Acta Math., 192 (2004), 95–109

The order of the top Chern class of the Hodge bundle on the moduli space of abelian varieties

by

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1. Introduction

Let $\mathcal{A}_g/\mathbf{Z}$ denote the moduli stack of principally polarized abelian varieties of dimension g. This is an irreducible algebraic stack of relative dimension $\frac{1}{2}g(g+1)$ with irreducible fibres over $\mathbf{Z}$. The stack $\mathcal{A}_g$ carries a locally free sheaf $\mathbf{E}$ of rank g, the Hodge bundle, defined as follows. If A/S is an abelian scheme over S with 0-section s we get a locally free sheaf $s^*\Omega^1_{A/S}$ of rank g on S, and this is compatible with pullbacks. If $\pi: A\to S$ denotes the structure map it satisfies the property $\Omega^1_{A/S}=\pi^*(\mathbf{E})$, and we will consider its Chern classes $\lambda_i(A/S):=c_i(\Omega^1_{A/S})$ (in the Chow ring of S). These then are the pullbacks of the corresponding classes in the universal case $\lambda_i:=c_i(\mathbf{E})$. The Hodge bundle can be extended to a locally free sheaf (again denoted by) $\mathbf{E}$ on every smooth toroidal compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ of the type constructed in [9], see Chapter VI, §4 there. By a slight abuse of notation we will continue to use the notation λ_i for its Chern classes.

The classes λ_i are defined over $\mathbf{Z}$ and give for each fibre $\mathcal{A}_g\otimes k$ rise to classes, also denoted λ_i, in the Chow ring $\mathrm{CH}^*(\mathcal{A}_g\otimes k)$, and in $\mathrm{CH}^*(\tilde{\mathcal{A}}_g\otimes k)$. They generate subrings ($\mathbf{Q}$-subalgebras) of $\mathrm{CH}^*_{\mathbf{Q}}(\mathcal{A}_g\otimes k)$ and of $\mathrm{CH}^*_{\mathbf{Q}}(\tilde{\mathcal{A}}_g\otimes k)$ which are called the *tautological subrings.*

It was proved in [11] by an application of the Grothendieck–Riemann–Roch theorem that these classes in the Chow ring $\mathrm{CH}^*_{\mathbf{Q}}(\mathcal{A}_g)$ with rational coefficients satisfy the relation

$$(1+\lambda_1+\ldots+\lambda_g)(1-\lambda_1+\ldots+(-1)^g\lambda_g)=1. \tag{1.1}$$

Furthermore, it was proved that λ_g vanishes in the Chow group $\mathrm{CH}_{\mathbf{Q}}(\mathcal{A}_g)$ with rational coefficients. The class λ_g does not vanish on $\tilde{\mathcal{A}}_g$. This raises two questions. First, since λ_g is a torsion class on $\mathcal{A}_g$ we may ask what its order is. Second, since λ_g up to torsion comes from a class on the 'boundary' $\tilde{\mathcal{A}}_g \setminus \mathcal{A}_g$ we may ask for a description of this class. As an answer to the first question we give an upper bound on the order of λ_g in the third section and a non-vanishing result in the fourth section which implies a lower bound. That result is obtained as a consequence of a more precise result that determines the order of the Chern classes of the de Rham bundle up to a (multiplicative) factor 2. As an answer to the second question we shall generalize the well-known relation $12\lambda_1 = \delta$ for $g=1$ to higher g in a sequel to this paper [8].

The authors thank the referee for many useful remarks and thank the editors for their patience.

2. K-theory for stacks

We will need to extend some results on K-theory from schemes to stacks. Just as in the schematic case we denote by $K^0(X)$ the Grothendieck group of vector bundles on the algebraic stack X, and by $K_0(X)$ the Grothendieck group of coherent sheaves. We shall use [14] for the general notions concerning algebraic stacks, in particular the definition of coherent sheaf on an algebraic stack, cf. [14, 15.1]. We shall follow standard usage in that by an *algebraic stack* we mean a general algebraic stack—what is also called an Artin stack—while by Deligne–Mumford stack we mean an algebraic stack with an étale chart. The first result is the homotopy exact sequence (cf. [3, Exp. IX, Proposition 1.1]).

Remark 2.1. Our purpose is quite restricted, we only want results that can be used in the next section, and hence we will make no attempt at maximum generality. It will be clear that some of the results are in fact true in larger generality than stated.

We will in this section only consider algebraic stacks of finite type over a base field $\mathbf{k}$.

PROPOSITION 2.2. *Let X be a noetherian algebraic stack, $i: X' \to X$ a closed substack of X, and $j: U \to X$ the complement of X'. Then the sequence*

$$K_0(X') \xrightarrow{i_*} K_0(X) \xrightarrow{j^*} K_0(U) \longrightarrow 0$$

is exact.

Proof. The proof is the same as that of [3, Exp. IX, Proposition 1.1] once we know that every coherent sheaf on U extends to one on X and that every coherent subsheaf of an extended quasi-coherent sheaf extends as a coherent subsheaf. Such extensions are provided by [14, Corollary 15.5]. □

The next step is to prove the analogue of [3, Exp. IX, Proposition 1.6], the calculation of K_0 for a vector bundle.

PROPOSITION 2.3. *Let X be a noetherian Deligne–Mumford stack and $E \to X$ a vector bundle over X. Then the pullback map $K_0(X) \to K_0(E)$ is an isomorphism.*

Proof. As a general remark we note that K_0 is contravariant not just for flat maps but for all morphisms between noetherian stacks of finite Tor-dimension.

We first prove the proposition for the case when $E = X \times \mathbf{A}^1$, and for that we follow the proof of [3, Exp. IX, Proposition 1.6]. We start by noting that the 0-section of $X \times \mathbf{A}^1$ is of finite Tor-dimension, and hence we have a map $K_0(X \times \mathbf{A}^1) \to K_0(X)$ such that the composition $K_0(X) \to K_0(X \times \mathbf{A}^1) \to K_0(X)$ is the identity. This shows that $K_0(X) \to K_0(E)$ is injective, and it remains to show surjectivity.

We may now by noetherian induction assume that the corresponding map is an isomorphism for every proper closed substack. By Proposition 2.2 it is hence enough to show the result for some non-empty open subset of X. By [14, Proposition 6.1.1] there is a non-empty open subset of X which is the stack quotient of an action of a finite group on an affine scheme, and we consider that open subset, so that we may assume that X itself is such a quotient. If the affine scheme is $\operatorname{Spec} R$ and the group is G then the category of coherent $\mathcal{O}_X$-modules is equivalent to the category of finitely generated modules over the twisted group ring $R[G]$ ("twisted" because of the relation $g\lambda = \lambda^g g$ for $\lambda \in R$ and $g \in G$), and the category of coherent modules over $X \times \mathbf{A}^1$ is then equivalent to the category of finitely generated modules over $R[G][T]$, the polynomial ring over $R[G]$. Note also that if S is the ring of invariants of G on R, then S is noetherian as R is noetherian and $R[G]$ is a finite S-algebra. We then conclude by [2, Exp. XII, Theorem 4.1], which says exactly that for a finite S-algebra A, the map $G_0(A) \to G_0(A[T])$ is an isomorphism; $G_0(B)$ in the notation by Bass being the Grothendieck group of finitely generated B-modules.

In the general case, we will say that two maps $f, g: Y \to Z$ between algebraic stacks are *affine-homotopic* if there is a map $F: Y \times \mathbf{A}^1 \to Z$ which restricted to the 0-section is f and restricted to the 1-section is g. Note also that the 0- and 1-sections in $X \times \mathbf{A}^1$ are of finite Tor-dimension, which means that we may pull back elements of $K_0(Y \times \mathbf{A}^1)$ along them. By the special case of $E = X \times \mathbf{A}^1$ just proved we get that 0^* and 1^* are both inverses to the said isomorphism, and hence they are equal. This implies that if we have two maps $f, g: Y \to Z$ of finite Tor-dimension related by an affine homotopy of finite Tor-dimension we get that $f^* = g^*$ on $K_0(X)$.

This can now be applied to the identity map and the composition of the structure map $E \to X$ and the 0-section $X \to E$, which are affine-homotopic by the usual map $(v, t) \mapsto tv$. □

Remark 2.4. In [3, Exp. IX, Proposition 1.6] noetherian induction is used to allow one to assume that the scheme is reduced. Then a passage to the limit is made so that instead of reducing to an open affine subset one reduces to a field in which case one is dealing with regular schemes and can switch to K^0 instead where the result is easier to prove. This is not possible in our situation as even for a field K the twisted group ring $K[G]$ is not of finite global dimension when the characteristic divides the order of the subgroup of elements of G acting trivially on K. In [2] the same strategy as in [3, Exp. IX, Proposition 1.6] is used to reduce to the case of a finite-dimensional algebra over a field. There a further reduction is made by dividing out by the radical of the algebra, a step which corresponds to assuming that the scheme is reduced.

We will now need to consider the topological filtration on K-theory. Hence we define, for X a regular algebraic stack, $\mathrm{Fil}^{\geqslant i}\subseteq K_0(X)$ to be the subgroup generated by classes of sheaves with support of codimension $\geqslant i$. Using Proposition 2.3 we get an isomorphism $K_0(X)\to K_0(E)$ for every vector bundle $E\to X$. We now define $\mathrm{Fil}_b^{\geqslant i}(X)$ to be the limit of $\mathrm{Fil}^{\geqslant i}(E)$ for all vector bundles $E\to X$ and surjective vector bundle maps between them.

PROPOSITION 2.5. *Let X be a Deligne–Mumford stack which is the stack quotient of the action of a finite group G on a smooth quasi-projective* $\mathbf{k}$*-scheme Y.*

(i) *The forgetful map $K^0(X)\to K_0(X)$ is an isomorphism.*

(ii) *If $\dim X=n$ and $e\in K^0(X)$ maps to $\mathrm{Fil}_b^{\geqslant i}(X)$ then $c_j(e)\cap[X]\in A_{n-j}(X)$ is zero for $0<j<i$.*

(iii) *Using the isomorphism $K^0(X)\to K_0(X)$ to get a multiplication on $K_0(X)$ the filtration $\mathrm{Fil}_b^{\bullet}$ is multiplicative, i.e., $\mathrm{Fil}_b^i\cdot\mathrm{Fil}_b^j\subseteq\mathrm{Fil}_b^{i+j}$ for all i and j.*

Proof. For (i) we use the fact that the category of coherent $\mathcal{O}_X$-modules is equivalent to the category of coherent $\mathcal{O}_Y$-modules with a G-action compatible with the action of G on Y, i.e., a $\mathcal{O}_Y[G]$-module for the twisted group ring. What needs to be proved is, as in the spatial (i.e., algebraic space) case, that every coherent $\mathcal{O}_X$-module has a finite resolution by coherent locally free $\mathcal{O}_X$-modules. An $\mathcal{O}_X$-module is locally free exactly when the corresponding $\mathcal{O}_Y$-module is. As Y is regular and of finite type over $\mathbf{k}$ there is a bound for the global dimension of the local rings of Y. Hence it is enough to show that each coherent $\mathcal{O}_Y[G]$-module $\mathcal{F}$ is the quotient of a $\mathcal{O}_Y[G]$-module $\mathcal{E}$ that is coherent and locally free as $\mathcal{O}_Y$-module. By [3, Exp. II, Corollary 2.2.7.1], $\mathcal{F}$ is the quotient of a coherent locally free $\mathcal{O}_Y$-module $\mathcal{E}'$, and it is then the quotient of the $\mathcal{O}_Y[G]$-module $\mathcal{E}'[G]$.

As for (ii) it follows directly from (i), excision ([13, Proposition 2.3.6]), boundedness by dimension ([13, Proposition 3.4.2]), and the fact that vector bundle maps induce isomorphisms on Chow groups ([13, Corollary 2.4.9]).

Continuing with (iii), we may, after possibly replacing X by a vector bundle over it, reduce to showing that $\mathrm{Fil}^i\cdot\mathrm{Fil}^j\subseteq\mathrm{Fil}_b^{i+j}$. According to [13, Proposition 3.5.6] we may find a vector bundle $E\to X$ and an open subset $U\subseteq E$ whose complement has codimension greater than $i+j$ such that U is an algebraic space. For our purposes we will however need U to be quasi-projective. This is easily arranged by letting $\pi\colon E\to X$ be the stack quotient of G acting on $\mathcal{O}_Y[G]^n$ for n large ($n>i+j$ will do) and letting U be the quotient of the part where G acts freely. We now have an isomorphism $\pi^*\colon K_0(X)\to K_0(E)$, and it will be enough to show that $\pi^*\mathrm{Fil}^i(X)\cdot\pi^*\mathrm{Fil}^j(X)\subseteq\mathrm{Fil}^{i+j}(E)$, and for that it is enough that $\mathrm{Fil}^i(E)\cdot\mathrm{Fil}^j(E)\subseteq\mathrm{Fil}^{i+j}(E)$. By excision, Proposition 2.2, it is enough to show that $\mathrm{Fil}^i(U)\cdot\mathrm{Fil}^j(U)\subseteq\mathrm{Fil}^{i+j}(U)$. But U is a smooth quasi-projective variety, and then this is found in [3, Exp. 0, Appendix, Corollary 1 to Theorem 2.12] when the base field is algebraically closed. The only dependence on the assumption of algebraic closedness is for the moving lemma. However, the moving lemma is true over any field, cf. [16, §3, Theorem]. □

Remark 2.6. We have used the notation Fil for the filtration directly defined by a support condition as it is already well-established in the spatial case. However, we do not believe it is the "right" definition for a general algebraic stack. Consider for instance the case of the stack quotient $[*/G]$ of a finite group G acting trivially on $*=\mathrm{Spec}\,\mathbf{k}$. In that case $K_0([*/G])$ equals the representation ring of G-representations over $\mathbf{k}$ and $\mathrm{Fil}^1(*/G)=\{0\}$, which seems to be too small. On the other hand, consider the case when $G=\mathbf{Z}/2$ (and $\mathbf{k}$ is a field of characteristic different from 2). For any G-representation V on which G acts non-trivially we have that $\mathrm{Fil}_b^{\geqslant 1}K_0([V/G])$ equals the group I of virtual bundles of rank 0 (which is what one would like). Using the multiplicativity we get that $\mathrm{Fil}_b^{\geqslant n}\supseteq I^n$. On the other hand, using Proposition 2.5 (ii) one can show the opposite inclusion and hence $\mathrm{Fil}_b^{\geqslant n}=I^n$. This equals the topological filtration on $K^0(BG)$ making it seem quite reasonable.

As has been stated in the introduction one of our goals is to give an explicit integer killing λ_g on $\mathcal{A}_g$. To make such a statement as useful as possible one would like to be able to conclude from this that its pullback along any map $S\to\mathcal{A}_g$ vanishes. In the case of schemes such a conclusion is possible because the Chern classes lie in Chow cohomology groups which are contravariant functors. It is of course possible to formulate this contravariant character of Chern classes without introducing Chow cohomology groups, but it would be quite awkward particularly when it comes to expressing relations between them. We shall therefore very briefly introduce, by perfect analogy with the spatial case, Chow cohomology groups. We shall only prove the minimal results necessary to formulate and prove our result on λ_g. Recall ([13]) that an algebraic stack is said to be *filtered by global quotients* if it has a stratification by substacks such that each of them is the quotient of an algebraic space by an affine group scheme. Note that the refined Gysin maps (which will be used in our proof) currently (cf. [13, §5.1]) are defined (essentially) only in the case when the involved stacks are filtered by global quotients. We now follow [10] in defining for a map $X \to Y$ of algebraic stacks filtered by global quotients the *bivariant Chow groups* $A^p(X \to Y)$ consisting of collections of operations as in [10, Definition 17.1] fulfilling the conditions C1–C3 with the difference that in C3 the map $i: Z'' \to Z'$ is assumed to be representable, locally separated and whose normal cone stack is a vector bundle stack of constant rank. (These are on the one hand the conditions under which Kresch defines the refined Gysin map, on the other hand we shall want to use the Gysin map for the diagonal of an algebraic stack, which means that it will not be enough to require C3 for l.c.i. embeddings.) As in loc. cit. we define the Chow cohomology groups to be the bivariant groups for the identity maps.

PROPOSITION 2.7. (i) *Let X be an algebraic stack filtered by global quotients and E a vector bundle over X. If for a map $g: X' \to X$ we set $c_p(E)(\alpha) := c_p(g^*E) \cap \alpha$ then we obtain an element of $A^p(X)$.*

(ii) *Let X be a smooth algebraic stack of pure dimension n that is filtered by global quotients. The map $A^p(X) \to A_{n-p}(X)$ given by cupping an operation with the fundamental class is an isomorphism.*

Proof. We will follow very closely the relevant parts of [10, Chapter 17]. For the first part we need the compatibility of Chern classes with proper pushforwards, pullbacks and Gysin maps. The first two properties are proved in [13, §2.5], and the proof of the third follows quite directly:(1) Chern classes are polynomials in Segre classes, so it is enough to prove that Segre classes commute with Gysin maps. From the definition of Segre classes ([13, Definition 2.5.4]) we see that they in turn are expressed in terms of flat pullbacks, (iterated) top Chern class operations, and proper pushdowns. Commutation of Gysin maps with flat pullbacks and proper pushdowns are clear (cf. [13, §5.1]), so it remains to show that they commute with top Chern classes. Looking at the construction of the Gysin map (cf. [13, §5.1, §3.1]) one sees that it is expressed in terms of flat pullbacks, proper pushforwards, and intersection with a principal effective Cartier divisor (the normal cone $C_{F'}G'$ in the deformation space $M^o_{F'}G'$, cf. [10, §5.1]). As the other operations are already known to commute with the top Chern class one is reduced to proving commutation with intersection with a principal effective Cartier divisor (the operation introduced in [13, §2.2]). Looking at the definition of the top Chern class what is needed is the projection formula (cf. [10, Proposition 2.3 c]) for intersection with a divisor. This is proved as in the proof of [10, Proposition 2.3].

(1) We are grateful to Andrew Kresch for providing us with the following proof.

As for (ii), the proof is identical with [10, Theorem 17.4.2] (specialized to the case of the base being a point). (Note that in that proof we cannot assume that the diagonal map is an immersion, which is why we have to allow for more general maps in C3.) □

3. A bound on the order of the class λ_g

We will make some computations in the Chow group of $\mathcal{A}_g$. To make the previous section applicable (and for other reasons) all our algebraic stacks will be of finite type over a field. We begin with a lemma which is no doubt well known.

LEMMA 3.1. *If E is a vector bundle of rank g, the total Chern class of the graded vector bundle Λ^*E is zero in degrees 1 to $g-1$, and $-(g-1)!\,c_g(E)$ in degree g.*

Proof. As usual we note that the components of the total Chern class are universal polynomials with integer coefficients in the Chern classes $c_i:=c_i(E)$. Such a relation is true if and only if it is true for the tautological bundle on all grassmannians of g-dimensional subspaces, and we may hence let E be such a bundle. Then, as we may think of the coefficients in the universal polynomials as rational numbers, we can note that the Chern classes of degree 1 to $g-1$ of Λ^*E vanish if and only if the Newton polynomials of the same degrees do, i.e., if and only if $\mathrm{ch}(\Lambda^*E)$ vanishes in the same degrees. Thus the first part follows from the Borel–Serre formula (cf. [4])

$$\mathrm{ch}(\Lambda^*E)=(-1)^g c_g(E)\,\mathrm{Td}(E)^{-1}.$$

Furthermore, if the Chern classes of degree 1 to $g-1$ vanish for a bundle F, then it is clear that $(-1)^{g-1}gc_g(F)=s_g(F)$, as can be seen from Newton's formula. (Here $s_i(F)$ are the Newton polynomials in the (roots of the) Chern classes of F.) Hence in degree g we have $\mathrm{ch}(\Lambda^*E)=(-1)^{g-1}gc_g(\Lambda^*E)/g!$, and using the Borel–Serre formula again gives the desired formula. □

LEMMA 3.2. *Let p be a prime. Suppose that $\pi\colon A\to S$ is a family of abelian varieties of relative dimension g, where S is an algebraic stack that is the quotient of a smooth quasi-projective* **k***-scheme by a finite group. Assume that L is a line bundle on A of order p, on all fibres of π. Let E be the Hodge bundle of π, i.e., the pullback of $\Omega^1_{A/S}$ along the 0-section, and let e be the class in $K^0(S)$ of the graded bundle Λ^*E. If $p>2g$ then $pe\in\mathrm{Fil}_b^{\geqslant g+1}$, and in particular $p(g-1)!\,\lambda_g=0$. If S is actually a smooth quasi-projective variety, we have that the stronger conclusion $pe\in\mathrm{Fil}^{\geqslant g+1}$ is true provided only that $p>\min\{2g,\dim S+g\}$.*

Proof. That $pe\in\mathrm{Fil}_b^{\geqslant g+1}$ implies that $p(g-1)!\,\lambda_g=0$ follows from Lemma 3.1, Propositions 2.5 (ii) and 2.7.

As in the proof of Proposition 2.5 we can find a vector bundle $E \to S$ with an open set U that is a smooth quasi-projective variety whose complement has codimension greater than g. By Proposition 2.3 and the definition of Fil_b we may pull back π to U and hence assume that S is smooth and quasi-projective.

By twisting L by a line bundle on S so that it is trivial along the 0-section we may assume that it is of order p on A. Denoting the class of L in K_0 by $[L]$ we then either have that $p([L]-1)$ has support of codimension greater than $2g$ if $p>2g$, or is zero if $p>\dim S+g$. Indeed, from the relation

$$0=[L]^p-1=p([L]-1)\big(1+\tfrac{1}{2}(p-1)([L]-1)+\ldots\big)+([L]-1)^p$$

and the fact that $[L]-1$ is nilpotent (which in turn follows for instance from the multiplicativity of the topological filtration) we get that $p([L]-1)$ is $([L]-1)^p$ times a unit, and $([L]-1)^p$ is an element supported in codimension $\geqslant p$ by the multiplicativity of the topological filtration. Now in the first case the image under π of the support has codimension greater than g on S, and so we may safely remove it and assume that $p[L]=p$ in $K_0(A)$.

Consider now the Poincaré bundle $\mathcal{P}$ on $A \times_S \check{A}$, where $\check{A}$ denotes the dual abelian variety. By base change $R\pi_*\mathcal{O}_A$ is the (derived) pullback along the 0-section of $\check{A}$ of the sheaf $R\pi_*\mathcal{P}$. We have that $p[\mathcal{P}]=p[L\otimes\mathcal{P}]$, and so $p[R\pi_*\mathcal{P}]=p[R\pi_*(L\otimes\mathcal{P})]$. Now, a fibrewise calculation shows that $R\pi_*(L\otimes\mathcal{P})$ has support along the inverse of the section of $\check{A}$ corresponding to L. As that section is everywhere disjoint from the 0-section the pullback of $R\pi_*(L\otimes\mathcal{P})$ along the 0-section is 0, and thus $p[R\pi_*\mathcal{O}_A]=0$. Now, the well-known calculation of $R^i\pi_*\mathcal{O}_A$ shows that it is isomorphic to $\Lambda^i E$, so the lemma follows. □

DEFINITION–LEMMA 3.3. *For an integer g we let n_g be the largest common divisor of all $p^{2g}-1$, where p runs through all primes larger than a sufficiently large fixed number N (which may be taken to be $2g+1$). For an odd prime l, the exponent k of the exact power l^k of l that divides n_g is the largest k such that $l^{k-1}(l-1)$ divides $2g$ and 0 if $l-1$ does not divide $2g$. The exponent k of the exact power 2^k that divides n_g is the largest k such that 2^{k-2} divides $2g$.*

Proof. Note that l^k divides n_g if and only if the exponent of $(\mathbf{Z}/l^k)^*$ divides $2g$. The statement now follows directly from the structure of $(\mathbf{Z}/l^k)^*$ and Dirichlet's prime number theorem. □

Example 3.4. We have $n_1=24$, $n_2=240$, $n_3=504$ and $n_4=480$.

THEOREM 3.5. *Suppose that $\pi\colon A\to\mathcal{A}_g$ is the universal family of principally polarized abelian varieties of relative dimension g. Then $(g-1)!\,n_g\lambda_g=0$ on $\mathcal{A}_g$.*

Proof. For any sufficiently large prime p we can apply Lemma 3.2 on the cover obtained by adding a line bundle everywhere of order p. Projecting down to $\mathcal{A}_g$ again and using that that cover has degree $p^{2g}-1$ (being equal to the number of line bundles of order p) gives $(g-1)!\,p(p^{2g}-1)\lambda_g=0$. We then finish by using Definition 3.3 (and noting that the factor p causes no trouble, as by using several primes we see that no prime greater than $2g$ can divide the smallest annihilating integer). □

Remark 3.6. For various mostly technical reasons we have been working over a field, where of course the prime fields are the optimal choices. If one would like to prove the statement over the integers then assuming that the technical details can be overcome we will still have to deal with the fact that one needs to invert a finite number of primes to use the trick of producing a line bundle of prime order by going to a covering. Though there is a large freedom in choosing the primes to which we could apply this trick, still we do not know how to handle this problem.

4. The order of the Chern classes of the de Rham bundle

We will now consider the Chern classes of the bundle of first relative de Rham cohomology of the universal abelian variety over $\mathcal{A}_g$. Over the complex numbers these Chern classes are the Chern classes of a flat bundle. By the Chern–Weil expression of Chern classes in terms of curvature for a smooth manifold, the Chern classes are torsion classes in integral cohomology. Grothendieck has given an arithmetic proof of this fact. His proof gives an explicit bound for the orders of the Chern classes which we will exploit. This bound depends on the field of definition of the representation of the fundamental group that is associated to the flat bundle. Luckily, in our case this representation is the natural representation of $\mathrm{Sp}_{2g}(\mathbf{Z})$, and hence the field of definition is $\mathbf{Q}$, which makes the bounds the best possible.

We will then give lower bounds for the order of these Chern classes. From the complex point of view, where the cohomology of $\mathcal{A}_g$ can be thought of as $H^*(\mathrm{Sp}_{2g}(\mathbf{Z}),\mathbf{Z})$, the idea is to find finite (cyclic) subgroups of $\mathrm{Sp}_{2g}(\mathbf{Z})$ and compute the order of the restriction of the Chern classes to the cohomology of such a subgroup. Now, any finite subgroup G of $\mathrm{Sp}_{2g}(\mathbf{Z})$ is actually the group of automorphisms of a principally polarized variety A (as such a subgroup has a fixed point on the Siegel upper half-space). Such an A gives not only a point of $\mathcal{A}_g$ but also a map from the stack quotient $[*/G]$ to $\mathcal{A}_g$. In particular it gives a map from the cohomology of $\mathcal{A}_g$ to that of G. This is an algebraic construction of the map given by the map from G to $\mathrm{Sp}_{2g}(\mathbf{Z})$. Even though it would be possible to give a purely arithmetic construction of the detecting subgroups that we will consider, it will be (at least) as easy to construct the principal abelian variety with a group action, and we shall do exactly that. We shall then also continue to use the stack language. An extra benefit of this way of presenting the argument is that we directly get the lower bound also in positive characteristic. In that case, if one knew that the specialization map in the cohomology of $\mathcal{A}_g$ induced an isomorphism, then the result would follow from the complex version. Though such a specialization result seems rather straightforward using the proper and smooth base change theorems, the toroidal compactifications of Chai and Faltings and an induction on g, we know of no reference. Our argument will avoid reference to such a specialization result.

Even though we shall apply our results to obtain information on the order of λ_g, the results we obtain should be of independent interest.

We start with some preliminary comments on the cohomology of stacks, and in particular how to extend results from the cohomology of schemes to that of algebraic stacks. If $\mathcal{A}\to S$ is an algebraic stack over a scheme S we may find a chart $U\to\mathcal{A}$, i.e., a smooth surjective map from a scheme U. We may then consider the simplicial S-scheme $U\times_{\mathcal{A}\bullet}$ which in degree n is $U\times_{\mathcal{A}}U\times_{\mathcal{A}}\dots\times_{\mathcal{A}}U$ ($n+1$ times) with the obvious face and degeneracy maps. We then have an augmentation map $U\times_{\mathcal{A}\bullet}\to\mathcal{A}$ which by smooth descent induces an isomorphism in cohomology. For any simplicial scheme $X_\bullet$ there is (cf. [1, Exp. $\mathrm{V}^{\mathrm{bis}}$]) a spectral sequence which at the E_1-term is the cohomology of X_n (with coefficients in the sheaf whose cohomology should be computed) converging to the cohomology of $X_\bullet$. We may therefore reduce a number of questions on the cohomology of stacks to that of schemes:

(1) If $S=\operatorname{Spec}\mathbf{C}$ we may use the comparison theorem between étale and classical cohomology with finite (or l-adic) coefficients to get a comparison theorem for the étale and classical cohomology of algebraic stacks.

(2) If S is a discrete valuation ring then for any smooth map $X\to S$ the smooth base change theorem gives us a specialization map (cf. [6, Exp. XIII] or [5, Chapter V, (1.6.1)] where it is called the "cospecialization map") $H^*(X_{\bar\eta},\mathbf{Z}/n)\to H^*(X_{\bar s},\mathbf{Z}/n)$, where n is invertible on S, $\bar s$ is a geometric point over the special fibre of S, and $\bar\eta$ is a geometric point over the generic. This extends immediately to smooth simplicial S-schemes and hence to smooth S-stacks.

(3) If $S=\operatorname{Spec}\mathbf{C}$ we get that the de Rham–Chern classes of a flat vector bundle over a smooth stack are trivial. In fact, the Chern–Weil construction has been done for simplicial manifolds (cf. [7]).

When we will be talking about l-adic cohomology for an algebraic stack $\pi\colon\mathcal{A}\to\operatorname{Spec}\mathbf{k}$ over a field we shall always mean the higher direct images of π, or equivalently the cohomology of the pullback of $\mathcal{A}$ to a separable closure of $\mathbf{k}$.

Note now that the *de Rham bundle*, $H^1_{\mathrm{dR}}:=R^1\pi_*\Omega^\bullet_{\mathcal{X}_g/\mathcal{A}_g}$, where $\pi\colon\mathcal{X}_g\to\mathcal{A}_g$ is the universal family, is indeed provided with an integrable connection. Hence its Chern classes in integral cohomology are torsion classes. By the comparison and specialization theorems they are thus also torsion classes in l-adic cohomology,(2) which we will denote $r_i \in H^{2i}(\mathcal{A}_g, \mathbf{Z}_l(i))$. In this section we shall determine their exact order up to a factor of 2.

We begin by using a result of Grothendieck to get an upper bound for the order of r_i.

PROPOSITION 4.1. *We have*

(i) $r_i = 0$ *for odd* i;

(ii) $n_i r_{2i} = 0$.

Proof. The first part follows immediately because H^1_{dR} is a symplectic vector bundle.

As for the second part we may assume that the characteristic is 0, since the case of positive characteristic follows from the case of characteristic 0 by specialization as the Chern classes are compatible with specialization maps. In that case we may further reduce to the case of the base field being the complex numbers. We may also prove the annihilation of $n_i r_{2i}$ in l-adic cohomology for a specific (but arbitrary) prime l.

Now, the existence of the Gauss–Manin connection on H^1_{dR} means that H^1_{dR} has a discrete structure group. More precisely, the (classical) fundamental group of the algebraic stack $\mathcal{A}_g$ is $\mathrm{Sp}_{2g}(\mathbf{Z})$, and H^1_{dR} is the vector bundle associated to the representation of it given by the natural inclusion of $\mathrm{Sp}_{2g}(\mathbf{Z})$ in $\mathrm{Sp}_{2g}(\mathbf{C})$. This complex representation is obviously defined over the rational numbers, so we may apply [12, 4.8] with field of definition $\mathbf{Q}$. We thus conclude that $r_{2i} \in H^{2i}(\mathcal{A}_g, \mathbf{Z}_l(i))$ (for the analytic stack, and hence by the comparison theorem, for the algebraic stack) is killed by $l^{\alpha(i)}$, where $\alpha(i)$ is defined as

$$\inf_{\lambda \in H} v_l(\lambda^i - 1)$$

and $H \subseteq \mathbf{Z}_l^*$ is the image of the Galois group of the field of definition of the cyclotomic character. However, as the base field is $\mathbf{Q}$ this image is all of $\mathbf{Z}_l^*$, and the result follows from the definition of n_i. □

We now aim to show that this upper bound is the precise order of the r_i, using, as was explained, actions of finite cyclic groups on principally polarized abelian varieties.

PROPOSITION 4.2. *Assume that* $i \leqslant g$. *The order of* r_{2i} *is divisible by* $\frac{1}{2} n_i$ *over* $\mathbf{C}$. *In general, for each prime* l *different from the characteristic of the base field,* r_{2i} *in* l*-adic cohomology has order divisible by the* l*-part of* $\frac{1}{2} n_i$.

(2) The number l being as usual a prime different from the characteristic.

Proof. The l-adic part implies the integral cohomology part, so we may pick a prime l different from the characteristic of the base field and look at r_{2i} in l-adic cohomology. What we want to show is that if l is odd, $l-1|2i$ and k is the largest integer such that $l^{k-1}(l-1)|2i$, then r_{2i} has order at least l^k, and similarly for $l=2$. We will do this by defining a map $B\mathbf{Z}/l^k \to \mathcal{A}_g$ such that the inverse image of the r_{2i} to $H^{4i}(\mathbf{Z}/l^k, \mathbf{Z}_l)$ has order l^k. Now, $H^{4i}(\mathbf{Z}/l^k, \mathbf{Z}_l)$ is isomorphic to $\mathbf{Z}/l^k$, and hence an element in it has order l^k if and only if its reduction modulo l is non-zero. As $H^{4i}(\mathbf{Z}/l^k, \mathbf{Z}_l)/l$ injects into $H^{4i}(\mathbf{Z}/l^k, \mathbf{Z}/l)$ it is enough to show that the pullback of r_{2i} is non-zero in $H^{4i}(\mathbf{Z}/l^k, \mathbf{Z}/l)$. Note that a map $B\mathbf{Z}/l^k \to \mathcal{A}_g$ consists of a principally polarized g-dimensional abelian variety A over the base field together with an action on it (preserving the polarization) by $\mathbf{Z}/l^k$. The pullback of r_{2i} is then obtained as follows: The group $\mathbf{Z}/l^k$ acts on $H^1_{\mathrm{dR}}(A)$. A representation of $\mathbf{Z}/l^k$ has Chern classes([3]) in $H^*(\mathbf{Z}/l^k, \mathbf{Z}_l)$, and the pullback of r_{2i} is the $2i$th Chern class of $H^1_{\mathrm{dR}}(A)$.

Assume now to begin with that l is odd. Consider any Galois cover $C \to \mathbf{P}^1$ with Galois group $\mathbf{Z}/l^k$ which is ramified at 0, 1 and ∞ with ramification group of order l^k, l^k and l, respectively (the existence of such a cover follows directly from Kummer theory). By the Hurwitz formula the genus of such a covering fulfills the relation $2g-2= -2l^k+2(l^k-1)+l^{k-1}(l-1)$, i.e., $2g=l^{k-1}(l-1)$.

We claim that the action of $\mathbf{Z}/l^k$ on $H^1_{\mathrm{dR}}(C)$ is isomorphic to the sum of all primitive([4]) characters of $\mathbf{Z}/l^k$. Admitting that for the moment, we can go on with computing its total Chern class. Choose a primitive l^kth root of unity ζ and use it in particular to identify $\mathbf{Z}/l$ with μ_l. Let $x \in H^2(\mathbf{Z}/l^k, \mathbf{Z}/l)$ be the generator given as the first Chern class of the character that takes 1 to ζ. Then the first Chern class of the character of $\mathbf{Z}/l^k$ that takes 1 to ζ^i equals ix, and hence the total Chern class of that character is $1+ix$. By the multiplicativity of the total Chern class we get that the total Chern class of $H^1_{\mathrm{dR}}(C)$ equals $\prod_{(i,l)=1}(1+ix)$, which in turn is equal to $(1+x^{l-1})^{l^{k-1}}=1+x^{l^{k-1}(l-1)}$, and hence the $l^{k-1}(l-1)$st Chern class is non-zero. If $2i$ instead is a proper multiple, $2i=ml^{k-1}(l-1)$, we look at the mth power of the jacobian of C (with the diagonal action of $\mathbf{Z}/l^k$). This gives the non-triviality when $g=i$, and when $g>i$ we simply add a principally polarized factor on which $\mathbf{Z}/l^k$ acts trivially.

It remains to show that the action of $\mathbf{Z}/l^k$ on $H^1_{\mathrm{dR}}(C)$ is as claimed. We start with a remark on Chern classes of representations of a finite group. They only depend on the corresponding elements in the representation ring, and those elements in turn are determined by their character (i.e., the traces of the actions of the group elements). We

([3]) In the classical case only for a complex representation, but Grothendieck (cf. [12]) extended it to a representation over any field. In any case our representations will be ordinary and can thus be lifted to characteristic 0.

([4]) That is, of order exactly l^k.

shall therefore speak of the Chern classes of a character.

The Lefschetz fixed-point formula gives a formula for the character in terms of the number of fixed points([5]) of the elements of $\mathbf{Z}/l^k$. A proof using this formula is certainly possible, but the following argument requires less computations. As the Lefschetz fixed-point formula is also valid for l-adic cohomology, the character looked for is the same as that for the action on $H^1(C, \mathbf{Q}_l)$, or more precisely, the character of the representation on $H^1_{\mathrm{dR}}(C)$ is the reduction modulo p of the character on $H^1(C, \mathbf{Q}_l)$. As the action of $\mathbf{Z}/l^k$ on C is faithful, its action on $H^1(C, \mathbf{Q}_l)$ is faithful as well, so that at least one primitive character has to appear in $H^1(C, \mathbf{Q}_l)$. On the other hand, the l^kth cyclotomic polynomial is irreducible over $\mathbf{Q}_l$, so if one primitive character appears they must all appear. Now, the number of such characters is $l^{k-1}(l-1)$, which equals the dimension of $H^1(C, \mathbf{Q}_l)$, so that $H^1(C, \mathbf{Q}_l)$ consists of each primitive character exactly once.

When $l=2$ we may assume that $k>2$, as the lower bound to be proved for $k\leqslant 2$ is implied by the one for $k=3$. We then make essentially the same construction, a Galois cover with group $\mathbf{Z}/2^k$ of $\mathbf{P}^1$ ramified at three points with ramification groups of order 2^k, 2^k and 2, respectively, of genus $g=2^{k-3}$. The rest of the argument is identical to the odd case. □

Remark 4.3. (i) It follows from (1.1) that the r_i are torsion classes already in the Chow groups. Our result gives a lower bound for this order, but we do not know if this bound is sharp.

(ii) From the complex point of view our geometric construction can be seen simply as constructing an element of order l^k in $\mathrm{Sp}_{l^{k-1}(l-1)}(\mathbf{Z})$. This can be done directly; in the odd case one may consider the ring of l^kth roots of unity $R=\mathbf{Z}[\zeta]$ with the obvious action of $\mathbf{Z}/l^k$ and the symplectic form $\langle\alpha,\bar\beta\rangle:=\mathrm{Tr}(\alpha\beta(\zeta-\zeta^{-1})^{-l^k+l^{k-1}+1})$. This is obviously a symplectic invariant form, and that it is indeed an integer-valued perfect pairing follows from the fact that the different of R is the ideal generated by $(\zeta-\zeta^{-1})^{l^k-l^{k-1}-1}$.

(iii) When $g=\frac{1}{2}l^k(l-1)$ we actually get a lower bound for the top Chern class of the de Rham cohomology of the universal curve over $\mathcal{M}_g$. However, there is no direct analogue of the trick of adding a factor with trivial action, so this does not give a lower bound for all $g\geqslant\frac{1}{2}l^k(l-1)$.

THEOREM 4.4. *We have that $r_{2i+1}=0$ for all i and that the order of r_{2i} in integral (resp. l-adic) cohomology equals (resp. equals the l-part of) $\frac{1}{2}n_i$ or n_i for $i\leqslant g$.*

Proof. This follows immediately from Propositions 4.2 and 4.1. □

([5]) Note that as the orders of these elements are prime to the characteristic, each fixed point is counted with multiplicity 1.

COROLLARY 4.5. *The order of λ_g is divisible by $\frac{1}{2}n_g$.*

Proof. The top Chern class of H^1_{dR} is λ_g^2. □

Remark 4.6. Our upper and lower bounds for r_{2i} are off by a (multiplicative) factor of 2. Furthermore, when $g=1$ the lower bound is the correct order. It is tempting to believe that it is the lower bound that is the correct one for all g, and furthermore that that should be related to the fact that we have a symplectic rather than a general linear representation.

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Received October 29, 2001
Received in revised form February 7, 2003