

FRÉCHET-VOLTERRA VARIATIONAL EQUATIONS, BOUNDARY VALUE PROBLEMS, AND FUNCTION SPACE INTEGRALS

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Introduction

In this paper we study some connections among (a) boundary value problems arising in partial differential equations, (b) function space integrals (stochastic process expectations), and (c) what we have decided to call Fréchet-Volterra (F.V.) variational equations—equations where an unknown functional appears under operations involving F.V. derivatives. Before giving an explicit example to illustrate the kind of connections we mean, let us first recall briefly the definition of a F.V. derivative.⁽¹⁾

By a functional we mean a complex valued function $u(q)$ defined on a space of functions $q = q(t)$ where $t \in T$, an open interval of R . By the F.V. derivative of the functional $u(q)$ at the point τ , denoted by $\delta u / \delta q(\tau)$, we mean the limit (in a suitable sense) of

$$\frac{u(q + \varphi_n) - u(q)}{\int \varphi_n(t) dt},$$

where $\{\varphi_n\}$ is a sequence of functions of t with support $[\tau - \varepsilon_n, \tau + \varepsilon'_n]$, $\varepsilon_n, \varepsilon'_n \rightarrow 0$, $\max |\varphi_n| \rightarrow 0$.⁽¹⁾ We also define F.V. derivatives of higher order.

As an illustrative example of the connections referred to above, let us consider first the function space integral. Let $C(0, t)$ be the space of continuous functions $z(\sigma)$ on $0 \leq \sigma \leq t$ with $z(0) = 0$. We will denote by $E_z^w\{F[z]\}$ the Wiener integral (Brownian motion expectation) of a functional $F[z]$ defined on $C(0, t)$, i.e., the integral based on the Wiener measure (Brownian motion stochastic process measure) on the space $C(0, t)$. By $E_z^w\{F[z]; x < z(t) < x + \varepsilon\}$ we mean the integral of $F[z]$ taken over the subspace of $C(0, t)$ consisting of functions $z(\sigma)$ for which $x < z(t) < x + \varepsilon$. Finally we will find it useful to consider $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} E_z^w\{F(z); x < z(t) < x + \varepsilon\}$, which we will denote by $E_z^w\{F[z] \delta(z(t) - x)\}$.

Now in particular consider

$$u(x, t; q) \equiv E_z^w\left\{\exp\left(i \int_0^t z(\sigma) q(\sigma) d\sigma - \frac{1}{2} \int_0^t z^2(\sigma) d\sigma\right) \delta(z(t) + x)\right\}, \quad (1.1)$$

⁽¹⁾ For a precise definition of the F.V. derivative, see chapter I, section 1, of this paper.

where $q(\sigma)$ (a parametric function in this integral) is continuous on $[0, t]$. Using probabilistic techniques it is not difficult to explicitly calculate this function space integral and indeed one obtains

$$\begin{aligned}
 u(x, t; q) = & (2\pi \sinh t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2 \tanh t} - \frac{ix}{\tanh t} \int_0^t R(t, \sigma; -1) q(\sigma) d\sigma\right) \\
 & \times \exp\left(\frac{1}{2} \int_0^t \int_0^t R(\sigma, \xi; -1) q(\sigma) q(\xi) d\sigma d\xi\right) \\
 & \times \exp\left(\frac{1}{2 \tanh t} \int_0^t \int_0^t R(t, \sigma; -1) R(t, \xi; -1) q(\sigma) q(\xi) d\sigma d\xi\right),
 \end{aligned} \tag{1.2}$$

where $R(\sigma, \xi; \mu)$ is the resolvent kernel on $[0, t]$ of $\min(\sigma, \xi)$, i.e.,

$$R(\sigma, \xi; -1) = \begin{cases} \frac{-\cosh(t-\xi) \sinh \sigma}{\cosh t} & \sigma \leq \xi, \\ \frac{-\cosh(t-\sigma) \sinh \xi}{\cosh t} & \sigma \geq \xi. \end{cases}$$

From a well-known theorem of Kac [12, 13] (see also Rosenblatt [20], Cameron [1], Darling and Siegert [4]) it follows that $u(x, t; q)$ as defined in (1.1) is the solution of

$$\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = -\frac{x^2}{2} u + ixq(t)u \quad u(x, t; q) \rightarrow 0, \quad x \rightarrow \pm \infty, \quad u(x, t; q) \rightarrow \delta(x) \quad t \rightarrow 0. \tag{1.3}$$

Now one can obtain (1.2) by solving this system directly as well as by calculating the function space integral above. The motivation for this paper is now illustrated by the observation that $u(x, t; q)$ as defined in (1.1) also satisfies a F.V. variational equation, specifically (1.1) is the unique solution of

$$\left. \begin{aligned}
 \frac{\delta u}{\delta q(\tau)} &= \left(- \int_0^t \min(\tau, s) q(s) ds \right) u - \int_0^t \min(\tau, s) \frac{\delta u}{\delta q(s)} ds - i\tau \frac{\partial u}{\partial x}, \quad 0 < \tau < t, \\
 \lim_{\tau \rightarrow t} \frac{\delta u}{\delta q(\tau)} &= ixu, \\
 \frac{\partial u(x, t, 0)}{\partial t} - \frac{1}{2} \frac{\partial^2 u(x, t, 0)}{\partial x^2} &= -\frac{x^2}{2} u(x, t, 0), \quad u(x, t, 0) \rightarrow \delta(x) \quad t \rightarrow 0.
 \end{aligned} \right\} \tag{1.4}$$

It will be shown in Chapter II, section 14, that (1.2) can also be obtained from the system (1.4) by using techniques appropriate to such a system—in this very simple example, by using a F.V. series expansion of the unknown functional $u(x, t; q)$ and determining the coefficients (functions in this case) by recurrence formulae and the other conditions in (1.4).

Thus we have three quite different problems, (1.1), (1.3), and (1.4), each solvable by its own particular technique and all leading to (1.2). As indicated above, the relation between the function space integral and the partial differential equation has been well explored and generalized. We are chiefly interested here in the connection between the systems (1.3) and (1.4) and the general relation between boundary value problems and F.V. variational equations that this particular example suggests.

Let us now see how one can obtain the system (1.4) by operating formally with the function space integral (1.1). We have from (1.1), for any point $0 < \tau < t$,

$$\frac{\delta u}{\delta q(\tau)} = i E_z^w \{ z(\tau) \exp \left(i \int_0^\tau z(\sigma) q(\sigma) d\sigma - \frac{1}{2} \int_0^\tau z^2(\sigma) d\sigma \right) \delta(z(t) - x) \}. \quad (1.5)$$

Noting that

$$E_z^w \{ F[z] \delta(z(t) - x) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} E_z^w \{ F[z] e^{i\mu z(t)} \} d\mu,$$

we have

$$\frac{\delta u}{\delta q(\tau)} = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} E_z^w \left\{ z(\tau) \exp \left(i \int_0^\tau z(\sigma) q(\sigma) d\sigma \right) \times \exp \left(-\frac{1}{2} \int_0^\tau z^2(\sigma) d\sigma + i\mu z(t) \right) \right\} d\mu. \quad (1.6)$$

To proceed it is essential to exploit the relation between the integral in function space and the derivative in function space, i.e. between the Wiener expectation and the F.V. derivative—we must integrate by parts in function space. For this we have the relation (cf. Cameron [2])

$$\int_0^t \min(\tau, s) E_z^w \left\{ \frac{\delta F}{\delta z(s)} \right\} ds = E_z^w \{ z(\tau) F[z] \}. \quad (1.7)$$

Using this we get from (1.6),

$$\begin{aligned} \frac{\delta u}{\delta q(\tau)} = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} \int_0^t \min(\tau, s) E_z^w \left\{ \frac{\delta}{\delta z(s)} \left[\exp \left(i \int_0^\tau z(\sigma) q(\sigma) d\sigma \right) \right. \right. \\ \left. \left. \times \exp \left(-\frac{1}{2} \int_0^\tau z^2(\sigma) d\sigma + i\mu z(t) \right) \right] \right\} ds d\mu. \end{aligned} \quad (1.8)$$

As is easily seen⁽¹⁾ $\delta z(t)/\delta z(s) = \delta_t(s)$, the Dirac measure in t at the point s . Thus we have

(1) We will show this in chapter I, example 1.1.

$$\begin{aligned}
 \frac{\delta u}{\delta q(\tau)} &= \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} \int_0^t \min(\tau, s) E_z^w \left\{ (iq(s) - z(s) + i\mu \delta_t(s)) \right. \\
 &\quad \left. \times \exp \left(i \int_0^t z(\sigma) q(\sigma) d\sigma - \frac{1}{2} \int_0^t z^2(\sigma) d\sigma + i\mu z(t) \right) \right\} ds d\mu \\
 &= \left(- \int_0^t \min(\tau, s) q(s) ds \right) E_z^w \left\{ \exp \left(i \int_0^t z(\sigma) q(\sigma) d\sigma - \frac{1}{2} \int_0^t z^2(\sigma) d\sigma \right) \delta(z(t) - x) \right\} \\
 &\quad - i \int_0^t \min(\tau, s) E_z^w \left\{ z(s) \exp \left(i \int_0^t z(\sigma) q(\sigma) d\sigma - \frac{1}{2} \int_0^t z^2(\sigma) d\sigma \right) \delta(z(t) - x) \right\} ds \\
 &\quad - \frac{\tau}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} E_z^w \left\{ \exp \left(i \int_0^t z(\sigma) q(\sigma) d\sigma - \frac{1}{2} \int_0^t z^2(\sigma) d\sigma + i\mu z(t) \right) \right\} d\mu, \tag{1.9}
 \end{aligned}$$

and finally, therefore, the Volterra variational equation,

$$\frac{\delta u}{\delta q(\tau)} = \left(- \int_0^t \min(\tau, s) q(s) ds \right) u - \int_0^t \min(\tau, s) \frac{\delta u}{\delta q(s)} ds - i\tau \frac{\partial u}{\partial x}. \tag{1.10}$$

The condition in (1.4) that $\lim_{\tau \rightarrow t} \frac{\delta u}{\delta q(\tau)} = i\tau u$ follows from (1.5), and the other conditions in (1.4) from the fact that (1.1) with $q(\sigma) \equiv 0$ satisfies (1.3) with $q(\sigma) \equiv 0$.

Thus we see that the variational equation in (1.4) arises from “differentiating” (1.1) with respect to the parametric function q and that what we have done is the function space analogue of the usual one-dimensional technique of obtaining a differential equation for a Fourier transform by differentiation with respect to the parameter of the transform. From this point of view, the differential equation “boundary condition” in (1.4) is then quite natural as it determines the transform when the parameter is zero.

It should be remarked that the technique used in this particular example of obtaining the F.V. variational equation from (1.1) can be used in more general situations. First of all one can consider the more general function space integral

$$u(x, t; q) = E_z^w \left\{ \exp \left(i \int_0^t z(\sigma) q(\sigma) d\sigma - \int_0^t v(z(\sigma)) d\sigma \right) \delta(z(t) - x) \right\}, \tag{1.11}$$

where $v(x) \geq 0$ is continuous on $(-\infty, \infty)$, (1.1) being the special case $v(x) = \frac{1}{2} x^2$. Assuming further that $v(x)$ is differentiable and denoting its derivative by $v'(x)$ we obtain formally, in the same way as above, the F.V. variational equation

$$\frac{\delta u}{\delta q(\tau)} = \left(- \int_0^t \min(\tau, s) q(s) ds \right) u - i \int_0^t \min(\tau, s) v' \left(-i \frac{\delta}{\delta q(s)} \right) u ds - i \tau \frac{\partial u}{\partial x}, \quad (1.12)$$

where again $0 < \tau < t$ and where the operator $v'(-i\delta/\delta q(s))$ must be defined appropriately⁽¹⁾—it is clear what the operator means when $v(x)$ is a polynomial.

Just as in the special case (1.1), it follows from Kac [12] that (1.11) is the solution (for this we need only that $v(x) \geq 0$ is continuous) of

$$\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = -v(x)u + ixq(t)u, \quad u(x, t; q) \rightarrow 0 \quad x \rightarrow \pm \infty, \quad u(x, t; q) \rightarrow \delta(x) \quad t \rightarrow 0. \quad (1.13)$$

And in this case we are again interested in the connection between the system (1.13) and the F.V. variational system consisting of (1.12) augmented by the conditions

$$\left. \begin{aligned} \lim_{\tau \rightarrow t^-} \frac{\delta u}{\delta q(\tau)} &= ixu \\ \frac{\delta u(x, t; 0)}{\partial t} - \frac{1}{2} \frac{\partial^2 u(x, t; 0)}{\partial x^2} &= -v(x)u(x, t; 0), \quad u(x, t; 0) \rightarrow \delta(x) \quad t \rightarrow 0. \end{aligned} \right\} \quad (1.14)$$

The first of the conditions in (1.14) comes again directly from (1.11) and the differential equation condition in (1.14) follows from the fact that (1.11) satisfies (1.13) when $q(\sigma) \equiv 0$.

From the point of view of differential equations the interesting observation is that *the solution of (1.13) considered as a functional of the parametric function q satisfies a F.V. variational equation*. Although in the particular example (1.1) and also in (1.11) the relationship between the differential equation boundary value problem and the F.V. variational system was through the space integral, it is clear that a more general problem presents itself which we now describe.

Consider a linear⁽²⁾ boundary value problem

$$\Lambda u(x) = 0, \quad x \in \Omega, \quad \text{an open set of } R^n, \quad (1.15)$$

where Λ is a linear partial differential operator with some boundary conditions

$$B_j u(x) = 0 \quad j = 0, 1, 2, \dots, \mu, \quad (1.16)$$

where $x \in \Gamma$, the boundary of Ω , and where B_j is a linear partial differential operator.

⁽¹⁾ We do this in chapter II, section 4.

⁽²⁾ The same problems arise in non-linear cases but nothing seems to be known in this direction at this time.

We introduce, in association with this boundary value problem,⁽¹⁾ a family of boundary value problems:

$$\Lambda u(x) = B(q)u, \quad (1.17)$$

where $B(q)$ is a family of linear partial differential operators depending on the parametric function q and where we impose on (1.17) the boundary conditions (1.16).

Now in (1.17) the solution u depends on q

$$u(x) = u(x; q), \quad (1.18)$$

i.e., it is a *functional* of q and we ask the following questions:

1. For what $B(q)$ does the F.V. derivative $\delta u / \delta q(\tau)$ exist? Do higher order F.V. derivatives exist? And what is important,
2. Can one *choose* $B(q)$ in such a way that $u(x; q)$ satisfies a F.V. variational equation?
3. If the answer to the preceding question is "yes", then the natural next question is, does this variational equation augmented by certain "boundary conditions" have $u(x; q)$ as its unique solution? That is, are the differential equation boundary value problem and the F.V. variational equation system equivalent?

We find these questions interesting because, as already noted in the explicit example (1.1), the tools which seem natural for attacking F.V. variational equations, e.g., F.V. series, function space integral transforms,⁽²⁾ etc., are essentially different from the natural or known tools used in partial differential equations.

In chapter I we will see that question 1 above can be answered quite generally but, as stated above, questions 2 and 3 seem difficult. We shall study these questions for some *mixed problems* in chapter I and in order to obtain anything like complete results we shall have to confine ourselves to some *Cauchy problems* in chapter II. Our approach will be quite abstract since we are looking for *general methods* to apply in the case of various differential operators (cf. Remark 9.5 in chapter I). We will, however, show that the solution $u(x, t; q)$ of (1.13) does satisfy the F.V. variational equation (1.12) and moreover that $u(x, t; q)$ is the unique solution of (1.12) satisfying the conditions (1.14). The *same methods* used to show this will also show that the fundamental solution of the Schrödinger equation,

⁽¹⁾ We are purposely being brief here—we implicitly assume all the boundary value problems are well set.

⁽²⁾ And other methods yet to be found.

$$\frac{\partial u}{\partial t} - \frac{i}{2} \frac{\partial^2 u}{\partial x^2} = -i V(x)u + i x q(t)u \quad (1.19)$$

satisfies the F.V. variational equation

$$\frac{\delta u}{\delta q(\tau)} = \left(-i \int_0^t \min(\tau, s) q'(s) ds \right) u - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u ds + \tau \frac{\partial u}{\partial x}. \quad (1.20)$$

Now in the Schrödinger equation case there is no direct representation of $u(x, t; q)$, the solution of (1.19), as a function space integral.^{(1)(*)} However, using a conjecture of Donsker expressing $u(x, t; q)$ as the limit of a certain Wiener expectation one can formally derive that $u(x, t; q)$ satisfies the F.V. variational equation (1.20). Actually, whether (1.20) is obtained, formally or not, using function space integrals or just guessing by analogy with (1.12), the point is that once we know the equation (1.20) we *prove* that $u(x, t, q)$, the solution of (1.19), satisfies (1.20), using our general methods not involving function space integrals.

Our proof of uniqueness, i.e., that $u(x, t; q)$ is the *only* solution of the F.V. variational equation satisfying certain side conditions does involve the use of function space integrals, and an inversion formula for function space transforms (cf. Cameron and Donsker [3]). Only in very special cases (cf. chapter II, section 14) can we prove uniqueness without the function space integral. It would be of great interest, in the general setting of this paper, to prove the equivalence of the differential equation boundary value problem and the F.V. variational system without recourse to function space integration, since in certain cases there is, intrinsically at least, no function space integral involved.

On the other hand,⁽²⁾ and what is of some interest from the point of view of stochastic processes, the relation between function space integrals and F.V. variational equations exists without the corresponding partial differential equation. To be specific, consider, for example, a Gaussian stochastic process $\{y_\sigma, 0 \leq \sigma \leq t\}$ with mean function zero and covariance function $\varrho(\sigma, \xi)$. Let us denote expectations on this process by $E_y^e\{\cdot\}$. It is possible to show that in this case (1.7) can be replaced by

$$\int_0^t \varrho(\tau, s) E_y^e \left\{ \frac{\delta F}{\delta y(s)} \right\} ds = E_y^e \{ y(\tau) F[y] \} \quad (1.21)$$

and that operating formally as before,

⁽¹⁾ Except by the so-called "Feynman integral" (see Gelfand and Yaglom [8]).

⁽²⁾ We do not pursue this point in the present paper.

^(*) (Added in proof.) Cf. also Nelson, Colloque C.N.A.S. Paris, June 1962.

$$u(x, t; q) \equiv E_y^q \left\{ \exp \left(i \int_0^t y(\sigma) q(\sigma) d\sigma - \int_0^t v(y(\sigma)) d\sigma \right) \delta(y(t) - x) \right\} \quad (1.22)$$

satisfies the F.V. variational equation

$$\frac{\delta u}{\delta q(\tau)} = \left(- \int_0^t \rho(\tau, s) q(s) ds \right) u - i \int_0^t \rho(\tau, s) v' \left(-i \frac{\delta}{\delta q(s)} \right) u ds - i \rho(t, \tau) \frac{\partial u}{\partial x}. \quad (1.23)$$

Now one of the reasons that in the special case $\rho(\sigma, \xi) = \min(\sigma, \xi)$ (the Wiener process), the function space integral also satisfies a partial differential equation is because the Wiener process is *Markovian*. The F.V. variational equation (1.23) holds for $u(x, t; q)$ defined by (1.22) whether the Gaussian process $\{y_\sigma, 0 \leq \sigma \leq t\}$ is Markovian or not. However, in the non-Markovian case the determination of just what boundary conditions on the solution of (1.23) specify the function space integral (1.22) as the *unique* solution seems difficult. It is not difficult to see that if $\rho(\sigma, \xi)$ is the Green function of a Sturm-Liouville differential equation (in this case $\{y_\sigma, 0 \leq \sigma \leq t\}$ is Markovian), then one has again a differential equation as in (1.3) but where the differential operator $-\frac{1}{2}\partial/\partial x^2$, which is inverse to $\min(\sigma, \xi)$, is replaced by the corresponding Sturm-Liouville differential operator.

The study of F.V. variational equations in their own right should prove useful. In this connection see Lévy [16] for a discussion of certain variational equations. In a paper of Hopf [11] a F.V. variational equation is considered in conjunction with a corresponding partial differential equation boundary value problem; however, the relation indicated formally there seems quite different from the type of correspondence considered here. Connections between F.V. variational equations, boundary value problems, and function space integrals of the type we consider here exist in a formal way in the literature of quantum field theory, although there the context is much more complicated. From that point of view the results in this paper are only the beginnings of what is needed (cf. Schwinger [24], [25], Kristensen [15]). The authors are extremely grateful to Professor Povl Kristensen, professor of Physics at Aarhus University, for generously informing them and discussing with them these latter results.

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CHAPTER I

F.V. derivatives of certain functionals

1. Definition of the F.V. derivative

Let E be a vector topological space, locally convex, complete,⁽¹⁾ and let T be an open interval, bounded or not, of R ($T=R$ is possible). By $\mathcal{D}(T)$ (or $C(T)$) we mean the space of infinitely differentiable (or continuous) functions on T with compact support provided with the topology of Schwartz (cf. Schwartz [22]) (or with the topology of uniform convergence on every compact set of T). The functions of $\mathcal{D}(T)$ and $C(T)$ are complex-valued.

A *functional* will be a mapping $q \rightarrow \Phi(q)$ from $\mathcal{D}(T)$ or $C(T)$ into E with the following properties:

The mapping $q \rightarrow \Phi(q)$ is *continuous*. (1.1)

For every q and q_1 belonging to $\mathcal{D}(T)$ (or $C(T)$), the function $\xi \rightarrow \Phi(q + \xi q_1)$ is entire analytic with values in E . (1.2)

Let q be a fixed element in $\mathcal{D}(T)$. For every $\psi \in \mathcal{D}(T)$ we define

$$\delta\Phi(q; \psi) = \left. \frac{d}{d\xi} \Phi(q + \xi\psi) \right|_{\xi=0}. \quad (1.3)$$

This is an element of E , hence a mapping $\psi \rightarrow \delta\Phi(q; \psi)$ from $\mathcal{D}(T)$ into E . Let us check that this mapping is *linear*. First of all it is obvious from the definition that

$$\delta\Phi(q; k\psi) = k\delta\Phi(q; \psi) \quad \text{for every } k \in C.$$

Therefore, we want to show for every $\psi_1, \psi_2 \in \mathcal{D}(T)$ that

$$\delta\Phi(q; \psi_1 + \psi_2) = \delta\Phi(q; \psi_1) + \delta\Phi(q; \psi_2). \quad (1.4)$$

Now if $e' \in E'$, the dual space of E , then the scalar function

$$\xi_1, \xi_2 \rightarrow \langle \Phi(q + \xi_1\psi_1 + \xi_2\psi_2), e' \rangle = \Psi(\xi_1, \xi_2)$$

is partially differentiable in ξ_1, ξ_2 . Thus

⁽¹⁾ We do not look for the most general hypotheses under which what follows is correct.

$$\Psi(\xi_1, \xi_2) = \Psi(0,0) + \xi_1 \frac{\partial \Psi(0,0)}{\partial \xi_1} + \xi_2 \frac{\partial \Psi(0,0)}{\partial \xi_2} + o(|\xi|),$$

where $|\xi| = (|\xi_1|^2 + |\xi_2|^2)^{\frac{1}{2}}$. But $\partial \Psi(0,0)/\partial \xi_i = \langle \delta \Phi(q; \psi_i), e' \rangle, i = 1, 2$ so that

$$\langle \Phi(q + \xi(\psi_1 + \psi_2)) - \Phi(q), e' \rangle = \xi \langle \delta \Phi(q; \psi_1) + \delta \Phi(q; \psi_2), e' \rangle + o(|\xi|).$$

Dividing by ξ and letting $\xi \rightarrow 0$, we obtain

$$\langle \delta \Phi(q; \psi_1 + \psi_2), e' \rangle = \langle \delta \Phi(q; \psi_1) + \delta \Phi(q; \psi_2), e' \rangle$$

for every $e' \in E'$ so that (1.4) follows.⁽¹⁾

We now check that the mapping $\psi \rightarrow \delta \Phi(q; \psi)$ is *continuous* from $\mathcal{D}(T)$ into E . Indeed from definition (1.3) and Cauchy's theorem⁽²⁾ it follows that

$$\delta \Phi(q; \psi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(q + e^{i\theta} \psi) e^{-i\theta} d\theta, \tag{1.5}$$

this integral being taken in E . Let V be a convex neighborhood of 0 in E . We look for a neighborhood U of 0 in $\mathcal{D}(T)$ such that

$$\delta \Phi(q; \psi) \in V \quad \text{whenever } \psi \in U. \tag{1.6}$$

From (1.1) there exists a neighborhood U of 0 in $\mathcal{D}(T)$ such that $\Phi(q + \chi) - \Phi(q) \in V$ whenever $\chi \in U$, and also $e^{i\theta} \chi \in U$ if $\chi \in U$.

Hence, since

$$\delta \Phi(q; \psi) - \delta \Phi(q; 0) = \delta \Phi(q; \psi) = \frac{1}{2\pi} \int_0^{2\pi} (\Phi(q + e^{i\theta} \psi) - \Phi(q)) e^{-i\theta} d\theta$$

we have (1.6) if $\psi \in U$.

Therefore we have obtained a *continuous linear mapping*

$$\psi \rightarrow \delta \Phi(q; \psi) \tag{1.7}$$

from $\mathcal{D}(T)$ into E . By definition (cf. Schwartz [23]) this means the mapping (1.7) defines a distribution $\delta \Phi(q)/\delta q(\tau)$ on T with values in E . This distribution verifies

$$\int_T \frac{\delta \Phi(q)}{\delta q(\tau)} \psi(\tau) d\tau = \delta \Phi(q; \psi). \tag{1.8}$$

⁽¹⁾ This reasoning is well known and we recall it here for the convenience of the reader (see Hille-Phillips [10] and the bibliography mentioned there).

⁽²⁾ Which holds for vector-valued analytic functions (see for instance Grothendieck [9]).

DEFINITION 1.1. The distribution $\delta\Phi(q)/\delta q(\tau) \in \mathcal{D}'(T; E)$, the space of distributions in T with values in E , is the F.V. derivative of the functional Φ .

Remark 1.1. If the functional $q \rightarrow \Phi(q)$ is given continuous on $C(T)$, then the same reasoning as above proves that $\psi \rightarrow \delta\Phi(q; \psi)$ is linear continuous from $C(T)$ into E and hence a measure with values in E for the F.V. derivative $\delta\Phi(q)/\delta q(\tau)$.

Example 1.1. Let $E = \mathcal{D}'(\Omega)$, distributions on Ω , where $\Omega = \{x, t | x \in \mathbb{R}, t > 0\}$ and let $T = (0, \infty)$. For $q \in \mathcal{D}(T)$ let $\Phi(q)$ be the function $x, t \rightarrow q(t)$ so that $\delta\Phi(q; \psi)$ is the function $x, t \rightarrow \psi(t)$. It follows that $\delta\Phi(q)/\delta q(\tau)$ is the function

$$\begin{aligned} \tau &\rightarrow 1_x \otimes \delta_t(\tau) \\ \tau &\geq 0 \rightarrow \mathcal{D}'(\Omega), \end{aligned}$$

where $\delta_t(\tau)$ is the Dirac measure in t at the point τ (so that $\langle 1_x \otimes \delta_t(\tau), u(x)v(t) \rangle = (\int_{-\infty}^{\infty} u(x)dx)v(\tau)$, u and v being test functions). To show this we need only note that

$$\int_0^{\infty} \langle 1_x \otimes \delta_t(\tau), u(x)v(t) \rangle \psi(\tau) d\tau = \left(\int_{-\infty}^{\infty} u(x)dx \right) \left(\int_0^{\infty} v(t)\psi(t)dt \right)$$

and
$$\langle \delta\Phi(q; \psi), u(x)v(t) \rangle = \left(\int_{-\infty}^{\infty} u(x)dx \right) \left(\int_0^{\infty} v(t)\psi(t)dt \right).$$

Example 1.2. Let E and T be as in Example 1.1 and for $q \in \mathcal{D}(T)$ let $\Phi(q)$ be the constant $x, t \rightarrow q(s)$, where s is fixed. Thus $\delta\Phi(q; \psi)$ is the function, $x, t \rightarrow \psi(s)$, from which it follows that $\delta\Phi(q)/\delta q(\tau)$ is now a distribution, $\delta_\tau(s) \otimes 1_{x,t}$. This is clear since if ψ is given in $\mathcal{D}(T)$, then

$$\int \delta_\tau(s) \otimes 1_{x,t} \psi(\tau) d\tau = \psi(s),$$

an element of $\mathcal{D}'_{x,t}$ and hence the result.

If we assume in (1.8) that $\tau \rightarrow \delta\Phi(q)/\delta q(\tau)$ is actually a continuous function from T to E , then one can define the value of this function at the point $\tau_0 \in T$ by

$$\frac{\delta\Phi(q)}{\delta q(\tau_0)} = \lim_n \delta\Phi(q; \psi_n),$$

where $\psi_n \in \mathcal{D}(T)$ and the support of ψ_n are $[\tau_0 - \alpha_n, \tau_0 + \beta_n]$, $\alpha_n \beta_n \rightarrow 0, \psi_n \geq 0, \int \psi_n(\tau) d\tau = 1$, and where $\psi_n \rightarrow \delta_{\tau_0}$ for the weak topology of measures on T . One says that ψ_n is

a regularising sequence at τ_0 . Assuming that

$$\frac{1}{\xi} [\Phi(q + \xi\psi) - \Phi(q)] \rightarrow \delta\Phi(q; \psi) \text{ in } E \text{ as } \xi \rightarrow 0$$

uniformly for ψ , an element of a regularizing sequence, one obtains

$$\frac{\delta\Phi(q)}{\delta q(\tau_0)} = \lim_{n, \xi} \left(\frac{\Phi(q + \xi\psi_n) - \Phi(q)}{\xi} \right), \quad \begin{matrix} \xi \rightarrow 0 \\ \psi_n \rightarrow \delta_{\tau}(\tau_0) \end{matrix} \tag{1.9}$$

We can state this results as follows; we introduce the

DEFINITION 1.2. A sequence $\varphi_n \in \mathcal{D}(T)$ is a *F.V. sequence at the point τ* if:

- (i) The support of φ_n is $[\tau + \alpha_n, \tau + \beta_n]$, $\alpha_n \beta_n \rightarrow 0$ (of arbitrary sign).
- (ii) $\max_t |\varphi_n(t)| \rightarrow 0$.

Then $\xi\varphi_n = \varphi_n$ is a F.V. sequence at the point τ and

$$\frac{\delta\Phi(q)}{\delta q(\tau)} = \lim \left(\frac{\Phi(q + \varphi_n) - \Phi(q)}{\int_{\tau} \varphi_n(t) dt} \right), \tag{1.10}$$

where $\{\varphi_n\}$ is a F.V. sequence at τ .

This is the original definition of Volterra [26]. One can show (cf. Volterra [26]) that, conversely, if a functional $\Phi(q)$, verifying (1.1) and (1.2) admits a F.V. derivative (given by (1.10)) for every $\tau \in T$, and that $\tau \rightarrow \delta\Phi(q)/\delta q(\tau)$ is, for example, piece-wise continuous with values in E , then $\tau \rightarrow \delta\Phi(q)/\delta q(\tau)$ defines a distribution which coincides with the distribution defined by (1.8).⁽¹⁾ We shall need

LEMMA 1.1. *Let $\Phi(q)$ be a functional verifying (1.1) and (1.2) such that*

$$\frac{\delta\Phi(q)}{\delta q(\tau)} = 0 \text{ in } \mathcal{D}'(T, E), \text{ for every } q \in \mathcal{D}(T). \tag{1.11}$$

Then $\Phi(q)$ does not depend on q .

Proof. It follows from (1.3) that

$$\frac{d}{d\xi} \Phi(q + \xi\psi) = \delta\Phi(q + \xi\psi; \psi), \tag{1.12}$$

⁽¹⁾ Our more general definition allows us to consider simultaneously the ordinary and the "exceptional" points of Volterra (cf. Volterra [26]).

since
$$\frac{d}{d\xi} \Phi(q + \xi\psi) = \frac{d}{dz} \Phi(q + (\xi + z)\psi)|_{z=0}.$$

But
$$\delta\Phi(q + \xi\psi; \psi) = \int_x \frac{\delta\Phi(q + \xi\psi)}{\delta q(\tau)} \psi(\tau) d\tau = 0,$$

by (1.11) so that $\Phi(q + \xi\psi)$ does not depend on ξ and this for every q and ψ in $\mathcal{D}(T)$ from which the result follows.

Remark 1.2. Let us apply definition (1.10) to example 1.1. We must consider

$$\frac{\Phi(q + \varphi_n) - \Phi(q)}{\int_0^\infty \varphi_n(t) dt},$$

where $\{\varphi_n\}$ is a F.V. sequence at τ . We obtain

$$1_x \otimes \frac{\varphi_n(t)}{\int_0^\infty \varphi_n(t) dt},$$

which converges to $1_x \otimes \delta_t(\tau)$ as $n \rightarrow \infty$.

2. A functional associated with a Cauchy problem

Let $A = A(\partial/\partial x)$ be a partial differential operator with *constant coefficients* on R . We consider the Cauchy problem:

$$A_x u(x, t) + \frac{\partial}{\partial t} u(x, t) + w(x)u(x, t) = ixq(t)u(x, t), \tag{2.1}$$

$$u(x, 0) = f(x), \tag{2.2}$$

where $x \in R, t > 0, f(x)$ is given and with growth conditions on $f(x)$ and $u(x, t)$ as $x \rightarrow \pm \infty$. We will assume $q(t)$ real and continuous and $w(x)$ continuous and complex-valued.

We now make precise assumptions. Let \mathcal{A} be a vector topological space, locally convex, complete, of functions or distributions on R_x . We assume

$$\left. \begin{array}{l} \text{For every } a \in \mathcal{A}, wa(x \rightarrow w(x)a(x)) \text{ and } xa \text{ are defined as elements of } \mathcal{D}'(R), \text{ distributions on } R; \text{ the mappings } a \rightarrow wa \text{ and } \\ a \rightarrow xa \text{ being continuous.} \end{array} \right\} \tag{C.1}$$

Given $f \in \mathcal{A}$, there exists one and only one function, $t \rightarrow u(\cdot, t)$, continuous from $t \geq 0 \rightarrow \mathcal{A}$ which is a solution of (2.1) and (2.2). We assume further that the mapping $f \rightarrow u$ is continuous from $\mathcal{A} \rightarrow C(0, \infty; \mathcal{A})$, the space of continuous functions from $t \geq 0 \rightarrow \mathcal{A}$ with the topology of uniform convergence on every compact set.⁽¹⁾ } (C.2)

We define in this way a functional

$$q \rightarrow u(q) = u(x, t, q) = u(q; x, t)$$

defined for $q(t) \in C(T)$ (continuous functions in $t \geq 0$) and with values in $C(0, \infty; \mathcal{A})$. We need two more assumptions.

The solution of (2.1) and (2.2) in $C(0, \infty; \mathcal{A})$ is *stable* in q , i.e., $q \rightarrow u(q; x, t)$ is continuous from $C(T)$ to $C(0, \infty; \mathcal{A})$. } (C.3)

Let $\mathcal{M}(\mathcal{A})$ be the space $\mathcal{L}(C(R), \mathcal{A})$ of continuous linear mappings from $C(R)$, continuous functions on R , into \mathcal{A} . $\mathcal{M}(\mathcal{A})$ is a subspace of $\mathcal{D}'(\mathcal{A})$, distributions with values in \mathcal{A} . We will assume

Given $g \in \mathcal{M}(\mathcal{A})$, $g = 0$ for $t < 0$, there exists a unique element $u \in \mathcal{M}(\mathcal{A})$, $u = 0$ for $t < 0$, which is a solution of } (C.4)

$$\left(A_x + \frac{\partial}{\partial t} + w(x) - ixq(t) \right) u = g.$$

The mapping $g \rightarrow u$ is continuous from $\mathcal{M}(\mathcal{A})$ into itself.⁽²⁾

Our purpose in this section is to study the Volterra derivative of the function $q \rightarrow u(q)$. One has

THEOREM 2.1. *Assuming (C.1) – (C.4), the functional $q \rightarrow u(q; x, t) = u(x, t; q)$ from $C(T)$, $T = [0, \infty)$, into $\mathcal{M}(\mathcal{A})$ admits a Volterra derivative for every $\tau > 0$. This Volterra derivative $\delta u(x, t; q) | \delta q(\tau)$ is characterized by the following properties:*

$$\frac{\delta u(x, t; q)}{\delta q(\tau)} = 0 \quad \text{for } t < \tau, \tag{2.3}$$

⁽¹⁾ In other words the Cauchy problem is “well set” in the sense of Hadamard. Notice also that the space \mathcal{A} contains the conditions at ∞ in x .

⁽²⁾ In this statement the initial Cauchy condition is contained in the second member g (Sobolev-Schwartz method).

for $t > \tau$, $\delta u(x, t; q) / \delta q(\tau)$ is the solution of the Cauchy problem,

$$\left. \begin{aligned} \left(A_x + \frac{\partial}{\partial t} + w(x) - ixq(t) \right) \frac{\delta u(x, t; q)}{\delta q(\tau)} &= 0 \\ \frac{\delta u(x, \tau; q)}{\delta q(\tau)} &= ixu(x, \tau; q) \\ t \rightarrow \frac{\delta u(\cdot, t; q)}{\delta q(\tau)} &\text{ is continuous from } t \geq \tau \rightarrow \mathcal{A}.^{(1)} \end{aligned} \right\} \quad (2.4)$$

Proof. Let

$$\Lambda = A_x + \frac{\partial}{\partial t} + w(x) \quad (2.5)$$

and let ϱ_n be a F.V. sequence at τ . We consider

$$v_n = v_n(x, t) = \frac{u(q + \varrho_n; x, t) - u(q; x, t)}{\int \varrho_n(t) dt}$$

By definition

$$\Lambda u(q + \varrho_n; x, t) = ix(q + \varrho_n) u(q + \varrho_n; x, t),$$

$$u(q + \varrho_n; x, 0) = f(x).$$

Therefore,

$$\Lambda v_n - ixq(t)v_n = ixu(q + \varrho_n; x, t) \frac{\varrho_n(t)}{\int \varrho_n(t) dt}, \quad (2.6)$$

$$v_n(x, 0) = 0. \quad (2.7)$$

Thus, v_n is the solution of the non-homogeneous Cauchy problem (as in (C.4)) with second member

$$g_n(x, t) = ixu(q + \varrho_n; x, t) \frac{\varrho_n(t)}{\int \varrho_n(t) dt}. \quad (2.8)$$

As has already been observed

$$\frac{\varrho_n(t)}{\int \varrho_n(t) dt} \rightarrow \delta_t(\tau)$$

⁽¹⁾ This Cauchy problem with initial value at $t = \tau$ is well set under hypotheses (C.1)–(C.4) (replace t by $t + \tau$ and note by (C.1) that $x \rightarrow u(x, \tau; q)$ belongs to \mathcal{A}).

weakly in the space of Radon measures on R_t , and according to (C.3) and (C.1), $ixu(q + q_n; x, t) \rightarrow ixu(q; x, t)$ in the space $C(0, \infty; \mathcal{A})$. Therefore

$$g_n \rightarrow ixu(q; x, t) \delta_t(\tau) = ixu(q; x, \tau) \delta_t(\tau) \tag{2.9}$$

in $\mathcal{M}(\mathcal{A})$. By (C.4) it follows that v_n converges in $\mathcal{M}(\mathcal{A})$ to the *unique* solution v in $\mathcal{M}(\mathcal{A})$ of

$$\Lambda v - ixq(t)v = ixu(x, t; q) \delta_t(\tau). \tag{2.10}$$

$$v(x, t; q) = 0 \text{ for } t < 0. \tag{2.11}$$

This proves already that $\delta u(x, t; q)/\delta q(\tau)$ exists (in $\mathcal{M}(\mathcal{A})$) and that

$$v = \frac{\delta u(x, t; q)}{\delta q(\tau)}. \tag{2.12}$$

Let us now introduce the *unique* function $t \rightarrow U(\cdot, t; q)$, continuous from $t \geq \tau \rightarrow \mathcal{A}$, with

$$\Lambda U - ixq(t)U = 0, \quad t > \tau, \tag{2.13}$$

$$U(x, \tau; q) = ixu(x, \tau; q). \tag{2.14}$$

If $\tilde{U}(x, t; q)$ is defined as $U(x, t; q)$ for $t \geq \tau$ and 0 otherwise, then we have

$$\Lambda \tilde{U} - ixq(t)\tilde{U} = ixu(x, \tau; q) \delta_t(\tau), \tag{2.15}$$

and, of course,

$$\tilde{U} = 0, \quad t < 0. \tag{2.16}$$

By comparing (2.10), (2.11) with (2.15), (2.16) and using the uniqueness in (C.4) we obtain that $v = U$. This completes the proof of the theorem.

Remark 2.1. Taking $E = \mathcal{M}(\mathcal{A})$ we see that the functional $q \rightarrow u(q)$ verifies (1.1) and (1.2). Indeed, (1.1) follows from (C.3) (and more precisely $q \rightarrow u(q)$ is continuous with values in $C(0, \infty; \mathcal{A})$). One proves (1.2) by the same reasoning as above in the proof of Theorem 2.1. In this way one obtains that $\delta u(x, t; q; \psi)$ is the solution of

$$\Lambda \delta u(x, t; q; \psi) - ixq(t)\delta u(x, t; q; \psi) = ix\psi(t)u(x, t; q),$$

$$\delta u(x, 0; q; \psi) = 0,$$

and then by the same reasoning as in Example (1.1) one obtains Theorem 2.1.

3. A Volterra variational equation

Our aim in this section is to see to what extent properties (2.3) and (2.4) "characterize" $u(x, t; q)$. More precisely, let $\Phi(x, t; q)$ be a functional verifying

$$\left. \begin{aligned} \Phi(x, t; q) \in C(0, \infty; \mathcal{A}), q \rightarrow \Phi \text{ being continuous from } \\ C(T), T = [0, \infty), \text{ into } C(0, \infty; \mathcal{A}), \end{aligned} \right\} \quad (3.1)$$

$$\left. \begin{aligned} \text{and if } E = \mathcal{M}(\mathcal{A}), (1.2) \text{ holds and we can define } \\ \delta\Phi(x, t; q)/\delta q(\tau) \text{ in } \mathcal{M}(\mathcal{A}) \text{ for every } \tau > 0. \end{aligned} \right\} \quad (3.2)$$

Let us denote by $\Psi(x, t; q)$ the solution of the Cauchy problem:

$$\left. \begin{aligned} (\Lambda - ixq(t))\Psi = 0 \text{ for } t > \tau, \\ \Psi(x, \tau; q) = ix\Phi(x, \tau; q), \\ \Psi(\cdot, t; q) \text{ continuous from } t \geq \tau \rightarrow \mathcal{A}. \end{aligned} \right\} \quad (3.3)$$

Let $\tilde{\Psi}(x, t; q) = \Psi(x, t; q)$ if $t \geq \tau$ and 0 otherwise. We now further assume about Φ that it verifies the following Volterra variational equation:

$$\frac{\delta\Phi(x, t; q)}{\delta q(\tau)} = \tilde{\Psi}(x, t; q) \quad (3.4)$$

with the *boundary conditions*:

$$\Phi(x, 0; q) = f(x), f(x) \text{ given in } \mathcal{A}, \quad (3.5)$$

$$\Lambda(x, t; 0) = 0. \quad (1) \quad (3.6)$$

We want to prove now

THEOREM 3.1. *Assuming that (C.1)–(C.4) (of section 2) hold, there exists one and only one functional $\Phi(q)$ verifying (3.1), (3.2), (3.4), (3.5) and (3.6). This functional Φ is the unique solution of the Cauchy problem:*

$$(\Lambda - ixq(t))\Phi = 0, \quad (3.7)$$

$$\Phi(x, 0; q) = f(x), \quad (3.5)$$

$$t \rightarrow \Phi(\cdot, t; q) \text{ is continuous from } t \geq 0 \rightarrow \mathcal{A}. \quad (3.8)$$

Proof. Let us introduce the new functional

$$R(x, t; q) = \Lambda\Phi(x, t; q) - ixq(t)\Phi(x, t; q). \quad (3.9)$$

(1) This is a new kind of "boundary condition".

Considered as a functional with values in $\mathcal{D}'(R_x \times R_t)$ this functional verifies (1.1) and (1.2). As is easily checked, it follows from (3.9) that

$$\frac{\delta R(q)}{\delta q(\tau)} = (\Lambda - ixq) \frac{\delta \Phi(q)}{\delta q(\tau)} - ix\Phi(q)\delta_t(\tau).$$

From (3.3) and (3.4) it follows that

$$\frac{\delta R(q)}{\delta q(\tau)} = 0. \tag{3.10}$$

But Lemma 1.1 and (3.10) imply

$$R(q) = R(x, t; q) = R(x, t; 0), \tag{3.11}$$

and (3.6) and (3.9) imply that $R(x, t; 0) = 0$. Thus $R(q) = 0$ and we have

$$(\Lambda - ixq(t))\Phi = 0.$$

Therefore, *if a solution exists*, it is necessarily given by the unique solution of the Cauchy problem (3.7), (3.5), and (3.8). Since, by Theorem 2.1, we know that this solution actually verifies (3.1), (3.2), (3.4), (3.5) and (3.6), the theorem is proved.

4. A functional associated with a mixed problem

We introduce the following notations. Let H and K be two Hilbert spaces. If $f, g \in H$ (and $u, v \in K$), (f, g) (and $((u, v))$) denote the scalar product of f and g (and u and v in H and K respectively). We assume that $K \subset H$, the injection $K \rightarrow H$ being continuous, and K being dense in H . We set $|f| = (f, f)^{\frac{1}{2}}, \|u\| = ((u, u))^{\frac{1}{2}}$.

We assume that t varies in $(-\infty, t_0)$ where $t_0 > 0$ is fixed. Let $a(t; u, v)$ be a family of continuous sesquilinear forms⁽¹⁾ on $K \times K$. We assume that $a(t; u, v)$ is given for $t \leq t_0$ and that the function $t \rightarrow a(t; u, v)$ is continuous on $(-\infty, t_0]$, for every $u, v \in K$, with

$$|a(t; u, v)| \leq M \|u\| \|v\|,$$

where M is a constant independent of t .⁽²⁾ If X is a Banach space, by $L^2(\alpha, \beta; X)$

⁽¹⁾ $a(t; u, v)$ is a linear in u , semi- (or anti)-linear in v and $|a(t; u, v)| \leq c(t)\|u\| \|v\|$.

⁽²⁾ We are not looking for the most general hypotheses here. We notice that the behaviour of $a(t; u, v)$ for $t < 0$ is irrelevant for what follows. For this kind of problem the reader is referred to [17] (where $K = V$).

we mean the space of the (classes of) functions in (α, β) which are square integrable with values in X . If X is a Hilbert space (scalar product $(f, g)_X$), then $L^2(\alpha, \beta; X)$ is a Hilbert space for the scalar product

$$\int_{\alpha}^{\beta} (f(t), g(t))_X dt.$$

We make now the first hypothesis.

Given $f \in H$ and $g \in L^2(-\infty, t_0; K)$, with $g = 0$ for $t < 0$, there exists an unique function $u \in L^2(-\infty, t_0; K)$, with $u = 0$ for $t < 0$, such that⁽¹⁾

$$a(t; u(t), v) + \frac{d}{dt} (u(t), v) - iq(t) (Bu(t), v) = ((g(t), v)) + (f, v)\delta \tag{M.1}$$

for every $v \in K$, where⁽²⁾ $B \in \mathcal{L}(K; H)$, q is given continuous in $(-\infty, t_0]$ ⁽³⁾ and where δ is the Dirac measure at the origin.

From (M.1) and the closed graph theorem, it follows that $\{f, g\} \rightarrow u$ is continuous from $H \times L^2(0, t_0; K) \rightarrow L^2(0, t_0; K)$.⁽⁴⁾ We define in this way a functional $q \rightarrow u(q) = u(t; q) = u(q; t)$ from $C(T)$, $T = [0, t_0]$ to $L^2(0, t_0; K)$.⁽⁵⁾

We assume (for this kind of stability condition the reader is referred to [17, chapter IV]):

$$\text{The mapping } q \rightarrow u(q; t) \text{ is continuous from } C(0, t_0) \rightarrow L^2(0, t_0; K). \tag{M.2}$$

Let us compare these hypotheses with those made in section 2. Here the space \mathcal{A} is replaced to a certain extent, by the two spaces K and H , and the space of *continuous* functions with values in \mathcal{A} is replaced by $L^2(0, t_0; K)$.⁽⁶⁾ The fact that in section 2, $a \rightarrow wa$ is continuous from $\mathcal{A} \rightarrow \mathcal{A}$ is contained here in the hypothesis that $a(t; u, v)$ is continuous on $K \times K$ (see section 9) and the fact that in section 2, $a \rightarrow xa$

⁽¹⁾ d/dt is taken in the sense of distributions in $(-\infty, t_0)$

⁽²⁾ In general we write $\mathcal{L}(X; Y)$ for the space of continuous linear mappings from X into Y .

⁽³⁾ Or in $[0, t_0]$ since one can then extend q arbitrarily for $t < 0$.

⁽⁴⁾ One can, of course, identify $L^2(0, t_0; K)$ with the subspace of $L^2(-\infty, t_0; K)$, consisting of the functions which are 0 for $t < 0$.

⁽⁵⁾ $q \in L^\infty(0, t_0)$ would in general be enough (cf. [17]).

⁽⁶⁾ The fact that we replace "continuity" by "square integrability" is the main advantage of this presentation since $\delta u/\delta q(\tau)$ has essentially a discontinuity at $t = \tau$.

is continuous from $\mathcal{A} \rightarrow \mathcal{A}$ is replaced, again to some extent (cf. section 9), by the hypothesis $B \in \mathcal{L}(K; H)$. Then (M.1) replaces (C.2) and (M.2) replaces (C.3). We shall also need:

$$\left. \begin{array}{l} \text{It is possible to choose } f \in H_1, \text{ a subspace of } H, \text{ such that} \\ \text{the corresponding solution of the mixed problem in (M.1),} \\ \text{with } g=0, \text{ is continuous from } t \geq 0 \rightarrow K. \end{array} \right\} \quad (\text{M.3})_1$$

We can now prove

THEOREM 4.1. *Assume that (M.1), (M.2) and (M.3)₁ hold and that f is given in H_1 and $g=0$. Let $u = u(q; t)$ be the solution of the mixed problem in (M.1). The functional $q \rightarrow u(q; t)$ admits, as a functional with values in $L^2(0, t_0; K)$, a Volterra derivative at $\tau, \tau > 0$. This derivative $\delta u(t; q) | \delta q(\tau)$ is characterized by*

$$\frac{\delta u(t; q)}{\delta q(\tau)} \in L^2(-\infty, t_0; K), \quad \frac{\delta u}{\delta q(\tau)} = 0 \quad \text{for } t < \tau, \quad (4.1)$$

$$\begin{aligned} a\left(t; \frac{\delta u(t; q)}{\delta q(\tau)}, v\right) + \frac{d}{dt} \left(\frac{\delta u(t; q)}{\delta q(\tau)}, v \right) - iq(t) \left(B \frac{\delta u(t; q)}{\delta q(\tau)}, v \right) \\ = i(Bu(\tau; q), v) \delta_t(\tau) \quad \text{for every } v \in K. \end{aligned} \quad (4.2)$$

Proof. The proof follows the same lines as that of Theorem 2.1. Let q_n be a Volterra sequence at the point τ . Setting

$$\varphi_n(t) = \frac{u(t; q + q_n) - u(t; q)}{\int_0^t q_n(t) dt}, \quad (4.3)$$

we have

$$a(t; \varphi_n(t), v) + \frac{d}{dt} (\varphi_n(t), v) - iq(t) (B\varphi_n(t), v) = i \frac{q_n(t)}{\int_0^t q_n(t) dt} (Bu(t; q + q_n), v). \quad (4.4)$$

Now let $\xi(t)$ be a once continuously differentiable function on $[0, t_0]$ with $\xi(t_0) = 0$. It follows from (4.4) that

$$\begin{aligned} \int_0^{t_0} \{ a(t; \varphi_n(t), \psi(t)) - (\varphi_n(t), \psi'(t)) - iq(t) (B\varphi_n(t), \psi(t)) \} dt \\ = \frac{i}{\int_0^{t_0} q_n(t) dt} \int_0^{t_0} q_n(t) (Bu(t; q + q_n), \psi(t)) dt, \end{aligned} \quad (4.5)$$

where here $\psi(t) = \xi(t)v$. Hence (4.5) holds for $\psi(t) = \sum_{i=1}^v \xi_i(t)v_i$, and passing to the limit (cf. details, for instance, in [17]), (4.5) holds for every $\psi \in L^2(0, t_0; K)$, with $\psi' \in L^2(0, t_0; K)$, $\psi(t_0) = 0$.

Now as $n \rightarrow \infty$, the second member in (4.5) converges to $(Bu(\tau; q), \psi(\tau))$ where we use here (M.3)₁ and $f \in H_1$. By using stability results given in [17], it follows that $\varphi_n \rightarrow \varphi$ in $L^2(0, t_0; K)$ which already proves that $\delta u(q)/\delta q(\tau)$ exists and indeed actually equals φ . Moreover φ verifies

$$\int_0^{t_0} \{a(t; \varphi(t), \psi(t)) - (\varphi(t), \psi'(t)) - iq(t)(B\varphi(t), \psi(t))\} dt = i(Bu(\tau; q), \psi(\tau))$$

and this is equivalent (cf. [17]) with

$$a(t; \varphi(t), v) + \frac{d}{dt}(\varphi(t), v) - iq(t)(B\varphi(t), v) = i(Bu(\tau; q), v) \delta_t(\tau),$$

where $v \in K$, (extending φ by 0 for $t < 0$) and this proves the theorem.

Remark 4.1. It is possible to give here a converse property analogous to the one given in section 3 but we do not give details.

Remark 4.2. The functional $q \rightarrow u(q; t)$ verifies (1.1) and (1.2) if $q \in C[0, t_0]$ and $E = L^2(0, t_0; K)$. Same proof as in Theorem 4.1 (cf. Remark 2.1).

5. F.V. derivatives of higher order

Let us again consider a functional $q \rightarrow \Phi(q)$ as in section 1. For every

$$\{\psi_1, \psi_2, \dots, \psi_n\} \in \mathcal{D}(T)^n = \mathcal{D}(T) \times \dots \times \mathcal{D}(T)$$

(n times), we define

$$\delta^n \Phi(q; \psi_1, \psi_2, \dots, \psi_n) = \frac{\partial^n}{\partial \xi_1 \partial \xi_2 \dots \partial \xi_n} \Phi(q + \xi_1 \psi_1 + \dots + \xi_n \psi_n) \Big|_{\xi_i=0}. \tag{5.1}$$

We define in this way an n -linear mapping $\psi_1, \psi_2, \dots, \psi_n \rightarrow \delta^n \Phi(q; \psi_1, \psi_2, \dots, \psi_n)$, from $\mathcal{D}(T)^n \rightarrow E$. This mapping is continuous and hence defines a distribution,

$$\frac{\delta^n \Phi(q)}{\delta q(\tau_1) \dots \delta q(\tau_n)} \in \mathcal{D}'(T^n; E). \tag{5.2}$$

One has

$$\delta^n \Phi(q; \psi_1, \dots, \psi_n) = \int_{T^n} \frac{\delta^n \Phi(q)}{\delta q(\tau_1) \dots \delta q(\tau_n)} \psi_1(\tau_1) \dots \psi_n(\tau_n) d\tau_1 \dots d\tau_n. \tag{5.3}$$

It is easy to check that

$$\left. \frac{d^n}{d\xi^n} \Phi(q + \xi\psi) \right|_{\xi=0} = \frac{\partial^n}{\partial \xi_1 \dots \partial \xi_n} \Phi(q + \xi_1 \psi + \dots + \xi_n \psi) \Big|_{\xi_i=0}, \tag{5.4}$$

hence

$$\left. \frac{d^n}{d\xi^n} \Phi(q + \xi\psi) \right|_{\xi=0} = \int_{T^n} \frac{\delta^n \Phi(q)}{\delta q(\tau_1) \dots \delta q(\tau_n)} \psi(\tau_1) \psi(\tau_2) \dots \psi(\tau_n) d\tau_1 \dots d\tau_n. \tag{5.5}$$

Since $\xi \rightarrow \Phi(q + \xi\psi)$ is an entire analytic function, one has (in the space E)

$$\Phi(q + \xi\psi) = \sum_{n \geq 0} \frac{\xi^n}{n!} \delta^n \Phi(q; \psi, \dots, \psi),$$

and using (5.5) we obtain

$$\Phi(q + \xi\psi) = \sum_{n \geq 0} \frac{\xi^n}{n!} \int_{T^n} \frac{\delta^n \Phi(q)}{\delta q(\tau_1) \dots \delta q(\tau_n)} \psi(\tau_1) \dots \psi(\tau_n) d\tau_1 \dots d\tau_n. \tag{5.6}$$

This is the *F.V. series* of the functional (cf. Volterra [26]). *The distributions* $\delta^n \Phi(q) / \delta q(\tau_1) \dots \delta q(\tau_n)$ are the *F.V. derivatives of higher order*. In what follows we are especially interested in the case when the $\delta^n \Phi(q) / \delta q(\tau_1) \dots \delta q(\tau_n)$ are *functions* and it will be essential to define $\delta^n \Phi(q) / \delta q(\tau_1) \dots \delta q(\tau_n)$ on the diagonal $\tau_1 = \tau_2 = \dots = \tau_n = \tau$. But the functions $\delta^n \Phi(q) / \delta q(\tau_1) \delta q(\tau_n)$ are *not continuous* (in our case) so that one has to define with care $\delta^n \Phi(q) / \delta q(\tau_1) \dots \delta q(\tau_n)$ when $\tau_1 = \tau_2 = \dots = \tau_n = \tau$. We take the following

DEFINITION 5.1.

$$\frac{\delta^n \Phi(q)}{\delta q(\tau)^n} \equiv \lim_{\substack{\tau^* \rightarrow \tau \\ \tau^* > \tau}} \frac{\delta}{\delta q(\tau^*)} \left(\frac{\delta^{n-1} \Phi}{\delta q(\tau)^{n-1}} \right), \quad n = 2, 3, \dots \tag{5.7}$$

(of course, when this limit exists).

Remark 5.1. One has the obvious generalization of Lemma 1.1 by replacing $\delta / \delta q(\tau)$ with $\delta^n / \delta q(\tau_1) \dots \delta q(\tau_n)$ but no generalization when replacing $\delta / \delta q(\tau)$ by $\delta^n / \delta q(\tau)^n$.

6. The higher order F.V. derivatives of the functional $u(q)$ defined in section 4

We shall need in this section a stronger hypothesis than (M.3)₁ of section 4.

It is possible to choose $f \in H_m$, a subspace of H , such that the corresponding solution of (M.1) (section 4), when $g=0$, verifies the conditions: $u(t), Bu(t), \dots, B^{(m-1)}u(t)$ are all continuous from $[0, t_0]$ into K . (M.3)_m

We can now prove

THEOREM 6.1. *We assume that (M.1), (M.2), and (M.3)_m hold, f being given in H_m and $g=0$. Then $\delta^m u(q)/\delta q(\tau)^m$ (cf. definition 5.1) exists and is characterized by:*

$$\frac{\delta^m u(t; q)}{\delta q(\tau)^m} \in L^2(-\infty, t_0; K), \quad \frac{\delta^m u(t; q)}{\delta q(t)^m} = 0 \text{ for } t < \tau \tag{6.1}$$

and

$$\begin{aligned} a\left(t; \frac{\delta^m u(t; q)}{\delta q(\tau)^m}, v\right) + \frac{d}{dt} \left(\frac{\delta^m u(t; q)}{\delta q(\tau)^m}, v\right) - iq(t) \left(B \frac{\delta^m u(t; q)}{\delta q(\tau)^m}, v\right) \\ = ((iB)^m u(\tau; q), v) \delta_t(\tau) \text{ for } v \in K. \end{aligned} \tag{6.2}$$

Proof. The proof is by induction on m . First of all the result is true for $m=1$ (Theorem 4.1). Assume then that $\delta^{m-1} u(q)/\delta q(\tau)^{m-1} = u_{m-1}(q) = u_{m-1}(t; q)$ exists and is the unique solution in $L^2(-\infty, t_0; K)$, which is 0 for $t < 0$, of

$$a(t; u_{m-1}(t; q), v) + \frac{d}{dt} (u_{m-1}(t; q), v) - iq(t) (Bu_{m-1}(t; q), v) = ((iB)^{m-1} u(\tau; q), v) \delta_t(\tau). \tag{6.3}$$

We now deduce the result for m . Let ϱ_n be a Volterra sequence at $\tau^*, \tau^* > \tau$. Let us set

$$\varphi_n(t) = \frac{1}{\int \varrho_n(t) dt} (u_{m-1}(t; q + \varrho_n) - u_{m-1}(t; q)). \tag{6.4}$$

We obtain

$$\begin{aligned} a(t; \varphi_n(t), v) + \frac{d}{dt} (\varphi_n(t), v) - iq(t) (B\varphi_n(t), v) \\ = \frac{i}{\int \varrho_n(t) dt} \varrho_n(t) (Bu_{m-1}(t; q + \varrho_n), v) + \frac{1}{\int \varrho_n(t) dt} ((iB)^{m-1} (u(\tau; q + \varrho_n) - u(\tau; q)), v) \delta_t(\tau). \end{aligned}$$

Multiplying by $\bar{\xi}(t) \in C'(0, t_0)$, $\bar{\xi}(t_0) = 0$ and integrating in t we obtain (as in Theorem (4.1)),

$$\int_0^{t_0} \{a(t; \varphi_n(t), \psi(t)) - (\varphi_n(t), \psi'(t)) - iq(t) (B\varphi_n(t), \psi(t))\} dt$$

$$= \frac{i}{\int \varrho_n(t) dt} \int_0^{t_0} (Bu_{m-1}(t; q + \varrho_n), \psi(t)) \varrho_n(t) dt + \left((Bi)^{m-1} \frac{u(\tau; q + \varrho_n) - u(\tau; q)}{\int \varrho_n(t) dt}, \psi(\tau) \right) \quad (6.5)$$

for every $\varphi \in L^2(0, t_0; K)$, $\psi' \in L^2(0, t_0; H)$, $\psi(t_0) = 0$. But as $n \rightarrow \infty$,

$$\eta_n \equiv \frac{u(\tau; q + \varrho_n) - u(\tau; q)}{\int \varrho_n(t) dt} \rightarrow \delta u(\tau; q) / \delta q(\tau^*) = \eta \text{ in } K$$

and, moreover, $B\eta_n \rightarrow B\eta, \dots, B^{m-1}\eta_n \rightarrow B^{m-1}\eta$ in K since $f \in H_m$. But (cf. Theorem 4.1) $\delta u(\tau; q) / \delta q(\tau^*) = 0$, since $\tau < \tau^*$ and therefore

$$\frac{\delta u_{m-1}}{\delta q(\tau^*)} = \frac{\delta}{\delta q(\tau^*)} \frac{\delta^{m-1} u(q)}{\delta q(\tau)^{m-1}} = U_{\tau^*}$$

exists and is characterized as the unique solution in $L^2(-\infty, t_0; K)$ which is 0 for $t < 0$, of

$$a(t; U_{\tau^*}(t), v) + \frac{d}{dt} (U_{\tau^*}(t), v) - iq(t) (BU_{\tau^*}(t), v) = ((iB)u_{m-1}(\tau^*), v) \delta_t(\tau^*) \text{ for } v \in K.$$

It then follows (cf. also section 8) that, as $\tau^* \rightarrow \tau$, $U_{\tau^*} \rightarrow U$ in $L^2(0, t_0; K)$, where U is the unique solution in $L^2(-\infty, t_0; K)$, which is 0 for $t < 0$, of

$$a(t; U(t), v) + \frac{d}{dt} (U(t), v) - iq(t) (BU(t), v) = \left((iB) \frac{\delta^{m-1} u(\tau; q)}{\delta q(\tau)^{m-1}}, v \right) \delta_t(\tau). \quad (6.7)$$

But the induction hypothesis implies that

$$\frac{\delta^{m-1} u(\tau; q)}{\delta q(\tau)^{m-1}} = (iB)^{m-1} u(\tau; q)$$

and hence the theorem follows.

Remark 6.1. According to Remark 5.1 there is here (when $m > 1$) no converse property analogous to the one of section 3.

7. The operational calculus

Under the hypotheses of Theorem 6.1, one can define

$$P\left(-i\frac{\delta}{\delta q(\tau)}\right)u = \sum_{k=0}^m a_k \left(-i\frac{\delta}{\delta q(\tau)}\right)^k u, \tag{7.1}$$

where
$$P(\lambda) = \sum_{k=0}^m a_k \lambda^k, \quad a_k \in C. \tag{7.2}$$

If now we assume that (M.3)_m holds for every m (and taking $f \in \bigcap_m H_m$) we can define $P(-i\delta/\delta q(\tau))u$ for every polynomial and, consequently, we can define $F(-i\delta/\delta q(\tau))u$ for suitable functions $F(\lambda)$.

We shall assume

$$\left. \begin{array}{l} \text{(M.3)}_m \text{ holds for every } m \text{ and it is possible to choose a} \\ \text{sequence of polynomials } P_m(\lambda) \text{ such that } P_m(\lambda) \rightarrow F(\lambda) \text{ in such} \\ \text{a way that for } f \in H_F \subset \bigcap_m H_m \subset H, \text{ the corresponding solution} \\ \text{of (M.1) (section 4), with } g=0, \text{ verifies the condition: } P_m(B)u(t) \\ \text{converges to a limit in } H, \text{ uniformly on every compact set,} \\ \text{the limit being called } F(B)u(t). \end{array} \right\} \tag{M.3}_F$$

We have then

THEOREM 7.1. *We assume that (M.1), (M.2), and (M.3)_F hold, and we take $f \in H_F$ and $g=0$. Then, the sequence⁽¹⁾ $P_m(-i\delta/\delta q(\tau))u(t; q)$ converges in $L^2(0, t_0; K)$ to a limit called $F(-i\delta/\delta q(\tau))u(t; q)$. This limit is characterized by*

$$F\left(-i\frac{\delta}{\delta q(\tau)}\right)u(t; q) \in L^2(-\infty, t_0; K), \quad F\left(-i\frac{\delta}{\delta q(\tau)}\right)u(t; q) = 0 \text{ for } t < 0 \tag{7.3}$$

and

$$\begin{aligned} a\left(t; F\left(-i\frac{\delta}{\delta q(\tau)}\right)u(t; q), v\right) + \frac{d}{dt}\left(F\left(-i\frac{\delta}{\delta q(\tau)}\right)u(t; q), v\right) - iq(t)\left(BF\left(-i\frac{\delta}{\delta q(\tau)}\right)u(t; q), v\right) \\ = (F(B)u(\tau; q), v)\delta_i(\tau) \text{ for } v \in K. \end{aligned} \tag{7.4}$$

Proof. We observe that from Theorem 6.1 it follows that

⁽¹⁾ The P_m are the polynomials introduced in (M.3)_F.

$$\begin{aligned}
 & a\left(t; P_m\left(-i \frac{\delta}{\delta q(\tau)}\right) u(t; q), v\right) + \frac{d}{dt}\left(P_m\left(-i \frac{\delta}{\delta q(\tau)}\right) u(t; q), v\right) \\
 & - i q(t)\left(B P_m\left(-i \frac{\delta}{\delta q(\tau)}\right) u(t; q), v\right) = \left(P_m(B) u(\tau; q), v\right) \delta_t(\tau).
 \end{aligned}
 \tag{7.5}$$

From (M.3)_F and the stability properties of the solution of the mixed problem we see that Theorem 7.1 follows from (7.5).

8. The function $\tau \rightarrow \frac{\delta^m u(t; q)}{\delta q(\tau)^m}$

We want to prove in this section

THEOREM 8.1. *Under the hypotheses of Theorem 6.1, the function $\tau \rightarrow \delta^m u(t; q) / \delta q(\tau)^m$ is continuous from $[0, t_0] \rightarrow L^2(0, t_0; K)$*

Proof. This follows from equations (6.1), (6.2) and the stability properties of mixed problems given in [17]. In the same way we have

THEOREM 8.2. *Under the hypotheses of Theorem 7.1, the function $\tau \rightarrow F(-i\delta/\delta q(\tau))u(t; q)$ is continuous from $[0, t_0] \rightarrow L^2(0, t_0; K)$.*

9. Example 1

We want now to apply the considerations above to the *Cauchy problem* for the *parabolic operator*

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - i x q(t),$$

where $V(x) \geq 0$ is a given function continuous on $(-\infty, \infty)^{(1)}$. Let us introduce the spaces H and K . We take $H = L^2(-\infty, \infty) = L^2(\mathbb{R})$ so that if $f, g \in H$,

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

as usual. For K we take the space of functions $u \in H$ such that

$$(1 + V(x))^{\frac{1}{2}} u \in L^2(\mathbb{R}) \text{ and } \frac{du}{dx} \in L^2(\mathbb{R}).^{(2)}$$

⁽¹⁾ Actually *measurable* is enough here.

⁽²⁾ du/dx is taken in the sense of distributions on \mathbb{R} .

For $u, v \in K$, we set

$$((u, v)) = \int_{-\infty}^{\infty} (1 + V(x))u(x)\overline{v(x)}dx + \int_{-\infty}^{\infty} \frac{du}{dx} \frac{d\bar{v}}{dx} dx,$$

noting that for this scalar product, K becomes a Hilbert space. For $u, v \in K$ we set

$$a(t; u, v) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{dx} \frac{d\bar{v}}{dx} dx + \int_{-\infty}^{\infty} V(x)u \bar{v} dx. \quad (9.1)$$

The operator B (using the notation of section 4) is the operator $B: f \rightarrow xf$ of multiplication by x . This defines $B \in \mathcal{L}(K; H)$ if $V(x) \geq c|x|^2$. With simple changes the same situation obtains if $V(x) \geq c|x|$ (cf. Remark 9.1). Actually if $V(x)$ is only assumed ≥ 0 all of what follows is correct but with less obvious changes in the proofs (cf. Remark 9.3 at the end of this section), therefore, we will first assume that

$$V(x) \geq c|x|, \quad c > 0. \quad (9.2)$$

If we set
$$\pi(t; u, v) = a(t; u, v) - iq(t) \int_{-\infty}^{\infty} x u \bar{v} dx, \quad (9.3)$$

then we define a *continuous* sesquilinear form on $K \times K$ (using (9.2)) and since $q(t)$ is real:

$$\operatorname{Re} \pi(t; v, v) + \lambda |v|^2 = \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{dv}{dx} \right|^2 dx + \int_{-\infty}^{\infty} (\lambda + V(x)) |v|^2 dx \geq \alpha \|v\|^2, \quad (9.4)$$

where $\alpha > 0$, $v \in K$ and $\lambda > 0$. Therefore, by [17, chapter IV] there exists a unique $u \in L^2(-\infty, t_0; K)$ such that $u = 0$ for $t < 0$ and such that for every $v \in K$

$$\pi(t; u(t), v) + \frac{d}{dt} (u(t), v) = \int_{-\infty}^{\infty} g(x, t) \overline{v(x)} dx + \left(\int_{-\infty}^{\infty} f(x) \overline{v(x)} dx \right) \delta_t, \quad (9.5)$$

where f is given in H and

$$(1 + V(x))^{-\frac{1}{2}} g(x, t) \in L^2(0, t_0; H). \quad (9.6)$$

Taking derivatives in the distribution sense we can write (9.5) as

(¹) So that $a(t; u, v) = a(u, v)$ does not depend on t . All our considerations apply, however, with easy changes to the case where $V(x, t)$ depends on x and t (and in that case $a(t; u, v)$ depends on t).

(²) I.e. $\int_0^{t_0} \int_{-\infty}^{\infty} \frac{1}{1 + V(x)} |g(x, t)|^2 dx dt < \infty$.

$$-\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + V(x)u + \frac{\partial u}{\partial t} - ixq(t)u = g(x, t) + f(x) \otimes \delta_t. \tag{9.7}$$

This solves the Cauchy problem. The solution depends continuously on $q(t)$ (cf. [17]), when q varies in $C(0, t_0)$. Moreover, (M.1) and (M.2) hold. It is known ([17, Chapter IV]) that $t \rightarrow u(\cdot; t)$ is almost everywhere equal to a continuous function from $[0, t_0] \rightarrow H$ with $u(\cdot; 0) = f$. We define now the subspace H_m of H . Let, for m a positive integer,

$$H_m = \{f \mid (1 + x^2)^m f \in H\}^{(1)} \tag{9.8}$$

and let

$$H_\infty = \bigcap_{m \geq 0} H_m. \tag{9.9}$$

We now prove

PROPOSITION 9.1. *If $f \in H_1$ and $g = 0$,⁽²⁾ the solution u of the Cauchy problem (9.7) verifies the condition:*

$$t \rightarrow u(\cdot; t) \text{ is continuous from } [0, t_0] \rightarrow H_1. \tag{9.10}$$

Proof. For $u, v \in K$, let us set

$$\begin{aligned} a_1(t; u, v) = a_1(u, v) &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{du \, d\bar{v}}{dx \, dx} \, dx \\ &+ \int_{-\infty}^{\infty} V(x)u(x)v(x) \, dx + \int_{-\infty}^{\infty} \frac{x}{1+x^2} \left(\frac{du}{dx} \bar{v} - u \frac{d\bar{v}}{dx} \right) \, dx - \int_{-\infty}^{\infty} \frac{2x^2}{(1+x^2)^2} u \bar{v} \, dx, \end{aligned} \tag{9.11}$$

which defines a continuous sesquilinear form on $K \times K$. As was done above one now checks that there exists⁽³⁾ a unique $u_1 \in L^2(-\infty, t_0; K)$ such that $u_1 = 0$ for $t < 0$ and such that for every $v \in K$ (we use here the fact that $(1 + x^2)f \in H$),

$$a_1(t; u_1(t), v) - iq(t) \int_{-\infty}^{\infty} x u_1(t) \bar{v} \, dx + \frac{d}{dt} (u_1(t), v) = \int_{-\infty}^{\infty} (1 + x^2) f(x) \bar{v}(x) \, dx. \tag{9.12}$$

(1) I.e. $\int_{-\infty}^{\infty} (1 + x^2)^{2m} |f(x)|^2 \, dx < \infty$. For convenience let $H_0 = H$.

(2) More generally, $\int_0^{t_0} \int_{-\infty}^{\infty} \frac{(1+x^2)^2}{1+V(x)} |g(x, t)|^2 \, dx \, dt < \infty$.

(3) In the inequality which corresponds to (9.4) one must now take λ "large enough".

Equation (9.12) can be written in the form

$$-\frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} + V(x)u_1 - ixq(t)u_1 + \frac{x}{1+x^2} \frac{\partial u_1}{\partial x} + \frac{\partial}{\partial x} \left(\frac{x}{1+x^2} u_1 \right) - \frac{2x^2}{(1+x^2)^2} u_1 = (1+x^2) f(x) \delta_t,$$

and therefore

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \left(\frac{u_1}{1+x^2} \right) + V(x) \left(\frac{u_1}{1+x^2} \right) - ixq(t) \left(\frac{u_1}{1+x^2} \right) = f(x) \delta_t.$$

From the uniqueness property we have

$$\frac{u_1(x, t)}{1+x^2} = u(x, t). \quad (9.13)$$

But since $t \rightarrow u_1(\cdot, t)$ is continuous from $[0, t_0] \rightarrow H$, we have that (9.10) follows from (9.13) and the definition of H_1 .

This, although not equivalent, has the force of (M.3)₁ by noticing the

Remark 9.1. Under the hypothesis of Proposition 9.1, $u(t)$ does not necessarily belong to K and $B: f \rightarrow xf$ does not map K into H , i.e., we are not exactly in the situation of (M.3)₁ but $t \rightarrow u(t)$ is continuous from $[0, t_0] \rightarrow H_1$ and $B \in \mathcal{L}(H_1; H)$.

Applying Theorem 4.1 we obtain

PROPOSITION 9.2. *If (9.2) holds and if $f \in H_1$, then the solution u of the Cauchy problem (9.7) (with $g=0$) admits a Volterra derivative $\delta u(x, t; q) / \delta q(\tau)$ which is characterized as the unique solution in $L^2(-\infty, t_0; K)$, which is zero for $t < 0$,⁽¹⁾ of*

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) \frac{\delta u(x, t; q)}{\delta q(\tau)} = ixu(x, \tau; q) \delta_t(\tau). \quad (9.14)$$

Remark 9.2. We are by no means looking for the largest space H_1 where the conclusion of Proposition 9.2 is valid, e.g., a simple generalization is obtained by replacing $1+x^2$ by $(1+x^2)^\frac{1}{2}$.

If $f \in H_m$, we can use (M.3)_m and Theorem 6.1, hence

PROPOSITION 9.3. *If (9.2) holds and if $f \in H_m$, the solution u of the Cauchy problem (9.7) (with $g=0$) admits a Volterra derivative $\delta^m u(x, t; q) / \delta q(\tau)^m$ which is characterized as the unique solution in $L^2(-\infty, t_0; K)$, which is zero for $t < 0$ ⁽¹⁾ of*

⁽¹⁾ And actually $t < \tau$.

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t)\right) \frac{\delta^m u(x, t; q)}{\delta q(\tau)^m} = (ix)^m u(x, \tau; q) \delta_t(\tau). \tag{9.15}$$

From this last proposition we have for $f \in H_\infty$,

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t)\right) P\left(-i \frac{\delta}{\delta q(\tau)}\right) u(x, t; q) = P(x) u(x, \tau; q) \delta_t(\tau) \tag{9.16}$$

for every polynomial $P(\lambda)$. We note that (9.16) characterizes $P(-i\delta/\delta q(\tau))u(x, t; q)$ by adding that $P(-i\delta/\delta q(\tau))u(x, t; q) \in L^2(-\infty, t_0; K)$ and should be zero for $t < 0$.⁽¹⁾ In chapter II we will pass to the limit and replace $P(-i\delta/\delta q(\tau))$ by $F(-i\delta/\delta q(\tau))$ where $F(x)$ is a suitable restricted function.

Remark 9.3. We want now to drop hypothesis (9.2) assuming only that $V(x)$ is continuous (or measurable) with

$$V(x) \geq 0. \tag{9.17}$$

The difficulty arises from the fact that

$$u, v \rightarrow iq(t) \int_{-\infty}^{\infty} x u \bar{v} dx$$

is not continuous on $K \times K$.⁽²⁾

We then modify the preceding observations as follows. Let us consider the class of functions φ which satisfy

$$\left. \begin{aligned} \varphi \in L^2(0, t_0; K), \quad \varphi' \in L^2(0, t_0; H), \quad \varphi(t_0) = 0 \\ \text{and } x(1 + V(x))^{-\frac{1}{2}} \varphi \in L^2(0, t_0; H). \end{aligned} \right\} \tag{9.18}$$

For $u \in L^2(0, t_0; K)$ and φ satisfying (9.18) we set

$$E(u, \varphi) = \int_0^{t_0} \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \frac{\partial u}{\partial x} \frac{\partial \bar{\varphi}}{\partial x} + V(x) u \bar{\varphi} - ixq(t) u \bar{\varphi} - u \frac{\partial \bar{\varphi}}{\partial t} \right\} dx dt. \tag{9.19}$$

By calculating $\text{Re } E(\varphi, \varphi)$, which eliminates the term in $ixq\varphi\bar{\varphi}$, we see that we can apply [17], Chapter 3 and therefore obtain:

Given g with $(1 + V(x))^{-\frac{1}{2}} g \in L^2(0, t_0; H)$ and $f \in H$, there exists $u \in L^2(0, t_0; K)$ such that

⁽¹⁾ And actually $t < \tau$.

⁽²⁾ And even not defined in general.

⁽³⁾ This condition is implied by the preceding ones in case $V(x) \geq c|x|$.

$$E(u, \varphi) = \int_0^{t_0} \int_{-\infty}^{\infty} g(x, t) \overline{\varphi(x, t)} dx dt + \int_{-\infty}^{\infty} f(x) \overline{\varphi(x, 0)} dx \tag{9.20}$$

for every φ satisfying (9.18). The *uniqueness* does not follow from the general theorem but is true and can be checked as follows:

Let us consider u with $E(u, \varphi) = 0$ for every φ satisfying (9.18). Then (extending u by 0 for $t < 0$) we have

$$u \in L^2(-\infty, t_0; K), \quad u = 0 \text{ for } t < 0, \tag{9.21}$$

$$-\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + V(x)u + \frac{\partial u}{\partial t} - ixq(t)u = 0. \tag{9.22}$$

This implies $u \equiv 0$. To see this let a be a function of $\mathcal{D}(R)$ (*real* infinitely differentiable functions with compact support) with $a(x) = 1$ in a neighborhood of zero and let $a_r(x) = a(x/r)$. We will let $r \rightarrow \infty$ in what follows. Now let α be a regularizing sequence of *even* functions of t and let $\theta_n(t)$ be defined by

$$\theta_n(t) = \begin{cases} 1, & t \leq t_0 - \frac{2}{n}, \\ n \left(t_0 - \frac{1}{n} - t \right), & t_0 - \frac{2}{n} \leq t \leq t_0 - \frac{1}{n}, \\ 0, & t \geq t_0 - \frac{1}{n}. \end{cases}$$

Multiplying (9.22) by $((a_r(x)\theta_n(t)\bar{u}) *_{(t)} \alpha) \theta_n^{(1)}$ and integrating over $R \times (0, t_0)$ we obtain

$$\begin{aligned} & \frac{1}{2} \operatorname{Re} \int \int \frac{\partial}{\partial x} (\theta_n u) \frac{\partial}{\partial x} ((a_r \theta_n \bar{u}) * \alpha) dx dt + \operatorname{Re} \int \int V(x) (\theta_n u) ((a_r \theta_n \bar{u}) * \alpha) dx dt \\ & - \operatorname{Re} \int \int (\theta_n u) \frac{\partial}{\partial t} ((a_r \theta_n \bar{u}) * \alpha) dx dt - \operatorname{Re} \int \int u \theta_n' ((a_r \theta_n \bar{u}) * \alpha) dx dt \\ & - \operatorname{Re} \left(i \int \int xq(t) \theta_n u ((a_r \theta_n \bar{u}) * \alpha) dx dt \right) = 0. \end{aligned} \tag{9.23}$$

Now the particular term

$$\operatorname{Re} \int \int (\theta_n u) \frac{\partial}{\partial t} ((a_r \theta_n \bar{u}) * \alpha) dx dt = 0,$$

(¹) Cf. [17] for similar techniques. A modification of this process gives uniqueness in non-linear problems, cf. Lions-Prodi [18].

and hence letting $\alpha \rightarrow \delta$ in (9.23) we obtain

$$\begin{aligned} & \frac{1}{2} \operatorname{Re} \iint \frac{\partial}{\partial x} (\theta_n u) \frac{\partial}{\partial x} (a_r \theta_n \bar{u}) dx dt + \operatorname{Re} \iint V(x) a_r |\theta_n u|^2 dx dt \\ & - \operatorname{Re} \iint \theta_n \theta_n' a_r |u|^2 dx dt - \operatorname{Re} \left(i \iint x q(t) \theta_n^2 a_r |u|^2 dx dt \right) = 0. \end{aligned}$$

Noticing that the last term on the left of this equation is itself 0 we obtain on letting $r \rightarrow \infty$,

$$\frac{1}{2} \operatorname{Re} \iint \left| \frac{\partial}{\partial x} (\theta_n u) \right|^2 dx dt + \operatorname{Re} \iint V(x) |\theta_n u|^2 dx dt - \operatorname{Re} \iint \theta_n \theta_n' |u|^2 dx dt = 0$$

But now $\theta_n \theta_n' \leq 0$ and hence $\theta_n u = 0$. Since n is arbitrary this implies $u = 0$.

We are now provided with a Theorem of existence and uniqueness and can therefore proceed here exactly as in Propositions 9.1-9.3. *The same results are valid.*

Remark 9.4. All the results of this section are valid if we replace $V(x)$ by $V(x, t)$ with $V(x, t) \geq 0$ and continuous (or even measurable).

Remark 9.5. We have used Hilbert space methods because this gives rise to many generalizations (cf. section 11) and works without essential changes in the case of the Schrödinger equation (next section). In the case of the parabolic operator treated in this section other techniques are available, especially integral equations. One can apply by a suitable adaption⁽¹⁾ the reasoning of several authors. We refer to Dressel [5], Fortet [7], Kac [12], Rosenblatt [20], Rosenbloom [21], and the bibliography in the latter.

10. Example 2

We want now to consider, from the same point of view as in section 9, the *Cauchy problem for the Schrödinger operator*,

$$-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t),$$

where $V(x)$ is given ≥ 0 and continuous (to fix the ideas). Here we will take the spaces H and K exactly the same as in section 9, and we will use the same notations as used

⁽¹⁾ One must take care of the term $ixq(t)u$.

there. Again we define $a(t; u, v)$ by (9.1), and we assume that (9.2) holds. Applying [17, chapter 8], we have that there exists a unique function $u \in L^2(-\infty, t_0; K)$, which is zero for $t < 0$, and such that

$$ia(t; u(t), v) + \frac{d}{dt}(u(t), v) - iq(t) \int_{-\infty}^{\infty} xu\bar{v}dx = \int_{-\infty}^{\infty} g(x, t)\overline{v(x)}dx \text{ for every } v \in K, \quad (10.1)$$

where $g(x, t)$ is given satisfying

$$\int_0^{t_0} \int_{-\infty}^{\infty} \frac{1}{1+V(x)} \left(|g(x, t)|^2 + \left| \frac{\partial g(x, t)}{\partial t} \right|^2 \right) dxdt < \infty, \quad (10.2)$$

and where
$$q \in C^1(0, t_0), \quad q(0) = 0. \quad (10.3)$$

We notice that in (10.1) there is no term $(\int_{-\infty}^{\infty} f(x)\bar{v}(x)dx)\delta_t$ which means that $u(x, 0) = 0$. It seems impossible in general to consider a term of this form assuming only that $f \in H^{-L^2}(R)$. But let us set in general

$$\left. \begin{aligned} Tf &= (1+V(x))^{-\frac{1}{2}} (-\frac{1}{2}f'' + V(x)f), \\ f'' &= \frac{d^2f}{dx^2} \text{ taken in the sense of distributions on } R. \end{aligned} \right\} \quad (10.4)$$

We have then

PROPOSITION 10.1. *Assume that*

$$V(x) \geq c|x|, \quad c > 0, \quad (10.5)$$

and (10.2), (10.3) hold. Let f be given in H with

$$Tf \in H = L^2(R). \quad (10.6)$$

Then, there exists a unique $u \in L^2(-\infty, t_0; K)$ with $u = 0$ for $t < 0$, such that

$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t) \right) u = g(x, t) + f(x)\delta_t. \quad (10.7)$$

Proof. Let $\theta(t)$ be twice continuously differentiable in R with compact support, and let $\theta(0) = 1$. Introduce the new unknown, $u^* = u - \theta(t)f(x)$, so that $u^* \in L^2(-\infty, t_0; K)$, $u^* = 0$ for $t < 0$, and

$$\begin{aligned} & \left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t) \right) u^* \\ & = g(x, t) - i \left(-\frac{1}{2} f'' + V(x)f \right) \theta(t) - \theta'(t)f(x) - ixq(t)\theta(t)f(x) \end{aligned} \quad (10.8)$$

Now equation (10.8) is *equivalent* to equation (10.1) with g replaced by g^* , where g^* is defined as the second member of (10.8). But the hypotheses imply that this new function g^* verifies

$$\int_0^{t_0} \int_{-\infty}^{\infty} \frac{1}{1 + V(x)} \left(|g^*|^2 + \left| \frac{\partial g^*}{\partial t} \right|^2 \right) dx dt < \infty,$$

and therefore u^* exists and is unique so that the proposition follows.

Let us now introduce a space H_m (different from the analogous space introduced in section 9). For m an integer > 0 , let

$$H_m = \{f \mid (1 + x^2)^m f \in H, T((1 + x^2)^m f) \in H\}, \quad (10.9)$$

$$H_\infty = \bigcap_m H_m. \quad (10.10)$$

The same reasoning as in Proposition 9.1 leads to

PROPOSITION 10.2. *Under the hypotheses of Proposition 10.1, with $g=0$ and $f \in H_m$, the solution u verifies*

$$t \rightarrow (1 + x^2)^m u(x, t) \text{ is continuous from } [0, t_0] \rightarrow L^2(R_x). \quad (10.11)$$

We have, therefore, the result analogous to the one of Proposition 9.3, namely

PROPOSITION 10.3. *Under the hypotheses of Proposition 10.2, the functional $q \rightarrow u(x, t; q)$ admits a F.V. derivative $\delta^m u(x, t; q) / \delta q(\tau)^m$ which is characterized as the unique solution in $L^2(-\infty, t_0; K)$, equal to zero for $t < 0$, of*

$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t) \right) \frac{\delta^m u(x, t; q)}{\delta q(\tau)^m} = (ix)^m u(x, \tau; q) \delta_t(\tau). \quad (10.12)$$

Remark 10.1. There is a difficulty similar to the one encountered in section 9 when assuming only that $V(x) \geq 0$. However, by the same kind of method ($E(u, \varphi)$

is now more complicated—cf. [17, chapter 8, section 5]) we can prove that all the preceding results hold assuming only $V(x) \geq 0$. One can also make the same observation here as in Remark 9.4.

11. Example 3

The observations of sections 9 and 10 are by no means restricted to operators of order 2 in x . We can in general consider

$$a(t; u, v) = \sum_{k=0}^m \int_{-\infty}^{\infty} a_k(x, t) D^k u D^k \bar{v} dx, \quad D = \frac{d}{dx}. \quad (11.1)$$

With suitable hypotheses (we do not want to give details here), the preceding results will extend to operators

$$\sum_{k=0}^m (-1)^k D^k (a_k(x, t) D^k) + \frac{\partial}{\partial t} - ixq(t). \quad (11.2)$$

As a very simple special case, we can consider

$$(-1)^n D_x^{2n} + V(x) + \frac{\partial}{\partial t}. \quad (11.3)$$

Here the space H remains unchanged and we define K by

$$K = \{u \mid \sqrt{1 + V(x)}u \in L^2(R), D^n u \in L^2(R)\}. \quad (11.4)$$

It should also be observed that the results above apply also to *mixed problems*, i.e. problems where R_x is replaced by Ω , an open set in R_x . Here conditions at infinity in x are replaced by suitable *boundary conditions*. Unfortunately however in the case of mixed problems we meet a difficulty in chapter II.

In the preceding the fact that the dimension in x is one is essentially irrelevant. However, if $x \in R^n$, then one must replace q by a system of n parametric functions. This is the purpose of the next section.

12. The multi-dimensional case

We start with some general remarks. With the notations of section 1 we consider a functional

(1) The potential $V(x)$ is now contained in the term $a_0(x, t)$.

$q = \{q_1, q_2, \dots, q_n\} \rightarrow \Phi(q_1, q_2, \dots, q_n) = \Phi(q)$ from $\mathcal{D}(T) \times \dots \times \mathcal{D}(T) = \mathcal{D}(T)^n$ into E and we

assume:
$$q = \{q_1, q_2, \dots, q_n\} \rightarrow \Phi(q) \tag{12.1}$$

is continuous from $\mathcal{D}(T)^n$ into E , and:

$$\left. \begin{array}{l} \text{For every } q, q^* \in \mathcal{D}(T)^n, \text{ the function} \\ \xi_1, \xi_2, \dots, \xi_n \rightarrow \Phi(q_1 + \xi_1 q_1^*, \dots, q_n + \xi_n q_n^*) = \Phi(q + \xi q^*) \\ \text{is entire analytic in } C^n \text{ with values in } E. \end{array} \right\} \tag{12.2}$$

Then

$$\psi_j \rightarrow \frac{\partial}{\partial \xi_j} \Phi(q + \xi \psi) \Big|_{\xi=0} = \frac{d}{d\xi_j} \Phi(q_1, q_2, \dots, q_{j-1}, q_j + \xi_j \psi, q_{j+1}, \dots, q_n) \Big|_{\xi_j=0}$$

is a linear and continuous mapping from $\mathcal{D}(T)$ into E (cf. section 1), and therefore defines a *distribution*

$$\frac{\delta \Phi(q)}{\delta q_j(\tau)} \in \mathcal{D}'(T; E), \tag{12.3}$$

which is called the *F.V. derivative with respect to q_j* and which verifies

$$\int_T \frac{\delta \Phi(q)}{\delta q_j(\tau)} \psi(\tau) d\tau = \frac{d}{d\xi_j} \Phi(q_1, \dots, q_{j-1}, q_j + \xi_j \psi, q_{j+1}, \dots, q_n) \Big|_{\xi_j=0}. \tag{12.4}$$

In the case when $\delta \Phi(q)/\delta q_j(\tau)$ is a *function*, we have

$$\frac{\delta \Phi(q)}{\delta q_j(\tau)} = \lim_{\varphi_\nu} \left(\frac{\Phi(q_1, \dots, q_{j-1}, q_j + \varphi_\nu, q_{j+1}, \dots, q_n) - \Phi(q)}{\int_T \varphi_\nu(t) dt} \right), \tag{12.5}$$

where $\{\varphi_\nu\}$ is a F.V. sequence at τ .

We can then define

$$\frac{\delta^m \Phi(q)}{\delta q_j(\tau)^m} = \lim_{\substack{\tau^* \rightarrow \tau \\ \tau^* > \tau}} \frac{\delta}{\delta q_j(\tau^*)} \left(\frac{\delta^{m-1} \Phi(q)}{\delta q_j(\tau)^{m-1}} \right) \tag{12.6}$$

whenever the limit exists in E .

Of course, we can also define mixed F.V. derivatives. Let us notice (cf. Lemma 1.1) that if $\Phi(q)$ verifies (12.1) and (12.2) and

$$\frac{\delta \Phi(q)}{\delta q_j(\tau)} = 0 \text{ in } \mathcal{D}'(T; E) \text{ for every } q \text{ and } j \tag{12.7}$$

then $\Phi(q)$ does not depend on q .

Let us look now at some mixed problems in this multi-dimensional case. We use the notations of section 4. Let B_1, B_2, \dots, B_n be a family of operators linear and continuous from K to $H^{(1)}$ and assume:

$$\left. \begin{aligned} &\text{There exists a unique function } u \in L^2(-\infty, t_0; K), \\ &\text{which is zero for } t < 0, \text{ and satisfying} \\ &a'(t; u(t), v) + \frac{d}{dt}(u(t), v) - i \sum_{j=1}^n q_j(t)(B_j u(t), v) = (f, v) \delta, \end{aligned} \right\} \quad (12.8)$$

where $v \in K, f$ given in H , and $q_j \in C(0, t_0)$.⁽²⁾ In this way we define a functional,

$$q = \{q_1, q_2, \dots, q_n\} \rightarrow u(t; q) = u(t; q_1, q_2, \dots, q_n),$$

about which we assume:

$$q \rightarrow u(t; q) \text{ is continuous from } C(0, t_0)^n \text{ into } L^2(-\infty, t_0; K) \quad (12.9)$$

and moreover

$$\left. \begin{aligned} &\text{there exists a subspace } H_m \text{ of } H \text{ such that for every } f \in H_m \\ &\text{the solution } u \text{ of (12.8) verifies} \\ &t \rightarrow u(t), B_j u(t), \dots, B_j^{m-1} u(t), j = 1, 2, \dots, n, \\ &\text{are continuous from } [0, t_0] \rightarrow K. \end{aligned} \right\} \quad (12.10)$$

Now we prove in the same way as Theorem 6.1,

THEOREM 12.1. *We assume that (12.8), (12.9), (12.10)_m hold and that f is given in H_m . Then $\delta^m u(q) / \delta q_j(\tau)^m$ exists and is characterized by*

$$\frac{\delta^m u'(q)}{\delta q_j(\tau)^m} \in L^2(-\infty, t_0; K) \text{ and } = 0 \text{ for } t < 0; \quad (12.11)$$

moreover,

$$\begin{aligned} a\left(t; \frac{\delta^m u(t; q)}{\delta q_j(\tau)^m}, v\right) + \frac{d}{dt}\left(\frac{\delta^m u(t; q)}{\delta q_j(\tau)^m}, v\right) - i \sum_{j=1}^n q_j(t) \left(B_j \frac{\delta^m u(t; q)}{\delta q_j(\tau)^m}, v\right) \\ = ((iB_j)^m u(\tau; q), v) \delta_t(\tau) \text{ for } v \in K. \end{aligned} \quad (12.12)$$

We could also consider here mixed F.V. derivatives.

⁽¹⁾ A slight change will be used in the examples as already seen in sections 9 and 10.

⁽²⁾ In some examples (cf. section 9) $q \in L^\infty(0, t_0)$ would be enough; in others, more restrictions are needed on the q_j (cf. section 10). We extend q arbitrarily for $t < 0$.

⁽³⁾ As we have noted previously, in some applications this condition appears in a slightly different way.

Remark 12.1. We can pass to the limit along the same lines as in section 7. Let us look now at the *Cauchy problem*:

$$-\frac{1}{2} \Delta_x u + V(x)u + \frac{\partial u}{\partial t} - i \sum_{j=1}^n x_j q_j(t)u = 0, \tag{12.13}$$

$$u(x, 0) = f(x), \tag{12.14}$$

where $x = \{x_1, x_2, \dots, x_n\} \in R^n$, $V(x)$ is a given non-negative continuous function in R^n , $\Delta_x = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \dots + \partial^2/\partial x_n^2$ and where $f(x)$ is a given function. This then is the generalization of section 9. The preceding considerations apply with

$$H = L^2(R^n), K = \{f \mid (1 + V(x))^{\frac{1}{2}} f \in L^2(R^n), \partial f/\partial x_j \in L^2(R^n), j = 1, \dots, n\}. \tag{1}$$

If $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$, we define $H_m = \{f \mid (1 + x^2)^m f \in H\}$. In this way the results of section 9 apply to this case. There are analogous generalizations for the Schrödinger equation.

13. Supplements

In this section we give some results supplementary to those of sections 9 and 10 (the same would apply to sections 11 and 12). These results will be useful in chapter II. First we prove the following uniqueness theorem.

THEOREM. 13.1 *Let u be given with*

$$u \in L^2(-\infty, t_0; H), \quad (H = L^2(R)) \text{ and } u = 0 \text{ for } t < 0, \tag{13.1}$$

such that

$$\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + V(x)u - iax'(t)u = 0, \tag{13.2}$$

where $V(x)$ is ≥ 0 . Then u is identically zero.

Proof. Let ϱ and θ be given in $\mathcal{D}(R)$ (infinitely differentiable functions with compact support). Assume that ϱ is real and even and that $\theta \geq 0$. Multiply (13.2) by $((u * \varrho)\theta) * \varrho$, where the convolution $*$ is taken with respect to the x variable. Inte-

(1) If $u, v \in K$, then

$$((u, v)) = \int_{R^n} (1 + V(x))u(x)\overline{v(x)}dx + \sum_{i=1}^n \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} dx.$$

grate over $Q_t = R \times [0, t]$ (the integrals are meaningful and note that (13.2) yields information about $\partial u / \partial t$). Since,

$$\int_R f(\bar{g} * \varrho) dx = \int_R (f * \varrho) \bar{g} dx,$$

we obtain

$$\begin{aligned} & \iint_{Q_t} \left(\frac{\partial}{\partial t} (u * \varrho) \right) ((\bar{u} * \varrho) \theta) dx dt + \frac{1}{2} \iint_{Q_t} \left(\frac{\partial}{\partial x} (u * \varrho) \right) \frac{\partial}{\partial x} ((\bar{u} * \varrho) \theta) dx dt \\ & + \iint_{Q_t} ((Vu) * \varrho) (\bar{u} * \varrho) \theta dx dt - i \iint_{Q_t} q(t) ((xu) * \varrho) ((\bar{u} * \varrho) \theta) dx dt = 0. \end{aligned}$$

Taking twice the real part of both members, we obtain

$$\begin{aligned} & \int_R \theta(x) |(u * \varrho)(x, t)|^2 dx + \iint_{Q_t} \theta \left| \frac{\partial}{\partial x} (u * \varrho) \right|^2 dx dt + \frac{1}{2} \iint_{Q_t} \theta' \frac{\partial}{\partial x} |u * \varrho|^2 dx dt \\ & + 2\text{Re} \iint_{Q_t} ((Vu) * \varrho) (\bar{u} * \varrho) \theta dx dt - 2\text{Re} i \iint_{Q_t} q(t) ((xu) * \varrho) ((\bar{u} * \varrho) \theta) dx dt = 0 \end{aligned}$$

for almost all t . Since the second integral on the left is ≥ 0 , we obtain

$$\begin{aligned} & \int_R \theta(x) |(u * \varrho)(x, t)|^2 dx - \frac{1}{2} \iint_{Q_t} \theta'' |u * \varrho|^2 dx dt \\ & + 2\text{Re} \iint_{Q_t} ((Vu) * \varrho) (\bar{u} * \varrho) dx dt - 2\text{Re} i \iint_{Q_t} q(t) ((xu) * \varrho) (\bar{u} * \varrho) \theta dx dt \leq 0. \end{aligned}$$

Now, taking a sequence of ϱ such that $\varrho \rightarrow \delta$, we obtain

$$\begin{aligned} & \int_R \theta(x) |u(x, t)|^2 dx - \frac{1}{2} \iint_{Q_t} \theta'' |u|^2 dx dt + 2\text{Re} \iint_{Q_t} V |u|^2 dx dt \\ & - 2\text{Re} i \iint_{Q_t} q(t) x |u|^2 \theta dx dt \leq 0. \quad (13.3) \end{aligned}$$

Since the third term on the left in (13.3) is ≥ 0 and the last one equals zero, we have

$$\int_R \theta(x) |u(x, t)|^2 dx - \frac{1}{2} \iint_{Q_t} \theta'' |u(x, t)|^2 dx dt \leq 0. \quad (13.4)$$

We now take a sequence $\theta = \theta_m(x) = a(x/m)$, where $a \in \mathcal{D}$, $a \geq 0$, and $a = 1$ in a neighborhood of zero. From (13.4) with $\theta = \theta_m$ and letting $m \rightarrow \infty$, we obtain

$$\int_R |u(x, t)|^2 dx = 0$$

for almost all t , from which the theorem follows.

The same method may be applied in the case of the Schrödinger operator to prove

THEOREM 13.2. *Let u be given satisfying (13.1) and*

$$\frac{\partial u}{\partial t} - \frac{i}{2} \frac{\partial^2 u}{\partial x^2} + iV(x)u - ixq(t)u = 0, \tag{13.5}$$

where $V(x) \geq 0$. Then u is identically zero.

Actually this result holds when V is assumed only to be real.

Now we prove an existence theorem.

THEOREM 13.3. *Let $f(x)$ be given satisfying*

$$f(x)(1 + V(x))^{-\frac{1}{2}} \in H = L^2(R), \tag{13.6}$$

where $V(x)$ is a given continuous function with $V(x) \geq c|x|^2$. Then, there exists a unique function u satisfying (13.1) and

$$\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + V(x)u - ixq(t)u = f(x)\delta. \tag{13.7}$$

Proof. The uniqueness follows from Theorem 13.1. In the space H , the unbounded operator,

$$\Lambda: u \rightarrow -\frac{1}{2} \frac{d^2 u}{dx^2} + V(x)u,$$

with domain $D(\Lambda) = \{u \mid u \in K, \Lambda u \in H\}$, is *self-adjoint* and ≥ 0 . We diagonalize Λ into the multiplication by λ over a measurable sum h over $(0, \infty)$. Let X be the unitary mapping from H onto h which diagonalizes Λ . If $u \in D(\Lambda)$, then $Xu \in h$, $\lambda Xu \in h$ and $X(\Lambda u) = \lambda Xu$. We set

$$u_1(x, t) = X^{-1}(e^{-\lambda t} Xf) \text{ for } t > 0 \text{ and } 0 \text{ for } t < 0. \tag{13.8}$$

We now check that

$$u_1 \in L^2(-\infty, t_0; H), \quad u_1 = 0 \text{ for } t < 0, \tag{13.9}$$

and

$$\frac{\partial u_1}{\partial t} - \frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} + V(x)u_1 = f(x)\delta. \tag{13.10}$$

Note that (13.6) means that $f \in D(\Lambda^{-\frac{1}{2}})$. Therefore, $\lambda^{-\frac{1}{2}} Xf \in h$ and since $\int_0^{t_0} e^{-2\lambda t} dt \leq c/\lambda$, we obtain (13.9) and (13.10) follows immediately.

Now, let $w = u - u_1$ (13.11)

we obtain
$$\frac{\partial w}{\partial t} - \frac{1}{2} \frac{\partial^2 w}{\partial x^2} + V(x)w - ixq(t)w = ixq(t)u_1. \quad (13.12)$$

From section 9 we know the existence of $w \in L^2(-\infty, t_0; K)$, $w = 0$ for $t < 0$, satisfying (13.12) providing

$$xq(t)u_1(1 + V(x))^{-\frac{1}{2}} \in L^2(0, t_0; H).$$

But this last follows from (13.9) and $V(x) \geq c|x|^2$. Thus $u = u_1 + w$ is a solution of (13.7) and (13.1) is verified. Thus Theorem 13.3 is proved.

This method does *not* work in the case of the Schrödinger operator because in (13.8) one has to replace $e^{-\lambda t}$ by $e^{-i\lambda t}$.

CHAPTER II

The F.V. variational equations

1. The general method

We want to derive a F.V. variational equation for the functional $u(t; q)$, the solution of the mixed problem described in chapter I, section 4. As we saw there, $\delta u(t; q)/\delta q(\tau)$ is characterized as the unique solution of a well-set mixed problem. In this chapter our primary aim is to find a second expression, apparently different, for the solution of this latter mixed problem, say $\Psi_u(q)$. By the uniqueness property, we then have

$$\frac{\delta u(t; q)}{\delta q(\tau)} = \Psi_u(q) = \Psi_u(t; q). \quad (1.1)$$

This is the F.V. variational equation we are looking for. Our next goal in this chapter will be to study to what extent the only solution of the F.V. variational equation (1.1), with some "boundary conditions" to be found,⁽¹⁾ is the solution of the mixed problem, $u(t; q)$.

The second expression $\Psi_u(q)$ is obtained by means of some algebraic properties and this fact explains why in what follows we are obliged to consider only the problems corresponding to the examples of sections 9–11 of chapter I.

⁽¹⁾ We have already seen some examples of such "boundary conditions" in chapter I, section 3.

2. The parabolic case (I)

We consider the situation of chapter I, section 9, with the additional hypothesis that $V(x)$ is a polynomial of degree $\leq m$. To be specific, assume

$$V(x) \text{ is a polynomial of degree } \leq m, V(x) \geq 0, m \text{ even.} \tag{2.1}$$

We at first for convenience make another assumption but later in section 3, after replacing condition (2.1) by a more general condition, we shall actually prove that this assumption holds. Thus, assume for now

$$\left. \begin{aligned} &\text{one can choose } f \in H_m^* \subset H_m \text{ (} H_m \text{ is defined by (9.8) of chapter} \\ &\text{I) in such a way that the corresponding solution of the} \\ &\text{Cauchy problem (9.7) verifies} \\ &\frac{\partial u}{\partial x} \in L^2(0, t_0; K) \text{ (}^1\text{)} \end{aligned} \right\} \tag{2.2}$$

Let us now introduce $\Phi_f(x, t; q)$ the solution of

$$\Phi_f \in L^2(-\infty, t_0; K), \quad \Phi_f = 0 \text{ for } t < 0 \tag{2.3}$$

and
$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t)\right) \Phi_f = ix f(x) \delta_t. \tag{2.4}$$

Since $f \in H_m$, it is easily checked that $xf \in H_{m-1}$ (in particular). Therefore Φ_f exists, is unique, and $t \rightarrow \Phi_f(x, t; q) (1 + x^2)^{m-1}$ is continuous from $[0, t_0] \rightarrow H$.

Since giving f is equivalent to giving u , we can also set

$$\Phi_f(x, t; q) = Lu(x, t; q), \tag{2.5}$$

where $u \rightarrow Lu$ is a linear operator in u . We now have

THEOREM 2.1. *We assume that (2.1) holds and that $f \in H_m^*$, defined in (2.2). Let us define $w(x, t, \tau; q)$ by*

$$\begin{aligned} w(x, t, \tau; q) = & -i\tau \frac{\partial u(x, t; q)}{\partial x} - \left(\int_0^t \min(\tau, s) q(s) ds \right) u(x, t; q) \\ & - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds + \Phi_f(x, t; q) \text{ if } t > \tau, \end{aligned} \tag{2.6}$$

(1) Notations are those of chapter I, section 9. Therefore $u \in L^2(-\infty, t_0; K)$, $u = 0$ for $t < 0$, and

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t)\right) u = f(x) \delta_t.$$

and
$$w(x, t, \tau; q) = 0 \text{ if } t < \tau. \tag{2.7}$$

Then
$$t \rightarrow w(\cdot, t, \tau; q) \in L^2(-\infty, t_0; K) \tag{2.8}$$

and
$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t)\right) w(x, t, \tau; q) = ixu(x, \tau; q) \delta_t(\tau). \tag{2.9}$$

We will prove this shortly but let us first notice by comparison with Proposition 9.2, chapter I and according to the general remarks of section 1 that we obtain

THEOREM 2.2. *Under the hypotheses of Theorem 2.1, the solution of the Cauchy problem*

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t)\right) u(x, t; q) = f(x) \delta_t. \tag{2.10}$$

$$u(\cdot, t; q) \in L^2(-\infty, t_0; K) \text{ and } = 0 \text{ for } t < 0, \tag{2.11}$$

satisfies the F.V. variational equation

$$\begin{aligned} \frac{\delta u(x, t; q)}{\delta q(\tau)} + i\tau \frac{\partial u(x, t; q)}{\partial x} + \left(\int_0^t \min(\tau, s) q(s) ds\right) u(x, t; q) \\ + i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\partial q(s)}\right) u(x, t; q) ds = \Phi_f(x, t; q) \text{ for } t > \tau, \end{aligned} \tag{2.12}$$

where Φ_f is defined by (2.3) and (2.4).

Also before proving Theorem 2.1, let us notice

Remark 2.1. The F.V. variational equation (2.12) is quite unsatisfactory as it stands, since the right side $\Phi_f(x, t; q) = Lu(x, t; q)$ cannot be explicitly expressed in terms of V and q .⁽²⁾ However, as we shall see, equation (2.12) becomes a much simpler equation for the kernel of the mapping $f \rightarrow u$ i.e., for the fundamental solution of the operator

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t).$$

Proof of Theorem 2.1. We first check that w defined by (2.6) and (2.7) verifies (2.8). This is true for the first term in (2.6), $-i\tau \delta u(x, t; q) / \partial x$, because of (2.2). Ob-

(1) And, of course, $= 0$ for $t < 0$ since $\tau \geq 0$.

(2) Except by using function space integrals (cf. Introduction).

viously the same is true for $\left(\int_0^t \min(\tau, s)q(s)ds\right)u(x, t; q)$ and also for Φ_f (cf. (2.3)). By Proposition 9.3 and Theorem 8.1, chapter I, the function $s \rightarrow V'(-i\delta/\delta q(s))u(x, t; q)$ is continuous from $[0, t_0]$ to $L^2(0, t_0; K)$ and therefore

$$\int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds$$

defines an element of $L^2(\tau, t; K)$ and thus (2.8) is proved. It remains to prove (2.9). To simplify the writing let us set

$$\Lambda = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t}.$$

Since $\Lambda u = ixq(t)u$ we have by differentiation in x ,

$$(\Lambda - ixq(t)) \frac{\partial u}{\partial x} + V'(x)u - iq(t)u = 0.$$

Hence for $t > \tau$,

$$(\Lambda - ixq(t)) \left(-i\tau \frac{\partial u}{\partial x} \right) = i\tau V'(x)u + \tau q(t)u. \tag{2.14}$$

On the other hand, for $t > \tau$

$$(\Lambda - ixq(t)) \left(\left(-\int_0^t \min(\tau, s)q(s)ds \right) u \right) = -\tau q(t)u. \tag{2.15}$$

Moreover,

$$\begin{aligned} (\Lambda - ixq(t)) \left(-i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds \right) \\ = -i\tau V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) \Big|_{s=t} \\ = -i\tau V'(x)u. \tag{2.16} \end{aligned}$$

Since by definition of Φ_f , $(\Lambda - ixq(t)) \Phi_f = 0$ for $t > \tau$ (cf. (2.4)), one obtains from (2.14), (2.15) and (2.16) that $(\Lambda - ixq(t))w = 0$ for $t > \tau$. Thus in order to prove (2.9) we must only check that

$$w(x, t, \tau; q) \rightarrow ixu(x, \tau; q) \text{ as } t \rightarrow \tau. \tag{2.17}$$

(¹) By Proposition 9.3 $(\delta^m u(x, t; q) / \delta q(\tau)^m) |_{\tau=t} = (ix)^m u(x, t; q)$, so that

$$V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) \Big|_{s=t} = V'(x)u(x, t; q).$$

(²) For instance, in H .

Now condition (2.17) is equivalent to

$$-i\tau \frac{\partial u(x, \tau; q)}{\partial x} - \left(\int_0^\tau sq'(s) ds \right) u(x, \tau; q) - i \int_0^\tau sV' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, \tau; q) ds + \Phi_f(x, \tau; q) = ixu(x, \tau; q) \quad (2.18)$$

and this relation (2.18) has to be true for every $\tau > 0$. Replacing τ by t , and setting

$$w_1(x, t; q) = -it \frac{\partial u(x, t; q)}{\partial x} - \left(\int_0^t sq(s) ds \right) u(x, t; q) - i \int_0^t sV' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds + \Phi_f(x, t; q), \quad (2.19)$$

one has to check that

$$w_1(x, t; q) = ixu(x, t; q) \text{ for } t > 0. \quad (2.20)$$

Since, by hypothesis, $f \in H_m^*$, both terms in (2.20) belong to $L^2(-\infty, t_0; K)$ and are $\equiv 0$ for $t < 0$. Therefore, in order to prove (2.20) it is enough to prove

$$(\Lambda - ixq(t))w_1 = (\Lambda - ixq(t))(ixu) \quad (2.21)$$

and

$$w_1(x, 0; q) = ixu(x, 0; q). \quad (2.22)$$

The latter equality is easy since $w_1(x, 0; q) = \Phi_f(x, 0; q)$ and $\Phi_f(x, 0; q) = ix f(x)$ (cf. (2.4)), and moreover $u(x, 0; q) = f(x)$. We now verify (2.21). First one has

$$(\Lambda - ixq(t)) \left(-it \frac{\partial u}{\partial x} - ixu \right) = -i(itq(t)u - tV'(x)u)$$

and since $(\Lambda - ixq(t))\Phi_f = 0$, there remains only to check that

$$tq(t)u + itV'(x)u - (\Lambda - ixq(t)) \left(\int_0^t sq'(s) ds \right) u - i(\Lambda - ixq(t)) \left(\int_0^t sV' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds \right) = 0.$$

But

$$(\Lambda - ixq(t)) \left(\int_0^t sq'(s) ds \right) u = tq(t)u$$

and $(\Lambda - ixq(t)) \left(\int_0^t sV' \left(-i \frac{\delta}{\delta q(s)} \right) u ds \right) = tV' \left(-i \frac{\delta}{\delta q(s)} \right) u \Big|_{s=t} = tV'(x)u(x, t; q),$

so that the result follows. The proof of Theorem 2.1 is completed.

3. Verification of hypothesis (2.2)

Let u be the solution in $L^2(-\infty, t_0; K)$ which is zero for $t < 0$ of

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - iq(t)\right) u = f(x)\delta_t. \tag{3.1}$$

We now apply the method of finite differences in x (cf. for this very general method L. Nirenberg [19]). Let us set

$$w_h(x, t) = \frac{1}{h} (u(x+h, t) - u(x, t)),$$

$$f_h(x) = \frac{1}{h} (f(x+h) - f(x)),$$

$$V_h(x) = \frac{1}{h} (V(x+h) - V(x)).$$

Then w_h verifies

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - iq(t)\right) w_h = -V_h(x)u(x+h, t) + iq(t)u(x+h, t) + f_h(x)\delta_t. \tag{3.2}$$

Let us assume that $f' = df/dx \in L^2(R) = H, \tag{3.3}$

and

$$V \text{ is once continuously differentiable with } V'(x)/(1+V(x)) \text{ bounded.} \tag{3.4}$$

As $h \rightarrow 0, iq(t)u(x+h, t) \rightarrow iq(t)u(x, t)$ in $L^2(0, t_0; K)$ and $f_h \rightarrow f'$ in $L^2(R) = H$. It follows that w_h will belong to a bounded set of $L^2(0, t_0; K)$ as $h \rightarrow 0$ if $V_h(x)u(x+h, t)$ remains in a bounded set of $L^2(0, t_0; K')$.⁽¹⁾ This last is true if

$$V_h(1+V)^{-\frac{1}{2}}u(x+h, t)$$

remains in a bounded set of $L^2(0, t_0; H)$. But $\sqrt{1+V}u(x+h, t)$ remains in a bounded set of this space and therefore

$$V_h(1+V)^{-\frac{1}{2}}u(x+h, t) = \frac{V_h}{1+V} (1+V)^{\frac{1}{2}}u(x+h, t)$$

remains in a bounded set of $L^2(0, t_0; H)$. Hence w_h belongs to a bounded set of $L^2(0, t_0; K)$. We can extract $h_i \rightarrow 0$ such that $w_{h_i} \rightarrow w$ in $L^2(0, t_0; K)$ weakly and since $w_{h_i} \rightarrow \partial u / \partial x$ in the sense of distributions, one has $\partial u / \partial x = w \in L^2(0, t_0; K)$. Hence we have

⁽¹⁾ K' is the dual of K .

PROPOSITION 3.1. *If we assume that $V(x) \geq 0$ satisfies (3.4) and if we take $f \in H$ satisfying (3.3), then the solution of the Cauchy problem (3.1) verifies (2.2), i.e.,*

$$\frac{\partial u}{\partial x} \in L^2(0, t_0; K). \quad (3.5)$$

Remark 3.1. If we consider, instead of a Cauchy problem, a mixed problem, i.e. $x \in (a, b)$ a or b finite, with $u(x, t)$ verifying some *boundary conditions* for $x = a$ or b , then condition (3.5) will *never* be satisfied. Consequently the problem of finding a F.V. variational equation somewhat analogous to (2.12) for a mixed problem is open.

Remark 3.2. With some more hypotheses on $V(x)$ we can obtain by the same method, similar results about $\partial^2 u / \partial x^2$, $\partial^3 u / \partial x^3$, etc. For instance, if V verifies (3.4) and we assume further

$$V' \text{ is once continuously differentiable with } V''/(1+V) \text{ bounded,} \quad (3.6)$$

then if $f, f', f'' \in H$ we have

$$\frac{\partial^2 u}{\partial x^2} \in L^2(0, t_0; K). \quad (3.7)$$

We now prove

PROPOSITION 3.2. *We assume that $V(x) \geq 0$ and that $V(x)$ verifies (3.4) and (3.6). If f is given with*

$$(1+V)f \in H = L^2(R), \quad (3.8)$$

then (u being the solution of (3.1)), $(1+V)u \in L^2(0, t_0; K)$ and the function $t \rightarrow (1+V)u(\cdot, t)$ is continuous from $[0, t_0] \rightarrow H$.

Proof. Using (3.6) one sees that there exists a unique w in $L^2(-\infty, t_0; K)$ which is zero for $t < 0$ such that

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) w + V' \frac{\partial}{\partial x} \left(\frac{w}{1+V} \right) + \frac{1}{2} \frac{V''}{1+V} w = (1+V)f(x)\delta_t.$$

But, if we set $u^* = w/(1+V)$, then $u^* \in L^2(-\infty, t_0; K)$, $u^* = 0$ for $t < 0$ and

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) u^* = f(x)\delta_t.$$

Hence $u^* = u$ and the result follows.

4. Stability in V

We consider a family of functions $V_m(x) \geq 0$, with

$$V_m \rightarrow V \text{ uniformly on every compact set.}^{(1)} \tag{4.1}$$

We define a space K_m by

$$K_m = \left\{ u \mid (1 + V_m)^{\frac{1}{2}} u \in L^2(R), \frac{du}{dx} \in L^2(R) \right\}. \tag{4.2}$$

Let u_m (or respectively u) be the unique solution in $L^2(-\infty, t_0; K_m)$ (or $L^2(-\infty, t_0; K)$) which is zero for $t < 0$, of

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V_m(x) + \frac{\partial}{\partial t} - ixq(t) \right) u_m(x, t) = f(x)\delta_t, \tag{4.3}$$

or of
$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) u(x, t) = f(x)\delta_t. \tag{4.4}$$

PROPOSITION 4.1. *Let V_m, V be ≥ 0 continuous functions satisfying (4.1). Then*

$$u_m \rightarrow u, \quad V_m^{\frac{1}{2}} u_m \rightarrow V^{\frac{1}{2}} u, \quad \frac{\partial u_m}{\partial x} \rightarrow \frac{\partial u}{\partial x},$$

in $L^2(R \times (0, t_0))$ weakly.

Proof. If we set $u_m(x, t) = e^t w_m(x, t)$, then w_m verifies

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + (1 + V_m) + \frac{\partial}{\partial t} - ixq(t) \right) w_m = f(x)\delta_t.$$

This implies (multiplying by \bar{w}_m and integrating by parts) that

$$\int_{-\infty}^{\infty} \int_0^{t_0} \left(\frac{1}{2} \left| \frac{\partial w_m}{\partial x} \right|^2 + (1 + V_m(x)) |w_m|^2 \right) dx dt \leq \frac{1}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

and therefore

$$\int_{-\infty}^{\infty} \int_0^{t_0} \left(\frac{1}{2} \left| \frac{\partial u_m}{\partial x} \right|^2 + (1 + V_m) |u_m|^2 \right) dx dt \leq c. \tag{4.5}$$

Consequently we can extract a subsequence u_{m_i} such that

$$u_{m_i} \rightarrow u^*, \quad \frac{\partial u_{m_i}}{\partial x} \rightarrow \frac{\partial u^*}{\partial x}, \quad V_{m_i}^{\frac{1}{2}} u_{m_i} \rightarrow u^{**},$$

⁽¹⁾ Practically, the V_m will be polynomials but for the moment this hypothesis is useless.

weakly in the space $L^2(R \times (0, t_0))$. But using (4.1) one sees that $V_m^{\frac{1}{2}} u_m \rightarrow V^{\frac{1}{2}} u^*$ in the sense of distributions, hence $u^{**} = V^{\frac{1}{2}} u^*$ and therefore $u^* \in L^2(-\infty, t_0; K)$ and $u^* = 0$ for $t < 0$. Passing to the limit in (4.3) one sees that

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) u^*(x, t) = f(x) \delta_t.$$

Comparing this with (4.4) one concludes that $u^* = u$ so u^* does not depend on the subsequence and proposition 4.1 follows.

PROPOSITION 4.2. *Assume the functions V_m, V are twice continuously differentiable with*

$$V_m \rightarrow V, V'_m \rightarrow V', V''_m \rightarrow V'' \text{ uniformly on every compact set,} \tag{4.6}$$

$$\left. \begin{aligned} & (|V'_m| + |V''_m|)/(1 + V_m) \text{ bounded (for each } m) \text{ and } |V'_m|/(1 + V_m) \text{ bounded} \\ & \text{uniformly in } m, \end{aligned} \right\} \tag{4.7}$$

$$\text{and } (|V'| + |V''|)/(1 + V) \text{ bounded.} \tag{4.8}$$

Let f be given with $f \in H, (1 + x^2)(1 + V)^2 f \in H, (1 + x^2)(1 + V_m)^2 f \in H$, for every m and remain in a bounded set of that space.⁽¹⁾

Then

$$\left. \begin{aligned} & (1 + |V_m^{\frac{1}{2}}|)(1 + x^2)(1 + V_m)^2 u_m \rightarrow (1 + |V^{\frac{1}{2}}|)(1 + x^2)(1 + V)^2 u, \\ \text{and } & \frac{\partial}{\partial x} [(1 + x^2)(1 + V_m)^2 u_m] \rightarrow \frac{\partial}{\partial x} [(1 + x^2)(1 + V)^2 u] \end{aligned} \right\} \tag{4.9}$$

weakly in $L^2(R \times (0, t_0))$.

Proof. Set $y_m(x) = (1 + x^2)(1 + V_m(x))^2$ and consider the equation

$$-\frac{1}{2} y_m \frac{\partial^2}{\partial x^2} \left(\frac{w_m}{y_m} \right) + \left(V_m + \frac{\partial}{\partial t} - ixq \right) w_m = y_m f(x) \delta_t, \tag{4.10}$$

$w_m \in L^2(-\infty, t_0; K_m)$ and $w_m = 0$ for $t < 0$.⁽²⁾ Now if we set $w_m/y_m = u_m$, then $u_m \in L^2(-\infty, t_0; K_m)$ and $u_m = 0$ for $t < 0$, as well as satisfying (4.3), so that it coincides with the u_m previously introduced. In the same way, let w be the unique solution in $L^2(-\infty, t_0; K)$ which is 0 for $t < 0$ of

$$-\frac{1}{2} y \frac{\partial^2}{\partial x^2} \left(\frac{w}{y} \right) + \left(V + \frac{\partial}{\partial t} - ixq \right) w = y f(x) \delta_t,$$

⁽¹⁾ We are not looking for the most general hypotheses on f for which what follows is true, e.g. one can obviously replace $(1 + x^2)$ by $(1 + x^2)^{\frac{1}{2}}$. For our main result, section 6, it would be enough to take f infinitely differentiable with compact support.

⁽²⁾ The solution exists and is unique. This fact is a simple variant of the case $y_m = 1$.

where $y(x) = (1 + x^2)(1 + V)^2$. Then as above $w/y = u$. If we replace w_m by $e^{kt}w_m$, then (4.10) is replaced by the same equation but with $(\partial/\partial t + k)$ instead of $\partial/\partial t$. We obtain

$$\frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \int_0^{t_0} \frac{\partial}{\partial x} \left(\frac{w_m}{y_m} \right) \frac{\partial}{\partial x} (\overline{w_m} y_m) dx dt + \int_{-\infty}^{\infty} \int_0^{t_0} (V_m(x) + k) |w_m|^2 dx dt \leq \frac{1}{2} \int_{-\infty}^{\infty} |y_m f(x)|^2 dx,$$

hence

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_0^{t_0} \frac{\partial}{\partial x} |w_m|^2 dx dt + \int_{-\infty}^{\infty} \int_0^{t_0} \left(V_m(x) + k - \frac{y'_m(x)}{y_m(x)} \right)^2 |w_m|^2 dx dt \leq \text{constant}.$$

Choosing k large enough and using the fact that y'_m/y_m is bounded, it follows that

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_0^{t_0} \frac{\partial}{\partial x} |w_m|^2 dx dt + \int_{-\infty}^{\infty} \int_0^{t_0} V_m(x) |w_m|^2 dx dt \leq \text{constant},$$

and we complete the proof as in Proposition 4.1.

If we use Proposition 3.1, we obtain, always by the same method,

PROPOSITION 4.3. *Assume that V_m, V are once continuously differentiable, $V'/(1 + V), V'_m/(1 + V_m)$ are bounded, $V_m \rightarrow V, V'_m \rightarrow V'$ uniformly on every compact set. Suppose f is given with $f, f' \in H$. Then*

$$(1 + V_m)^{\frac{1}{2}} \frac{\partial u_m}{\partial x} \rightarrow (1 + V)^{\frac{1}{2}} \frac{\partial u}{\partial x}, \quad \frac{\partial^2 u_m}{\partial x^2} \rightarrow \frac{\partial^2 u}{\partial x^2},$$

weakly in $L^2(R \times (0, t_0))$.

PROPOSITION 4.4. *Assume the hypotheses of Propositions 4.2 and 4.3. Then $(1 + V_m)u_m(\cdot, t) \rightarrow (1 + V)u(\cdot, t)$ weakly in H for every fixed $t > 0$.*

Proof. It follows from (4.3) and (4.4) that in the open set $R \times (0, t_0)$

$$\frac{\partial u_m}{\partial t} = \frac{1}{2} \frac{\partial^2 u_m}{\partial x^2} - V_m u_m + i x q u_m,$$

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - V u + i x q u.$$

Using the results of Propositions 4.2 and 4.3, this implies $\partial u_m/\partial t \rightarrow \partial u/\partial t$ weakly in $L^2(0, t_0; L^2(R))$. Let $s > 0$ be fixed and let $\varphi(x) \in \mathcal{D}(R)$ (infinitely differentiable functions with compact support). Furthermore, let $\Phi(x, t) \in \mathcal{D}(R_x \times R_t)$ with $\Phi(x, s) = \varphi(x)$. Then,

$$\int_{-\infty}^{\infty} (1 + V_m)u_m(x, s)\overline{\varphi(x)}dx = \int_{-\infty}^{\infty} \int_0^s \frac{\partial}{\partial t}(1 + V_m)(u_m\overline{\Phi})dxdt + \int_{-\infty}^{\infty} (1 + V_m)f(x)\overline{\Phi(x, 0)}dx,$$

and since $(1 + V_m)u_m$ and $\partial u_m/\partial t$ converge respectively to $(1 + V)u$ and $\partial u/\partial t$ in $L^2(R \times (0, t_0))$ weakly we obtain

$$\int_{-\infty}^{\infty} (1 + V_m)u_m(x, s)\overline{\varphi(x)}dx \rightarrow \int_{-\infty}^{\infty} (1 + V)u(x, s)\overline{\varphi(x)}dx.$$

On the other hand,

$$\int_{-\infty}^{\infty} (1 + V_m)^2 |u_m(x, s)|^2 dx = \int_{-\infty}^{\infty} \int_0^s \frac{\partial}{\partial t} [(1 + V_m)^2 u_m \overline{u_m}] dxdt + \int_{-\infty}^{\infty} (1 + V_m)^2 |f(x)|^2 dx$$

and this is *bounded*. Therefore one can extract a subsequence u_{m_i} such that $(1 + V_{m_i})u_{m_i}(x, s) \rightarrow g_s$ weakly in $L^2(R)$. But since

$$\int_{-\infty}^{\infty} (1 + V_{m_i})u_{m_i}(x, s)\overline{\varphi(x)}dx \rightarrow \int_{-\infty}^{\infty} (1 + V)u(x, s)\overline{\varphi(x)}dx,$$

one has $g_s = (1 + V(x))u(x, s)$ and therefore $(1 + V_m)u_m(x, s) \rightarrow (1 + V)u(x, s)$ weakly in $L^2(R)$. Thus proposition 4.4 is proved.

We pass now to the stability of Volterra derivatives. By Proposition 9.2 (chapter I) we know that $\partial u_m/\partial q(\tau)$ (or respectively $\delta u/\delta q(\tau)$) is characterized when $f \in H_1$ as the unique element in $L^2(-\infty, t_0; K_m)$ (or $L^2(-\infty, t_0; K)$) which is zero for $t < 0$ (in fact for $t < \tau$) and is the solution of

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V_m(x) + \frac{\partial}{\partial t} - ixq(t) \right) \frac{\delta u_m(x, t; q)}{\delta q(\tau)} = ixu_m(x, \tau; q)\delta_t(\tau) \tag{4.11}$$

or of
$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) \frac{\delta u(x, t; q)}{\delta q(\tau)} = ixu(x, \tau; q)\delta_t(\tau). \tag{4.12}$$

When $f \in H_1$ one has by a variant of Proposition 4.4, $ixu_m(x, \tau; q) \rightarrow ixu(x, \tau; q)$ weakly in $L^2(R)$ and by Proposition 4.1⁽¹⁾ we get

PROPOSITION 4.5. *Under hypotheses of Proposition 4.1 and with $f \in H_1$,*

⁽¹⁾ Where $t=0$ is replaced by $t=\tau$ and in (4.3) f is replaced by f_m where $f_m \rightarrow f$ weakly in H . This does not change the result.

$$\frac{\delta u_m}{\delta q(\tau)} \rightarrow \frac{\delta u}{\delta q(\tau)}, \quad \sqrt{V_m} \frac{\delta u_m}{\delta q(\tau)} \rightarrow \sqrt{V} \frac{\delta u}{\delta q(\tau)}, \quad \frac{\partial \delta u_m}{\partial x \delta q(\tau)} \rightarrow \frac{\partial \delta u}{\partial x \delta q(\tau)} \tag{4.13}$$

weakly in $L^2(R \times (0, t_0))$.

Assume now that $f \in H$ and henceforth that V_m is a polynomial. By Proposition 9.3, (chapter I) we obtain

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V_m(x) + \frac{\partial}{\partial t} - ixq(t) \right) V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m = V'_m(x) u_m(x, s) \delta_t(s), \tag{4.14}$$

with $V'_m(-i\delta/\delta q(s))u_m \in L^2(-\infty, t_0; K)$ and $\equiv 0$ for $t < 0$ (in fact for $t < s$). We want now to pass to the limit. We make the fundamental hypothesis:

$$\left. \begin{array}{l} V \text{ is twice continuously differentiable, } V(x) \geq 0, \text{ and} \\ \frac{|V'(x)| + |V''(x)|}{1 + V(x)} \leq M < \infty. \\ \text{One can find a sequence of polynomials } V_m(x), V_m(x) \geq 0, \\ \text{such that} \\ V_m \rightarrow V, V'_m \rightarrow V', V''_m \rightarrow V'' \\ \text{uniformly on every compact set and with } |V'_m|/(1 + V_m) \leq c, \\ \text{where } c \text{ is a constant independent of } m. \end{array} \right\} \tag{4.15}$$

Under the hypotheses of Proposition 4.2 one has (cf. Proposition 4.4)

$$V'_m(x)u_m(x, s) = \frac{V'_m}{1 + V_m} (1 + V_m)u_m(x, s) \rightarrow V'u(x, s)$$

weakly in H , and this remark combined with (4.14) and the proof of Proposition 4.1 gives.

THEOREM 4.1. *Assume that V and V_m verify (4.15) and that the function f is given such that*

$$f \in H_\infty, \quad f', f'' \in H, \quad (1 + x^2)(1 + V)^2 f \in H, \quad (1 + x^2)(1 + V_m)^2 f \in H \tag{4.16}$$

and remains bounded in that space. Then, $V'_m(-i\delta/\delta q(s))u_m$ converges to a limit weakly in $L^2(R \times (0, t_0))$. By definition this limit is called $V'(-i\delta/\delta q(s))u$. One has

$$\left. \begin{aligned} V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m &\rightarrow V' \left(-i \frac{\delta}{\delta q(s)} \right) u, & V_m^\dagger V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m &\rightarrow V_m^\dagger V' \left(-i \frac{\delta}{\delta q(s)} \right) u, \\ \frac{\partial}{\partial x} \left(V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m \right) &\rightarrow \frac{\partial}{\partial x} \left(V' \left(-i \frac{\delta}{\delta q(s)} \right) u \right), & \text{all weakly in } L^2(\mathbb{R} \times (0, t_0)). \end{aligned} \right\} \quad (4.17)$$

Moreover, $V' \left(-i \frac{\delta}{\delta q(s)} \right) u$ is the unique element in $L^2(-\infty, t_0; K)$ which is $\equiv 0$ for $t < s$ and satisfies

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) V' \left(-i \frac{\delta}{\delta q(s)} \right) u = V'(x)u(x, s)\delta_t(s). \quad (4.18)$$

Also the limit in (4.17) behaves in s in such a way that

$$\int_0^t \min(\tau, s) V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m ds \rightarrow \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u ds$$

weakly in $L^2(0, t_0; H)$. (4.19)

Proof. There remains to prove only (4.19). Let us set

$$w_m^{(s)} = V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m, \quad w^{(s)} = V' \left(-i \frac{\delta}{\delta q(s)} \right) u.$$

By (4.14), we have

$$\int_{-\infty}^{\infty} \int_0^{t_0} \left(\frac{1}{2} \left| \frac{\partial w_m^{(s)}}{\partial x} \right|^2 + V_m(x) |w_m^{(s)}|^2 \right) dx dt \leq \frac{1}{2} \int_{-\infty}^{\infty} |V'_m(x)|^2 |u_m(x, s)|^2 dx.$$

But
$$\int_{-\infty}^{\infty} |V'_m(x)|^2 |u_m(x, s)|^2 dx \leq c_1 \int_{-\infty}^{\infty} (1 + V_m(x))^2 |u_m(x, s)|^2 dx$$

and according to the proof of Proposition 4.4, this is $\leq c_2$. Therefore we obtain in particular $s \rightarrow w_m^{(s)} \in L^\infty(0, t_0; L^2(0, t_0; H))$ (bounded measurable functions in $(0, t_0)$ with values in $L^2(0, t_0; H)$), and $w_m^{(s)}$ remains in a bounded set of this space. As a consequence, we can extract a subsequence m_i such that $w_{m_i}^{(s)}$ converges weakly in $L^\infty(0, t_0; L^2(0, t_0; H))$. Necessarily this limit is $w^{(s)}$, hence $w_{m_i}^{(s)} \rightarrow w^{(s)}$ weakly in $L^\infty(0, t_0; L^2(0, t_0; H))$ and this implies (4.19).

By using the results of section 13, chapter I, we are now going to prove that essentially Theorem 4.1 remains true under similar hypotheses but with V assumed only *once* continuously differentiable instead of twice.

We need first some more propositions.

PROPOSITION 4.6. Assume that q is once continuously differentiable with $q(0) = 0$ and that $V(x) \geq c|x|^2, c > 0$. Let f be given in K with $-(1/2) f'' + V(x)f \in H$. Then the solution u of (4.4) verifies

$$\frac{\partial u}{\partial t} \in L^2(0, t_0; K). \tag{4.20}$$

Proof. Using the notations of (13.2) (section 13, chapter I) let us consider

$$u_1 = X^{-1}(e^{-\lambda t} Xf) \quad (\text{and } = 0 \text{ for } t < 0).$$

By hypothesis $f \in D(\wedge)$ so that $\lambda Xf \in h$ and therefore

$$\frac{\partial u_1}{\partial t} \in L^2(0, t_0; K). \tag{4.21}$$

Since $u_1(1 + V)^{\frac{1}{2}}$ belongs to $L^2(0, t_0; H)$ and $V(x) \geq c|x|^2$, we have

$$xu_1(1 + V)^{-\frac{1}{2}} \in L^2(0, t_0; H). \tag{4.22}$$

Now if we set $w = u - u_1$, then w belongs to $L^2(-\infty, t_0; K)$ and satisfies

$$\frac{\partial w}{\partial t} - \frac{1}{2} \frac{\partial^2 w}{\partial x^2} + V(x)w - ixq(t)w = ixq(t)u_1. \tag{4.23}$$

But since q is once continuously differentiable and $q(0) = 0$, then setting $g(x, t) = ixq(t)u_1(x, t)$ we have g and $\partial g/\partial t$ belonging to $L^2(-\infty, t_0; K')$ where K' is the dual of K . Therefore by [17, chapter V, Th. 3.1], we have

$$\frac{\partial w}{\partial t} \in L^2(0, t_0; K)$$

and this combined with (4.21) gives the result.

PROPOSITION 4.7. We assume that q and V are given as in Proposition 4.6 and that $f \in \mathcal{D}(\mathbb{R})$. Let V_m be a sequence of polynomials with $V_m(x) \geq c_m|x|^2, c_m \geq c_0 > 0$ and $V_m \rightarrow V$ on every compact set. Let u_m be the solution of (4.3). We have

$$u_m(x, s)(1 + V_m(x))^{\frac{1}{2}} \rightarrow u(x, s)(1 + V(x))^{\frac{1}{2}} \text{ weakly in } L^2(\mathbb{R}).$$

Proof. Let $\varphi_m = \partial u_m/\partial t$ and we have

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} + V_m(x) - ixq \right) \varphi_m = ixq' u_m + \left(\frac{1}{2} f'' - V_m f \right) \delta_t.$$

It follows that (where $Q_{t_0} = R \times (0, t_0)$)

$$\begin{aligned} \frac{1}{2} \int_R |\varphi_m(x, t_0)|^2 dx + \iint_{Q_{t_0}} \left(\frac{1}{2} \left| \frac{\partial \varphi_m}{\partial x} \right|^2 + V_m(x) |\varphi_m|^2 \right) dx dt \\ = \frac{1}{2} \int_R \left| \frac{1}{2} f'' - V_m f \right|^2 dx + \operatorname{Re} \left(i \iint_{Q_{t_0}} x q'(t) u_m \overline{\varphi_m} dx dt \right). \end{aligned} \quad (4.24)$$

But

$$\iint_{Q_{t_0}} x q' u_m \overline{\varphi_m} dx dt = \iint_{Q_{t_0}} \left(\frac{x}{1 + V_m(x)} \right) q'(t) (u_m (1 + V_m)^{\frac{1}{2}}) (\varphi_m (1 + V_m)^{\frac{1}{2}}) dx dt,$$

and since $u_m (1 + V_m)^{\frac{1}{2}}$ is bounded in $L^2(Q_{t_0})$, it follows that, in particular, $(1 + V_m)^{\frac{1}{2}} \varphi_m$ is bounded in $L^2(Q_{t_0})$ and therefore

$$(1 + V_m)^{\frac{1}{2}} \frac{\partial u_m}{\partial t} \rightarrow (1 + V)^{\frac{1}{2}} \frac{\partial u}{\partial t} \text{ weakly in } L^2(Q_{t_0}).$$

From this and Proposition 4.1 we infer (4.24).

THEOREM 4.2. *Assume the hypotheses of Proposition 4.7 and moreover that $|V'_m|/(1 + V_m) \leq \text{constant}$ independent of m . Then*

$$V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m \rightarrow V' \left(-i \frac{\delta}{\delta q(s)} \right) u \text{ weakly in } L^2(0, t_0; H) \quad (4.25)$$

and

$$\left. \begin{aligned} \int_0^t \min(\tau, s) V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m ds \rightarrow \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u ds \\ \text{weakly in } L^2(0, t_0; H). \end{aligned} \right\} \quad (4.26)$$

Proof. Set $w_m^{(s)} = V'_m(-i \delta / \delta q(s)) u_m$. From section 13, chapter I, we know that $w_m^{(s)}$ is the unique solution in $L^2(0, t_0; H)$ which is 0 for $t < s$ of

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} + V_m(x) - i x q(t) \right) w_m^{(s)} = V'_m(x) u_m(x, s) \delta_t(s).$$

Let $W_m^{(s)}$ be the solution in $L^2(-\infty, t_0; H)$ which is zero for $t < 0$ of

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} + V_m(x) \right) W_m^{(s)} = V'_m(x) u_m(x, s) \delta_t(s)$$

(cf. chapter I, section 13). We have

$$\iint_{Q_{t_0}} |W_m^{(s)}|^2 dx dt \leq c \int_R |V'_m u_m(x, s)|^2 \frac{1}{1 + V_m} dx = c \int_R \left(\frac{V'_m}{1 + V_m} \right)^2 (1 + V_m) |u_m(x, s)|^2 dx,$$

where c is independent of m . Thus $W_m^{(s)}$ remains in a bounded set of $L^2(Q_{t_0})$. Using the proof of Theorem 13.3, chapter I, we see that $w_m(s)$ belongs to a bounded set of $L^2(Q_{t_0})$ when m and s vary. We complete the proof as in Theorem 4.1

5. The parabolic case (II)

THEOREM 5.1 *Under the hypotheses of Theorem 4.1 or of Theorem 4.2, the solution u of the Cauchy problem*

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \frac{\partial}{\partial t} - ixq(t) \right) u(x, t; q) = f(x) \delta_t, \tag{5.1}$$

$$u(\cdot, t; q) \in L^2(-\infty, t_0; K) \text{ and } = 0 \text{ for } t < 0, \tag{5.2}$$

satisfies the Volterra variational equation

$$\begin{aligned} \frac{\delta u(x, t; q)}{\delta q(\tau)} + i\tau \frac{\partial u(x, t; q)}{\partial x} + \left(\int_0^t \min(\tau, s) q(s) ds \right) u(x, t; q) \\ + i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds = \Phi_f(x, t; q), \end{aligned} \tag{5.3}$$

where Φ_f is defined as in (2.3) and (2.4).

Proof. Assuming first that we are under hypotheses of Theorem 4.1, we consider V_m a sequence of non-negative polynomials verifying the conditions which appear in (4.15). Let u_m be the corresponding solution of the Cauchy problem. Now V_m verifies (cf. Theorem 2.2) the Volterra variational equation analogous to (5.3). By Propositions 4.1, 4.3, 4.5 and Theorem 4.1 we can pass to the limit in the left-hand side. With obvious notations one checks by the same method as in Proposition 4.1 that $\Phi_f^{(m)}(x, t; q) \rightarrow \Phi_f(x, t; q)$ weakly in $L^2(R \times (0, t_0))$ and hence the theorem follows. If we assume the hypotheses of Theorem 4.2, we see that the same proof obtains.

6. The parabolic case (III): The F.V. variational equation for the kernel

We consider the mapping $f \rightarrow u$. It is, *in particular*, a mapping from $\mathcal{D}(R)$ (space of functions infinitely differentiable on R with compact support) into $\mathcal{D}'(R \times (-\infty, t_0))$. This mapping is defined, by the Schwartz kernel theorem (see Schwartz [23]), by a kernel $Q(x, y, t; q)$ which is a *distribution* on $R_x \times R_y \times (-\infty, t_0)$ and is $\equiv 0$ for $t < 0$ and is, moreover, such that

$$u(x, t; q) = \int_{-\infty}^{\infty} Q(x, y, t; q) f(y) dy. \quad (6.1)$$

The distribution $Q(x, y, t; q)$ is the *fundamental* solution of

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t),$$

i.e.
$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t} - ixq(t) \right) Q(x, y, t; q) = \delta_x(y) \otimes \delta_t(0) \quad (6.2)$$

and, for every y and t , $x \rightarrow Q(x, y, t; q)$ satisfies growth conditions *equivalent to the fact* that the operator (6.1) maps $L^2(R)$ into $L^2(0, t_0; K)$.⁽¹⁾ Using the kernel $Q(x, y, t; q)$, one can write

$$\Phi_f(x, t; q) = \int_{-\infty}^{\infty} Q(x, y, t; q) iyf(y) dy. \quad (6.3)$$

Admit, for the time being, the

LEMMA 6.1. *Let X and Y be two open sets in R^n and R^m respectively. Denote by $\mathcal{D}(Y)$ (or $\mathcal{D}'(X)$) the space of functions infinitely differentiable in Y with compact support (or of distributions on X). Let $\mathcal{L}(\mathcal{D}(Y), \mathcal{D}'(X))$ be the space of continuous linear mappings from $\mathcal{D}(Y)$ into $\mathcal{D}'(X)$ provided with the topology of uniform convergence on bounded sets of $\mathcal{D}(Y)$. Let T be an open interval of R_t . Let $q \rightarrow M(q)$ be a mapping from $\mathcal{D}(T)$ into $\mathcal{L}(\mathcal{D}(Y), \mathcal{D}'(X))$ which verifies:*

$$\left. \begin{array}{l} \text{For every } \varphi \in \mathcal{D}(Y), q \rightarrow M(q)\varphi \text{ is continuous from } \mathcal{D}(T) \text{ into} \\ \mathcal{D}'(X) \text{ uniformly for } \varphi \text{ belonging to a bounded set of } \mathcal{D}(Y). \end{array} \right\} \quad (6.4)$$

⁽¹⁾ In the present case, one has in particular $Q(x, y, t; q) \rightarrow 0$ as $x \rightarrow \pm \infty$.

For every $\varphi \in \mathcal{D}(Y)$ and every $q, q_1 \in \mathcal{D}(T)$, the function $\xi \rightarrow M(q + \xi q_1)\varphi$ is entire analytic from C into $\mathcal{D}'(X)$. (6.5)

Let $K(x, y; q)$ be the kernel (in the sense of Schwartz) of $M(q)$, i.e.

$$M(q)\varphi = \int_Y K(x, y; q)\varphi(y)dy \text{ for } \varphi \in \mathcal{D}(Y). \tag{6.6}$$

Then

$q \rightarrow K(x, y; q)$ is continuous from $\mathcal{D}(T) \rightarrow \mathcal{D}'(X \times Y)$,
 $\xi \rightarrow K(x, y; q + \xi q_1)$ is entire analytic from $C \rightarrow \mathcal{D}'(X \times Y)$ (6.7)

and

$\frac{\delta K(x, y; q)}{\delta q(\tau)} \in \mathcal{D}'(T; \mathcal{D}'(X \times Y)) = \mathcal{D}'(T \times X \times Y)$
satisfies $\frac{\delta M(q)}{\delta q(\tau)} \varphi = \int_Y \frac{\delta K(x, y; q)}{\delta q(\tau)} \varphi(y)dy.$ (6.8)

We apply this lemma to the kernel $Q(x, y, t; q)^{(1)}$ of the mapping $f \rightarrow u$. We assume that V satisfies (4.15). Then

$$V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m(x, t; q) \rightarrow V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) \tag{6.9}$$

weakly in $L^2(R \times (0, t_0))$ (in particular, cf. section 4).

Let $Q_m(x, y, t; q)$ be the kernel of $f \rightarrow u_m$. By (6.8)

$$V'_m \left(-i \frac{\delta}{\delta q(s)} \right) u_m(x, t; q) = \int_{-\infty}^{\infty} V'_m \left(-i \frac{\delta}{\delta q(s)} \right) Q_m(x, y, t; q) f(y) dy \tag{6.10}$$

and, on the other hand, the mapping

$$f \rightarrow V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) \tag{6.11}$$

has a kernel (applying once more Schwartz's Kernel Theorem). But the kernels depend continuously on the mappings (stability of the kernels) and therefore the kernel

⁽¹⁾ $\{x, t\}$ plays the role of x in the lemma.

$$V'_m\left(-i\frac{\delta}{\delta q(s)}\right)Q_m(x, y, t; q)$$

has a limit (for every fixed s) in $\mathcal{D}'(R_x \times R_y \times (-\infty, t_0))$. This limit is the kernel of the mapping (6.11). By definition this limit kernel is denoted by

$$V'\left(-i\frac{\delta}{\delta q(s)}\right)Q(x, y, t; q).$$

We notice also that the kernel of the mapping

$$f \rightarrow i \int_0^t \min(\tau, s) V'\left(-i\frac{\delta}{\delta q(s)}\right)u(x, t; q)ds$$

is
$$i \int_0^t \min(\tau, s) V'\left(-i\frac{\delta}{\delta q(s)}\right)Q(x, y, t; q)ds.$$

From these remarks and from Theorem 5.1 we deduce

THEOREM 6.1. *Assume that V and V_m verify (4.15). Let $Q(x, y, t; q)$ be the fundamental solution of the operator $-1/2 \partial^2/\partial x^2 + V(x) + \partial/\partial t - ixq(t)$ (cf. (6.1), (6.2)). One defines in this way a functional $q \rightarrow Q(x, y, t; q)$ from $C(0, t_0) \rightarrow \mathcal{D}'(R_x \times R_y \times (-\infty, t_0))$. This functional verifies the Volterra variational equation*

$$\begin{aligned} \frac{\delta Q(x, y, t; q)}{\delta q(\tau)} + i\tau \frac{\partial Q(x, y, t; q)}{\partial x} + \left(\int_0^t \min(\tau, s)q(s)ds\right)Q(x, y, t; q) \\ + i \int_0^t \min(\tau, s) V'\left(-i\frac{\delta}{\delta q(s)}\right)Q(x, y, t; q)ds = iyQ(x, y, t; q) \text{ for } t > \tau. \end{aligned} \tag{6.12}$$

We note that the same result holds under the hypotheses of Theorem 4.2.

Remark 6.1. The interest of equation (6.12) is that now all the expressions which appear in (6.12) are known once V is given.

Remark 6.2. We leave open the problem of determining the best conditions on V for which (6.12) holds. Now there remains only to prove the lemma.

Proof of Lemma 6.1. First of all (6.7) follows from general properties of vector-valued distributions (cf. Schwartz [23]). For (6.8), we calculate

$$\frac{d}{d\xi} M(q + \xi q_1) \varphi = \frac{d}{d\xi} \int_Y K(x, y; q + \xi q_1) \varphi(y) dy = \int_Y \frac{d}{d\xi} K(x, y; q + \xi q_1) \varphi(y) dy$$

and taking $\xi = 0$ and applying the definitions of Chapter I, section 1, we have the result.

7. The Schrödinger case (I)

We consider the situation in section 10 of chapter I. To begin with assume

$$V(x) \geq 0 \text{ is a polynomial of degree } m. \tag{7.1}$$

We consider $u(x, t; q)$, solution of

$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t) \right) u = f(x) \delta_t, \tag{7.2}$$

where $u \in L^2(-\infty, t_0; K)$ and $u = 0$ for $t < 0$. (7.3)

This solution exists and is unique if

$$q \in C^1(0, t_0), \quad q(0) = 0 \tag{7.4}$$

and f and Tf ($= (1 + V)^{-\frac{1}{2}}(-1/2 f'' + V(x)f)$) belong to $H = L^2(R)$. (7.5)

The space K is unchanged, i.e.

$$K = \left\{ u \mid (1 + V)^{\frac{1}{2}} u \in H, \frac{du}{dx} \in H \right\}.$$

The same proof as in Proposition 3.1 gives

PROPOSITION 7.1. *If V is once continuously differentiable with*

$$\frac{|V'(x)|}{1 + V(x)} \leq M < \infty, \tag{7.6}$$

and if f verifies (7.5) and

$$f', Tf' \in H, \tag{7.7}$$

then the solution u of 7.2 and 7.3 satisfies

$$\frac{\partial u}{\partial x} \in L^2(0, t_0; K). \tag{7.8}$$

We shall now prove

THEOREM 7.1. *Under the hypotheses of Proposition 7.1 and with $f \in H_m$ (cf. (10.9), chapter I), the solution u of the Cauchy problem (7.2), (7.3), satisfies the Volterra variational equation*

$$\begin{aligned} \frac{\partial u(x, t; q)}{\partial q(\tau)} - \tau \frac{\partial u(x, t; q)}{\partial x} + i \left(\int_0^t \min(\tau, s) q(s) ds \right) u(x, t; q) \\ - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds = \Phi_f(x, t; q) \end{aligned} \quad (7.9)$$

for $t > \tau$ and where Φ_f is the solution in $L^2(-\infty, t_0; K)$ which is $\equiv 0$ for $t < 0$ of

$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t) \right) \Phi_f(x, t; q) = ix f(x) \delta_t. \quad (7.10)$$

Proof. The proof is along the same lines as that of Theorem 2.1. We have to check that

$$\begin{cases} w = \tau \frac{\partial u}{\partial x} - i \left(\int_0^t \min(\tau, s) q(s) ds \right) u(x, t; q) + i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds + \Phi_f \\ \quad \text{for } t > \tau, \\ w = 0 \text{ for } t < \tau, \end{cases}$$

satisfies $w \in L^2(0, t_0; K)$ (7.11)

$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t) \right) w = ixu(x, t) \delta_t(\tau). \quad (7.12)$$

Condition (7.11) follows from (7.8) and the results of section 10, chapter I. In order to prove (7.12) we first check that this relation holds for $t > \tau$. Set

$$\Lambda = -\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t}. \quad (7.13)$$

From $(\Lambda - ixq(t))u = 0$, we deduce

$$(\Lambda - ixq) \left(\tau \frac{\partial u}{\partial x} \right) = -i\tau V'(x)u + i\tau qu. \quad (7.14)$$

Now $(\Lambda - ixq) \left(\left(-i \int_0^t \min(\tau, s) q(s) ds \right) u \right) = -i\tau q(t)u,$

and

$$(\Lambda - ixq) \left(i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds = i\tau V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) \Big|_{s=t} = i\tau V'(x)u. \right.$$

Thus since by definition $(\Lambda - ixq)\Phi_f = 0$, these relations combined with (7.14) prove that $(\Lambda - ixq)w = 0$ for $t > \tau$. Consequently in order to prove (7.12) there remains only to check that $w = w(x, t, \tau; q) \rightarrow ix f(x)$ as $t \rightarrow \tau$ or, replacing τ by t , that

$$t \frac{\partial u}{\partial x} - i \left(\int_0^t sq(s) ds \right) u(x, t; q) + i \int_0^t s V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds + \Phi_f(x, t; q) = ixu(x, t; q). \tag{7.15}$$

In order to verify (7.15) we note that both members belong to $L^2(-\infty, t_0; K)$, are zero for $t < 0$ and are equal for $t = 0$ (where both sides equal $ix f(x)$). Now applying $(\Lambda - ixq)$ to both members gives the same result, for

$$\begin{aligned} (\Lambda - ixq) \left(t \frac{\partial u}{\partial x} - ixu \right) &= itq(t)u - itV'(x)u, \\ (\Lambda - ixq) \left(-i \left(\int_0^t sq(s) ds \right) u \right) &= -itq(t)u, \\ (\Lambda - ixq) \left(i \int_0^t s V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds \right) &= itV' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) \Big|_{s=t} \\ &= itV'(x)u, \end{aligned}$$

and hence the conclusion follows.

8. The Schrödinger case (II)

We can pass to the limit in Theorem 7.1 along the same lines as in sections 4 and 5. We obtain

THEOREM 8.1. *Assume that V_m, V verify (4.15). Let f be given with*

$$\left. \begin{aligned} f &\in H_m^{(1)} \\ f', (TF)', f'', (Tf)'' &\in H, \\ (1+x^2)(1+V)^2 f, T((1+x^2)(1+V)^2 f), (1+x^2)(1+V_m)^2 f, T((1+x^2)(1+V_m)^2 f) \\ \text{all in } H \text{ and remaining bounded in } H. \end{aligned} \right\} \tag{8.1}^{(2)}$$

⁽¹⁾ Cf. (10.9), chapter I.

⁽²⁾ For our purposes here we can take $f \in \mathcal{D}(R)$ so that these hypotheses are harmless.

Let u be the solution of the Cauchy problem

$$u \in L^2(-\infty, t_0; K) \text{ with } u = 0 \text{ for } t < 0 \quad (8.2)$$

and
$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t)\right)u = f(x)\delta_t. \quad (8.3)$$

Then $V'_m(-i\delta/\delta q(s))u_m$ converges to a limit weakly in $L^2(R \times (0, t_0))$. This limit is called $V'(-i\delta/\delta q(s))u$. The functional u verifies the Volterra variational equation,

$$\begin{aligned} \frac{\delta u(x, t; q)}{\delta q(\tau)} - \tau \frac{\partial u}{\partial x}(x, t; q) + i \left(\int_0^t \min(\tau, s) q(s) ds \right) u(x, t; q) \\ - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u(x, t; q) ds = \Phi_f(x, t; q) \end{aligned} \quad (8.4)$$

for $t > \tau$, where Φ_f is the solution in $L^2(-\infty, t_0; K)$, which is zero for $t < 0$, of

$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t)\right)\Phi_f = ix f(x)\delta_t. \quad (8.5)$$

We consider now the kernel $Q(x, y, t; q)$ of the mapping $f \rightarrow u$, i.e.

$$u(x, t; q) = \int_{-\infty}^{\infty} Q(x, y, t; q) f(y) dy, \quad (8.6)$$

where $Q(x, y, t; q)$ is the fundamental solution of the operator

$$-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t);$$

$$Q(x, y, t; q) \in \mathcal{D}'(R_x \times R_y \times (-\infty, t_0)), \quad Q = 0 \text{ for } t < 0, \quad (8.7)$$

and
$$\left(-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t)\right)Q(x, y, t; q) = \delta_x(y) \otimes \delta_t(0) \quad (8.8)$$

with growth conditions as $x \rightarrow \pm \infty$. These latter growth conditions are equivalent to the fact that $f \rightarrow u$, given by (8.6), maps the space of functions f for which f and Tf belong to $L^2(R)$ into $L^2(0, t_0; K)$. By the same considerations as in section 6, we obtain

THEOREM 8.2. *Assume that V and V_m satisfy (4.15). Let $Q(x, y, t; q)$ be the fundamental solution of the operator*

$$-\frac{i}{2} \frac{\partial^2}{\partial x^2} + iV(x) + \frac{\partial}{\partial t} - ixq(t)$$

(cf. (8.7), (8.8)). In this way one defines a functional $q \rightarrow Q(x, y, t; q)$ from the space of $q \in C^1(0, t_0)$ with $q(0) = 0$ into $\mathcal{D}'(R_x \times R_y \times (-\infty, t_0))$.¹ This functional satisfies the Volterra variational equation,

$$\begin{aligned} \frac{\delta Q(x, y, t; q)}{\delta q(\tau)} - \tau \frac{\partial Q(x, y, t; q)}{\partial x} + i \left(\int_0^t \min(\tau, s) q(s) ds \right) Q(x, y, t; q) \\ - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) Q(x, y, t; q) ds \\ = iyQ(x, y, t; q) \text{ for } t > \tau. \end{aligned} \tag{8.9}$$

Remark 8.1. As in Remark 6.1, we notice that all the expressions appearing in equation (8.9) are known, once V is given.

9. The multi-dimensional case

We consider now the situation in chapter I, section 12, i.e., we consider u , the solution of

$$-\frac{1}{2} \Delta_x u + V(x)u + \frac{\partial u}{\partial t} - i \sum_{j=1}^n x_j q_j(t)u = f(x)\delta_t, \tag{9.1}$$

where

$$\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2},$$

with

$$u \in L^2(-\infty, t_0; K), \quad u = 0 \text{ for } t < 0 \tag{9.2}$$

and

$$K = \left\{ u \mid (1 + V)^{\frac{1}{2}} u \in L^2(R^n), \frac{\partial u}{\partial x_j} \in L^2(R^n) \right\}, \quad H = L^2(R^n).$$

Now, if $f \in H$, u exists and is unique. By the same methods as in sections 2 and 3, we prove

THEOREM 9.1. *Asssume V is a polynomial ≥ 0 of degree m . Assume f is given with*

$$(1 + x^2)^m f \in H, \quad \frac{\partial f}{\partial x_j} \in H, \quad j = 1, 2, \dots, n. \tag{9.3}$$

Then, for $j = 1, 2, \dots, n$, one has

$$\begin{aligned} \frac{\delta u(x, t; q)}{\delta q_j(\tau)} + i\tau \frac{\partial u(x, t; q)}{\partial x_j} + \left(\int_0^t \min(\tau, s) q_j(s) ds \right) u(x, t; q) \\ + i \int_0^t \min(\tau, s) \left(\frac{\partial V}{\partial x_j} \left(-i \frac{\delta}{\delta q_j(s)} \right) \right) u(x, t; q) ds = \Phi_f^{(j)} \end{aligned} \tag{9.4}$$

for $t > \tau$ and where $\Phi_f^{(j)}$ is the solution in $L^2(-\infty, t_0; K)$, which is zero for $t < 0$, of

$$\left(-\frac{1}{2} \Delta_x + V(x) + \frac{\partial}{\partial t} - i \sum_{j=1}^n x_j q_j(t) \right) \Phi_f^{(j)} = i x_j f(x) \delta_t. \tag{9.5}$$

One can pass to the limit along the same lines as in sections 4 and 5. Let us denote by $Q(x, y, t; q_1, \dots, q_n) = Q(x, y, t; q)$ the kernel of the mapping $f \rightarrow u$, i.e.

$$u(x, t; q_1, q_2, \dots, q_n) = \int_{R^n} Q(x, y, t; q_1, q_2, \dots, q_n) f(y) dy. \tag{9.6}$$

We obtain the result

THEOREM 9.2. *Assume that $V \geq 0$ is given⁽¹⁾ in C^2 , that*

$$\frac{\left| \frac{\partial V}{\partial x_j} \right| + \left| \frac{\partial^2 V}{\partial x_j \partial x_k} \right|}{1 + V(x)} \leq M < \infty,$$

and that there exists a sequence of polynomials $V_m(x)$ such that $V_m \rightarrow V$ in C^2 , with

$$\frac{\left| \frac{\partial V_m(x)}{\partial x_j} \right|}{1 + V_m(x)} \leq \text{constant}.$$

Then, for $j = 1, 2, \dots, n$,

$$\begin{aligned} \frac{\delta Q(x, y, t; q)}{\delta q_j(\tau)} + i\tau \frac{\partial Q(x, y, t; q)}{\partial x_j} + \left(\int_0^t \min(\tau, s) q_j(s) ds \right) Q(x, y, t; q) \\ + i \int_0^t \min(\tau, s) \left(\frac{\partial V}{\partial x_j} \left(-i \frac{\delta}{\delta q_j(s)} \right) \right) Q(x, y, t; q) ds \\ = i y_j Q(x, y, t; q) \text{ for } t > \tau. \end{aligned} \tag{9.7}$$

The same remarks are valid in the multi-dimensional Schrödinger case.

⁽¹⁾ C^2 is the space of twice continuously differentiable functions. In that space $g_n \rightarrow g$ if

$$g_n \rightarrow g, \partial g_n / \partial x_j \rightarrow \partial g / \partial x_j, \partial^2 g_n / \partial x_i \partial x_j \rightarrow \partial^2 g / \partial x_i \partial x_j$$

uniformly on every compact set.

10. Case of example 3, chapter I

We do not here make a systematic study, along the same lines as above, for the general situation of section 11, chapter I. We only want to mention, however, the following result.

Let $u(x, t; q)$ be the solution in $L^2(-\infty, t_0; K)$ which is $\equiv 0$ for $t < 0$, of

$$\left((-1)^n \frac{1}{2n} D_x^{2n} + \frac{\partial}{\partial t} - ixq(t) \right) u = f(x)\delta_t, \tag{10.1}$$

where

$$K = \left\{ u \mid u, \frac{d^n u}{dx^n} \in H \right\}, \quad H = L^2(R), \quad f \in H.$$

Now, assuming that

$$f, f' \in H, \tag{10.2}$$

we have the F.V. variational equation

$$\frac{\delta u(x, t; q)}{\delta q(\tau)} + \left(\int_0^\tau \left(i \frac{\partial}{\partial x} + \int_\sigma^t q(u) du \right)^{(2n-1)} d\sigma \right) u = \Phi_f, \quad t > \tau, \tag{10.3}$$

where Φ_f is the solution in $L^2(-\infty, t_0; K)$ which is $\equiv 0$ for $t < 0$ of

$$\left(\frac{(-1)^n}{2n} \frac{\partial^{2n}}{\partial x^{2n}} + \frac{\partial}{\partial t} - ixq(t) \right) \Phi_f = ix f(x) \delta_t. \tag{10.4}$$

If

$$u(x, t) = \int_{-\infty}^\infty Q(x, y, t) f(y) dy, \tag{10.5}$$

then one obtains the desired F.V. variational equation,

$$\frac{\delta Q(x, y, t; q)}{\delta q(t)} + \left(\int_0^\tau \left(i \frac{\partial}{\partial x} + \int_\sigma^t q(u) du \right)^{(2n-1)} d\sigma \right) Q(x, y, t; q) = iy Q(x, y, t; q) \tag{10.6}$$

for $t > \tau$.

11. Lemmas

We recall the following Lemma (cf. Cameron [2]):

LEMMA 11.1. *Let $M(z)$ and $N(z)$ be two functionals, $z \in C(0, t)$, $z(0) = 0$. We assume $z \rightarrow M(z)$ is continuous with values in a vector topological space E (cf. section 1, chapter I), and admits a Volterra derivative $\delta M(z) / \delta z(s)$ which depends continuously on s in E .*

Also we assume $z \rightarrow N(z)$ is continuous with values in $\mathcal{L}(E; E)$,⁽¹⁾ and admits a Volterra derivative $\delta N(z)/\delta z(s)$ which depends continuously on s in $\mathcal{L}(E; E)$. Then

$$\int_0^t \min(\tau, s) \left[E_z^w \left\{ M \frac{\delta N}{\delta z(s)} \right\} + E_z^w \left\{ N \frac{\delta M}{\delta z(s)} \right\} \right] ds = E_z^w \{ z(\tau) MN \}. \tag{11.11}$$

LEMMA 11.2. Assume that for every continuous function $q \in C(0, t)$:

$$E_z^w \left\{ \exp \left(i \int_0^t z(\sigma) q(\sigma) d\sigma \right) iR(x, t; z) \right\} = 0, \quad t > \tau,$$

where $R(x, t; z) = \tau \frac{\partial \mathcal{G}}{\partial x} + \left(\int_0^t \min(\tau, s) V'(z(s)) ds \right) \mathcal{G} + \int_0^t \min(\tau, s) \frac{\delta \mathcal{G}(x, t; z)}{\delta z(s)} ds, \quad t > \tau.$

Then

$$R = 0. \tag{2}$$

Proof.

More precisely we now prove:

Let $F[z]$ be a functional defined on $C(0, t)$ which is bounded and continuous in the uniform topology. If for all $q(\sigma) \in C(0, t)$ we have

$$E_z^w \left\{ \exp \left(i \int_0^t q(\sigma) z(\sigma) d\sigma \right) F[z] \right\} = 0 \tag{11.1}$$

then $F[z] = 0$ for all $z(\sigma) \in C(0, t)$.

The proof of this fact makes use of certain techniques and calculations used in Cameron and Donsker [1]. We repeat some of these here so that the present proof will be self contained. In the proof we postpone to the end some of the calculations so that the simple idea of the proof will be clear.

Let λ and μ be positive constants and let $x(\sigma)$ be a fixed function defined on $0 \leq \sigma \leq t$ with $x(0) = 0$ and satisfying a Lip- α condition for some $\alpha > 0$. From assumption (11.1) we have for all $q(\sigma) \in C(0, t)$

$$\exp \left\{ -i\lambda\mu \int_0^t x(\sigma) q(\sigma) d\sigma \right\} E_z^w \left\{ \exp \left\{ i\lambda\mu \int_0^t q(\sigma) z(\sigma) d\sigma \right\} F[z] \right\} = 0 \tag{11.2}$$

Now let $R(\sigma, \xi; -\mu^2)$ be the resolvent kernel on $[0, t]$ of $\min(\sigma, \xi)$, i.e.,

⁽¹⁾ Continuous linear mappings from E to E .

⁽²⁾ The authors wish to thank G. E. Baxter for pointing out an error in an earlier version of Lemma 11.2 which is corrected in the present proof.

$$R(\sigma, \xi; -\mu^2) = \begin{cases} -\frac{\cosh \mu(t - \xi) \sinh \mu\sigma}{\mu \cosh \mu t} & \sigma \leq \xi \\ -\frac{\cosh \mu(t - \sigma) \sinh \mu\xi}{\mu \cosh \mu t} & \sigma \geq \xi. \end{cases}$$

The eigenvalues of $-R(\sigma, \xi; -\mu^2)$ are all positive (being $(k + \frac{1}{2})^2 \pi^2 / t^2 + \mu^2$, $k = 1, 2, 3, \dots$), and therefore $p_\mu(\sigma, \xi) \equiv -R(\sigma, \xi; -\mu^2)$ is a positive definite, symmetric function and we can form a Gaussian process, $\{q_\sigma, 0 \leq \sigma \leq t\}$, with mean function zero, covariance function $p_\mu(\sigma, \xi)$ and almost all the sample functions of which, $q(\sigma)$, vanish at $\sigma = 0$. Moreover, with this covariance function almost all sample functions $q(\sigma)$ are continuous.

Since (11.2) holds for *all* continuous $q(\sigma)$, it holds in particular for almost all sample functions of the Gaussian process just constructed, and therefore taking expectations with respect to this Gaussian process we get

$$E_q^{\nu\mu} \left\{ \exp \left\{ -i\lambda\mu \int_0^t x(\sigma)q(\sigma) d\sigma \right\} E_z^w \left\{ \exp \left\{ i\lambda\mu \int_0^t q(\sigma)z(\sigma) d\sigma \right\} F[z] \right\} \right\} = 0$$

But,

$$\begin{aligned} & E_q^{\nu\mu} \left\{ \exp \left\{ i\lambda\mu \int_0^t [z(\sigma) - x(\sigma)]q(\sigma) d\sigma \right\} \right. \\ &= \exp \left\{ -\frac{1}{2} \lambda^2 \mu^2 \int_0^t \int_0^t [z(\sigma) - x(\sigma)][z(\xi) - x(\xi)] p_\mu(\sigma, \xi) d\sigma d\xi \right\} \\ &= \exp \left\{ \frac{1}{2} \lambda^2 \mu^2 \int_0^t \int_0^t [z(\sigma) - x(\sigma)][z(\xi) - x(\xi)] R(\sigma, \xi; -\mu^2) d\sigma d\xi \right\} \end{aligned}$$

and therefore we have

$$E_z^w \left\{ \exp \left\{ \frac{1}{2} \lambda^2 \mu^2 \int_0^t \int_0^t [z(\sigma) - x(\sigma)][z(\xi) - x(\xi)] R(\sigma, \xi; -\mu^2) d\sigma d\xi \right\} F[z] \right\} = 0. \tag{11.3}$$

We will show later in this proof that if $y(\sigma) \in C(0, t)$ and furthermore satisfies some order Lipschitz condition, then

$$\lim_{\mu \rightarrow \infty} \mu^2 \int_0^t \int_0^t y(\sigma)y(\xi) R(\sigma, \xi; -\mu^2) d\sigma d\xi = - \int_0^t y^2(\sigma) d\sigma. \tag{11.4}$$

Since by assumption $x(\sigma)$ satisfies a Lip α -condition for some $\alpha > 0$, and since almost all sample functions, $z(\sigma)$, of the Wiener process satisfy a Lipschitz condition of order strictly less than $\frac{1}{2}$, we apply (11.4) and obtain from (11.3) on letting $\mu \rightarrow \infty$,

$$E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} F[z] \right\} = 0 \quad (11.5)$$

We will also demonstrate later in this proof that under the assumptions imposed here on $F[z]$ and on $x(\sigma)$,

$$\lim_{\lambda \rightarrow \infty} \frac{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} F[z] \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} = F[x]. \quad (11.6)$$

Thus, dividing both sides of (11.5) by

$$E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}.$$

and letting $\lambda \rightarrow \infty$, we obtain $F[x] = 0$. Now let $z(\sigma)$ be an arbitrary element of $C(0, t]$. Since $F[z]$ is continuous in the uniform topology and since every uniform neighborhood of $z(\sigma)$ contains a function $x(\sigma)$ satisfying a Lip α -condition, we have $F[z] = 0$. This completes the proof except we must now prove (11.4) and (11.6).

To prove (11.4), consider

$$\begin{aligned} & \int_0^t y^2(\sigma) d\sigma + \mu^2 \int_0^t \int_0^t y(\sigma) y(\xi) R(\sigma, \xi; -\mu^2) d\sigma d\xi \\ &= \int_0^t y^2(\sigma) \left[1 + \mu^2 \int_0^t R(\sigma, \xi; -\mu^2) d\xi \right] d\sigma - \frac{\mu^2}{2} \int_0^t \int_0^t R(\sigma, \xi; -\mu^2) [y(\sigma) - y(\xi)]^2 d\sigma d\xi \end{aligned} \quad (11.7)$$

Each of the terms in this last expression are positive and therefore to prove (11.4) it will suffice to show that each goes to 0 as $\mu \rightarrow \infty$. Now

$$1 + \mu^2 \int_0^t R(\sigma, \xi; -\mu^2) d\xi = \frac{\cosh \mu(t - \sigma)}{\cosh \mu t}$$

and therefore, letting h be the bound on $y(\sigma)$ on $[0, t]$,

$$\begin{aligned} \int_0^t y^2(\sigma) \left[1 + \mu^2 \int_0^t R(\sigma, \xi; -\mu^2) d\xi \right] d\sigma &= \frac{1}{\cosh \mu t} \int_0^t y^2(\sigma) \cosh \mu(t - \sigma) d\sigma \\ &\leq \frac{h^2}{\cosh \mu t} \int_0^t \cosh \mu(t - \sigma) d\sigma = \frac{h^2 \sinh \mu t}{\mu \cosh \mu t}. \end{aligned}$$

Since this last approaches 0 as $\mu \rightarrow \infty$, we see that the first term on the right in (11.7) goes to zero.

For the second term on the right of (11.7) we use first the assumed Lip α -condition on $y(\sigma)$, i.e.,

$$\begin{aligned} \left| \frac{\mu^2}{2} \int_0^t \int_0^t R(\sigma, \xi; -\mu^2) [y(\sigma) - y(\xi)]^2 d\sigma d\xi \right| &= \mu^2 \int_0^t \int_0^\xi \frac{\cosh \mu(t-\xi) \sinh \mu\sigma}{\mu \cosh \mu t} [y(\sigma) - y(\xi)]^2 d\sigma d\xi \\ &\leq \frac{\mu h^2}{\cosh \mu t} \int_0^t \int_0^\xi (\sigma - \xi)^{2\alpha} \cosh \mu(t-\xi) \sinh \mu\sigma d\sigma d\xi. \end{aligned}$$

By the Hölder inequality this last expression is less than or equal to

$$\begin{aligned} &\frac{\mu h^2}{\cosh \mu t} \left[\int_0^t \int_0^\xi (\sigma - \xi)^2 \cosh \mu(t-\xi) \sinh \mu\sigma d\sigma d\xi \right]^\alpha \left[\int_0^t \int_0^\xi \cosh \mu(t-\xi) \sinh \mu\sigma d\sigma d\xi \right]^{1-\alpha} \\ &= \frac{\mu h^2}{\cosh \mu t} \left[\frac{\cosh \mu t}{\mu^3} \left(1 + \frac{2}{\cosh \mu t} \right) \right]^\alpha \left[\frac{t \cosh \mu t}{2\mu} - \frac{\sinh \mu t}{2\mu^2} \right]^{1-\alpha} \\ &= \mu h^2 \left[\frac{1}{\mu^3} \left(1 + \frac{2}{\cosh \mu t} \right) \right]^\alpha \left[\frac{t}{2\mu} - \frac{\sinh \mu t}{2\mu^2 \cosh \mu t} \right]^{1-\alpha} \\ &\sim \mu h^2 \cdot \frac{1}{\mu^{3\alpha}} \cdot \frac{t^{1-\alpha}}{2^{1-\alpha} \mu^{1-\alpha}} = \frac{h^2 t^{1-\alpha}}{2^{1-\alpha} \mu^{2\alpha}} \end{aligned}$$

which approaches 0 as $\mu \rightarrow \infty$. Hence the second term on the right of (11.7) also goes to zero and we have proved (11.4).

We now prove (11.6), but for this we need only assume that $x(\sigma) \in C(0, t)$. The Lipschitz condition assumption on $x(\sigma)$ was used in applying (11.4) to (11.2) and is not needed here. Assume then that $x(\sigma)$ is a fixed function in $C(0, t)$ and that $F[z]$ is bounded and continuous in the uniform topology on $C(0, t)$. For $\varepsilon > 0$ let $\delta > 0$ be such that $|F[z] - F[x]| < \varepsilon$ whenever $\sup_{0 \leq \sigma \leq t} |z(\sigma) - x(\sigma)| < \delta$. Let $S_\delta = \{z(\sigma) : \sup_{0 \leq \sigma \leq t} |z(\sigma) - x(\sigma)| < \delta\}$. Now

$$\begin{aligned} &\frac{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} F[z] \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} - F[x] \\ &= \frac{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} [F[z] - F[x]] \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{E_{z \in S_\delta}^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} [F[z] - F[x]] \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} \\
 &+ \frac{E_{z \in S'_\delta}^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} [F[z] - F[x]] \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}},
 \end{aligned}$$

where S'_δ is the complement of S_δ . Using the continuity of $F[z]$ and letting M be the assumed bound on $F[z]$ we have

$$\begin{aligned}
 &\left| \frac{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} F[z] \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} - F[x] \right| \\
 &\leq \varepsilon + 2M \frac{E_{z \in S_\delta}^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}},
 \end{aligned}$$

and therefore to prove (11.6) it will suffice to show

$$\lim_{\lambda \rightarrow \infty} \frac{E_{z \in S_\delta}^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} = 0,$$

or what is equivalent

$$\lim_{\lambda \rightarrow \infty} \frac{E_{z \in S_\delta}^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} = 1 \tag{11.8}$$

The proof of (11.8) is somewhat delicate and is done in Cameron and Donsker [1]. The context there is more complicated and therefore to avoid confusion we now show (11.7) in detail. We need first the following transformation theorem: Let $x(\sigma) \in C(0, t)$ and let $L[z]$ be a functional such that $L[z-x]$ is measurable on the Wiener process. Then, for any positive number λ

$$\begin{aligned}
 E_z^w \left\{ L[z-x] \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\} \\
 = (\operatorname{sech} \lambda t)^{\dagger} \exp \left\{ \frac{1}{2} \int_0^t [q_0'(\sigma)]^2 d\sigma \right\} \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t x^2(\sigma) d\sigma \right\} E_p^w \{ L[\theta - x] \}, \quad (11.9)
 \end{aligned}$$

where

$$q_0(\sigma) = \lambda^2 \int_0^\sigma \operatorname{sech} \lambda(t-s) ds \int_s^t \cosh \lambda(t-\xi) x(\xi) d\xi$$

and

$$\begin{aligned}
 \theta(\sigma) &= \cosh \lambda(t-\sigma) \int_0^\sigma \operatorname{sech} \lambda(t-s) d[p(s) + q_0(s)] \\
 &= x(\sigma) + \cosh \lambda(t-\sigma) \int_0^\sigma \operatorname{sech} \lambda(t-s) dp(s) + \frac{\sinh \lambda \sigma}{\cosh \lambda t} \int_\sigma^t \sinh \lambda(t-s) dx(s) \\
 &\quad - \frac{\cosh \lambda(t-\sigma)}{\cosh \lambda t} \int_0^\sigma \cosh \lambda s dx(s).
 \end{aligned}$$

We will now prove (11.9) and use it to prove (11.8). To show (11.9) we make use of a result of Cameron and Martin to the effect that if $r(\sigma)$ is positive and continuous on $[0, t]$ and if $G[z]$ is measurable on the Wiener process, then

$$E_z^w \left\{ G[z] \exp \left\{ \frac{1}{2} \mu \int_0^t r(\sigma) z^2(\sigma) d\sigma \right\} \right\} = \left(\frac{f_\mu(t)}{f_\mu(0)} \right)^{\dagger} E_y^w \left\{ G \left[f_\mu(\cdot) \int_0^{(\cdot)} \frac{dy(s)}{f_\mu(s)} \right] \right\}, \quad (11.10)$$

where $f_\mu(\sigma)$ is a non-trivial solution of

$$\begin{cases} f_\mu''(\sigma) + \mu r(\sigma) f_\mu(\sigma) = 0 \\ f_\mu'(t) = 0 \end{cases} \quad (11.11)$$

and μ is less than the least eigenvalue of the system (11.11) augmented by the condition $f_\mu(0) = 0$. Equation (11.10) holds in the sense that the existence of either side implies that of the other and the equality. We apply (11.10) to the left side of (11.27) with

$$\begin{aligned}
 G[z] &= L[z-x] \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t x^2(\sigma) d\sigma \right\} \exp \left\{ \lambda^2 \int_0^t x(\sigma) z(\sigma) d\sigma \right\} \\
 r(\sigma) &= 1 \\
 \mu &= -\lambda^2
 \end{aligned}$$

and we choose $f_\mu(\sigma) = \cosh \lambda(t - \sigma)$ in accordance with (11.11). Then the left member of (11.9) becomes identical with the left member of (11.10) and is hence equal to

$$\begin{aligned} & (\operatorname{sech} \lambda t)^\dagger E_y^w \left\{ G \left[\cosh \lambda(t - (\cdot)) \int_0^{(\cdot)} \operatorname{sech} \lambda(t - s) dy(s) \right] \right\} \\ &= (\operatorname{sech} \lambda t)^\dagger \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t x^2(\sigma) d\sigma \right\} \\ & \cdot E_y^w \left\{ L \left[\cosh \lambda(t - (\cdot)) \int_0^{(\cdot)} \operatorname{sech} \lambda(t - s) dy(s) - x(\cdot) \right] \right\} \\ & \cdot \exp \left\{ \lambda^2 \int_0^t \cosh \lambda(t - \sigma) x(\sigma) d\sigma \int_0^\sigma \operatorname{sech} \lambda(t - s) dy(s) \right\}. \end{aligned} \quad (11.12)$$

To show that this last expression is the same as the right member of (11.9), i.e., to show that

$$\begin{aligned} & \exp \left\{ \frac{1}{2} \int_0^t [q_0'(\sigma)]^2 d\sigma \right\} E_p^w \{ L[\theta - x] \} \\ &= E_y^w \left\{ L \left[\cosh \lambda(t - (\cdot)) \int_0^{(\cdot)} \operatorname{sech} \lambda(t - s) dy(s) - x(\cdot) \right] \right\} \\ & \exp \left\{ \lambda^2 \int_0^t \cosh \lambda(t - \sigma) x(\sigma) d\sigma \int_0^\sigma \operatorname{sech} \lambda(t - s) dy(s) \right\}, \end{aligned} \quad (11.13)$$

we make use of the Cameron-Martin translation theorem which states that when $q_0(\sigma)$ is absolutely continuous, $q_0'(\sigma) \in L_2$, and H is a functional measurable on the Wiener process

$$E_p^w \{ H[p + q_0] \} = \exp \left\{ -\frac{1}{2} \int_0^t [q_0'(\sigma)]^2 d\sigma \right\} E_y^w \left\{ H[y] \exp \left\{ \int_0^t q_0'(\sigma) dy(\sigma) \right\} \right\}. \quad (11.14)$$

We apply (11.14) to the left side of (11.13) with $q_0(\sigma)$ as given just after (11.9) and with $H[y] = L[\cosh \lambda(t - (\cdot)) \int_0^{(\cdot)} \operatorname{sech} \lambda(t - s) dy(s) - x(\cdot)]$. This verifies (11.13) and proves (11.9). If we consider (11.9) in the special case where $L \equiv 1$ and divide (11.9) member by member by this special case we obtain that

$$\frac{E_z^w \left\{ L[z - x] \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}}{E_z^w \left\{ \exp \left\{ -\frac{1}{2} \lambda^2 \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\} \right\}} = E_p^w \{ L[\theta - x] \}. \quad (11.15)$$

Let $\varphi(u) = 1$ for $|u| \leq \delta$ and 0 otherwise. Define the functional $L[y] = \varphi(\sup_{0 \leq \sigma \leq t} |y(\sigma)|)$. Comparing (11.15) for this functional and the left side of (11.8) it is clear that to prove (11.8) and hence (11.4) we need to show

$$\lim_{\lambda \rightarrow \infty} E_p^w \{ \varphi(\sup_{0 \leq \sigma \leq t} | \theta(\sigma) - x(\sigma) |) \} = 1$$

where $\theta(\sigma)$ is given just after (11.9). For this it will suffice to show that for almost all $p(\sigma)$ (Wiener process measure) $\lim_{\lambda \rightarrow \infty} \theta(\sigma) = x(\sigma)$ uniformly for $\sigma \in [0, t]$. Looking at the second form for $\theta(\sigma)$ (which is obtained from the first form by repeated integrations by parts) we want to show that uniformly for $\sigma \in [0, t]$ and for almost all $p(\sigma)$

$$\begin{cases} \lim_{\lambda \rightarrow \infty} \cosh \lambda(t - \sigma) \int_0^\sigma \operatorname{sech} \lambda(t - s) dp(s) = 0 \\ \lim_{\lambda \rightarrow \infty} \frac{\sinh \lambda \sigma}{\cosh \lambda t} \int_\sigma^t \sinh \lambda(t - s) dx(s) = 0 \\ \lim_{\lambda \rightarrow \infty} \frac{\cosh \lambda(t - \sigma)}{\cosh \lambda t} \int_0^\sigma \cosh \lambda s dx(s) = 0. \end{cases} \tag{11.16}$$

Almost all sample functions of the Wiener process satisfy for some h depending on p the modified Hölder condition

$$|p(\sigma) - p(\sigma')| \leq h \left(|\sigma - \sigma'| \left| \log \frac{|\sigma - \sigma'|}{e} \right| \right)^{\frac{1}{2}}. \tag{11.17}$$

Let $\eta = (\log \lambda) / \lambda$. In order to estimate the first expression in (11.16) we note that

$$\begin{aligned} & \left| \cosh \lambda(t - \sigma) \int_0^{\sigma - \eta} \operatorname{sech} \lambda(t - s) dp(s) \right| \\ & \leq [\cosh \lambda(t - \sigma)] [\operatorname{sech} [\lambda(t - \sigma) + \log \lambda]] h \\ & < 2h e^{\lambda(t - \sigma)} e^{[-\lambda(t - \sigma) - \log \lambda]} = \frac{2h}{\lambda} \rightarrow 0 \end{aligned}$$

uniformly for $\sigma \in [0, t]$ as $\lambda \rightarrow \infty$. Also for sufficiently large λ

$$\begin{aligned} & \left| \cosh \lambda(t - \sigma) \int_{\sigma - \eta}^\sigma \operatorname{sech} \lambda(t - s) dp(s) \right| \\ & \leq [\cosh \lambda(t - \sigma)] [\operatorname{sech} \lambda(t - \sigma)] h \left(\eta \left| \log \frac{\eta}{e} \right| \right)^{\frac{1}{2}} \\ & = h \left\{ \frac{\log \lambda}{\lambda} (\log \lambda - \log \log \lambda + 1) \right\}^{\frac{1}{2}} \leq h \left(\frac{\log \lambda}{\lambda} \right)^{\frac{1}{2}} (2 \log \lambda)^{\frac{1}{2}} = 2h \left(\frac{(\log \lambda)^2}{\lambda} \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

uniformly in $\sigma \in [0, t]$ as $\lambda \rightarrow \infty$. Thus we have shown the first statement in (11.16). The next two statements in (11.16) are true because of the uniform continuity of $x(\sigma)$. The argument for the third statement is almost identical with that for the second and therefore we will demonstrate only the second. Since $x(\sigma)$ is uniformly continuous on $[0, t]$ there exists a continuous increasing function $\gamma(\varepsilon)$ such that $\gamma(0) = 0$ and $|x(\sigma) - x(\sigma')| \leq \gamma(|\sigma - \sigma'|)$. Again letting $\eta = (\log \lambda)/\lambda$, we have

$$\left| \frac{\sinh \lambda \sigma}{\cosh \lambda t} \int_{\sigma}^{\sigma+\eta} \sinh \lambda(t-s) dx(s) \right| \leq \frac{\sinh \lambda \sigma}{\cosh \lambda t} \sinh \lambda(t-\sigma) \gamma(\eta) \leq \gamma(\eta) \rightarrow 0$$

uniformly for $\sigma \in [0, t]$ as $\lambda \rightarrow \infty$. Also

$$\begin{aligned} \left| \frac{\sinh \lambda \sigma}{\cosh \lambda t} \int_{\sigma-\eta}^t \sinh \lambda(t-s) dx(s) \right| &\leq \frac{\sinh \lambda \sigma}{\cosh \lambda t} [\sinh \lambda(t-\sigma-\eta)] \gamma(t) \\ &\leq \gamma(t) \frac{e^{i\sigma} e^{\lambda(t-\sigma-\eta)}}{4 \cosh \lambda t} \leq \gamma(t) \frac{e^{\lambda(t-\eta)}}{2e^{\lambda t}} = \frac{\gamma(t)}{2\lambda} \rightarrow 0 \end{aligned}$$

again uniformly for $\sigma \in [0, t]$ as $\lambda \rightarrow \infty$.

12. The equivalence Problem; Parabolic case

THEOREM 12.1. *Assume that V and V_m verify (4.15). Let $u(x, t; q)$ be a functional which satisfies*

$$\frac{\delta u}{\delta q(\tau)} = \left(- \int_0^t \min(\tau, s) q(s) ds \right) u - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u ds - i \tau \frac{\partial u}{\partial x}, \quad (12.1)$$

$$\lim_{\tau \rightarrow t} \frac{\delta u}{\delta q(\tau)} = i x u \quad (12.2)$$

$$\frac{\partial u}{\partial t}(x, t; 0) - \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(x, t; 0) = -V(x) u(x, t; 0) \quad (12.3)$$

$$u(x, t; 0) \rightarrow \delta \text{ as } t \rightarrow 0. \quad (12.4)$$

Suppose that

$$u(x, t; q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} E_z^w \{ e^{i \int_0^t a(\sigma) z(\sigma) d\sigma} H(z) e^{i\mu z(t)} \} d\mu. \quad (12.5)$$

Then necessarily

$$H(z) = e^{-\int_0^t V(z(\sigma)) d\sigma} \tag{12.6}$$

Proof. From (12.5) it follows that ($0 < \tau < t$):

$$\frac{\delta u}{\delta q(\tau)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} E_z^w \{ iz(\tau) e^{i\int_0^t q(\sigma)z(\sigma) d\sigma} H(z) e^{i\mu z(t)} \} d\mu.$$

We apply now Lemma 11.1 with

$$M(z) = e^{i\int_0^t q(\sigma)z(\sigma) d\sigma + i\mu z(t)}, \quad N(z) = H(z),$$

and we compare the result with (12.1). We obtain

$$\begin{aligned} - \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u ds &= \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} \int_0^t \min(\tau, s) E_z^w \left\{ e^{i\int_0^t q(\sigma)z(\sigma) d\sigma} \frac{\delta H(z)}{\delta z(s)} e^{i\mu z(t)} \right\} ds d\mu. \end{aligned}$$

But
$$V' \left(-i \frac{\delta}{\delta q(s)} \right) u = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} E_z^w \{ V'(z(s)) e^{i\int_0^t q(\sigma)z(\sigma) d\sigma} H(z) e^{i\mu z(t)} \} d\mu,$$

hence

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} \int_0^t \min(\tau, s) E_z^w \left\{ e^{i\int_0^t q(\sigma)z(\sigma) d\sigma} \left[\frac{\delta H(z)}{\delta z(s)} + V'(z(s)) H(z) \right] e^{i\mu z(t)} \right\} ds d\mu = 0.$$

Since this relation holds for every x we have

$$\int_0^t \min(\tau, s) E_z^w \left\{ e^{i\int_0^t q(\sigma)z(\sigma) d\sigma} \left[\frac{\delta H(z)}{\delta z(s)} + V'(z(s)) H(z) \right] e^{i\mu z(t)} \right\} ds = 0.$$

Applying $-d^2/d\tau^2$ we obtain:

$$E_z^w \left\{ e^{i\int_0^t q(\sigma)z(\sigma) d\sigma} \left[\frac{\delta H(z)}{\delta z(\tau)} + V'(z(\tau)) H(z) \right] e^{i\mu z(t)} \right\} = 0.$$

for all μ , any $0 < \tau < t$, and all $q \in C(0, t)$.

Using Lemma 11.2 it follows that

$$\frac{\delta H(z)}{\delta z(\tau)} + V'(z(\tau)) H(z) = 0$$

from which

$$H(z) = Ke^{-\int_0^t V(z(\sigma))d\sigma}.$$

Using (12.3) (12.4), it follows that $K = 1$, which completes the proof of the theorem.

13. The equivalence problem: general case

The solution of the equivalence problem given in sections 11 and 12 does not apply to the Schrödinger case, where a representation of the kernel $Q(x, y, t; q)$ as a single Wiener integral does not exist. It would be of interest to find a proof of uniqueness for the solution of the F.V variational equations involved here *without any use of function space integrals*. Such a proof is given but only for very special $V(x)$ in section 14.

In these connections we would like to point out the following commutative property which obtains here. For a functional $F(x, y, t; q)$ with values in $\mathcal{D}(R_x \times R_y \times (-\infty, t_0))$ we set

$$\begin{aligned} \mathcal{V}_q F = & \frac{\delta F}{\delta q(\tau)} + i\tau \frac{\partial F}{\partial x} + \left(\int_0^\tau \min(\tau, s) q(s) ds \right) F \\ & + i \int_0^\tau \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) F ds - iq(t)F \text{ for } t > \tau, \end{aligned} \tag{13.1}$$

and
$$\Lambda_q = \Lambda - ixq(t), \quad \Lambda = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\partial}{\partial t}. \tag{13.2}$$

THEOREM 13.1 *Let F be a functional so that (13.1) exists and which verifies*

$$V' \left(-i \frac{\delta}{\delta q(s)} \right) F(x, y, t; q) \rightarrow V'(x) F(x, y, t; q) \tag{13.3}$$

in the sense of distributions as $s \rightarrow t$.⁽¹⁾

Then
$$\Lambda_q \mathcal{V}_q F = \mathcal{V}_q \Lambda_q F, \text{ for } t < \tau.$$

Proof. We calculate $\mathcal{V}_q \Lambda_q F$. One has

$$\frac{\delta}{\delta q(\tau)} (\Lambda_q F) = \Lambda_q \frac{\delta F}{\delta q(\tau)} - ix F \delta_t(\tau),$$

⁽¹⁾ We assume V is infinitely differentiable here.

hence
$$\frac{\delta(\Lambda_q F)}{\delta q(\tau)} = \Lambda_q \frac{\delta F}{\delta q(\tau)} \text{ for } t > \tau.$$

Next,
$$i\tau \frac{\partial}{\partial x} \Lambda_q F = i\tau \Lambda_q \frac{\partial F}{\partial x} + i\tau V'(x) F + \tau q(t) F.$$

But
$$\Lambda_q \left(\left(\int_0^t \min(\tau, s) q(s) ds \right) F \right) = \left(\int_0^t \min(\tau, s) q(s) ds \right) \Lambda_q F + \tau q(t) F,$$

$$\Lambda_q \left(i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) F ds \right) = i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) \Lambda_q F ds + i\tau V'(x) F,$$

using (13.3). Hence (13.4) follows.

14. Solution by F.V. series

In this section we show that in the case of very simple $V(x)$ one can solve the F.V. variational system:

$$\left. \begin{aligned} \frac{\delta u}{\delta q(\tau)} &= \left(- \int_0^t \min(\tau, s) q(s) ds \right) u - i \int_0^t \min(\tau, s) V' \left(-i \frac{\delta}{\delta q(s)} \right) u ds - i\tau \frac{\partial u}{\partial x}, \\ \lim_{\tau \rightarrow t} \frac{\delta u}{\delta q(\tau)} &= i x u, \\ \frac{\partial u(x, t; 0)}{\partial t} - \frac{1}{2} \frac{\partial^2 u(x, t; 0)}{\partial x^2} &= -V(x) u(x, t; 0) \quad u(x, t; 0) \rightarrow \delta(x), \quad t \rightarrow 0, \end{aligned} \right\} \quad (14.1)$$

by the use of F.V. series expansions of the unknown functional $u(x, t; q)$ (cf. chapter I, section 5 and the remarks made in the Introduction). It is of interest to note that this elementary technique also provides uniqueness proofs for the solution of (14.1) in these simple cases of $V(x)$ —uniqueness proofs different from those given in the preceding sections, since here no function space integrals are used.

Consider (14.1) with $V(x) = \frac{1}{2}x^2$ (this is the example considered in the Introduction) and the F.V. series expansion of $u(x, t; q)$,

$$u(x, t; q) = c_0(x, t) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{t(n)} \dots \int_0^t c_n(x, t; \tau_1, \tau_2, \dots, \tau_n) q(\tau_1) \dots q(\tau_n) d\tau_1 \dots d\tau_n. \quad (14.2)$$

Substituting (14.2) in the F.V. variational equation of (14.1) we obtain the recurrence formulae,

$$c_1(x, t; \tau) = - \int_0^t \min(\tau, s) c_1(x, t; s) ds - i\tau \frac{\partial c_0(x, t)}{\partial x}, \tag{14.3}$$

$$c_2(x, t; \xi, \tau) = -c_0(x, t) \min(\tau, \xi) - \int_0^t \min(\tau, s) c_2(x, t; \xi, s) ds - i\tau \frac{\partial c_1(x, t; \xi)}{\partial x} \text{ etc.} \tag{14.4}$$

Now we can solve the integral equation (14.3) and obtain

$$c_1(x, t; \tau) = -i \frac{\partial c_0(x, t)}{\partial x} \left[\tau + \int_0^t sR(\tau, s; -1) ds \right], \tag{14.5}$$

where $R(\tau, s; -1)$, the resolvent kernel of $\min(\tau, s)$ on $[0, t]$, is given explicitly in the Introduction. From the boundary condition in (14.1) involving the limit we obtain using again the F.V. series (14.2) that in particular

$$c_1(x, t; t) = ix c_0(x, t). \tag{14.6}$$

Putting (14.5) (with $\tau = t$) and (14.6) together we get the equation

$$ix c_0(x, t) = -i \frac{\partial c_0(x, t)}{\partial x} \left[t + \int_0^t sR(t, s; -1) ds \right] \tag{14.7}$$

and thus since

$$t + \int_0^t sR(t, s; -1) ds = \tanh t,$$

we get

$$c_0(x, t) = K(t) e^{-x^{1/2} \tanh t}. \tag{14.8}$$

Now the differential equation ‘‘boundary condition’’ in (14.1) and the last condition in (14.1) determine that

$$K(t) = (2\pi \sinh t)^{-1/2}.$$

From (14.5) we now can obtain explicitly and *uniquely* $c_1(x, t; \tau)$ and from (14.4) etc. we determine explicitly and uniquely the coefficients in the F.V. series expansion (14.2). The resulting series is exactly formula (1.2) of the Introduction.

An even simpler example is $V(x) \equiv 0$, where the technique above leads to the same result one would obtain from either the differential equation (Introduction (1.13) with $V(x) \equiv 0$) or by calculating the function space integral (Introduction (1.11) with $V(x) \equiv 0$). By all three techniques one obtains

$$u(x, t; q) = (2\pi t)^{-1/2} \exp \left[-\frac{x^2}{2t} + \frac{ix}{t} \int_0^t \sigma q(\sigma) d\sigma \right] \cdot \exp \left[-\frac{1}{2} \int_0^t \int_0^t \left(\min(\sigma, u) - \frac{\sigma u}{t} \right) q(\sigma) q(u) d\sigma du \right].$$

Again the uniqueness comes from the fact that the *coupled* recurrence formulae determine the coefficients uniquely.

Even in the case $V(x) = x^4/4$ the recurrence formulae are coupled in such a more complicated way that it seems very difficult to try to prove uniqueness by this method. The difficulty with this coupling is to be expected, since in the case $V(x) = x^4/4$ it is not possible to calculate the function space integral and also not possible to explicitly solve the differential equation boundary value problem.

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