f\| < 1$, and (7.61) says

$$\int \log |f| dm_2 < K \int \log |f| dm_1$$

that is,

$$|\Phi_2(f)| < |\Phi_1(f)|^K.$$

Let $\alpha = \Phi_2(f)$ and $\beta = \Phi_1(f)$. Let

$$g = \frac{f - \alpha}{1 - \bar{\alpha}f}.$$

Hence θ is a complex homomorphism of A which is bounded on $H^2(dm)$, i.e., θ is in D . Certainly $\hat{Z}(\theta) = \lambda$. Thus \hat{Z} maps D onto $|\lambda| < 1$.

Now let τ be the inverse map of \hat{Z} , i.e., if $|\lambda| < 1$, then $\tau(\lambda) = \theta$, the complex homomorphism of A defined by (7.43). We know that τ is one-one and maps the open unit disc onto D . It is also clear that τ is continuous. This simply says that, if we fix $f \in A$, the map $\lambda \rightarrow \tau(\lambda)(f)$ is continuous, i.e., that $f \circ \tau$ is continuous. But $f \circ \tau$ is an analytic function in the unit disc:

$$(f \circ \tau)(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n,$$

where $\{a_n\}$ is the bounded sequence of numbers defined by

$$a_n = \int \bar{Z}^n f dm.$$

That completes the proof.

We should now ask ourselves just which properties of a logmodular algebra were used in this argument. Actually, we employed only properties of a particular representing measure for the fixed complex homomorphism Φ . The properties were those which were necessary for the proof of the invariant subspace theorem (5.5), (i) $A + \bar{A}$ is dense in $L^2(dm)$, (ii) if μ is a representing measure for Φ which is absolutely continuous with respect to m , then $\mu = m$. Suppose we have a sup norm algebra A and a complex homomorphism Φ of A which has a representing measure m which satisfies (i), (ii), and (iii) there exists a complex homomorphism ψ of A which is distinct from Φ and is bounded on $H^2(dm)$. By the same argument, there exists an "analytic disc" in $M(A)$ which passes through Φ , i.e., there exists a one-one map τ of $|\lambda| < 1$ into $M(A)$ such that $\tau(0) = \Phi$ and $f \circ \tau$ is analytic for every f in A .

For the logmodular A , one can give a slightly more intrinsic characterization of the analytic disc D which occurs in Theorem 7.4. It is what Gleason termed the "part" of Φ . In [11], Gleason pointed out that the relation

$$\Phi_1 \sim \Phi_2 \Leftrightarrow \|\Phi_1 - \Phi_2\| < 2$$

is an equivalence relation on the set of complex homomorphisms of a sup norm algebra (or commutative Banach algebra). The equivalence classes for this relation he called the *parts* of $M(A)$. It is not immediately evident that the relation is transitive; however, it is not difficult to show that it is. For logmodular algebras, the transitivity will soon become evident.

LEMMA 7.5. *Let Φ_1 and Φ_2 be complex homomorphisms of the sup norm algebra A . These statements are equivalent.*

Then $\|g\| < 1$, $g \in A$, $\Phi_2(g) = 0$, and

$$|\Phi_1(g)| = \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| \geq \frac{|\beta| - |\alpha|}{1 - |\alpha||\beta|} \geq \frac{|\beta| - |\beta|^K}{1 - |\beta|^{K+1}}.$$

Now (7.61) will not be affected if we replace u by εu , $\varepsilon > 0$. As $\varepsilon \rightarrow 0$ we have $|\beta| \rightarrow 1$. Since

$$\lim_{r \rightarrow 1} \frac{r - r^K}{1 - r^{K+1}} = \frac{K-1}{K+1},$$

we conclude that

$$\frac{K-1}{K+1} \leq \sup_g |\Phi_1(g)| \quad (g \in A, \|g\| \leq 1, \Phi_2(g) = 0).$$

If $\|\Phi_1 - \Phi_2\| < 2$, Lemma 7.5 tells us that the supremum on the right is less than 1. Thus, for some sufficiently large K we must have

$$m_2 \leq Km_1.$$

Therefore (i) implies (ii).

It is evident that (ii) implies (i), and also that (ii) implies (iii). If (iii) holds, we can deduce (ii) immediately from part (ii) of Theorem 7.2. That completes the proof.

We can easily see (from this result) that $\|\Phi_1 - \Phi_2\| < 2$ is an equivalence relation on $M(A)$. This relation, i.e., "belonging to the same part", means that the representing measures m_1 and m_2 are mutually absolutely continuous, with bounded derivatives each way. If we combine the last theorem with Theorem 7.4, we have the following.

THEOREM 7.7. *Let A be a logmodular algebra on the space X . Let Φ be a complex homomorphism of A , and suppose that the (Gleason) part $P(\Phi)$ contains at least two points. Then there exists a one-one continuous map τ from the open unit disc into the maximal ideal space of A such that*

- (a) *the range of τ is the part $P(\Phi)$*
- (b) *for every f in A , the function $f \circ \tau$ is analytic.*

It is easy to see (for example, by Theorem 7.5) that each point of X constitutes a one-point part in $M(A)$. A point not in X (i.e., not on the Šilov boundary) may constitute a one-point part. One can raise the same questions for logmodular algebras that Wermer [27] raised for Dirichlet algebras. If $A \neq C(X)$ is X a proper subset of $M(A)$? If $A \neq C(X)$ must there exist a part in $M(A)$ which contains at least two points?

In the various examples of section 3, it is relatively easy to identify the parts, except in the case of the algebra H^∞ (Example 4). We shall discuss the parts for this algebra in a later paper. In Example 1, the open unit disc is one part, and the points of the unit

circle constitute one-point parts. For the "Big Disc" algebra A_G of Example 3 (G not isomorphic to the integers) the description of the parts is as follows. Each point of the boundary \hat{G} constitutes a one-point part. The "origin" of $M(A_G)$, i.e., the Haar homomorphism, is also a one-point part. The remaining parts are "analytic discs", each of which is dense in the entire maximal ideal space. These parts are easily described. If one regards A_G as an algebra of almost periodic functions in the upper half-plane, there is a natural injection τ of the half-plane into $M(A_G)$, "z goes into evaluation at z". The image of the open half-plane under τ is a part of $M(A_G)$. It consists of all points (r, α) in $M(A_G)$ such that $0 < r < 1$ and α belongs to a dense one-parameter subgroup of \hat{G} . This subgroup K is the image of the real axis under (the extended) τ . Any remaining (non-trivial) part is formed of the points (r, α) , $0 < r < 1$, where α ranges over a coset of the subgroup K . The description of the parts for the algebra B_G of Example 5 is similar. In this case there are no one-point parts of the Šilov boundary X .

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