SOME PROBLEMS OF 'PARTITIO NUMERORUM'; III: ON THE EXPRESSION OF A NUMBER AS A SUM OF PRIMES.

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1. Introduction.

1. 1. It was asserted by GOLDBACH, in a letter to EULER dated 7 June, 1742, that *every even number 2m is the sum of two odd primes*, and this proposition has generally been described as 'Goldbach's Theorem'. There is no reasonable doubt that the theorem is correct, and that the number of representations is large when m is large; but all attempts to obtain a proof have been completely unsuccessful. Indeed it has never been shown that every number (or every *large* number, any number, that is to say, from a certain point onwards) is the sum of 10 primes, or of 1 000 000; and the problem was quite recently classified as among those 'beim gegenwärtigen Stande der Wissenschaft unangreifbar'.[1]

In this memoir we attack the problem with the aid of our new transcendental method in 'additiver Zahlentheorie'.[2] We do not solve it: we do not

[1] E. LANDAU, 'Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion', *Proceedings of the fifth International Congress of Mathematicians*, Cambridge, 1912, vol. 1, pp. 93—108 (p. 105). This address was reprinted in the *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 21 (1912), pp. 208—228.

[2] We give here a complete list of memoirs concerned with the various applications of this method.

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1. 'Asymptotic formulae in combinatory analysis', *Comptes rendus du quatrième Congrès des mathematiciens Scandinaves à Stockholm*, 1916, pp. 45—53.

2. 'On the expression of a number as the sum of any number of squares, and in particular of five or seven', *Proceedings of the National Academy of Sciences*, vol. 4 (1918), pp. 189—193.

even prove that any number is the sum of 1 000 000 primes. In order to prove anything, we have to assume the truth of an unproved hypothesis, and, even on this hypothesis, we are unable to prove Goldbach's Theorem itself. We show, however, that the problem is not 'unangreifbar', and bring it into contact with the recognized methods of the Analytic Theory of Numbers.

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3. 'Some famous problems of the Theory of Numbers, and in particular Waring's Problem' (Oxford, Clarendon Press, 1920, pp. 1—34).

4. 'On the representation of a number as the sum of any number of squares, and in particular of five', *Transactions of the American Mathematical Society*, vol. 21 (1920), pp. 255—284.

5. 'Note on Ramanujan's trigonometrical sum $c_q(n)$', *Proceedings of the Cambridge Philosophical Society*, vol. 20 (1921), pp. 263—271.

G. H. Hardy and J. E. Littlewood.

1. 'A new solution of Waring's Problem', *Quarterly Journal of pure and applied mathematics*, vol. 48 (1919), pp. 272—293.

2. 'Note on Messrs. Shah and Wilson's paper entitled: On an empirical formula connected with Goldbach's Theorem', *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 245—254.

3. 'Some problems of 'Partitio numerorum'; I: A new solution of Waring's Problem', *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen* (1920), pp. 33—54.

4. 'Some problems of 'Partitio numerorum'; II: Proof that any large number is the sum of at most 21 biquadrates', *Mathematische Zeitschrift*, vol. 9 (1921), pp. 14—27.

G. H. Hardy and S. Ramanujan.

1. 'Une formule asymptotique pour le nombre des partitions de n', *Comptes rendus de l'Académie des Sciences*, 2 Jan. 1917.

2. 'Asymptotic formulae in combinatory analysis', *Proceedings of the London Mathematical Society*, ser. 2, vol. 17 (1918), pp. 75—115.

3. 'On the coefficients in the expansions of certain modular functions', *Proceedings of the Royal Society of London* (A), vol. 95 (1918), pp. 144—155.

E. Landau.

1. 'Zur Hardy-Littlewood'schen Lösung des Waringschen Problems', *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen* (1921), pp. 88—92.

L. J. Mordell.

1. 'On the representations of numbers as the sum of an odd number of squares', *Transactions of the Cambridge Philosophical Society*, vol. 22 (1919), pp. 361—372.

A. Ostrowski.

1. 'Bemerkungen zur Hardy-Littlewood'schen Lösung des Waringschen Problems', *Mathematische Zeitschrift*, vol. 9 (1921), pp. 28—34.

S. Ramanujan.

1. 'On certain trigonometrical sums and their applications in the theory of numbers', *Transactions of the Cambridge Philosophical Society*, vol. 22 (1918), pp. 259—276.

N. M. Shah and B. M. Wilson.

1. 'On an empirical formula connected with Goldbach's Theorem', *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 238—244.

Our main result may be stated as follows: *if a certain hypothesis* (a natural generalisation of Riemann's hypothesis concerning the zeros of his Zeta-function) *is true, then every large odd number n is the sum of three odd primes; and the number of representations is given asymptotically by*

$$\bar{N}_3(n) \backsim C_3 \frac{n^2}{(\log n)^3} \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p}-1)(\mathfrak{p}-2)}{\mathfrak{p}^2-3\mathfrak{p}+3}\right), \tag{1. 11}$$

where $\mathfrak{p}$ *runs through all odd prime divisors of* n, *and*

$$C_3 = \Pi\left(1+\frac{1}{(\varpi-1)^3}\right), \tag{1. 12}$$

the product extending over all odd primes ϖ.

Hypothesis R.

1. 2. We proceed to explain more closely the nature of our hypothesis. Suppose that q is a positive integer, and that

$$h = \varphi(q)$$

is the number of numbers less than q and prime to q. We denote by

$$\chi(n) = \chi_k(n) \qquad (k = 1, 2, \ldots, h)$$

one of the h Dirichlet's 'characters' to modulus q[1]: χ_1 is the 'principal' character.

By $\bar{\chi}$ we denote the complex number conjugate to χ: $\bar{\chi}$ is a character.

By $L(s, \chi)$ we denote the function defined for $\sigma > 1$ by

$$L(s) = L(\sigma+it) = L(s, \chi) = L(s, \chi_k) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Unless the contrary is stated the modulus is q. We write

$$\bar{L}(s) = L(s, \bar{\chi}).$$

By

$$\varrho = \beta + i\gamma$$

[1] Our notation, so far as the theory of L-functions is concerned, is that of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, book 2, pp. 391 *et seq.*, except that we use q for his k, k for his $\varkappa$, and ϖ for a typical prime instead of p. As regards the 'Farey dissection', we adhere to the notation of our papers *3* and *4*.

We do not profess to give a complete summary of the relevant parts of the theory of the L-functions; but our references to Landau should be sufficient to enable a reader to find for himself everything that is wanted.

we denote a typical zero of $L(s)$, those for which $\gamma = 0$, $\beta \leqq 0$ being excluded. We call these the *non-trivial* zeros. We write $N(T)$ for the number of ϱ's of $L(s)$ for which $0 \leqq \gamma \leqq T$.

The natural extension of Riemann's hypothesis is

HYPOTHESIS R. Every ϱ has its real part less than or equal to $\frac{1}{2}$.*[1]

We shall not have to use the full force of this hypothesis. What we shall in fact assume is

HYPOTHESIS R. There is a number $\Theta < \frac{3}{4}$ such that

$$\beta \leqq \Theta$$

for every ϱ of every $L(s)$.

The assumption of this hypothesis is fundamental in all our work; *all the results of the memoir, so far as they are novel, depend upon it*[2]; and we shall not repeat it in stating the conditions of our theorems.

We suppose that Θ has its smallest possible value. In any case $\Theta \geqq \frac{1}{2}$. For, if ϱ is a complex zero of $L(s)$, $\bar{\varrho}$ is one of $\bar{L}(s)$. Hence $1 - \bar{\varrho}$ is one of $\bar{L}(1-s)$, and so, by the functional equation[3], one of $L(s)$.

Further notation and terminology.

1. 3. We use the following notation throughout the memoir.

A is a positive absolute constant wherever it occurs, but not the same constant at different occurrences. B is a positive constant depending on the single parameter r. O's refer to the limit process $n \to \infty$, the constants which they involve being of the type B, and o's are uniform in all parameters *except* r.

ϖ is a prime. $\mathfrak{p}$ (which will only occur in connection with n) is an odd prime divisor of n. p is an integer. If $q = 1$, $p = 0$; otherwise

$$0 < p < q, \quad (p, q) = 1,$$

(m, n) is the greatest common factor of m and n. By $m \mid n$ we mean that n is divisible by m; by $m \nmid n$ the contrary.

$\Lambda(n)$, $\mu(n)$ have the meanings customary in the Theory of Numbers. Thus $\Lambda(n)$ is $\log \varpi$ if $n = \varpi^m$ and zero otherwise: $\mu(n)$ is $(-1)^k$ if n is a product of

[1] The hypothesis must be stated in this way because

(a) it has not been proved that no $L(s)$ has real zeros between $\frac{1}{2}$ and 1,

(b) the L-functions as ociated with *imprimitive* (uneigentlich) characters have zeros on the line $\sigma = 0$.

[2] Naturally many of the results stated incidentally do not depend upon the hypothesis.

[3] Landau, p. 489. All references to 'Landau' are to his *Handbuch*, unless the contrary is stated.

k different prime factors, and zero otherwise. The fundamental function with which we are concerned is

(1. 31) $$f(x)=\sum_{\varpi}\log\varpi\, x^{\varpi}.$$

To simplify our formulae we write

$$e(x)=e^{2\pi i x},\quad e_q(x)=e\left(\frac{x}{q}\right).$$

Also

(1. 32) $$c_q(n)=\sum_{p} e_q(np).$$

If χ_k is primitive,

(1. 33) $$\tau_k=\tau(\chi_k)=\sum_{p} e_q(p)\,\chi_k(p)=\sum_{m=1}^{q} e_q(m)\,\chi_k(m).$$[1]

This sum has the absolute value[2] $\sqrt{q}$.

The Farey dissection.

1. 4. We denote by Γ the circle

(1. 41) $$|x|=e^{-H}=e^{-\frac{1}{n}}.$$

We divide Γ into arcs $\xi_{p,q}$ which we call *Farey arcs*, in the following manner. We form the Farey's series of order

(1. 42) $$N=[\sqrt{n}],$$

the first and last terms being $\frac{0}{1}$ and $\frac{1}{1}$. We suppose that $\frac{p}{q}$ is a term of the series, and $\frac{p'}{q'}$ and $\frac{p''}{q''}$ the adjacent terms to the left and right, and denote by $j_{p,q}$ $(q>1)$ the intervals

$$\frac{p}{q}-\frac{1}{q(q+q')},\quad \frac{p}{q}+\frac{1}{q(q+q'')};$$

by $j_{0,1}$ and $j_{1,1}$ the intervals $\left(0,\frac{1}{N+1}\right)$ and $\left(1-\frac{1}{N+1},1\right)$. These intervals just

[1] $\chi_k(m)=0$ if $(m,q)>1$.

[2] Landau, p. 497.

fill up the interval (0, 1), and the length of each of the parts into which $j_{p,q}$ is divided by $\frac{p}{q}$ is less than $\frac{1}{qN}$ and not less than $\frac{1}{2qN}$. If now the intervals $j_{p,q}$ are considered as intervals of variation of $\frac{\theta}{2\pi}$, where $\theta = \arg x$, and the two extreme intervals joined into one, we obtain the desired dissection of Γ into arcs $\xi_{p,q}$.[1]

When we are studying the arc $\xi_{p,q}$, we write

$$(1.43)\qquad x = e^{\frac{2p\pi i}{q}} X = e_q(p) X = e_q(p) e^{-Y},$$

$$(1.44)\qquad Y = \eta + i\theta.$$

The whole of our work turns on the behaviour of $f(x)$ as $|x| \to 1$, $\eta \to 0$, and we shall suppose throughout that $0 < \eta \leqq \frac{1}{2}$. When x varies on $\xi_{p,q}$, X varies on a congruent arc $\zeta_{p,q}$, and

$$\theta = -\left(\arg x - \frac{2p\pi}{q}\right)$$

varies (in the inverse direction) over an interval $-\theta'_{p,q} \leqq \theta \leqq \theta_{p,q}$. Plainly $\theta_{p,q}$ and $\theta'_{p,q}$ are less than $\frac{2\pi}{qN}$ and not less than $\frac{\pi}{qN}$, so that

$$\bar{\theta}_{p,q} = \operatorname{Max}(\theta_{p,q}, \theta'_{p,q}) < \frac{A}{qN}.$$

In all cases $Y^{-s} = (\eta + i\theta)^{-s}$ has its principal value

$$\exp(-s \log(\eta + i\theta)),$$

wherein (since η is positive)

$$-\frac{1}{2}\pi < \Im \log(\eta + i\theta) < \frac{1}{2}\pi.$$

By $N_r(n)$ we denote the number of representations of n by a sum of r primes, attention being paid to order, and repetitions of the same prime being allowed, so that

$$(1.45)\qquad \sum_{n=2}^{\infty} N_r(n) x^n = \left(\sum_{\varpi} x^{\varpi}\right)^r.$$

[1] The distinction between major and minor arcs, fundamental in our work on Waring's Problem, does not arise here.

By $\nu_r(n)$ we denote the sum

(1. 46) $$\nu_r(n) = \sum_{\varpi_1+\varpi_2+\cdots+\varpi_r=n} \log\varpi_1 \log\varpi_2 \ldots \log\varpi_r,$$

so that

(1. 47) $$\sum_{n=2}^{\infty} \nu_r(n)\, x^n = (f(x))^r.$$

Finally S_r is the *singular series*

(1. 48) $$S_r = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n).$$

2. Preliminary lemmas.

2. 1. *Lemma 1. If* $\eta = \Re(Y) > 0$ *then*

(2. 11) $$f(x) = f_1(x) + f_2(x),$$

where

(2. 12) $$f_1(x) = \sum_{(q,n)>1} \Lambda(n)x^n - \sum_{\varpi} \log\varpi\,(x^{\varpi^2} + x^{\varpi^3} + \cdots),$$

(2. 13) $$f_2(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\,\Gamma(s)\,Z(s)\,ds,$$

Y^{-s} *has its principal value,*

(2. 14) $$Z(s) = \sum_{k=1}^{h} C_k \frac{L'_k(s)}{L_k(s)},$$

C_k *depends only on* p, q *and* χ_k,

(2. 15) $$C_1 = -\frac{\mu(q)}{h}$$

and

(2. 16) $$|C_k| \leq \frac{\sqrt{q}}{h}.$$

We have

$$f_2(x) = f(x) - f_1(x) = \sum_{(q,n)=1} \Lambda(n)\, x^n$$

$$= \sum_{1 \leq j \leq q,\,(q,j)=1} e_q(pj) \sum_{l=0}^{\infty} \Lambda(lq+j)\, e^{-(lq+j)Y}$$

$$= \sum_j e_q(pj) \sum_l \Lambda(lq+j) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)(lq+j)^{-s}\, ds,$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s) Z(s)\, ds,$$

where

$$Z(s) = \sum_j e_q(pj) \sum_l \frac{\Lambda(lq+j)}{(lq+j)^s}.$$

Since $(q, j) = 1$, we have[1]

$$\sum_l \frac{\Lambda(lq+j)}{(lq+j)^s} = -\frac{1}{h} \sum_{k=1}^{h} \bar{\chi}_k(j) \frac{L'_k(s)}{L_k(s)};$$

and so

$$Z(s) = \sum_{k=1}^{h} C_k \frac{L'_k(s)}{L_k(s)},$$

where

$$C_k = -\frac{1}{h} \sum_{j=1}^{q} e_q(pj)\, \bar{\chi}_k(j).$$

Since $\bar{\chi}_k(j) = 0$ if $(q, j) > 1$, the condition $(q, j) = 1$ may be omitted or retained at our discretion.

Thus[2]

$$C_1 = -\frac{1}{h} \sum_{1 \leq j \leq q,\,(q,j)=1} e_q(pj)$$

$$= -\frac{1}{h} \sum_{1 \leq m \leq q,\,(q,m)=1} e_q(m) = -\frac{\mu(q)}{h}.$$

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[1] Landau, p. 421.

[2] Landau, pp. 572—573.

Again, if $k > 1$ we have[1]

$$C_k = -\frac{1}{h}\sum_{j=1}^{q} e_q(pj)\,\bar{\chi}_k(j) = -\frac{\chi_k(p)}{h}\sum_{m=1}^{q} e_q(m)\,\bar{\chi}_k(m).$$

If $\bar{\chi}_k$ is a primitive character,

$$\sum_{m=1}^{q} e_q(m)\,\bar{\chi}_k(m) = \tau(q, \bar{\chi}_k),$$

$$|\tau(q, \bar{\chi}_k)| = \sqrt{q},^{2}$$

$$|C_k| = \frac{\sqrt{q}}{h}.$$

If $\bar{\chi}$ is imprimitive, it belongs to $Q = \frac{q}{d}$, where $d > 1$. Then $\bar{\chi}_k(m)$ has the period Q, and

$$\sum_{m=1}^{q} e_q(m)\,\bar{\chi}_k(m) = \sum_{n=1}^{Q} e_q(n)\,\bar{\chi}_k(n)\sum_{l=0}^{d-1} e_q(lQ).$$

The inner sum is zero. Hence $C_k = 0$, and the proof of the lemma is completed.[3]

2. 2. *Lemma 2. We have*

(2. 21)
$$|f_1(x)| < A(\log(q+1))^A \eta^{-\frac{1}{2}}.$$

We have

$$f_1(x) = \sum_{(q,n)>1} \Lambda(n)x^n - \sum_{\varpi} \log\varpi(x^{\varpi^2} + x^{\varpi^3} + \cdots) = f_{1,1}(x) - f_{1,2}(x).$$

But

$$|f_{1,1}(x)| \leq \sum_{\varpi | q} \log\varpi \sum_{r=1}^{\infty} |x|^{\varpi^r}$$

$$< A\log(q+1)\log q\sum_{r=1}^{\infty}|x|^{2^r} < A(\log(q+1))^2\sum_{r=1}^{\infty} e^{-\eta 2^r}$$

$$< A(\log(q+1))^A \log\frac{1}{\eta} < A(\log(q+1))^A \eta^{-\frac{1}{2}}.$$

[1] Landau, p. 485. The result is stated there only for a primitive character, but the proof is valid also for an imprimitive character when $(p, q) = 1$.

[2] Landau, pp. 485, 489, 492.

[3] See the additional note at the end.

Also

$$\sum_{r \geqq 2, \varpi^r \leqq \xi} \log \varpi < A\sqrt{\xi},$$

and so

$$|f_{1,2}(x)| \leqq \sum_{r \geqq 2, \varpi} \log \varpi |x|^{\varpi^r} < A(1-|x|)\sum_n \sqrt{n}|x|^n$$

$$< A(1-|x|)^{-\frac{1}{2}} < A\eta^{-\frac{1}{2}}.$$

From these two results the lemma follows.

2. 3. *Lemma 3. We have*

(2. 31) $$\frac{L'(s)}{L(s)} = -\frac{\mathfrak{b}}{s-1} + \frac{\mathfrak{d}-\mathfrak{b}}{s} + b - \frac{1}{2}\psi\left(\frac{s+\mathfrak{a}}{2}\right) + \sum_\varrho \left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right),$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

the $\mathfrak{a}$'s, $\mathfrak{b}$'s, $\mathfrak{d}$'s *and* b's *are constants depending upon* q *and* χ, $\mathfrak{a}$ *is* 0 *or* 1,

(2. 32) $$\mathfrak{b}_1 = 1, \qquad \mathfrak{b}_k = 0 \qquad (k > 1),$$

and

(2. 33) $$0 \leqq \mathfrak{d} < A \log(q+1).$$

All these results are classical except the last.[1]

The precise definition of $\mathfrak{d}$ is rather complicated and does not concern us. We need only observe that $\mathfrak{d}$ does not exceed the number of different primes that divide q,[2] and so satisfies (2. 33).

2. 41. *Lemma 4. If* $0 < \eta \leqq \frac{1}{2}$, *then*

(2. 411) $$f(x) = \frac{\mu(q)}{hY} + \sum_{k=1}^{h} C_k G_k + P,$$

where

(2. 412) $$G_k = \sum_{\varrho_k} \Gamma(\varrho) Y^{-\varrho},$$

[1] Landau, pp. 509, 510, 519.

[2] Landau, p. 511 (footnote).

(2. 413) $$|P| < A\sqrt{q}\,(\log(q+1))^A\left(\frac{1}{h}\sum_{k=1}^{h}|b_k| + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}}\delta^{-\frac{1}{2}}\right),$$

(2. 414) $$\delta = \operatorname{arc\,tan}\frac{\eta}{|\theta|}.$$

We have, from (2. 13) and (2. 14),

(2. 415) $$f_2(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)\,Z(s)\,ds$$

$$= \sum_{k=1}^{h}\frac{C_k}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)\frac{L'_k(s)}{L_k(s)}ds = \sum_{k=1}^{h} C_k f_{2,k}(x),$$

say. But[1]

(2.416) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}ds = -\frac{\mathfrak{b}}{Y} + R + \sum_{\varrho}\Gamma(\varrho)\,Y^{-\varrho} + \frac{1}{2\pi i}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}ds,$$

where

$$R = \left\{Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}\right\}_0,$$

$\{f(s)\}_0$ denoting generally the residue of $f(s)$ for $s = 0$.

Now[2]

$$\frac{L'(s)}{L(s)} = \log\frac{\pi}{Q} + \sum_{\nu=1}^{\mathfrak{c}}\frac{\varepsilon_\nu \log \varpi_\nu}{\varpi_\nu^s - \varepsilon_\nu} + \sum_{\nu=1}^{\mathfrak{c}}\frac{\bar{\varepsilon}_\nu \log \varpi_\nu}{\varpi_\nu^{1-s} - \bar{\varepsilon}_\nu}$$

$$-\frac{1}{2}\psi\left(\frac{s+\mathfrak{a}}{2}\right) - \frac{1}{2}\psi\left(\frac{1-s+\mathfrak{a}}{2}\right) - \frac{\bar{L}'(1-s)}{\bar{L}(1-s)},$$

where Q is the divisor of q to which χ belongs, $\mathfrak{c}$ is the number of primes which divide q but not Q, $\varpi_1, \varpi_2, \ldots$ are the primes in question, and ε_ν is a root of unity. Hence, if $\sigma = -\frac{1}{4}$, we have

[1] This application of Cauchy's Theorem may be justified on the lines of the classical proof of the 'explicit formulae' for $\psi(x)$ and $\pi(x)$: see Landau, pp. 333—368. In this case the proof is much easier, since $Y^{-s}\Gamma(s)$ tends to zero, when $|t| \to \infty$, like an exponential $e^{-a|t|}$ Compare pp. 134—135 of our memoir 'Contributions to the theory of the Riemann Zeta-function and the theory of the distribution of primes', *Acta Mathematica*, vol. 41 (1917), pp. 119—196.

[2] Landau, p. 517.

(2. 417) $$\left|\frac{L'(s)}{L(s)}\right| < A \log q + A\mathfrak{c} \log q + A \log(|t|+2) + A$$

$$< A(\log(q+1))^A \log(|t|+2).$$

Again, if $s = -\frac{1}{4} + it$, $Y = \eta + i\theta$, we have

$$|Y^{-s}| = |Y|^{\frac{1}{4}} \exp\left(t \operatorname{arc\,tan} \frac{\theta}{\eta}\right),$$

$$|Y^{-s}\Gamma(s)| < A|Y|^{\frac{1}{4}}(|t|+2)^{-\frac{3}{4}} \exp\left(-\left(\frac{1}{2}\pi - \operatorname{arc\,tan}\frac{|\theta|}{\eta}\right)|t|\right),$$

$$< A|Y|^{\frac{1}{4}} \frac{|t|^{-\frac{1}{2}}}{\log(|t|+2)} e^{-\delta|t|};$$

and so

(2. 418) $$\left|\frac{1}{2\pi i}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}ds\right| < A(\log(q+1))^A |Y|^{\frac{1}{4}} \int_0^\infty t^{-\frac{1}{2}} e^{-\delta t} dt$$

$$< A(\log(q+1))^A |Y|^{\frac{1}{4}} \delta^{-\frac{1}{2}}.$$

2. 42. We now consider R. Since

$$\sum\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) = 0 \qquad (s = 0),$$

we have

$$R = \{(\mathfrak{b}+b)\Gamma(s)\}_0 + \left\{\frac{\mathfrak{d}-\mathfrak{b}}{s} Y^{-s}\Gamma(s)\right\}_0 - \frac{1}{2}\left\{Y^{-s}\Gamma(s)\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right\}_0$$

$$= A_1(\mathfrak{b}+b) - (\mathfrak{d}-\mathfrak{b})(A_2 + A_3 \log Y) + C_1(\mathfrak{a}) + C_2(\mathfrak{a}) \log Y,$$

where each of the C's has one of two absolute constant values, according to the value of $\mathfrak{a}$. Since

$$0 \leqq \mathfrak{b} \leqq 1, \; 0 \leqq \mathfrak{d} < A \log(q+1), \; |\log Y| < A \log\frac{1}{\eta} < A\eta^{-\frac{1}{2}},$$

we have

(2. 421) $$|R| < A|b| + A \log(q+1)\eta^{-\frac{1}{2}}.$$

From (2. 415), (2. 416), (2. 418), (2. 421) and (2. 15) we deduce

$$f_{2,k}(x) = -\frac{\mathfrak{b}}{Y} + G_k + P_k,$$

$$|P_k| < A\,(\log(q+1))^A \left(|b| + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}}\delta^{-\frac{1}{2}}\right),$$

$$f_2(x) = -\frac{\mu(q)}{hY} + \sum_k C_k G_k + P, \tag{2. 422}$$

$$|P| < A\sqrt{q}\,(\log(q+1))^A \left(\frac{1}{h}\sum_k |b_k| + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}}\delta^{-\frac{1}{2}}\right). \tag{2. 423}$$

Combining (2. 422) and (2. 423) with (2. 11) and (2. 21), we obtain the result of Lemma 4.

2. 5. *Lemma 5. If $q > 1$ and χ_k is a primitive (and therefore non-principal*[1]*) character, then*

$$L(s) = \frac{a e^{bs}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \prod_\varrho \left(\left(1 - \frac{s}{\varrho}\right) e^{\frac{s}{\varrho}}\right), \tag{2. 51}$$

where

$$a = a(q, \chi) = a_k,$$

$$|L(1)| = \pi q^{-\frac{1}{2}} |L(0)| \qquad (\mathfrak{a} = 1), \tag{2. 521}$$

$$|L(1)| = 2 q^{-\frac{1}{2}} |L'(0)| \qquad (\mathfrak{a} = 0). \tag{2. 522}$$

Further

$$1 - \Theta \leq \Re(\varrho) \leq \Theta, \tag{2. 53}$$

and

$$\left|\frac{L'(1)}{L(1)}\right| < A\,(\log(q+1))^A. \tag{2. 54}$$

This lemma is merely a collection of results which will be used in the proof of Lemmas 6 and 7. They are of very unequal depth. The formula (2. 51) is classical.[2] The two next are immediate deductions from the functional equation for $L(s)$.[3] The inequalities (2. 53) follow from the functional equation and the

[1] Landau, p. 480.
[2] Landau, p. 507.
[3] Landau, pp. 496, 497.

absence (for primitive χ) of factors $1-\varepsilon_\nu \varpi_\nu^{-s}$ from L. Finally (2. 54) is due to GRONWALL.[1]

2. 61. *Lemma 6. If $M(T)$ is the number of zeros ϱ of $L(s)$ for which*

$$0 \leq T \leq |\gamma| \leq T+1,$$

then

$$M(T) < A(\log(q+1))^A \log(T+2). \tag{2. 611}$$

The ϱ's of an imprimitive $L(s)$ are those of a certain primitive $L(s)$ corresponding to modulus Q, where $Q|q$, together with the zeros (other than $s=0$) of certain functions

$$E_\nu = 1-\varepsilon_\nu \varpi_\nu^{-s},$$

where

$$|\varepsilon_\nu| = 1, \quad \varpi_\nu | q.$$

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[1] T. H. GRONWALL, 'Sur les séries de Dirichlet correspondant à des caractères complexes', *Rendiconti del Circolo Matematico di Palermo*, vol. 35 (1913), pp. 145—159. Gronwall proves that

$$\frac{1}{|L(1)|} < A \log q (\log \log q)^{\frac{3}{8}}$$

for every *complex* χ, and states that the same is true for *real* χ if hypothesis R (or a much less stringent hypothesis) is satisfied. LANDAU ('Über die Klassenzahl imaginär-quadratischer Zahlkörper', *Göttinger Nachrichten*, 1918, pp. 285—295 (p. 286, f. n. 2)) has, however, observed that, in the case of a real χ, Gronwall's argument leads only to the slightly less precise inequality

$$\frac{1}{|L(1)|} < A \log q \sqrt{\log \log q}.$$

Landau also gives a proof (due to HECKE) that

$$\frac{1}{|L(1)|} < A \log q$$

for the special character $\left(\frac{-q}{n}\right)$ associated with the fundamental discriminant $-q$.

The first results in this direction are due to Landau himself ('Über das Nichtverschwinden der Dirichletschen Reihen, welche komplexen Charakteren entsprechen', *Math. Annalen*, vol. 70 (1911), pp. 69—78). Landau there proves that

$$\frac{1}{|L(1)|} < A (\log q)^5$$

for complex χ.

It is easily proved (see p. 75 of Landau's last quoted memoir) that

$$|L'(1)| < A (\log q)^2,$$

so that any of these results gives us more than all that we require.

The number of ϖ_ν's is less than $A \log(q+1)$, and each E_ν has a set of zeros, on $\sigma = 0$, at equal distances

$$\frac{2\pi}{\log \varpi_\nu} > \frac{2\pi}{\log(q+1)}.$$

The contribution of these zeros to $M(T)$ is therefore less than $A(\log(q+1))^2$; and we need consider only a primitive (and therefore, if $q > 1$, non-principal) $L(s)$.

We observe:

(a) that $\mathfrak{a}$ is the same for $L(s)$ and $\bar{L}(s)$;

(b) that $L(s)$ and $\bar{L}(s)$ are conjugate for real s, so that the b corresponding to $\bar{L}(s)$ is $\bar{b}$, the conjugate of the b of $L(s)$;

(c) that the typical ϱ of $\bar{L}(s)$ may be taken to be either $\bar{\varrho}$ or (in virtue of the functional equation) $1-\varrho$, so that

$$S = \sum\left(\frac{1}{\varrho} + \frac{1}{1-\varrho}\right) = \sum\left(\frac{1}{\varrho} + \frac{1}{\bar{\varrho}}\right)$$

is real.

Bearing these remarks in mind, suppose first that $\mathfrak{a} = 1$. We have then, from (2. 51) and (2. 521),

$$\frac{\pi^2}{q} = \left|\frac{L(1)\bar{L}(1)}{L(0)\bar{L}(0)}\right| = A\left|e^b \prod\left(\left(1-\frac{1}{\varrho}\right)e^{\frac{1}{\varrho}}\right) e^{\bar{b}} \prod\left(\left(1-\frac{1}{1-\varrho}\right)e^{\frac{1}{1-\varrho}}\right)\right|$$

$$= A\, e^{2\Re(b)+S},$$

since

$$\left(1-\frac{1}{\varrho}\right)\left(1-\frac{1}{1-\varrho}\right) = 1.$$

Thus

(2. 612) $$|2\Re(b) + S| < A \log(q+1).$$

On the other hand, if $\mathfrak{a} = 0$, we have, from (2. 51) and (2. 522),

$$\frac{4}{q} = \left|\frac{L(1)\bar{L}(1)}{L'(0)\bar{L}'(0)}\right| = A\left|e^b \prod\left(\left(1-\frac{1}{\varrho}\right)e^{-\frac{1}{\varrho}}\right) e^{\bar{b}} \prod\left(\left(1-\frac{1}{1-\varrho}\right)e^{\frac{1}{1-\varrho}}\right)\right|,$$

and (2. 612) follows as before.

2. 62. Again, by (2. 31)

(2. 621) $$\frac{L'(1)}{L(1)} = \mathfrak{b} + b - \frac{1}{2}\psi\left(\frac{1+\mathfrak{a}}{2}\right) + \sum\left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right),$$

for every non-principal character (whether primitive or not). In particular, when χ is primitive, we have, by (2. 621), (2. 54), and (2. 33),

(2. 622) $$|\Re(b)+S| = \left|\Re\frac{L'(1)}{L(1)} - \mathfrak{b} + \frac{1}{2}\psi\left(\frac{1+\mathfrak{a}}{2}\right)\right| < A\,(\log(q+1))^{A}.$$

Combining (2. 612) and (2. 622) we see that

(2. 623) $$S < A\,(\log(q+1))^{A}$$

and

(2. 624) $$|\Re(b)| < A\,(\log(q+1))^{A}.$$

2. 63. If now $q > 1$, and χ is primitive (so that $\mathfrak{b} = 0$), and $s = 2 + iT$, we have, by (2. 31), (2. 33), and (2. 624),

$$0 < \sum\left(\frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} + \frac{\beta}{\beta^2+\gamma^2}\right) = \Re\sum\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)$$

$$= \Re\frac{L'(s)}{L(s)} - \Re\left(\frac{\mathfrak{b}}{s}\right) - \Re(b) + \frac{1}{2}\Re\left(\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right)$$

$$\leq \left|\frac{L'(s)}{L(s)}\right| + \left|\frac{\mathfrak{b}}{s}\right| + |\Re(b)| + \left|\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right|$$

$$< A + A\log(q+1) + A\,(\log(q+1))^{A} + A\log(|T|+2)$$

$$< A\,(\log(q+1))^{A}\log(|T|+2),$$

$$\sum_{|T-\gamma|\leq 1}\frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} < A\,(\log(q+1))^{A}\log(|T|+2).$$

Every term on the left hand side is greater than A, and the number of terms is not less than $M(T)$. Hence we obtain the result of the lemma. We have excluded the case $q = 1$, when the result is of course classical.[1]

2. 71. *Lemma 7. We have*

(2. 711) $$|b| < Aq\,(\log(q+1))^{A}.$$

Suppose first that χ is non-principal. Then, by (2. 621) and (2. 54),

(2. 712) $$|b| < A\,(\log(q+1))^{A} + \left|\sum\left(\frac{1}{1-\varrho}+\frac{1}{\varrho}\right)\right|.$$

[1] Landau, p. 337.

We write

$$\sum = \sum_1 + \sum_2, \tag{2. 713}$$

where $\sum_1$ is extended over the zeros for which $1 - \Theta \leqq \Re(\varrho) \leqq \Theta$ and $\sum_2$ over those for which $\Re(\varrho) = 0$. Now $\sum_1 = S'$, where S' is the S corresponding to a primitive $L(s)$ for modulus Q, where $Q \mid q$. Hence, by (2. 623),

$$\left|\sum_1\right| < A (\log (Q+1))^A < A (\log (q+1))^A. \tag{2. 714}$$

Again, the ϱ's of $\sum_2$ are the zeros (other than $s = 0$) of

$$\prod_\nu \left(1 - \frac{\varepsilon_\nu}{\varpi_\nu^s}\right),$$

the ϖ_ν's being divisors of q and ε_ν an m-th root of unity, where $m = \varphi(Q) < q$[1]; so that the number of ϖ_ν's is less than $A \log q$ and

$$\varepsilon_\nu = e^{2\pi i \omega_\nu},$$

where either $\omega_\nu = 0$ or

$$\frac{1}{q} \leqq |\omega_\nu| \leqq \frac{1}{2}.$$

Let us denote by ϱ_ν a zero (other than $s = 0$) of $1 - \varepsilon_\nu \varpi_\nu^{-s}$, by ϱ'_ν a ϱ_ν for which $|\varrho_\nu| \leqq 1$, and by ϱ''_ν a ϱ_ν for which $|\varrho_\nu| > 1$. Then

$$\left|\sum_2 \left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right)\right| \leqq \sum_\nu \left(\sum_{\varrho'_\nu} + \sum_{\varrho''_\nu}\right) \left|\frac{1}{1-\varrho} + \frac{1}{\varrho}\right|. \tag{2. 715}$$

Any ϱ_ν is of the form

$$\varrho_\nu = \frac{2\pi i (m + \omega_\nu)}{\log \varpi_\nu},$$

where m is an integer. Hence the number of zeros ϱ'_ν is less than $A \log \varpi_\nu$ or than $A \log (q+1)$; and the absolute value of the corresponding term in our sum is less than

$$\frac{A}{|\varrho|} < \frac{A \log \varpi_\nu}{|\omega_\nu|} < A q \log (q+1); \tag{2. 716}$$

[1] For (Landau, p. 482). $\varepsilon_\nu = X(\varpi_\nu)$, where X is a character to modulus Q.

so that

(2. 717) $$\left|\sum_{\varrho'_\nu}\right| < A q\,(\log (q+1))^2.$$

Also

(2. 718) $$\left|\sum_{\varrho''_\nu}\right| \leq \sum_{\varrho''_\nu}\left|\frac{1}{\varrho(1-\varrho)}\right| < \sum_{\varrho''_\nu}\frac{1}{|\varrho|^2}$$

$$< A\,(\log \varpi_\nu)^2 \sum_{m=1}^{\infty}\frac{1}{m^2} < A\,(\log (q+1))^2.$$

From (2. 715), (2. 717) and (2. 718) we deduce

(2. 719) $$\left|\sum\nolimits_2\right| < A q\,(\log (q+1))^A;$$

and from (2. 713), (2. 714) and (2. 719) the result of the lemma.

2. 72. We have assumed that χ is not a principal character: For the principal character (mod. q) we have[1]

$$L_1(s) = \prod_{\varpi \mid q}\left(1 - \frac{1}{\varpi^s}\right)\zeta(s).$$

Since $\mathfrak{a} = 0$, $\mathfrak{b} = 1$, we have

$$\sum_{\varpi \mid q}\frac{\log \varpi}{\varpi^s - 1} + \frac{\zeta'(s)}{\zeta(s)} = \frac{L'_1(s)}{L_1(s)}$$

$$= \frac{\mathfrak{b}-1}{s} - \frac{1}{s-1} + b - \frac{1}{2}\psi\left(\frac{1}{2}s\right) + \sum\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right),^{2}$$

$$\sum_{\varpi \mid q}\frac{\log \varpi}{\varpi - 1} + \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) = \mathfrak{b} - 1 + b - \frac{1}{2}\psi\left(\frac{1}{2}\right) + \sum\left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right),$$

$$|b| < A\log (q+1) + \left|\sum\left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right)\right|.$$

This corresponds to (2. 712), and from this point the proof proceeds as before.

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[1] Landau, p. 423.

[2] $\sum$ refers to the complex zeros of $L_1(s)$, not merely to those of $\zeta(s)$.

2.81. *Lemma 8. If* $0 < \eta \leqq \frac{1}{2}$ *then*

(2.811) $$f(x) = \frac{\mu(q)}{hY} + \sum_{k=1}^{h} C_k G_k + P,$$

where

(2.812) $$G_k = \sum_{\varrho_k} \Gamma(\varrho)\, Y^{-\varrho},$$

(2.813) $$|P| < A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}} \delta^{-\frac{1}{2}}\right),$$

(2.814) $$\delta = \operatorname{arc\,tan} \frac{\eta}{|\theta|}.$$

This is an immediate corollary of Lemmas 4 and 7.

2.82. *Lemma 9. If* $0 < \eta \leqq \frac{1}{2}$ *then*

(2.821) $$f(x) = \varphi + \Phi,$$

where

(2.822) $$\varphi = \frac{\mu(q)}{hY},$$

(2.823) $$|\Phi| < A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right)\right),$$

(2.824) $$\delta = \operatorname{arc\,tan} \frac{\eta}{|\theta|}.$$

We have

(2.825) $$|G_k| \leqq \sum\nolimits_1 |\Gamma(\varrho)\, Y^{-\varrho}| + \sum\nolimits_2 |\Gamma(\varrho)\, Y^{-\varrho}|,$$

where $\sum_1$ extends over ϱ_k's for which $|\gamma| \geqq 1$, $\sum_2$ over those for which $|\gamma| < 1$. In $\sum_1$ we have

$$|\Gamma(\varrho)\, Y^{-\varrho}| = |\Gamma(\beta + i\gamma)|\, |Y|^{-\beta} \exp\left(\gamma \operatorname{arc\,tan} \frac{\theta}{\eta}\right)$$

$$\leqq A|\gamma|^{\beta-\frac{1}{2}} |Y|^{-\beta} \exp\left(-\left(\frac{1}{2}\pi - \operatorname{arc\,tan} \frac{|\theta|}{\eta}\right)|\gamma|\right)$$

$$\leqq A|\gamma|^{\Theta-\frac{1}{2}} |Y|^{-\Theta} e^{-\delta|\gamma|}$$

(since $|Y| < A$ and, by hypothesis R, $\beta \leqq \Theta$). The number $M(T)$ of ϱ's for which $|\gamma|$ lies between T and $T+1$ $(T \geqq 0)$ is less than $A(\log(q+1))^A \log(T+2)$, by (2. 611). Hence

$$\sum_1 |\gamma|^{\Theta-\frac{1}{2}} e^{-\delta|\gamma|} \leqq A(\log(q+1))^A \sum_{n=0}^{\infty} (n+1)^{\Theta-\frac{1}{2}} \log(n+2)\, e^{-\delta n}$$

$$< A(\log(q+1))^A \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta}+2\right),$$

(2. 826) $$\sum_1 |\Gamma(\varrho)\, Y^{-\varrho}| < A(\log(q+1))^A |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta}+2\right).$$

2. 83. Again, once more by (2. 611), $\sum_2$ has at most $A(\log(q+1))^A$ terms. We write

(2. 831) $$\sum_2 = \sum_{2,1} + \sum_{2,2},$$

$\sum_{2,1}$ applying to zeros for which $1-\Theta \leqq \beta \leqq \Theta$, and $\sum_{2,2}$ to those for which $\beta = 0$. Now, in $\sum_2$,

$$|Y^{-\varrho}| = |Y|^{-\beta} \exp\left(\gamma \arctan \frac{\theta}{\eta}\right) < A|Y|^{-\beta};$$

and in $\sum_{2,1}$, $|\Gamma(\varrho)| < A$. Hence

(2. 832) $$\left|\sum_{2,1}\right| < A|Y|^{-\beta} \sum_{2,1} |\Gamma(\varrho)| < A|Y|^{-\Theta} \sum_{2,1} 1 < A(\log(q+1))^A |Y|^{-\Theta}.$$

Again, in $\sum_{2,2}$, $|Y| < A$ and

$$\frac{1}{|\varrho|} < Aq\log(q+1),$$

by (2. 716); so that

(2. 833) $$\left|\sum_{2,2}\right| < A\sum_{2,2} |\Gamma(\varrho)| = A\sum_{2,2} \frac{|\Gamma(1+\varrho)|}{|\varrho|}$$

$$< A\sum_{2,2} \frac{1}{|\varrho|} < Aq(\log(q+1))^A.$$

From (2. 825), (2. 826), (2. 831), (2. 832), and (2. 833), we obtain

(2. 834) $$|G_k| < A(\log(q+1))^A \left(q + |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta}+2\right)\right) = H_k,$$

say; and from (2. 811), (2. 812), (2. 813), (2. 821), (2. 822) and (2. 834) we deduce

$$|\Phi| = \left|\sum_{k=1}^{h} C_k G_k + P\right|$$

$$< \sum_{k=1}^{h} |C_k G_k| + A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}} \delta^{-\frac{1}{2}}\right)$$

$$< \frac{\sqrt{q}}{h} \sum_{k=1}^{h} H_k + A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right)\right)$$

$$< A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right)\right);$$

that is to say (2. 823).

2. 9. *Lemma 10. We have*

(2. 91) $$h = \varphi(q) > Aq\,(\log q)^{-A}.$$

We have in fact[1]

$$\varphi(q) > (1 - \delta)\, e^{-C} \frac{q}{\log\log q} \qquad (q > q_0(\delta))$$

for every positive δ, C being Euler's constant.

3. Proof of the main theorems.

Approximation to $\nu_r(n)$ by the singular series.

3. 11. **Theorem A.** *If r is an integer, $r \geqq 3$, and*

(3. 111) $$(f(x))^r = \sum \nu_r(n)\, x^n,$$

so that

(3. 112) $$\nu_r(n) = \sum_{\varpi_1 + \varpi_2 + \cdots + \varpi_r = n} \log \varpi_1 \log \varpi_2 \cdots \log \varpi_r,$$

then

(3. 113) $$\nu_r(n) = \frac{n^{r-1}}{(r-1)!} S_r + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)} (\log n)^B\right) \sim \frac{n^{r-1}}{(r-1)!} S_r,$$

[1] Landau, p. 217.

where

(3. 114)
$$S_r = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n).$$

It is to be understood, here and in all that follows, that O's refer to the limit-process $n \to \infty$, and that their constants are functions of r alone.

If $n \geqq 2$, we have

(3. 115)
$$\nu_r(n) = \frac{1}{2\pi i}\int (f(x))^r \frac{dx}{x^{n+1}},$$

the path of integration being the circle $|x| = e^{-H}$, where $H = \frac{1}{n}$, so that

$$1 - |x| = \frac{1}{n} + O\left(\frac{1}{n^2}\right) \sim \frac{1}{n}.$$

Using the Farey dissection of order $N = [\sqrt{n}]$, we have

(3. 116)
$$\begin{aligned}\nu_r(n) &= \sum_{q=1}^{N} \sum_{p<q,\,(p,q)=1} \frac{1}{2\pi i}\int_{\xi_{p,q}} (f(x))^r \frac{dx}{x^{n+1}} \\ &= \sum e_q(-np)\frac{1}{2\pi i}\int_{\zeta_{p,q}} (f(x))^r \frac{dX}{X^{n+1}} \\ &= \sum e_q(-np)\, j_{p,q},\end{aligned}$$

say. Now

$$\begin{aligned}|f^r - \varphi^r| &\leqq |\Phi|(|f^{r-1}| + |f^{r-2}\varphi| + \cdots + |\varphi^{r-1}|) \\ &< B(|\Phi f^{r-1}| + |\Phi\varphi^{r-1}|).\end{aligned}$$

Also $|X^{-n}| = e^{nH} < A$. Hence

(3. 117)
$$j_{p,q} = l_{p,q} + m_{p,q},$$

where

(3. 118)
$$l_{p,q} = \frac{1}{2\pi i}\int_{\zeta_{p,q}} \varphi^r \frac{dX}{X^{n+1}},$$

(3. 119)
$$|m_{p,q}| = O\left(\int_{-\theta'_{p,q}}^{\theta_{p,q}} (|\Phi f^{r-1}| + |\Phi\varphi^{r-1}|)\, d\theta\right).$$

3. 12. We have $\eta = H = \frac{1}{n}$ and $q \leqq \sqrt{n}$, and so, by (2. 823),

$$|\Phi| < An^{\frac{3}{4}} (\log n)^A + A (\log n)^A \sqrt{q} \, |Y|^{-\Theta} \delta^{-\Theta - \frac{1}{2}} \log \left(\frac{1}{\delta} + 2\right), \tag{3. 121}$$

where $\delta = \operatorname{arc\,tan} \frac{\eta}{|\theta|}$. We must now distinguish two cases. If $|\theta| \leqq \eta$, we have

$$|Y| > A\eta, \ \delta > A,$$

and

$$\sqrt{q} \, |Y|^{-\Theta} \delta^{-\Theta - \frac{1}{2}} \log \left(\frac{1}{\delta} + 2\right) < A n^{\frac{1}{4}} \eta^{-\Theta} = A n^{\Theta + \frac{1}{4}}. \tag{3. 122}$$

If on the other hand $\eta < |\theta| \leqq \theta_{p,q}$, we have

$$\delta > A \frac{\eta}{|\theta|} > \frac{A}{n}, \ |Y| > A |\theta|,$$

$$\sqrt{q} \, |Y|^{-\Theta} \delta^{-\Theta - \frac{1}{2}} \log \left(\frac{1}{\delta} + 2\right) < A \sqrt{q} \, . \, |\theta|^{-\Theta} \, . \, \eta^{-\Theta - \frac{1}{2}} |\theta|^{\Theta + \frac{1}{2}} . \log n \tag{3. 123}$$

$$= A n^{\Theta + \frac{1}{2}} \log n \, (q |\theta|)^{\frac{1}{2}} < A n^{\Theta + \frac{1}{2}} \log n \, . \, n^{-\frac{1}{4}} = A n^{\Theta + \frac{1}{4}} \log n,$$

since $q |\theta| \leqq q \theta_{p,q} < A n^{-\frac{1}{2}}$. Thus (3. 123) holds in either case. Also $\Theta \geqq \frac{1}{2}$ and so, by (3. 121),

$$|\Phi| < A n^{\Theta + \frac{1}{4}} (\log n)^A \tag{3. 124}$$

3. 13. Now, remembering that $r \geqq 3$, we have

$$\int_{-\theta'_{p,q}}^{\theta_{p,q}} |\varphi|^{r-1} d\theta < B h^{-(r-1)} \int_{-\theta'_{p,q}}^{\theta_{p,q}} |Y|^{-(r-1)} d\theta$$

$$< B h^{-(r-1)} \int_0^\infty (\eta^2 + \theta^2)^{-\frac{1}{2}(r-1)} d\theta$$

$$< B h^{-(r-1)} n^{r-2};$$

and so

(3. 131)
$$\sum_{p,q} \int_{-\theta'_{p,q}}^{\theta_{p,q}} |\Phi\varphi^{r-1}|\, d\theta < Bn^{r-2}\,(\mathrm{Max}\,|\Phi|)\sum_q h^{-(r-2)}$$

$$< Bn^{r-2+\Theta+\frac{1}{4}}(\log n)^B = Bn^{r-1+\left(\Theta-\frac{3}{4}\right)}(\log n)^B,$$

by (3. 124) and (2. 91).

3. 14. Again, if $\arg x = \psi$, we have

$$\sum \int_{-\theta'_{p,q}}^{\theta_{p,q}} |f|^2\, d\theta = \int_0^{2\pi} |f|^2\, d\psi$$

$$= 2\pi \sum_{\varpi} (\log \varpi)^2 |x|^{2\varpi} < A \sum_{m=2}^{\infty} \log m\, \varLambda(m)\, |x|^{2m}$$

$$< A(1-|x|^2)\sum_{m=2}^{\infty}\left(\sum_{k=2}^{m} \log k\, \varLambda(k)\right)|x|^{2m}$$

$$< A(1-|x|)\sum_{m=2}^{\infty} m \log m\, |x|^{2m}\,.$$

$$< \frac{A}{1-|x|}\log\left(\frac{1}{1-|x|}\right) < An\log n.$$

Similarly

$$|f| \leqq \sum_{\varpi} \log \varpi\, |x|^{\varpi} < \sum_m \varLambda(m)\,|x|^m < \frac{A}{1-|x|} < An.$$

Hence

(3. 141)
$$\sum_{p,q} \int_{-\theta'_{p,q}}^{\theta_{p,q}} |f|^{r-1} |\Phi|\, d\theta \leqq \mathrm{Max}\,|\Phi f^{r-3}| \int_0^{2\pi} |f|^2\, d\psi$$

$$< B\, n^{\Theta+\frac{1}{4}} \log n \,.\, n^{r-3} \,.\, n \log n$$

$$< Bn^{r-1+\left(\Theta-\frac{3}{4}\right)}(\log n)^B.$$

From (3. 116), (3. 117), (3. 119), (3. 131) and (3. 141) we deduce

(3. 142) $$\nu_r(n) = \sum e_q(-np)\, l_{p,q} + O\left(n^{r-1+\left(\theta-\frac{3}{4}\right)}(\log n)^B\right),$$

where $l_{p,q}$ is defined by (3. 118).

3. 15. In $l_{p,q}$ we write $X = e^{-Y}$, $dX = -e^{-Y}dY$, so that Y varies on the straight line from $\eta + i\theta_{p,q}$ to $\eta - i\theta'_{p,q}$. Then, by (2. 822) and (3. 118),

(3. 151) $$l_{p,q} = -\frac{1}{2\pi i}\left(\frac{\mu(q)}{h}\right)^r \int_{\eta+i\theta_{p,q}}^{\eta-i\theta'_{p,q}} Y^{-r} e^{nY} dY.$$

Now

(3. 152) $$-\int_{\eta+i\theta_{p,q}}^{\eta-i\theta'_{p,q}} = \int_{\eta-i\infty}^{\eta+i\infty} Y^{-r} e^{nY} dY + O\left(\int_{\theta_q}^{\infty} |\eta + i\theta|^{-r} d\theta\right)$$

$$= 2\pi i \frac{n^{r-1}}{(r-1)!} + O\left(\int_{\theta_q}^{\infty} |\eta + i\theta|^{-r} d\theta\right),$$

where

$$\theta_q = \operatorname*{Min}_{p<q} (\theta_{p,q}, \theta'_{p,q}) \geqq \frac{1}{2qN}.$$

Also

(3. 153) $$\int_{\theta_q}^{\infty} (\eta + i\theta)^{-r} d\theta < \int_{\theta_q}^{\infty} \theta^{-r} d\theta < B\theta_q^{1-r} < B(q\sqrt{n})^{r-1}.$$

From (3. 151), (3. 152) and (3. 153), we deduce

(3. 154) $$\sum e_q(-np)\, l_{p,q} = \frac{n^{r-1}}{(r-1)!} \sum_{p,q} \left(\frac{\mu(q)}{\varphi(q)}\right)^r e_q(-np) + Q,$$

where

(3. 155) $$|Q| < B \sum_{p,q} h^{-r} q^{r-1} n^{\frac{1}{2}(r-1)} < Bn^{\frac{1}{2}(r-1)} \sum_q \left(\frac{q}{h}\right)^{r-1}$$

$$< Bn^{\frac{1}{2}(r-1)} \sum_{q=1}^{N} (\log q)^B < Bn^{\frac{1}{2}r} (\log n)^B.$$

Since $r \geqq 3$ and $\Theta \geqq \frac{1}{2}$, $\frac{1}{2} r < r - 1 - \frac{1}{4} \leqq r - 1 + \left(\Theta - \frac{3}{4}\right)$; and from (3. 142), (3. 154), and (3. 155) we obtain

$$(3.\ 156) \qquad \nu_r(n) = \frac{n^{r-1}}{(r-1)!} \sum_{p,q} \left(\frac{\mu(q)}{\varphi(q)}\right)^r e_q(-np) + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)} (\log n)^B\right)$$

$$= \frac{n^{r-1}}{(r-1)!} \sum_{q \leqq N} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n) + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)} (\log n)^B\right).$$

3. 16. In order to complete the proof of Theorem A, we have merely to show that the finite series in (3. 156) may be replaced by the infinite series S_r. Now

$$\left| n^{r-1} \sum_{q > N} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n) \right| < B n^{r-1} \sum_{q > N} q^{1-r} (\log q)^B < B n^{\frac{1}{2} r} (\log n)^B,$$

and $\frac{1}{2} r < r - 1 + \left(\Theta - \frac{3}{4}\right)$. Hence this error may be absorbed in the second term of (3. 156), and the proof of the theorem is completed.

Summation of the singular series.

3. 21. *Lemma 11. If*

$$(3.\ 211) \qquad c_q(n) = \sum e_q(np),$$

where n is a positive integer and the summation extends over all positive values of p less than and prime to q, $p = 0$ being included when $q = 1$, but not otherwise, then

$$(3.\ 212) \qquad c_q(-n) = c_q(n);$$

$$(3.\ 213) \qquad c_{qq'}(n) = c_q(n)\, c_{q'}(n)$$

if $(q, q') = 1$; and

$$(3.\ 214) \qquad c_q(n) = \sum \delta \mu \left(\frac{q}{\delta}\right),$$

where δ is a common divisor of q and n.

The terms in p and $q - p$ are conjugate. Hence $c_q(n)$ is real. As $c_q(n)$ and $c_q(-n)$ are conjugate we obtain (3. 212).[1]

---

[1] The argument fails if $q = 1$ or $q = 2$; but $c_1(n) = c_1(-n) = 1$, $c_2(n) = c_2(-n) = -1$.

Again

$$c_q(n)\,c_{q'}(n)=\sum_{p,p'}\exp\left(2n\pi i\left(\frac{p}{q}+\frac{p'}{q'}\right)\right)=\sum_{p,p'}\exp\left(\frac{2nP\pi i}{qq'}\right),$$

where

$$P=pq'+p'q.$$

When p assumes a set of $\varphi(q)$ values, positive, prime to q, and incongruent to modulus q, and p' a similar set of values for modulus q', then P assumes a set of $\varphi(q)\varphi(q')=\varphi(qq')$ values, plainly all positive, prime to qq' and incongruent to modulus qq'. Hence we obtain (3. 213).

Finally, it is plain that

$$\sum_{d\mid q}c_d(n)=\sum_{h=0}^{q-1}e_q(nh),$$

which is zero unless $q\mid n$ and then equal to q. Hence, if we write

$$\eta(q)=q\quad(q\mid n),\qquad \eta(q)=0\quad(q\nmid n),$$

we have

$$\sum_{d\mid q}c_d(n)=\eta(q),$$

and therefore

$$c_q(n)=\sum_{d\mid q}\eta(d)\,\mu\left(\frac{q}{d}\right)$$

by the well-known inversion formula of Möbius.[1] This is (3. 214).[2]

3. 22. *Lemma 12. Suppose that* $r\geqq 2$ *and*

(3. 221) $$S_r=\sum_{q=1}^{\infty}\left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n).$$

Then

(3. 222) $$S_r=0$$

[1] Landau, p. 577.

[2] The formula (3. 214) is proved by RAMANUJAN ('On certain trigonometrical sums and their applications in the theory of numbers', *Trans. Camb. Phil. Soc.*, vol. 22 (1918), pp. 259—276 (p. 260)). It had already been given for $n=1$ by LANDAU (*Handbuch* (1909), p. 572: Landau refers to it as a known result), and in the general case by JENSEN ('Et nyt Udtryk for den talteoretiske Funktion $\sum\mu(n)=M(n)$', *Den 3. Skandinaviske Matematiker-Kongres, Kristiania 1913,* Kristiania (1915), p. 145). Ramanujan makes a large number of very beautiful applications of the sums in question, and they may well be associated with his name.

if n and r are of opposite parity. But if n and r are of like parity then

(3. 223) $$S_r = 2 C_r \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p}-1)^r + (-1)^r(\mathfrak{p}-1)}{(\mathfrak{p}-1)^r - (-1)^r}\right),$$

where $\mathfrak{p}$ is an odd prime divisor of n and

(3. 224) $$C_r = \prod_{\varpi=3}^{\infty} \left(1 - \frac{(-1)^r}{(\varpi-1)^r}\right).$$

Let

(3. 225) $$\left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n) = A_q.$$

Then

$$\mu(qq') = \mu(q)\mu(q'),\ \varphi(qq') = \varphi(q)\varphi(q'),\ c_{qq'}(-n) = c_q(-n)\,c_{q'}(-n)$$

if $(q, q') = 1$; and therefore (on the same hypothesis)

(3. 226) $$A_{qq'} = A_q A_{q'}.$$

Hence[1]

$$S_r = A_1 + A_2 + A_3 + \cdots = 1 + A_2 + \cdots = \prod_{\varpi} \chi_\varpi$$

where

(3. 227) $$\chi_\varpi = 1 + A_\varpi + A_{\varpi^2} + A_{\varpi^3} + \cdots = 1 + A_\varpi,$$

since $A_{\varpi^2}, A_{\varpi^3}, \cdots$ vanish in virtue of the factor $\mu(q)$.

3. 23. If $\varpi \nmid n$, we have

$$\mu(\varpi) = -1,\ \varphi(\varpi) = \varpi - 1,\ c_\varpi(n) = \mu(\varpi) = -1,$$

(3. 231) $$A_\varpi = -\frac{(-1)^r}{(\varpi-1)^r}.$$

If on the other hand $\varpi \mid n$, we have

$$c_\varpi(n) = \mu(\varpi) + \varpi\mu(1) = \varpi - 1$$

(3. 232) $$A_\varpi = \frac{(-1)^r}{(\varpi-1)^{r-1}}.$$

---

[1] Since $|c_q(n)| \leq \sum \delta$, where $\delta \mid n$, we have $c_q(n) = O(1)$ when n is fixed and $q \to \infty$. Also by Lemma 10, $\varphi(q) > A q (\log q)^{-A}$. Hence the series and products concerned are absolutely convergent.

Hence

$$S_r = \prod_{\varpi \mid n} \left(1 + \frac{(-1)^r}{(\varpi - 1)^{r-1}}\right) \prod_{\varpi \nmid n} \left(1 - \frac{(-1)^r}{(\varpi - 1)^r}\right). \tag{3. 233}$$

If n is even and r is odd, the first factor vanishes in virtue of the factor for which $\varpi = 2$; if n is odd and r even, the second factor vanishes similarly. Thus $S_r = 0$ whenever n and r are of opposite parity.

If n and r are of like parity, the factor corresponding to $\varpi = 2$ is in any case 2; and

$$S_r = 2 \prod_{\varpi = 3}^{\infty} \left(1 - \frac{(-1)^r}{(\varpi - 1)^r}\right) \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p} - 1)^r + (-1)^r (\mathfrak{p} - 1)}{(\mathfrak{p} - 1)^r - (-1)^r}\right),$$

as stated in the lemma.

Proof of the final formulae.

3. 3. **Theorem B.** *Suppose that $r \geqq 3$. Then, if n and r are of unlike parity,*

$$\nu_r(n) = o(n^{r-1}). \tag{3. 31}$$

But if n and r are of like parity then

$$\nu_r(n) \backsim \frac{2 C_r}{(r-1)!} n^{r-1} \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p} - 1)^r + (-1)^r (\mathfrak{p} - 1)}{(\mathfrak{p} - 1)^r - (-1)^r}\right), \tag{3. 32}$$

where $\mathfrak{p}$ is an odd prime divisor of n and

$$C_r = \prod_{\varpi = 3}^{\infty} \left(1 - \frac{(-1)^r}{(\varpi - 1)^r}\right). \tag{3. 33}$$

This follows immediately from Theorem A and Lemma 12.[1]

3. 4. *Lemma 13. If $r \geqq 3$ and n and r are of like parity, then*

$$\nu_r(n) > B n^{r-1},$$

for $n \geqq n_0(r)$.

---

[1] Results equivalent to these are stated in equations (5. 11)—(5. 22) of our note 2, but incorrectly, a factor

$$(\log n)^{-r}$$

being omitted in each, owing to a momentary confusion between $\nu_r(n)$ and $N_r(n)$. The $\nu_r(n)$ of 2 is the $N_r(n)$ of this memoir.

This lemma is required for the proof of Theorem C. If r is *even*

$$\prod \left(\frac{(\mathfrak{p}-1)^r+\mathfrak{p}-1}{(\mathfrak{p}-1)^r-1}\right) > 1.$$

If r is *odd*

$$\prod \left(\frac{(\mathfrak{p}-1)^r-\mathfrak{p}+1}{(\mathfrak{p}-1)^r+1}\right) > \prod \left(\frac{(\mathfrak{p}-1)^r-\mathfrak{p}}{(\mathfrak{p}-1)^r}\right) > \prod_{\varpi=3}^{\infty} \left(1-\frac{\varpi}{(\varpi-1)^3}\right) = A.$$

In either case the conclusion follows from (3. 32).

3. 5. **Theorem C.** *If $r \geqq 3$ and n and r are of like parity, then*

(3. 51)
$$N_r(n) \sim \frac{\nu_r(n)}{(\log n)^r}.$$

We observe first that

$$N_r(n) = \sum_{\varpi_1+\varpi_2+\cdots+\varpi_r=n} 1 \leqq \sum_{m_1+m_2+\cdots+m_r=n} 1 < Bn^{r-1}$$

and

(3. 511)
$$\nu_r(n) = \sum_{\varpi_1+\varpi_2+\cdots+\varpi_r=n} \log\varpi_1\cdots\log\varpi_r \leqq (\log n)^r N_r(n) < Bn^{r-1}(\log n)^r.$$

Write now

(3. 512)
$$\nu_r = \nu'_r + \nu''_r, \quad N_r = N'_r + N''_r,$$

where ν'_r and N'_r include all terms of the summations for which

$$\varpi_s \geqq n^{1-\delta} \quad (0<\delta<1,\ s=1, 2, \ldots, r).$$

Then plainly

(3. 513)
$$\nu'_r(n) \geqq (1-\delta)^r (\log n)^r N'_r(n).$$

Again

$$N''_r(n) \leqq r \sum_{\varpi_r<n^{1-\delta}} \left(\sum_{\varpi_1+\varpi_2+\cdots+\varpi_{r-1}=n-\varpi_r} 1\right)$$

$$< B\sum_{\varpi_r<n^{1-\delta}} N_{r-1}(n-\varpi_r) < Bn^{1-\delta}.n^{r-2} < Bn^{r-1-\delta},$$

$$\nu''_r(n) \leqq (\log n)^r N''_r(n) < Bn^{r-1-\delta}(\log n)^r.$$

But $\nu_r(n) > Bn^{r-1}$ for $n \geqq n_0(r)$, by Lemma 13; and so

(3. 514)
$$(\log n)^r N''_r(n) = o(\nu_r(n)), \quad \nu''_r(n) = o(\nu_r(n)),$$

for every positive δ.

From (3. 511), (3. 512), (3. 513), and (3. 514) we deduce

$$(1-\delta)^r (\log n)^r (N_r - N''_r) \leqq \nu_r - \nu''_r \leqq (\log n)^r N_r,$$

$$(1-\delta)^r (\log n)^r N_r \leqq \nu_r + o(\nu_r) \leqq (\log n)^r N_r,$$

$$(1-\delta)^r \leqq \underline{\lim} \frac{\nu_r}{(\log n)^r N_r}, \quad \overline{\lim} \frac{\nu_r}{(\log n)^r N_r} \leqq 1.$$

As δ is arbitrary, this proves (3. 51).

3. 6. **Theorem D.** *Every large odd number n is the sum of three odd primes. The asymptotic formula for the number of representations $\bar{N}_3(n)$ is*

$$\bar{N}_3(n) \sim C_3 \frac{n^2}{(\log n)^3} \prod \left(\frac{(\mathfrak{p}-1)(\mathfrak{p}-2)}{\mathfrak{p}^2 - 3\mathfrak{p} + 3}\right), \tag{3. 61}$$

where $\mathfrak{p}$ is a prime divisor of n and

$$C_3 = \prod_{\varpi=3}^{\infty} \left(1 + \frac{1}{(\varpi - 1)^3}\right). \tag{3. 62}$$

This is an almost immediate corollary of Theorems B and C. These theorems give the corresponding formula for $N_3(n)$. If not all the primes are odd, two must be 2 and $n-4$ a prime. The number of such representations is one at most.

Theorem E. *Every large even number n is the sum of four odd primes (of which one may be assigned.) The asymptotic formula for the total number of representations is*

$$\bar{N}_4(n) \sim \frac{1}{3} C_4 \frac{n^3}{(\log n)^4} \prod \left(\frac{(\mathfrak{p}-1)(\mathfrak{p}^2 - 3\mathfrak{p} + 3)}{(\mathfrak{p}-2)(\mathfrak{p}^2 - 2\mathfrak{p} + 2)}\right), \tag{3. 63}$$

where $\mathfrak{p}$ is an odd prime divisor of n and

$$C_4 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi - 1)^4}\right). \tag{3. 64}$$

This is a corollary of the same two theorems. We have only to observe that the number of representations by four primes which are not all odd is plainly $O(n)$. There are evidently similar theorems for any greater value of r.

4. Remarks on 'Goldbach's Theorem'.

4. 1. Our method fails when $r = 2$. It does not fail *in principle*, for it leads to a definite result which appears to be correct; but we cannot overcome the difficulties of the proof, even if we assume that $\Theta = \frac{1}{2}$. The best upper bound that we can determine for the error is too large by (roughly) a power $n^{\frac{1}{4}}$.

The formula to which our method leads is contained in the following

Conjecture A. *Every large even number is the sum of two odd primes. The asymptotic formula for the number of representatives is*

$$N_2(n) \backsim 2C_2 \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right) \tag{4. 11}$$

where $\mathfrak{p}$ *is an odd prime divisor of* n, *and*

$$C_2 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi-1)^2}\right). \tag{4. 12}$$

We add a few words as to the history of this formula, and the empirical evidence for its truth.[1]

The first definite formulation of a result of this character appears to be due to SYLVESTER[2], who, in a short abstract published in the *Proceedings of London Mathematical Society* in 1871, suggested that

$$N_2(n) \backsim \frac{2n}{\log n} \prod \left(\frac{\varpi-2}{\varpi-1}\right), \tag{4. 13}$$

where

$$3 \leqq \varpi < \sqrt{n}, \quad \varpi \nmid n.$$

Since

$$\prod_{\varpi<\sqrt{n}} \left(\frac{\varpi-2}{\varpi-1}\right) = \prod_{\varpi<\sqrt{n}} \left(1 - \frac{1}{(\varpi-1)^2}\right) \prod_{\varpi<\sqrt{n}} \left(1 - \frac{1}{\varpi}\right) \backsim C_2 \prod_{\varpi<\sqrt{n}} \left(1 - \frac{1}{\varpi}\right),$$

[1] As regards the earlier history of 'Goldbach's Theorem', see L. E. DICKSON, *History of the Theory of Numbers*, vol. 1 (Washington 1919), pp. 421—425.

[2] J. J. SYLVESTER, 'On the partition of an even number into two primes', *Proc. London Math. Soc.*, ser. 1, vol. 4 (1871), pp. 4—6 (*Math. Papers*, vol. 2, pp. 709—711). See also 'On the Goldbach-Euler Theorem regarding prime numbers', *Nature*, vol. 55 (1896—7), pp. 196—197, 269 (*Math. Papers*, vol. 4, pp. 734—737).

We owe our knowledge of Sylvester's notes on the subject to Mr. B. M. WILSON of Trinity College, Cambridge. See, in connection with all that follows, Shah and Wilson, 1, and Hardy and Littlewood, 2.

and[1]

$$\prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{\varpi}\right) \sim \frac{2e^{-C}}{\log n}, \tag{4.14}$$

where C is Euler's constant, (4. 13) is equivalent to

$$N_2(n) \sim 4e^{-C} C_2 \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right), \tag{4.15}$$

and contradicts (4. 11), the two formulae differing by a factor $2e^{-C} = 1.123\ldots$ We prove in 4. 2 that (4. 11) is the only formula of the kind that can possibly be correct, so that Sylvester's formula is erroneous. But Sylvester was the first to identify the factor

$$\prod \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right), \tag{4.16}$$

to which the *irregularities* of $N_2(n)$ are due. There is no sufficient evidence to show how he was led to his result.

A quite different formula was suggested by STÄCKEL[2] in 1896, viz.,

$$N_2(n) \sim \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p}}{\mathfrak{p}-1}\right).$$

This formula does not introduce the factor (4. 16), and does not give anything like so good an approximation to the facts; it was in any case shown to be incorrect by LANDAU[3] in 1900.

In 1915 there appeared an uncompleted essay on Goldbach's Theorem by MERLIN.[4] MERLIN does not give a complete asymptotic formula, but recognises (like Sylvester before him) the importance of the factor (4. 16).

About the same time the problem was attacked by BRUN[5]. The formula to which Brun's argument naturally leads is

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[1] Landau, p. 218.

[2] P. STÄCKEL, 'Über Goldbach's empirisches Theorem: Jede grade Zahl kann als Summe von zwei Primzahlen dargestellt werden', *Göttinger Nachrichten*, 1896, pp. 292—299.

[3] E. LANDAU, 'Über die zahlentheoretische Funktion $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz', *Göttinger Nachrichten*, 1900, pp. 177—186.

[4] J. MERLIN, 'Un travail sur les nombres premiers', *Bulletin des sciences mathématiques*, vol. 39 (1915), pp. 121—136.

[5] V. BRUN, 'Über das Goldbachsche Gesetz und die Anzahl der Primzahlpaare', *Archiv for Mathematik* (Christiania), vol. 34, part 2 (1915), no. 8, pp. 1—15. The formula (4. 18) is not actually formulated by Brun: see the discussion by Shah and Wilson, 1, and Hardy and Littlewood, 2. See also a second paper by the same author, 'Sur les nombres premiers de la forme $ap+b$', *ibid.*, part. 4 (1917), no. 14, pp. 1—9; and the postscript to this memoir.

(4. 17)
$$N_2(n) \sim 2Hn \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right),$$

where

(4. 171)
$$H = \prod_{3 \leq \varpi < \sqrt{n}} \left(1 - \frac{2}{\varpi}\right).$$

This is easily shown to be equivalent to

(4. 18)
$$N_2(n) \sim 8 e^{-2\gamma} C_2 \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right),$$

and differs from (4. 11) by a factor $4e^{-2C} = 1.263\ldots$ The argument of 4. 2 will show that this formula, like Sylvester's, is incorrect.

Finally, in 1916 STÄCKEL[1] returned to the subject in a series of memoirs published in the *Sitzungsberichte der Heidelberger Akademie der Wissenschaften*, which we have until very recently been unable to consult. Some further remarks concerning these memoirs will be found in our final postscript.

4. 2. We proceed to justify our assertion that the formulae (4. 15) and (4. 18) cannot be correct.

Theorem F. *Suppose it to be true that*[2]

(4. 21)
$$N_2(n) \sim A \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right)$$

if

$$n = 2^{\alpha} \mathfrak{p}^{a} \mathfrak{p}'^{a'} \ldots \qquad (\alpha > 0,\ a, a', \ldots > 0),$$

and

(4. 22)
$$N_2(n) = o\left(\frac{n}{(\log n)^2}\right)$$

if n is odd. Then

(4. 23)
$$A = 2C_2 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi-1)^2}\right).$$

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[1] P. STÄCKEL, 'Die Darstellung der geraden Zahlen als Summen von zwei Primzahlen', 8 August 1916; 'Die Lückenzahlen r-ter Stufe und die Darstellung der geraden Zahlen als Summen und Differenzen ungerader Primzahlen', I. Teil 27 Dezember 1917, II. Teil 19 Januar 1918, III. Teil 19 Juli 1918.

[2] Throughout 4.2 A is *the same* constant.

Write

$$\Omega(n) = A n \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right) \ (n \text{ even}), \ \Omega(n) = 0 \ (n \text{ odd}). \tag{4. 24}$$

Then, by (4. 21) and Theorem C, now valid in virtue of (4. 21),

$$\nu_2(n) = \sum_{\varpi+\varpi'=n} \log \varpi \log \varpi' \sim \Omega(n), \tag{4. 25}$$

it being understood that, when n is odd, this formula means

$$\nu_2(n) = o(n).$$

Further let

$$f(s) = \sum \frac{\Omega(n)}{n^s} = \sum \frac{\Omega(n)}{n^{1+u}},$$

these series being absolutely convergent if $\Re(s) > 2$, $\Re(u) > 1$. Then

$$\begin{aligned} f(s) &= A \sum_{n\equiv 0 \,(\mathrm{mod.}\, 2)} n^{-u} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right) \\ &= A \sum_{a>0} 2^{-au}\, \mathfrak{p}^{-au}\, \mathfrak{p}'^{-a'u} \cdots \frac{(\mathfrak{p}-1)(\mathfrak{p}'-1)\ldots}{(\mathfrak{p}-2)(\mathfrak{p}'-2)\ldots} \\ &= \frac{2^{-u}A}{1-2^{-u}} \prod_{\varpi=3}^{\infty} \left(1 + \frac{\varpi-1}{\varpi-2}\,\frac{\varpi^{-u}}{1-\varpi^{-u}}\right) = \frac{2^{-u}A}{1-2^{-u}}\,\xi(u), \end{aligned} \tag{4. 26}$$

say. Suppose now that $u \to 1$, and let

$$\eta(u) = \prod_{\varpi=3}^{\infty} \left(1 + \frac{\varpi^{-u}}{1-\varpi^{-u}}\right) = \prod_{\varpi=3}^{\infty} \left(\frac{1}{1-\varpi^{-u}}\right) = (1-2^{-u})\,\zeta(u).$$

Then

$$\begin{aligned} \frac{\xi(u)}{\eta(u)} &= \prod \left(\left(1 + \frac{\varpi-1}{\varpi-2}\,\frac{\varpi^{-u}}{1-\varpi^{-u}}\right) \Big/ \left(1 + \frac{\varpi^{-u}}{1-\varpi^{-u}}\right)\right) \\ &\to \prod \left(\left(1 + \frac{1}{\varpi-2}\right) \Big/ \left(1 + \frac{1}{\varpi-1}\right)\right) = \prod \left(\frac{(\varpi-1)^2}{\varpi(\varpi-2)}\right) \\ &= \prod \left(\frac{(\varpi-1)^2}{(\varpi-1)^2-1}\right) = \frac{1}{C_2}. \end{aligned}$$

Hence

$$f(s) \sim A\,\xi(u) \sim \frac{A}{C_2}\,\eta(u) \sim \frac{A}{2C_2}\,\zeta(u) \sim \frac{A}{2C_2(u-1)} = \frac{A}{2C_2(s-2)}. \tag{4. 27}$$

On the other hand, when $x \to 1$,

$$\sum \nu_2(n) x^n \sim \left(\sum \log \varpi\, x^{\varpi}\right)^2 \sim \frac{1}{(1-x)^2},$$

and so[1]

$$\nu_2(1) + \nu_2(2) + \cdots + \nu_2(n) \sim \frac{1}{2} n^2. \tag{4. 28}$$

It is an elementary deduction[2] that

$$g(s) = \sum \frac{\nu_2(n)}{n^s} \sim \sum \frac{1}{n^{s-1}} \sim \frac{1}{s-2}$$

when $s \to 2$; and hence[2] that (under the hypotheses (4. 21) and (4. 22))

$$f(s) \sim \frac{1}{s-2}. \tag{4. 29}$$

Comparing (4. 27) and (4. 29), we obtain the result of the theorem.

4. 3. The fact that both Sylvester's and Brun's formulae contain an erroneous constant factor, and that this factor is in each case a simple function of the number e^{-C}, is not so remarkable as it may seem.

In the first place we observe that any formula in the theory of primes, *deduced from considerations of probability*, is likely to be erroneous in just this way. Consider, for example, the problem '*what is the chance that a large number n should be prime?*' We know that the answer is that the chance is approximately $\frac{1}{\log n}$.

Now the chance that n should not be divisible by any prime less than a *fixed* number x is asymptotically equivalent to

$$\prod_{\varpi < x} \left(1 - \frac{1}{\varpi}\right);$$

[1] We here use Theorem 8 of our paper 'Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive', *Proc. London Math. Soc.*, ser. 2, vol. 13, pp. 174—192. This is the quickest proof, but by no means the most elementary. The formula (4. 28) is equivalent to the formula

$$\sum_1^n N_2(m) \sim \frac{n^2}{2(\log n)^2}$$

used by Landau in his note quoted on p. 33.

[2] For general theorems including those used here as very special cases, see K. Knopp, Divergenzcharactere gewisser Dirichlet'scher Reihen', *Acta Mathematica*, vol. 34, 1909, pp. 165—204 (*e. g.* Satz III, p. 176).

and it would be natural to infer[1] that the chance required is asymptotically equivalent to

$$\prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{\varpi}\right).$$

But[2]

$$\prod_{\varpi > \sqrt{n}} \left(1 - \frac{1}{\varpi}\right) \sim \frac{2e^{-C}}{\log n};$$

and our inference is incorrect, to the extent of a factor $2e^{-C}$.

It is true that Brun's argument is not stated in terms of probabilities[3]; but it involves a heuristic passage to the limit of exactly the same character as that in the argument we have just quoted. Brun finds first (by an ingenious use of the 'sieve of Eratosthenes') an asymptotic formula for the number of representations of n as the sum of two numbers, *neither divisible by any fixed number of primes.* This formula is correct and the proof valid. So is the first stage in the argument above; it rests on an enumeration of cases, and all reference to 'probability'[4] is easily eliminated. It is in the passage to the limit that error is introduced, and the nature of the error is the same in one case as in the other.

4. 4. SHAH and WILSON have tested Conjecture A extensively by comparison with the empirical data collected by CANTOR, AUBRY, HAUSSNER, and RIPERT. We reprint their table of results; but some preliminary remarks are required. In the first place it is essential, in a numerical test, to work with a formula $N_2(n)$, such as (4. 11), and not with one for $\nu_2(n)$, such as (4. 25). In our analysis, on the other hand, it is $\nu_2(n)$ which presents itself first, and the formula for $N_2(n)$ is secondary. In order to derive the asymptotic formula for $N_2(n)$, we write

$$\nu_2(n) = \sum_{\varpi + \varpi' = n} \log \varpi \log \varpi' \sim (\log n)^2 N_2(n).$$

The factor $(\log n)^2$ is certainly in error to an order $\log n$, and it is more natural[5] to replace $\nu_2(n)$ by

$$((\log n)^2 - 2 \log n + \cdots) N_2(n).$$

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[1] One might well replace $\varpi < \sqrt{n}$ by $\varpi < n$, in which case we should obtain a probability half as large. This remark is in itself enough to show the unsatisfactory character of the argument.

[2] Landau, p. 218.

[3] Whether Sylvester's argument was or was not we have no direct means of judging.

[4] *Probability* is not a notion of pure mathematics, but of philosophy or physics.

[5] Compare Shah and Wilson, *l. c.*, p. 238. The same conclusion may be arrived at in other ways.

For the *asymptotic* formula, naturally, it is indifferent which substitution we adopt. But, for purposes of *verification within the limits of calculation*, it is by no means indifferent, for the term in log n is by no means of negligible importance; and it will be found that is makes a vital difference in the plausibility of the results. Bearing these considerations in mind, Shah and Wilson worked, not with the formula (4. 11), but with the modified formula

$$N_2(n) \sim \varrho(n) = 2\,C_2 \frac{n}{(\log n)^2 - 2\log n} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right).$$

Failure to make allowances of this kind has been responsible for a good deal of misapprehension in the past. Thus (as is pointed out by Shah and Wilson[1]) Sylvester's erroneous formula gives, for values of n within the limits of Table I, decidedly *better* results than those obtained from the *unmodified* formula (4. 11).

There is another point of less importance. The function which presents itself most naturally in our analysis is not

$$f(x) = \sum \log \varpi \, x^{\varpi}$$

but

$$g(x) = \sum \Lambda(n)\, x^n = \sum_{\varpi, l} \log \varpi \, x^{\varpi^l}.$$

The corresponding numerical functions are not $\nu_2(n)$ and $N_2(n)$, but

$$g_2(n) = \sum_{m+m'=n} \Lambda(m)\,\Lambda(m'),\ Q_2(n) = \sum_{\varpi^l + \varpi'^{l'} = n} 1$$

(so that $Q_2(n)$ is the number of decompositions of n *into two primes or two powers of primes*). Here again, $N_2(n)$ and $Q_2(n)$ are asymptotically equivalent; the difference between them is indeed of lower order than errors which we are neglecting in any case; but there is something to be said for taking the latter as the basis for comparison, when (as is inevitable) the values of n are not very large.

In the table the decompositions into primes, and powers of primes, are reckoned separately; but it is the total which is compared with $\varrho(n)$. The value of the constant $2\,C_2$ is 1 . 3203. It will be seen that the correspondence between the calculated and actual values is excellent.

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[1] *l. c.*, p. 242.

Table I.

n	$Q_2(n)$	$\rho(n)$	$Q_2(n):\rho(n)$
$30 = 2 . 3 . 5$	$6 + 4 = 10$	22	0.45
$32 = 2^5$	$4 + 7 = 11$	8	1.38
$34 = 2 . 17$	$7 + 6 = 13$	9	1.44
$36 = 2^2 . 3^2$	$8 + 8 = 16$	17	0.94
$210 = 2 . 3 . 5 . 7$	$42 + 0 = 42$	49	0.85
$214 = 2 . 107$	$17 + 0 = 17$	16	1.07
$216 = 2^3 . 3^3$	$28 + 0 = 28$	32	0.88
$256 = 2^8$	$16 + 3 = 19$	17	1.10
$2,048 = 2^{11}$	$50 + 17 = 67$	63	1.06
$2,250 = 2 . 3^2 . 5^3$	$174 + 26 = 200$	179	1.11
$2,304 = 2^8 . 3^2$	$134 + 8 = 142$	136	1.04
$2,306 = 2 . 1153$	$67 + 20 = 87$	69	1.26
$2,310 = 2 . 3 . 5 . 7 . 11$	$228 + 16 = 244$	244	1.00
$3,888 = 2^4 . 3^5$	$186 + 24 = 210$	197	1.06
$3,898 = 2 . 1949$	$99 + 6 = 105$	99	1.06
$3,990 = 2 . 3 . 5 . 7 . 19$	$328 + 20 = 348$	342	1.02
$4,096 = 2^{12}$	$104 + 5 = 109$	102	1.06
$4,996 = 2^2 . 1249$	$124 + 16 = 140$	119	1.18
$4,998 = 2 . 3 . 7^2 . 17$	$228 + 20 = 308$	305	1.01
$5,000 = 2^3 . 5^4$	$150 + 26 = 176$	157	1.12
$8,190 = 2 . 3^2 . 5 . 7 . 13$	$578 + 26 = 604$	597	1.01
$8,192 = 2^{13}$	$150 + 32 = 182$	171	1.06
$8,194 = 2 . 17 . 241$	$192 + 10 = 202$	219	0.92
$10,008 = 2^3 . 3^2 . 139$	$388 + 30 = 418$	396	1.06
$10,010 = 2 . 5 . 7 . 11 . 13$	$384 + 36 = 420$	384	1.09
$10,014 = 2 . 3 . 1669$	$408 + 8 = 416$	396	1.05
$30,030 = 2 . 3 . 5 . 7 . 11 . 13$	$1,800 + 54 = 1854$	1795	1.03
$36,960 = 2^5 . 3 . 5 . 7 . 11$	$1,956 + 38 = 1994$	1937	1.03
$39,270 = 2 . 3 . 5 . 7 . 11 . 17$	$2,152 + 36 = 2188$	2213	0.99
$41,580 = 2^2 . 3^3 . 5 . 7 . 11$	$2,140 + 44 = 2184$	2125	1.03
$50,026 = 2 . 25013$	$702 + 8 = 710$	692	1.03
$50,144 = 2^5 . 1567$	$607 + 32 = 706$	694	1.02
$170,166 = 2 . 3 . 79 . 359$	$3,734 + 46 = 3780$	3762	1.00
$170,170 = 2 . 5 . 7 . 11 . 13 . 17$	$3,784 + 8 = 3792$	3841	0.99
$170,172 = 2^2 . 3^2 . 29 . 163$	$3,732 + 48 = 3780$	3866	0.98

5. Other problems.

5. 1. This last section is frankly conjectural, and is not to be judged by the same standards as §§ 1—3.

The problems to which we have applied our method may be divided roughly into three classes. The typical problem of the first class is Waring's Problem. Our solution of this problem is not yet as conclusive as we should desire, and we have not exhausted the possibilities of our method, even when allowance is made for still unpublished work; we cannot at present prove, for example, that every large number is the sum of 7 cubes or 16 biquadrates. But our proofs, so far as they go, are complete.

The typical problem of the second class is that considered in §§ 1—3. The arguments by which we prove our results are rigorous, but the results depend upon the unproved hypothesis R.

There is a third class of problems, of which Goldbach's Problem is typical. Here we are unable (with or without Hypothesis R) to offer anything approaching to a rigorous proof. What our method yields is a *formula*, and one which seems to stand the test of comparison with the facts. In this concluding section we propose to state a number of further formulae of the same kind. Our apology for doing so must be (1) that no similar formulae have been suggested before, and that the process by which they are deduced has at any rate a certain algebraical interest, and (2) that it seems to us very desirable that (in default of proof) the formulae should be checked, and that we hope that some of the many mathematicians interested in the computative side of the theory of numbers may find them worthy of their attention.

Conjugate problems: the problem of prime-pairs.

5. 2. The problems to which our method is applicable group themselves in pairs in an interesting manner which will be most easily understood by an example.

In Goldbach's Problem we have to study the integral

$$\frac{1}{2\pi i}\int (f(x))^2 \frac{dx}{x^{n+1}},$$

where

$$f(x) = \sum \log \varpi x^{\varpi}, \quad x = Re^{i\varphi} = e^{-\frac{1}{n}+i\varphi},$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\psi}) f(Re^{i\psi}) e^{1-ni\psi} d\psi. \tag{5. 21}$$

The formal transformations of this integral to which we are led may be stated shortly as follows. We divide up the range of integration into a large number of pieces by means of the Farey arcs $\xi_{p,q}$, ψ varying over the interval $\left(\frac{2p\pi}{q} - \theta'_{p,q}, \frac{2p\pi}{q} + \theta_{p,q}\right)$ when x varies over $\xi_{p,q}$. We then replace $f(x)$ by the appropriate approximation

$$\frac{\mu(q)}{\varphi(q)} \frac{1}{\log\left(\frac{e_q(p)}{x}\right)} = \frac{\mu(q)}{\varphi(q)} \frac{1}{\frac{1}{n} - i\left(\psi - \frac{2p\pi}{q}\right)},$$

$\psi - \frac{2p\pi}{q}$ by u, and the integral

$$e_q(-np)\int_{-\theta'_{p,q}}^{\theta_{p,q}} \frac{e^{1-niu}}{\left(\frac{1}{n} - iu\right)^2} du \tag{5. 22}$$

by

$$n\, e_q(-np)\int_{-\infty}^{\infty} \frac{e^{1-iw}}{(1-iw)^2} dw = 2\pi n e_q(-np). \tag{5. 23}$$

We are thus led to the singular series S_2.

Now suppose that, instead of the integral (5. 21), we consider the integral

$$J(R) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\psi}) f(Re^{-i\psi}) e^{ki\psi} d\psi, \tag{5. 24}$$

where now k is a *fixed* positive integer. Instead of (5. 22), we have now

$$e_q(kp)\int_{-\theta'_{p,q}}^{\theta_{p,q}} \frac{e^{kiu}}{\left(\frac{1}{n} - iu\right)\left(\frac{1}{n} + iu\right)} du \sim e_q(kp)\int_{-\infty}^{\infty} \frac{du}{\frac{1}{n^2} + u^2} = \pi n e_q(kp).$$

We are thus led to suppose that

$$(5.25)\qquad J(R) \sim \frac{1}{2} n \sum \left(\frac{\mu(q)}{\varphi(q)}\right)^2 e_q(kp)$$

when $R = e^{-\frac{1}{n}}$, $n \to \infty$.

The series here (which we call for the moment S'_2) is the singular series S_2 with $-k$ in the place of n. On the other hand

$$J(R) = \frac{1}{2\pi}\int_0^{2\pi} \sum \log \varpi\, R^{\varpi} e^{\varpi i \psi} . \sum \log \varpi\, R^{\varpi} e^{-\varpi i \psi} . e^{k i \psi} d\psi = R^k \sum a_{\varpi} R^{2\varpi},$$

where

$$a_{\varpi} = \log \varpi \log(\varpi + k)$$

if both ϖ *and* $\varpi + k$ *are prime*, and $a_{\varpi} = 0$ otherwise. Hence we obtain

$$\sum a_{\varpi} R^{2\varpi} \sim \frac{1}{1 - R^2} S'_2.$$

Here $R = e^{-\frac{1}{n}}$, but the result is easily extended to the case of continuous approach to the limit 1, and we deduce[1]

$$(5.26)\qquad \sum_{\varpi < n} a_{\varpi} \sim n S'_2.$$

And from this it would be an easy deduction that the number of prime pairs differing by k, and less than a large number n, is asymptotically equivalent to

$$\frac{n}{(\log n)^2} S'_2.$$

We are thus led to the following

Conjecture B. *There are infinitely many prime pairs*

$$\varpi,\ \varpi' = \varpi + k,$$

for every even k. *If* $P_k(n)$ *is the number of pairs less than* n, *then*

$$P_k(n) \sim 2 C_2 \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p} - 1}{\mathfrak{p} - 2}\right),$$

where C_2 *is the constant of* § 4 *and* $\mathfrak{p}$ *is an odd prime divisor of* k.

[1] We appeal again here to the Tauberian theorem referred to at the end of 4. 2 (f. n. 1). This time, of course, there is no question of an alternative argument.

[2] Note that $S'_2 = 0$ if k is odd, as it should be.

It will be observed that the analysis connected with Conjectures A and B, which deal respectively with the equations

$$n = \varpi + \varpi', \quad \varpi' = \varpi + k,$$

is substantially the same. It is pairs of problems connected in this manner that we call *conjugate* problems.

Numerical verifications.

5. 31. For $k = 2, 4, 6$ we obtain

(5. 311) $$P_2(n) \sim \frac{2\,C_2 n}{(\log n)^2},$$

(5. 312) $$P_4(n) \sim \frac{2\,C_2 n}{(\log n)^2},$$

(5. 313) $$P_6(n) \sim \frac{4\,C_2 n}{(\log n)^2},$$

Thus there should be approximately equal numbers of prime-pairs differing by 2 *and by* 4, *but about twice as many differing by* 6. The actual numbers of pairs, below the limits

100, 500, 1000, 2000, 3000, 4000, 5000

are

9	24	35	61	81	103	125
9	26	41	63	86	107	121
16	47	73	125	168	201	241

The correspondence is as accurate as could be desired.

5. 32. The first formula (5. 311) has been verified much more systematically. A little caution has to be exercised in undertaking such a verification. The formula (5. 26) is equivalent, when $k = 2$, to

(5. 321) $$\sum_{m<n} \Lambda(m)\,\Lambda(m+2) \sim 2\,C_2 n;$$

and, when we pass from this formula to one for the number of prime-pairs, the formula which arises most naturally is not (5. 311) but[1]

[1] This formula follows from (5. 321) in exactly the same way that

$$\pi(x) \sim Li\,x$$

follows from

$$\sum_{m<x} \Lambda(m) \sim x.$$

(5. 322) $$P_2(n) \backsim 2\,C_2 \int^{n} \frac{dx}{(\log x)^2};$$

indeed it is not unreasonable to expect this approximation to be a really good one, and much better than the formulae of 4. 4. The formula (5. 322) is naturally equivalent to (5. 311). But

$$\int^{n} \frac{dx}{(\log x)^2} = \frac{n}{(\log n)^2}\left(1 + \frac{2\,!}{\log n} + \frac{3\,!}{(\log n)^2} + \cdots\right),$$ [1]

and the second factor on the right hand side is (for such values of n as we have to consider) far from negligible. It is for this reason that Brun, when he attempted to deduce a value of the constant in (5. 311) from the statistical results, was led to a value seriously in error.

We therefore take the formula (5. 322) as our basis for comparison, choosing the lower limit to be 2. For our statistics as to the actual number of prime-pairs we are indebted to (*a*) a count up to 100,000 made by Glaisher in 1878 [2] and (*b*) a much more extensive count made for us recently by Mrs. G. A. Streatfeild. The results obtained by Mrs. Streatfeild are as follows.

n	$P_2(n)$	$2\,C_2 \int_2^n \frac{dx}{(\log x)^2}$	Ratio
100000	1224	1246.3	1.018
200000	2159	2179.5	1.009
300000	2992	3035.4	1.015
400000	3801	3846.1	1.012
500000	4562	4625.6	1.014
600000	5328	5381.5	1.010
700000	6058	6118.7	1 010
800000	6763	6840.2	1.011
900000	7469	7548.6	1.011
1000000	8164	8245.6	1.010

[1] The series is of course divergent, and must be regarded as closed after a finite number of terms, with an error term of lower order than the last term retained.

[2] J. W. L. Glaisher, 'An enumeration of prime-pairs', *Messenger of Mathematics*, vol. 8 (1878), pp. 28—33. Glaisher counts (1, 3) as a pair, so that his figure exceeds ours by 1.

5. 33. Similar reasoning leads us to the following more general results.

Conjecture C. *If a, b are fixed positive integers and $(a, b) = 1$, and $N(n)$ is the number of representations of n in the form*

$$n = a\varpi + b\varpi',$$

then

$$N(n) = o\left(\frac{n}{(\log n)^2}\right)$$

unless $(n, a) = 1$, $(n, b) = 1$, and one and only one of n, a, b is even.[1] *But if these conditions are satisfied then*

$$N(n) \sim \frac{2C_2}{ab} \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p} - 1}{\mathfrak{p} - 2}\right),$$

where C_2 is the constant of § 4, and the product extends over all odd primes $\mathfrak{p}$ which divide n, a, or b.

Conjecture D. *If $(a, b) = 1$ and $P(n)$ is the number of pairs of solutions of*

$$a\varpi' - b\varpi = k$$

such that $\varpi' < n$, then

$$P(n) = o\left(\frac{n}{(\log n)^2}\right)$$

unless $(k, a) = 1$, $(k, b) = 1$, and just one of k, a, b is even. But if these conditions are satisfied then

$$P(n) \sim \frac{2C_2}{a} \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p} - 1}{\mathfrak{p} - 2}\right),$$

where $\mathfrak{p}$ is an odd prime factor of k, a, or b.

It should be clear that the theorems proved in §§ 1—3 must be capable of a similar generalisation. Thus we might investigate the number of representations of n in the form

$$n = a\varpi + b\varpi' + c\varpi'';$$

and here proof would be possible, though only with the assumption of hypothesis R. We have not performed the actual calculations.

[1] This is trivial. If n and a have a common factor, it divides $b\varpi'$, and must therefore be ϖ', which is thus restricted to a finite number of values. If n, a, b are all odd, ϖ or ϖ' must be 2.

Primes of the forms m^2+1, am^2+bm+c.

5. 41. Of the four problems mentioned by Landau in his Cambridge address, two were Goldbach's problem and the problem of the prime-pairs. The third was that of *the existence of an infinity of primes of the form* m^2+1.[1]

Our method is applicable to this problem also. We have now to consider the integral

$$J(R)=\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\psi})\,\vartheta(Re^{-i\psi})\,e^{-i\psi}\,d\psi,$$

where $f(x)$ is the same function as before and

$$\vartheta(x)=\sum_{m=1}^{\infty} x^{m^2}.$$

The approximation for $\vartheta(\bar{x})=\vartheta(Re^{-i\psi})$ on $\xi_{p,q}$ is

$$\vartheta(Re^{-i\psi})\sim\frac{1}{2}\sqrt{\pi}\,\frac{\bar{S}_{p,q}}{q}\left(\frac{1}{n}+i\left(\psi-\frac{2p\pi}{q}\right)\right)^{-\frac{1}{2}},$$

where

$$S_{p,q}=\sum_{h=1}^{q} e_q(h^2p)$$

and $\bar{S}_{p,q}$ is the conjugate of $S_{p,q}$; and we find, as an approximation for $J(R)$,

$$\frac{1}{4\sqrt{\pi}}\sum_{p,q}\frac{\mu(q)}{q\,\varphi(q)}\bar{S}_{p,q}\,e_q(-p)\int_{-\theta'_{p,q}}^{\theta_{p,q}}\frac{e^{-iu}\,du}{\left(\frac{1}{n}-iu\right)\sqrt{\frac{1}{n}+iu}}.$$

We replace the ntegral here by

$$\int_{-\infty}^{\infty}\frac{du}{\left(\frac{1}{n}-iu\right)\sqrt{\frac{1}{n}+iu}}=\pi\sqrt{2n};$$

[1] The fourth was that of the existence of a prime between n^2 and $(n+1)^2$ for every $n>0$.

The problem of primes am^2+bm+c must not be confused with the much simpler (though still difficult) problem of primes included in the definite quadratic form $ax^2+bxy+cy^2$ in two independent variables. This problem, of course, was solved in the classical researches of DE LA VALLÉE POUSSIN. Our method naturally leads to de la Vallée Poussin's results, and the formal verification of them in this manner is not without interest. Here, however, our method is plainly not the right one, and could lead at best to a proof encumbered with an unnecessary hypothesis and far more difficult than the accepted proof.

and we are led to the formula

$$J(R) \sim \frac{1}{4} \sqrt{2\pi n}\, S, \tag{5. 411}$$

where S is the singular series

$$S = \sum_{p,q} \frac{\mu(q)}{q\,\varphi(q)} \bar{S}_{p,q}\, e_q(-p). \tag{5. 412}$$

Repeating the arguments of § 5. 2, we conclude that *the number $P(n)$ of primes of the form m^2+1 and less than n is given asymptotically by*

$$P(n) \sim \frac{\sqrt{n}}{\log n} S. \tag{5. 413}$$

5. 42. The singular series (5. 412) may be summed by the method of § 3. 2. Writing

$$S = \sum A_q = 1 + A_2 + A_3 + \cdots,$$

there is no difficulty in proving that $A_{qq'} = A_q A_{q'}$ if $(q, q') = 1$. Hence we write[1]

$$S = \prod \chi_{\varpi},$$

where

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + \cdots = 1 + A_{\varpi}.$$

If $\varpi = 2$, $A_{\varpi} = 0$, $\chi_{\varpi} = 1$. If $\varpi > 2$,[2]

$$S_{p,\varpi} = \left(\frac{p}{\varpi}\right) i^{\frac{1}{4}(\varpi-1)^2} \sqrt{\varpi},\ \bar{S}_{p,\varpi} = \left(\frac{p}{\varpi}\right) i^{-\frac{1}{4}(\varpi-1)^2} \sqrt{\varpi},$$

and

$$A_{\varpi} = -\frac{1}{(\varpi-1)\sqrt{\varpi}} i^{-\frac{1}{4}(\varpi-1)^2} \sum_{p=1}^{\varpi-1} \left(\frac{p}{\varpi}\right) e_{\varpi}(-p)$$

$$= -\frac{(-1)^{\frac{1}{2}(\varpi-1)}}{\varpi-1} = -\frac{1}{\varpi-1}\left(\frac{-1}{\varpi}\right).$$

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[1] Even this is a formal process, for (5. 412) is not absolutely convergent.
[2] See DIRICHLET-DEDEKIND, *Vorlesungen über Zahlentheorie*, ed. 4 (1894), pp. 293 *et seq.*

Thus finally we are led to

Conjecture E. *There are infinitely many primes of the form* m^2+1. *The number* $P(n)$ *of such primes less than* n *is given asymptotically by*

$$P(n) \backsim C\frac{\sqrt{n}}{\log n},$$

where

$$C = \prod_{\varpi=3}^{\infty}\left(1-\frac{1}{\varpi-1}\left(\frac{-1}{\varpi}\right)\right).$$

5.43. Generalising the analysis of §§ 5.41, 5.42, we arrive at

Conjecture F. *Suppose that* a, b, c *are integers and* a *is positive; that* $(a, b, c) = 1$; *that* $a+b$ *and* c *are not both even; and that* $D = b^2 - 4ac$ *is not a square. Then there are infinitely many primes of the form* am^2+bm+c. *The number* $P(n)$ *of such primes less than* n *is given asymptotically by*

$$P(n) \backsim \frac{\varepsilon C}{\sqrt{a}}\frac{\sqrt{n}}{\log n}\prod\left(\frac{\mathfrak{p}}{\mathfrak{p}-1}\right),$$

where $\mathfrak{p}$ *is a common odd prime divisor of* a *and* b, ε *is* 1 *if* $a+b$ *is odd and* 2 *if* $a+b$ *is even, and*

(5.4321) $$C = \prod_{\varpi \geqq 3,\, \varpi \nmid a}\left(1-\frac{1}{\varpi-1}\left(\frac{D}{\varpi}\right)\right).$$

It is instructive here to observe the genesis of the exceptional cases. If $(a, b, c) = d > 1$, there can obviously be at most one prime of the form required. In this case χ_ϖ vanishes for every ϖ for which $\varpi \mid d$. If $a+b$ and c are both even, am^2+bm+c is always even: in this case χ_2 vanishes. If $D = k^2$, then

$$4a(am^2+bm+c) = (2am+b)^2-k^2,$$

and

$$4a\varpi = (2am+b)^2-k^2$$

involves $2am+b\pm k \mid 4a$, which can be satisfied by at most a finite number of values of m. In this case no factor χ_ϖ vanishes, but the product (5.4321) diverges to zero.

5.44. The conjugate problem is that of the expression of a number n in the form

(5.441) $$n = am^2+bm+\varpi.$$

Here we are led to

Conjecture G. *Suppose that a and b are integers, and $a > 0$, and let $N(n)$ be the number of representations of n in the form $am^2 + bm + \varpi$. Then if n, a, b have a common factor, or if n and $a+b$ are both even, or if $b^2 + 4an$ is a square, then*

$$N(n) = o\left(\frac{\sqrt{n}}{\log n}\right). \tag{5.442}$$

In all other cases

$$N(n) \sim \frac{\varepsilon}{\sqrt{a}} \frac{\sqrt{n}}{\log n} \prod \left(\frac{\mathfrak{p}}{\mathfrak{p}-1}\right) \prod_{\varpi \geq 3,\, \varpi \nmid a} \left(1 - \frac{1}{\varpi - 1}\left(\frac{b^2 + 4an}{\varpi}\right)\right), \tag{5.443}$$

where $\mathfrak{p}$ is a common odd prime divisor of a and b, and ε is 1 if $a+b$ is odd and 2 if $a+b$ is even.

The following are particularly interesting special cases of this proposition.

Conjecture H. *Every large number n is either a square or the sum of a prime and a square. The number $N(n)$ of representations is given asymptotically by*

$$N(n) \sim \frac{\sqrt{n}}{\log n} \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{\varpi - 1}\left(\frac{n}{\varpi}\right)\right). \tag{5.444}$$

There does not seem to be anything in mathematical literature corresponding to this conjecture: probably because the idea that *every* number is a square, or the sum of a prime and a square, is refuted (even if 1 is counted as a prime) by such immediate examples as 34 and 58. But the problem of the representation of an *odd* number in the form $\varpi + 2m^2$ has received some attention; and it has been verified that the only odd numbers less than 9000, and not of the form desired, are 5777 and 5993.[1]

Conjecture I. *Every large odd number n is the sum of a prime and the double of a square. The number $N(n)$ of representations is given asymptotically by*

$$N(n) \sim \frac{\sqrt{2n}}{\log n} \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{\varpi - 1}\left(\frac{2n}{\varpi}\right)\right). \tag{5.445}$$

[1] By STERN and his pupils in 1856. See Dickson's *History* (referred to on p. 32) p. 424. The tables constructed by Stern were preserved in the library of Hurwitz, and have been very kindly placed at our disposal by Mr. G. Pólya. These manuscripts also contain a table of decompositions of primes $q = 4m + 3$ into sums $q = p + 2p'$, where p and p' are primes of the form $4m + 1$, extending as far as $q = 20983$. The conjecture that such a decomposition is always possible (1 being counted as a prime) was made by Lagrange in 1775 (see Dickson, *l. c.*, p. 424).

5. 45 We may equally work out the number of representations of n as the sum of a prime and any number of squares. Thus, for example, we find

Conjecture J. *The numbers of representations of n in the forms*

$$n = \varpi + m_1^2 + m_2^2, \quad n = \varpi + m_1^2 + m_2^2 + m_3^2 + m_4^2,$$

are given asymptotically by the formulae

(5. 451) $$N(n) \sim C\pi n \prod_{\mathfrak{p} \equiv 1 \pmod 4} \left(\frac{(\mathfrak{p}-1)^2}{\mathfrak{p}^2-\mathfrak{p}+1}\right) \prod_{\mathfrak{p} \equiv 3 \pmod 4} \left(\frac{\mathfrak{p}^2-1}{\mathfrak{p}^2-\mathfrak{p}-1}\right),$$

where

(5. 4511) $$C = \sum_{\varpi=3}^{\infty} \left(1 + \frac{1}{\varpi(\varpi-1)}\left(\frac{-1}{\varpi}\right)\right);$$

and

(5. 452) $$N(n) \sim \frac{1}{2} C\pi^2 n^2 \prod \left(\frac{(\mathfrak{p}-1)^2(\mathfrak{p}+1)}{\mathfrak{p}^3-\mathfrak{p}^2+1}\right),$$

where

(5. 4521) $$C = \prod_{\varpi=3}^{\infty} \left(1 + \frac{1}{\varpi^2(\varpi-1)}\right),$$

Here $\mathfrak{p}$ is an odd prime divisor of n, and representations which differ only in the sign or order of the numbers $m_1, m_2, \ldots$ are counted as distinct.

The last pair of formulae should be capable of rigorous proof.

Problems with cubes.

5. 5. The corresponding problems with cubes have, so far as we are aware, never been formulated. The problem which suggests itself first is that of the existence of an infinity of primes of the form $m^3 + 2$ or, more generally, $m^3 + k$, where k is any number other than a (positive or negative) cube.

Here again our method may be used, but the algebraical transformations, depending, as obviously they must, on the theory of cubic residuacity, are naturally a little more complex. As there is in any case no question of proof, we content ourselves with stating a few of the results which are suggested.

Conjecture K. *If k is any fixed number other than a (positive or negative) cube, then there are infinitely many primes of the form $m^3 + k$. The number $P(n)$ of such primes less than n is given asymptotically by*

(5. 51) $$P(n) \sim \frac{n^{\frac{1}{3}}}{\log n} \prod_{\varpi} \left(1 - \frac{2}{\varpi - 1}(-k)_{\varpi}\right),$$

where

$$\varpi \equiv 1 \pmod{3}, \quad \varpi \nmid k,$$

and $(-k)_\varpi$ *is equal to* 1 *or to* $-\frac{1}{2}$ *according as* $-k$ *is or is not a cubic residue of* ϖ.

Conjecture L. *Every large number n is either a cube or the sum of a prime and a (positive) cube. The number* $N(n)$ *of representations is given asymptotically by*

$$N(n) \sim \frac{n^{\frac{1}{3}}}{\log n} \prod_{\varpi} \left(1 - \frac{2}{\varpi - 1} (n)_\varpi\right),$$

the range of values of ϖ *being defined as in* K.

Conjecture M. *If k is any fixed number other than zero, there are infinitely many primes of the form* $l^3 + m^3 + k$, *where l and m are both positive. The number* $P(n)$ *of such primes less than n, every prime being counted multiply according to its number of representations, is given asymptotically by*

$$P(n) \sim \frac{\left(\Gamma\left(\frac{4}{3}\right)\right)^2}{\Gamma\left(\frac{5}{3}\right)} \frac{n^{\frac{2}{3}}}{\log n} \prod_{\mathfrak{p}} \left(1 - \frac{2}{\mathfrak{p}}\right) \prod_{\varpi} (1 + A_\varpi),$$

where $\mathfrak{p}$ *and* ϖ *are odd primes of the form* $3r + 1$, $\mathfrak{p} \mid k$, $\varpi \nmid k$, *and*

$$A_\varpi = -\frac{A - 2}{\varpi(\varpi - 1)}$$

if $-k$ *is a cubic residue of* ϖ,

$$A_\varpi = \frac{\frac{1}{2} A \pm \frac{9}{2} B - 2}{\varpi(\varpi - 1)}$$

in the contrary case. The positive sign is to be chosen if

$$\left(\frac{-k}{\omega}\right)_3 = e^{\frac{2}{3}\pi i} = \varrho,$$

$\omega = a + b\varrho$ *being that complex prime factor of* ϖ *for which* $a \equiv -1$, $b \equiv 0 \pmod{3}$; *the negative in the contrary event. And A and B are defined by*

$$A = 2a - b, \quad 3B = b, \quad 4\varpi = A^2 + 27 B^2.$$

In particular, when $k = 1$, the number of primes $l^3 + m^3 + 1$ is given asymptotically by

$$P(n) \sim \frac{\left(\Gamma\left(\frac{4}{3}\right)\right)^2}{\Gamma\left(\frac{5}{3}\right)} \frac{n^{\frac{2}{3}}}{\log n} \prod_{\varpi} \left(1 - \frac{A - 2}{\varpi(\varpi - 1)}\right),$$

primes susceptible of multiple representation being counted multiply.

Conjecture N. *There are infinitely many primes of the form $k^3 + l^3 + m^3$, where k, l, m are all positive. The number $P(n)$ of such primes less than n, primes susceptible of multiple representation being counted multiply, is given asymptotically by*

$$P(n) \sim \left(\Gamma\left(\frac{4}{3}\right)\right)^3 \frac{n}{\log n} \prod_{\varpi} \left(1 - \frac{A}{\varpi^2}\right),$$

where ϖ is a prime of the form $3m + 1$, and A has the meaning explained under M.

Triplets and other sequences of primes.

5. 61. It is plain that the numbers $\varpi, \varpi + 2, \varpi + 4$ cannot all be prime, for at least one of the three is divisible by 3. But it is possible that $\varpi, \varpi + 2, \varpi + 6$ or $\varpi, \varpi + 4, \varpi + 6$ should all be prime. It is natural to enquire whether our method is applicable in principle to the investigation of the distribution of triplets and longer sequences.

The general case raises very interesting questions as to the density of the distribution of primes, and it will be convenient to begin by discussing them.

We write

(5. 611) $$\varrho(x) = \overline{\lim_{n \to \infty}} (\pi(n + x) - \pi(n)),$$

so that $\varrho(x) = \varrho([x])$ is the greatest number of primes that occurs indefinitely often in a sequence $n + 1, n + 2, \ldots, n + [x]$ of $[x]$ consecutive integers. The existence of an infinity of primes shows that $\varrho(x) \geqq 1$ for $x \geqq 1$, and nothing more than this is known; but of course Conjecture B involves $\varrho(x) \geqq 2$ for $x \geqq 3$. It is plain that the determination of a lower bound for $\varrho(x)$ is a problem of exceptional depth.

The problem of an upper bound has greater possibilities. We proceed to prove, by a simple extension of an argument due to Legendre[1],

[1] See Landau, p. 67.

Theorem G. *If* $\varepsilon > 0$ *then*

$$\varrho(x) < (1+\varepsilon)\, e^{-C} \frac{x}{\log\log x} \quad (x > x_0 = x_0(\varepsilon)),$$

where C *is Euler's constant. More generally, if* $N(x, n)$ *is the number of the integers* $n+1, n+2, \ldots, n+[x]$ *that are not divisible by any prime less than or equal to* $\log x$, *then*

$$\sigma(x) = \overline{\lim_{n \to \infty}}\, N(x, n) < (1+\varepsilon)\, e^{-C} \frac{x}{\log\log x} \quad (x > x_0(\varepsilon)).$$

It is well-known that the number of the integers $1, 2, \ldots, [y]$, not divisible by any one of the primes $p_1, p_2, \ldots, p_\nu$, is

$$[y] - \sum \left[\frac{y}{p_r}\right] + \sum \left[\frac{y}{p_r p_s}\right] - \cdots$$

where the i-th summation is taken over all combinations of the ν primes i at a time. Since the number of terms in the total summation is 2^ν, this is

$$y - \sum \frac{y}{p_r} + \sum \frac{y}{p_r p_s} - \cdots + O(2^\nu) = y\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_\nu}\right) + O(2^\nu).$$

We now take $p_1, p_2, \ldots, p_\nu$ to be the first ν primes, write $n + x$ and n successively for y, subtract, and take the upper limit of the difference as $n \to \infty$. We obtain

$$\sigma(x) \leqq x \prod_{r=1}^{\nu} \left(1 - \frac{1}{p_r}\right) + O(2^\nu).$$

But

$$\prod_{p \leqq y} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-C}}{\log y}$$

as $y \to \infty$.[1] If we take $y = \log x$, and p_ν to be the greatest prime not less than y, we have

$$\nu < p_\nu \leqq \log x, \quad 2^\nu = o\left(\frac{x}{\log\log x}\right),$$

$$\sigma(x) < (1+\varepsilon)\, e^{-C} \frac{x}{\log\log x} \quad (x > x_0(\varepsilon)),$$

the desired result.

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[1] Landau, p. 140.

An examination of the primes less than 200 suggests forcibly that

$$\varrho(x) \leqq \pi(x) \qquad (x \geqq 2).$$

But although the methods we are about to explain lead to striking conjectural lower bounds, they throw no light on the problem of an upper bound. We have not succeeded in proving, even with our additional hypothesis, more than the »elementary» Theorem G. We pass on therefore to our main topic.

5. 62. We consider now the problem of the occurrence of groups of primes of the form

$$n,\ n+a_1,\ n+a_2,\ \ldots,\ n+a_m,$$

where $a_1, a_2, \ldots, a_m$ are distinct positive integers. We write for brevity

$$f_m(x) = \sum_{\varpi=2}^{\infty} \Lambda(\varpi)\,\Lambda(\varpi+a_1)\ldots\Lambda(\varpi+a_m)\,x^{\varpi}.$$

Then, if $(h, k) = 1$, we have

$$(5.\ 621) \qquad r^{a_m} f_m(r^2 e_k(h)) = \sum \Lambda(\varpi)\,\Lambda(\varpi+a_1)\ldots\Lambda(\varpi+a_m) r^{2\varpi+a_m} e_k(\varpi h)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum \Lambda(\varpi)\ldots\Lambda(\varpi+a_{m-1})\,r^{\varpi} e^{\varpi i\varphi} e_k(\varpi h)\,.\sum \Lambda(\varpi) r^{\varpi} e^{-\varpi i\varphi}\,.\,e^{a_m i\varphi}\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f_{m-1}\left(r e^{i\left(\varphi+\frac{2h\pi}{k}\right)}\right) f(re^{-i\varphi})\,e^{a_m i\varphi}\,d\varphi.$$

If $\varphi = \frac{2p\pi}{q}+\theta$, $r\to 1$, $\theta\to 0$, and θ is sufficiently small in comparison with $1-r$, then

$$f(re^{-i\varphi}) \sim \frac{\chi(q)}{1-re^{-i\theta}},$$

where

$$\chi(q) = \frac{\mu(q)}{\varphi(q)}.$$

Let us assume for the moment that

$$f_{m-1}(re^{i\psi}) \sim g_{m-1}\left(\frac{p'}{q'}\right)\frac{1}{1-re^{i\theta}}$$

if $\psi = \frac{p'}{q'}+\theta$, $r\to 1$, and θ is sufficiently small. Then our method leads us to write

$$\frac{1}{2\pi}\int_0^{2\pi} f_{m-1}\left(r e^{i\left(\varphi+\frac{2h\pi}{k}\right)}\right) f(re^{-i\varphi})\, e^{a_m i\varphi}\, d\varphi$$

$$\sim \sum_{p\,q} \chi(q)\, g_{m-1}\left(\frac{p}{q}+\frac{h}{k}\right) e\left(\frac{a_m p}{q}\right)\frac{1}{2\pi}\int_{\xi_{p,q}} \frac{d\theta}{(1-re^{i\theta})(1-re^{-i\theta})}$$

$$\sim \frac{1}{1-r^2}\sum_{p,q}\chi(q)\, g_{m-1}\left(\frac{p}{q}+\frac{h}{k}\right) e\left(\frac{a_m p}{q}\right),$$

on replacing the integral by one extended from $-\pi$ to π. Thus (5. 621) suggests that

(5. 622) $$f_m(r) \sim \frac{g_m(0)}{1-r},$$

where g_m is determined by the recurrence formula

(5. 623) $$g_m\left(\frac{h}{k}\right) = \sum_{p,q}\chi(q)\, g_{m-1}\left(\frac{p}{q}+\frac{h}{k}\right) e\left(\frac{a_m p}{q}\right)$$

and

(5. 624) $$g_0\left(\frac{h}{k}\right) = \chi(k).$$

From this recurrence formula we obtain without difficulty

(5. 625) $$g_m(0) = S_m = \sum_{p_1, q_1, \ldots, p_m, q_m} \prod_{r=1}^{m} \chi(q_r)\, \chi(Q)\, e\left(\sum_{r=1}^{m} \frac{a_r p_r}{q_r}\right),$$

where q_r runs through all positive integral values, p_r through all positive values less than and prime to q_r, and Q is the number such that

$$\frac{P}{Q} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \cdots + \frac{p_m}{q_m}, \qquad (P, Q) = 1.$$

If we sum with respect to the p's, we obtain a result which we shall write in the form

(5. 6251) $$S_m = \sum_{q_1, q_2, \ldots, q_m} A(q_1, q_2, \ldots, q_m).$$

We shall see presently that the multiple series (5. 6251) is absolutely convergent.

For greater precision of statement we now introduce a definite hypothesis.

Hypothesis X. *If* $m \geqq 0$, *and* $r \to 1$, *then*

(5. 626) $$f_m(r) \sim \frac{S_m}{1-r},$$

where S_m *is given by* (5. 625) *and* (5. 6251).

Our deductions from this hypothesis will be made rigorously, and we shall describe the results as Theorems X 1, X 2, ...

5. 63. From (5. 626) it follows, by the argument of 4. 2, that

(5. 631) $$P(x;\, 0,\, a_1,\, a_2,\, \ldots,\, a_m) \sim S_m \frac{x}{(\log x)^m},$$

as $x \to \infty$; *where the left-hand side denotes the number of groups of* $m+1$ *primes* $n, n+a_1, \ldots, n+a_m$ *of which all the members are less than* x.

We proceed to evaluate S_m. In the first place we observe that $A(q_1, q_2, \ldots, q_m)$ is zero if any q has a square factor. Next we have

(5. 632) $$A(q_1 q'_1,\, q_2 q'_2,\, \ldots,\, q_m q'_m) = A(q_1, q_2, \ldots q_m)\, A(q'_1, q'_2, \ldots, q'_m),$$

provided $(q_r, q'_s) = 1$ for all values of r and s. For, if we write

$$\frac{p_r}{q_r} + \frac{p'_r}{q'_r} = \frac{p_r q'_r + p'_r q_r}{q_r q'_r} = \frac{\mathfrak{p}_r}{\mathfrak{q}_r},$$

so that $\mathfrak{q}_r = q_r q'_r$, and suppose that p_r and p'_r run through complete sets of residues prime to q_r (or q'_r) and incongruent to modulus q_r (or q'_r), then $\mathfrak{p}_r$ runs through a similar set of residues for modulus $\mathfrak{q}_r$. Also $(Q, Q') = 1$ and so $(PQ' + P'Q, QQ') = 1$. Hence, since

$$\sum \frac{\mathfrak{p}_r}{\mathfrak{q}_r} = \frac{P}{Q} + \frac{P'}{Q'} = \frac{PQ' + P'Q}{QQ'},$$

the Q associated with $\sum \frac{\mathfrak{p}_r}{\mathfrak{q}_r}$ is QQ'. Since $\chi(qq') = \chi(q)\chi(q')$ if $(q, q') = 1$, (5. 632) follows at once.

Assuming then the absolute convergence, more conveniently established later, of the series and the product, we have

(5. 633) $$S_m = \sum A(q_1, q_2, \ldots, q_m) = \Pi X(\varpi) = \Pi X_m(\varpi) = \Pi X_m(\varpi;\, a_1, \ldots, a_m),$$

where

$$(5.634)\quad X(\varpi) = 1 + \sum_1 A(\varpi, 1, 1, \ldots, 1) + \sum_2 A(\varpi, \varpi; 1, \ldots, 1) + \cdots$$
$$+ \sum_r A(\varpi, \varpi, \varpi, \ldots 1) + \cdots + A(\varpi, \varpi, \varpi, \ldots, \varpi),$$

and where $\sum_r$ is extended over all A's in which r of the m places are filled by ϖ's and the remaining $m-r$ by 1's.

Our next step is to evaluate the A's corresponding to a prime ϖ. Writing $x = \chi(\varpi) = -\frac{1}{\varpi - 1}$, we have first, when only one place, say the first, is filled by a ϖ,

$$q_1 = \varpi,\ q_r = 1\ (r > 1),\ p_r = 0\ (r > 1),\ Q = \varpi,$$

and so

$$(5.635)\qquad A(\varpi, 1, 1, \ldots, 1) = (\chi(\varpi))^2 \sum_{(p,\varpi)=1} e_\varpi(a_1 p) = x^2 c_\varpi(a_1).$$

When $r > 1$ places, say the first r, are filled by ϖ's, we have similarly

$$A(\varpi, \varpi, \varpi, \ldots, 1) = x^r \sum_{p_1, p_2, \ldots, p_r} e_\varpi(a_1 p_1 + \cdots + a_r p_r)\, \chi(Q),$$

where the p's run through the numbers $1, 2, \ldots, \varpi - 1$, and Q is determined by

$$(P, Q) = 1,\quad \frac{P}{Q} = \frac{p_1 + p_2 + \cdots + p_r}{\varpi}.$$

Clearly

$$Q = 1\ \left(\sum p \equiv 0\ (\mathrm{mod.}\ \varpi)\right),\ Q = \varpi\left(\sum p \not\equiv 0\ (\mathrm{mod.}\ \varpi)\right).$$

Hence

$$(5.636)\quad A(\varpi, \varpi, \varpi, \ldots, 1) = x^{r+1}\left[\sum_{p_1, \ldots, p_r} e_\varpi\left(\sum_{s=1}^{r} a_s p_s\right) + \right.$$
$$\left. + \frac{\chi(1) - \chi(\varpi)}{\chi(\varpi)} \sum_{p_1 + p_2 + \cdots \equiv 0} e_\varpi\left(\sum_{s=1}^{r} a_s p_s\right)\right]$$

$$= x^{r+1}\left[\prod_{s=1}^{r} c_\varpi(a_s) - \varpi \sum_{p_1 + p_2 + \cdots \equiv 0} e_\varpi\left(\sum a_s p_s\right)\right].$$

Now

$$B=\sum_{p_1+p_2+\cdots\equiv 0} e\varpi\left(\sum a_s p_s\right)$$

is evidently a function of $\varpi, a_1, \ldots, a_r$ which is unaltered by a permutation of $a_1, \ldots, a_r$. We denote it (dropping the reference to ϖ) by $B_m(a_1, a_2, \ldots, a_r)$, the suffix m being used to recall that $a_1, a_2, \ldots, a_r$, or rather the a's that replace them in the general case, are selected from $a_1, a_2, \ldots, a_m$.

Then

$$(5.637)\quad B_m(a_1, a_2, \ldots, a_r)=\left(\sum_{p_2, p_3, \ldots, p_r} - \sum_{p_2+p_3+\cdots\equiv 0}\right) e\varpi(a_2p_2+\cdots+a_rp_r-a_1(p_2+\cdots+p_r))$$

$$=\sum_{p_2,\ldots,p_r} e\varpi((a_2-a_1)p_2+\cdots+(a_r-a_1)p_r)-\sum_{p_3,\ldots,p_r} e\varpi((a_3-a_2)p_3+\cdots+(a_r-a_2)p_r)+\cdots$$

$$=\prod_{s=2}^{r} c\varpi(a_s-a_1)-\prod_{s=3}^{r} c\varpi(a_s-a_2)+\cdots$$

Here we are supposing $r \geqq 2$. We shall adopt the convention $B_m(a_s)=0$.

5. 64. We now digress for a moment to establish the absolute convergence of our product and multiple series. We have

$$(5.641)\qquad c\varpi(k)=\varpi-1 \quad (\varpi \mid k), \quad c\varpi(k)=-1 \quad (\varpi \nmid k).$$

Hence, when ϖ is large, every $c\varpi$ occurring in (5. 635), (5. 636), or (5. 637) is equal to -1.[1] It follows that

$$|A(\varpi, \varpi, \varpi, \ldots, 1)|<Kx^2<\frac{K}{\varpi^2} \quad (r \geqq 1);$$

and so, since $A(q_1, q_2, \ldots)$ is the product of A's each involving only a single prime ϖ, that the multiple series and the product in (5. 633) are absolutely convergent.

5. 65. Returning now to $X(\varpi)$, we have, for $r \geqq 1$,

$$A(\varpi, \varpi, \varpi, \ldots 1)=x^{r+1}\left(\prod_{s=1}^{r} c\varpi(a_s)-\varpi B_m(a_1, a_2, \ldots, a_r)\right),$$

the result being true for $r=1$ in virtue of (5. 635) and our convention as to $B_m(a_s)$. Hence

[1] It is here that we use the condition $a_r \ne a_s$.

$$(5.651)\quad X_m(\varpi) = 1 + \sum_{r=1}^{m} x^{r+1} \prod_{s=1}^{r} c\varpi(a_s) - \varpi \sum_{r=2}^{m} x^{r+1} \sum\nolimits_r B_m(a_1, a_2, \ldots, a_r)$$

$$= 1 + x\left(\prod_{r=1}^{m}(1 + xc\varpi(a_r)) - 1\right) - \varpi x \sum_{r=2}^{m} x^r C_{m,r}$$

$$= Y_m - \varpi x Z_m,$$

say, where

$$(5.652)\quad C_{m,r} = \sum\nolimits_r B_m(a_1, a_2, \ldots, a_r),$$

the summation being taken over all combinations (without reference to order) of $a_1, \ldots, a_m$ taken r at a time.

Now

$$(5.653)\quad Y_{m+1} - (1-x)Y_m = 1 - x - (1-x)^2 + x\prod_{r=1}^{m}(1 + xc\varpi(a_r))(1 + xc\varpi(a_{m+1}) - 1 + x)$$

$$= x(1-x) + x^2(1 + c\varpi(a_{m+1}))\prod_{r=1}^{m}(1 + xc\varpi(a_r)).$$

Also

$$C_{m+1,r} = C_{m,r} + \sum{}' B(a_{m+1}, a_1, a_2, \ldots, a_{r-1}) \quad (r \geqq 2).$$

Here $\sum'$ denotes a sum taken over the combinations of $a_1, a_2, \ldots, a_m$, $r-1$ at a time; and the equation holds even for $r = m+1$ if we interpret $C_{m,m+1}$ as zero. Hence, by (5.637),

$$C_{m+1,r} = C_{m,r} + \sum{}'\left(\prod_{s=1}^{r} c\varpi(a_s - a_{m+1}) - B_m(a_1, a_2, \ldots, a_{r-1})\right)$$

$$= C_{m,r} + \sum{}'\prod_{s=1}^{r} c\varpi(a_s - a_{m+1}) - C_{m,r-1};$$

and therefore

$$(5.654)\quad Z_{m+1} = \sum_{r=2}^{m+1} x^r C_{m+1,r} = Z_m + \sum_{r=2}^{m+1} x^r \sum{}' \prod_{s=1}^{r-1} c\varpi(a_s - a_{m+1}) - \sum_{r=2}^{m+1} x^r C_{m,r-1}$$

$$= (1-x)Z_m + x\left(\prod_{r=1}^{m}(1 + xc\varpi(a_r - a_{m+1})) - 1\right).$$

Using (5. 651), (5. 653), and (6. 654), and observing that $x(1-x) = -\varpi x^2$, we obtain

(5. 655) $$X_{m+1}(\varpi) - (1-x)\, X_m(\varpi) = x^2(1 + c_\varpi(a_{m+1})) \prod_{r=1}^{m} (1 + x c_\varpi(a_r)) - $$
$$- \varpi x^2 \prod_{r=1}^{m} (1 + x c_\varpi(a_r - a_{m+1})).$$

To this recurrence formula we add the value of $X_m(\varpi)$ for $m = 1$, viz.

(5. 656) $$X_1(\varpi) = 1 + A(\varpi) = 1 + x^2 c_\varpi(a_1).$$

5. 66. We can now deduce an exceedingly simple formula for $X_m(\varpi)$, viz.

(5. 661) $$X_m(\varpi) = \left(\frac{\varpi}{\varpi - 1}\right)^m \frac{\varpi - \nu}{\varpi - 1},$$

where

(5. 662) $$\nu = \nu_m = \nu(\varpi;\, 0, a_1, a_2, \ldots, a_m)$$

is the number of distinct residues of $0, a_1, a_2, \ldots, a_m$ (mod. ϖ).

This is readily proved by induction. Let us denote the right hand side of (5. 661) by X'_m; and let us consider first the case $m = 1$.

If $a_1 \equiv 0$ (mod. ϖ) we have $\nu = 1$, $c_\varpi(a_1) = \varpi - 1$; if $a_1 \not\equiv 0$ we have $\nu = 2$, $c_\varpi(a_1) = -1$. In either case $X_1 = X'_1$.

Now suppose the result true for m, and consider X_{m+1}. There are three cases:

(i) $a_{m+1} \equiv 0$ (mod. ϖ). Here

$$\nu_{m+1} = \nu_m,\quad X'_{m+1} = \frac{\varpi}{\varpi - 1} X'_m = (1-x)\, X'_m.$$

On the other hand $1 + c_\varpi(a_{m+1}) = \varpi$, $c_\varpi(a_r - a_{m+1}) = c_\varpi(a_r)$; the right hand side of (5. 655) vanishes; and so

$$X_{m+1} = (1-x)\, X_m = (1-x)\, X'_m = X'_{m+1}.$$

(ii) $a_{m+1} \equiv a_{r_1} \not\equiv 0$ for some $r_1 \leqq m$. Here again $\nu_{m+1} = \nu_m$. On the one hand we have, as before, $X'_{m+1} = (1-x) X'_m$. On the other

$$1 + c_\varpi(a_{m+1}) = 0,\ 1 + x c_\varpi(a_{r_1} - a_{m+1}) = 1 - \frac{1}{\varpi - 1} c_\varpi(0) = 0;$$

the right hand side of (5. 665) vanishes, and $X_{m+1} = X'_{m+1}$ as before.

(iii) $a_{m+1} \neq 0$, $a_{m+1} \neq a_r (r \leq m)$. Here $\nu_{m+1} = \nu_m + 1 = \nu + 1$. Also all the c's concerned are equal to -1. Hence

$$X_{m+1} - (1-x) X_m = -\varpi x^2 (1-x)^m = x(1-x)^{m+1},$$

or, since $X_m = X'_m$,

$$\begin{aligned} X_{m+1} &= (1-x) \cdot (1-x)^m (1 + (\nu - 1) x) + x(1-x)^{m+1} \\ &= (1-x)^{m+1} (1 + \nu x) = X'_{m+1}. \end{aligned}$$

This completes the proof.

We now restate our conclusions in a more symmetrical form.

Theorem X 1.[1] *Let $b_1, b_2, \ldots, b_m$ be m distinct integers, and $P(x; b_1, b_2, \ldots, b_m)$ the number of groups $n + b_1, n + b_2, \ldots\ldots, n + b_m$ between 1 and x and consisting wholly of primes. Then*

(5. 663) $$P(x) \sim G(b_1, b_2, \ldots, b_m) \, Li_m(x)$$

when $x \to \infty$, where

(5. 664) $$G(b_1, b_2, \ldots, b_m) = \prod_{\varpi \geq 2} \left(\left(\frac{\varpi}{\varpi - 1} \right)^{m-1} \frac{\varpi - \nu}{\varpi - 1} \right),$$

$\nu = \nu(\varpi; b_1, b_2, \ldots, b_m)$ is the number of distinct residues of $b_1, b_2, \ldots, b_m$ to modulus ϖ, and

$$Li_m(x) = \int_2^x \frac{du}{(\log u)^m}.$$

Further

(5. 665) $$G(b_1, b_2, \ldots, b_m) = C_m H(b_1, b_2, \ldots, b_m)$$

where

(5. 666) $$C_m = \prod_{\varpi > m} \left(\left(\frac{\varpi}{\varpi - 1} \right)^{m-1} \frac{\varpi - m}{\varpi - 1} \right),$$

(5. 667) $$H(b_1, b_2, \ldots, b_m) = \prod_{\varpi \leq m} \left(\left(\frac{\varpi}{\varpi - 1} \right)^{m-1} \frac{\varpi - \nu}{\varpi - 1} \right) \prod_{\substack{\varpi \mid \Delta \\ \varpi > m}} \left(\frac{\varpi - \nu}{\varpi - m} \right),$$

and Δ is the product of the differences of the b's.

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[1] To avoid any possible misunderstanding, we repeat that these theorems are *consequences of Hypothesis X.*

The change from $0, a_1, \ldots, a_m$ to $b_1, b_2, \ldots, b_m$ is obtained by writing $n - b_1$ for n and m for $m+1$. The expression of G as the product of the constant C_m (depending only on m) and the finite expression H follows immediately from the fact that $\nu = m$ if $\varpi \nmid \Delta$, $\varpi > m$.

5.67. There are plainly many directions in which it would be possible to extend these investigations. We may ask, for example, whether there are indefinitely recurring pairs, triplets, or longer sequences of primes subject to further restrictions, such as that of belonging to specified quadratic forms. We have considered one problem of this character only, which is interesting in that it combines those contemplated in Conjectures B and E. Is there an infinity of pairs of primes of the forms m^2+1, m^2+3? The result suggested is as follows.

Conjecture P. *There are infinitely many prime pairs of the form m^2+1, m^2+3. The number of such pairs less than n is given asymptotically by*

$$Q(n) \sim \frac{3C\sqrt{n}}{(\log n)^2} = \frac{3\sqrt{n}}{(\log n)^2} \prod_{\varpi \geq 5} \frac{\varpi(\varpi - \nu)}{(\varpi - 1)^2},$$

where ν is 0, 2, or 4 according as neither, one, or both of -1 and -3 are quadratic residues of ϖ.

Numerical verifications.

5.68. A number of our conjectures have been tested numerically by Mrs. Streatfeild, Dr. A. E. Western, and Mr. O. Western. We state here a few of their results, reserving a fuller discussion of them for publication elsewhere.

The first of these numerical tests apply to conjectures E and P. In applying such tests we work (for reasons similar to those explained in 4.4 and 5.32) not with the actual formulae stated in the enunciations of those conjectures, but with the asymptotically equivalent formulae

$$P(n) \sim \frac{1}{2} C \int^{n} \frac{dx}{\sqrt{x} \log x} \sim \frac{1}{2} C \, Li\sqrt{n} \tag{5.681}$$

and

$$Q(n) \sim \frac{3}{2} C \int^{n} \frac{dx}{\sqrt{x}(\log x)^2} \sim \frac{3}{4} C \, Li_2 \sqrt{n}. \tag{5.682}$$

The number of primes less than 9000000 and of the prime form m^2+1 is 301. The number given by (5.681) is 302.6. The ratio is 1.005, and the agreement is all that could be desired.

The number of prime-pairs m^2+1 and m^2+3, both of whose members are less than 9000000, is 57. The value given by (5.682) is 48.9. The ratio is

858. The numbers concerned are naturally rather small, but the result is perhaps a little disappointing.

A more systematic test has been applied to the formulae for triplets and quadruplets of primes, the particular groups considered being

$$\varpi,\ \varpi+2,\ \varpi+6;\ \varpi,\ \varpi+4,\ \varpi+6;$$

$$\varpi,\ \varpi+2,\ \varpi+6,\ \varpi+8;\ \varpi,\ \varpi+4,\ \varpi+6,\ \varpi+10.$$

The two kinds of triplets should occur with the same frequency. On the other hand there should be twice as many quadruplets of the second type as of the first. For 0, 2, 6, 8 have 4 distinct residues to modulus 5 and 0, 4, 6, 10 but 3, while for all other moduli the number of residues is the same; and the ratio 5—3: 5—4 is 2. The actual results are shown in the following table.

Triplets.

x	$P_3(x;\,0,2,6)$	$\frac{9}{2}C_3\,Li_3(x)$	Ratio	$P_3(x;\,0,4,6)$	Ratio
10^5	260	270.78	1.041	249	1.087
2.10^5	417	440.71	1.057	425	1.037
3.10^5	566	589.89	1.042	588	1.003
4.10^5	718	727.43	1.013	748	0.972
5.10^5	833	857.10	1.029	881	0.973
6.10^5	950	980.92	1.033	1008	0.973
7.10^5	1073	1100.16	1.025	1133	0.971
8.10^5	1195	1215.64	1.017	1231	0.988
9.10^5	1295	1327.97	1.025	1331	0.998
10^6	1398	1437.59	1.028	1443	0.996

Quadruplets.

x	$P_4(x;\,0,2,6,8)$	$\frac{27}{2}C_4\,Li_4(x)$	Ratio	$P_4(x;\,0,4,6,10)$	$27\,C_4\,Li_4(x)$	Ratio
10^5	38	40.41	1.063	80	80.82	1.010
2.10^5	52	61.18	1.177	124	122.35	0.987
3.10^5	70	78.62	1.123	160	157.24	0.983
4.10^5	87	94.28	1.084	194	188.55	0.972
5.10^5	103	108.75	1.056	219	217.50	0.993
6.10^5	117	122.36	1.045	239	244.71	1.024
7.10^5	133	135.29	1.017	263	270.59	1.029
8.10^5	141	147.69	1.047	285	295.39	1.036
9.10^5	156	159.64	1.023	299	319.29	1.068
10^6	166	171.21	1.031	316	342.42	1.084

Here C_2 and C_4 are the constants of Theorem X 1. The results are on the whole very satisfactory, though there is a curious deficiency of quadruplets of the second type between 800000 and 1000000.

5. 691. We return to the problems connected with the function $\varrho(x) = \overline{\lim}_{n\to\infty} (\pi(n+x) - \pi(n))$. We shall adhere to the notation of Theorem X 1, and shall suppose in addition that x is integral and that $0 \leqq b_1 < b_2 < \cdots < b^m$. It follows from Theorem X 1 that, if $H(b_1, b_2, \ldots, b_m) \neq 0$, groups $n+b_1, n+b_2, \ldots, n+b_m$ consisting wholly of primes continually recur, and we shall say, when this happens, that $b_1, b_2, \ldots, b_m$ is a *possible* m-group of b's. We say also that the $n+b_1, \ldots, n+b_m$ is an m-group of primes. If, in a possible group, $m = \varrho(x)$, where $x = b_m - b_1 + 1$, we shall call the group, either of primes or of b's, a *maximum* group. A set of x consecutive positive integers we call an x-sequence; and we say that the group $n+b_1, \ldots, n+b_m$ is *contained in* the $(b_m - b_1 + 1)$-sequence $b_1 \leqq c \leqq b_m$, and that its *length* is $b_m - b_1 + 1$.

Theorem X 2. *If* $b_1, b_2, \ldots, b_m$ *have a missing residue* (mod. ϖ) *for each* $\varpi \leqq m$, *then these* b'*s form a possible group.*

This is an immediate consequence of Theorem X 1, since $\nu \leqq \varpi - 1$ for $\varpi > m$.

Theorem X 3. *Let* $M(x, n)$ *be the number of distinct integers* $c_1, c_2, \ldots, c_M$, *in the interval* $n < c \leqq n + x$, *which are not divisible by any prime less than or equal to*

$$\bar{\varrho}(x) = \varrho(x) + \left[\frac{x}{\varrho(x)}\right] + 1,$$

and let

$$\varrho_1(x) = \operatorname*{Max}_{(n)} M(x, n).$$

Then

$$\varrho_1(x) = \varrho(x).$$

Let $\varrho(x, \mu)$ be the number obtained in place of $\varrho_1(x)$ when the $\bar{\varrho}(x)$ that occurs in the definition of $\varrho_1(x)$ is replaced by μ. Clearly we have

$$\varrho(x, \mu - 1) \geqq \varrho(x, \mu) \geqq \varrho(x) \tag{5.6911}$$

and

$$\varrho(x, \mu) \geqq \varrho(x, \mu - 1) - \left[\frac{x}{\mu}\right] - 1. \tag{5.6912}$$

Further,

(5. 6913) $$\tau = \varrho(x, \mu) \leqq \mu$$

implies

$$\varrho(x, \mu) = \varrho(x).$$

For let $d_1, d_2, \ldots, d_\tau$ be an increasing set of positive integers with the properties (characteristic of a set of $\tau = \varrho(x, \mu)$ such integers) that (*a*) there is an n such that $n + d_1, \ldots, n + d_\tau$ are not divisible by any prime less than or equal to μ, and (*b*) $d_\tau - d_1 + 1 \leqq x$. Then $n + d_1, \ldots, n + d_\tau$ form a 'possible' group of *b*'s, since they lack the residue 0 for every prime less than or equal to τ. Hence $\varrho(x) \geqq \tau = \varrho(x, \mu)$, and so, by (5. 6911), $\varrho(x) = \varrho(x, \mu)$.

Next we observe that $\varrho(x, \mu) = \varrho(x)$ for $\mu = x$, since the inequality $\tau \leqq \mu$ is clearly satisfied in this case. Let now μ_0 be the least μ, greater than or equal to $\varrho(x)$, for which $\varrho(x, \mu_0) = \varrho(x)$. Then $\varrho(x) \leqq \mu_0 \leqq x$. We have then

(5. 6914) $$\varrho(x, \mu_0) = \varrho(x), \quad \varrho(x, \mu_0 - 1) > \varrho(x),$$

and so

$$\varrho(x, \mu_0 - 1) > \mu_0 - 1,$$

by (5. 6913). Thus

$$\mu_0 \leqq \varrho(x, \mu_0 - 1) \leqq \varrho(x, \mu_0) + \left[\frac{x}{\mu_0}\right] + 1 = \varrho(x) + \left[\frac{x}{\mu_0}\right] + 1$$

$$\leqq \varrho(x) + \left[\frac{x}{\varrho(x)}\right] + 1 = \bar{\varrho}(x).$$

Hence

$$\varrho(x) = \varrho(x, \mu_0) \geqq \varrho(x, \bar{\varrho}(x)) = \varrho_1(x).$$

But it is evident that $\varrho_1(x) \geqq \varrho(x)$, and therefore $\varrho_1(x) = \varrho(x)$.

It follows from the theorem that, *in a maximum group of primes of length x, the remaining numbers of the x-sequence are all divisible by primes less than or equal to $\bar{\varrho}(x)$*. We shall see presently that (on hypothesis X) $\bar{\varrho}(x) \leqq \varrho(x) + \log x$ for large values of x.

5. 692. We consider now the problem of a lower bound for $\varrho(x)$. Let p_s denote the s-th prime.

Theorem X 4. *Let $r = r(n)$ be defined, for every value of n, by*

$$p_r \leqq n < p_{r+1}.$$

Then $p_{r+1}, p_{r+2}, \ldots, p_{r+n}$ is a possible n-group of b's.

For the primes less than or equal to n are $p_1, p_2, \ldots, p_r$ and the b's lack the residue 0 for each of them.

From Theorem X 4 we deduce at once

Theorem X 5. *If* $x = p_{r+n} - p_{r+1} + 1$, $p_r \leqq n < p_{r+1}$, *then*

$$\varrho(x) \geqq n.$$

As a numerical example, let $n = 76501$. We have $p_{7525} = 76493$, $p_{7526} = 76507$. Hence

$$r = 7525,\ n + r = 84026,\ p_{n+r} = 1076503$$

$$x = 1076503 - 76507 + 1 = 999997.$$

Thus

$$\varrho(1000000) \geqq 76501.$$

We may compare this with the numbers of primes in the first, second, and third millions, viz.

$$78498,\ 70433,\ 67885.$$

Theorem X 5 provides a lower limit for $\varrho(x)$ when x has a certain form: we proceed to consider the case when x is unrestricted.

Theorem X 6. *We have*

$$\varrho(x) > \frac{x}{\log x}$$

for sufficiently large values of x.

When m is large

$$p_m = m(\log m + \log\log m) - m + O\left(\frac{m \log\log m}{\log m}\right).$$

Let

$$r = \left[\frac{y}{(\log y)^2}\left(1 + \frac{\log\log y}{\log y}\right)\right].$$

Then we have, by straightforward calculations,

$$p_r = \frac{y}{\log y}\left(1 - \frac{1}{\log y} + O\left(\frac{\log\log y}{\log y}\right)^2\right).$$

Take $n = p_r$. Then

$$n + r = \frac{y}{\log y}\left(1 + O\left(\frac{\log\log y}{\log y}\right)^2\right).$$

$$p_{n+r} = y\left(1 - \frac{1}{\log y} + O\left(\frac{\log\log y}{\log y}\right)^2\right)$$

$$x = p_{n+r} - p_{r+1} + 1 < p_{n+r} - p_r$$

$$= y\left(1 - \frac{2}{\log y} + O\left(\frac{\log\log y}{\log y}\right)^2\right) < y - \frac{3}{2}\frac{y}{\log y} = z,$$

when y is large. Thus

$$\varrho(z) \geqq \varrho(x) \geqq n = p_r = \frac{y}{\log y} - \frac{y}{(\log y)^2} + O\left(\frac{y(\log\log y)^2}{(\log y)^3}\right)$$

$$> \frac{z}{\log z}.$$

Since y is arbitrary, so is z, and the theorem is proved.

5. 693. We conclude our discussion of $\varrho(x)$ with an account of one or two particular cases. For a given x it is, of course, theoretically possible to determine the maximum number of integers in an x-sequence that are not divisible by any prime less than x. On hypothesis X, this number is $\varrho(x)$. Thus L. Aubry[1] has shown that 30 consecutive *odd* integers cannot contain more than 15 primes (or more than 15 numbers not divisible by 2, 3, 5, or 7). Thus $\varrho(59) \leqq 15$. On the other hand if we take, in Theorem X 5, $n = 15$, $r = 6$, we see that the 15 primes from 17 to 73 give a possible group of b's. Hence, on hypothesis X,

$$\varrho(59) \geqq \varrho(57) = \varrho(73 - 17 + 1) \geqq 15;$$

and so $\varrho(59) = 15$. The value of $\pi(59)$ is 17.

Similarly a 35-sequence cannot contain more than 10 numbers not divisible by 2, 3, or 5, but the 10 primes from 13 to 47, and therefore the numbers 0, 4, 6, 10, 16, 18, 24, 28, 30, 34, form a possible 10-group of b's, so that $\varrho(35) = 10 = \pi(35) - 1$. A striking example of a maximum prime group $n + b_1, \ldots, n + b_{10}$, corresponding to this group of b's, is provided by $n = 113143$.

The best example of a close approach by $\varrho(x)$ to $\pi(x)$ occurs when $x = 97$. Consider the 24 primes from 17 to 113. They are a possible group of b's if they have a missing residue for each prime less than 24. We have only to test 17, 19, 23, and we find that 17 lacks the residue 8, 19 lacks 1 and 11, and 23 lacks 3, 12, 16, and 22. Hence on hypothesis X, $\varrho(97) \geqq 24$. On the other hand it

[1] L. E. Dickson, *l. c.*, vol. 1, p. 355.

may be shown that a 97-sequence cannot contain 25 numbers not divisible by 2, 3, 5, 7, 11, or 13. Let us denote the range $n \leqq x \leqq n+96$ by R_n. There is one and only one value of n, not greater than $2.3.5.7 = 210$, for which R_n contains 25 numbers not divisible by 2, 3, 5, or 7, viz. $n = 101$. If then $n \leqq 2.3.5.7.11$, and R_n contains 25 numbers not divisible by 2, 3, 5, 7, or 11, n must be one of the numbers $101 + 210\,m$ ($m = 0, 1, \ldots, 10$); and on examination it proves that we may exclude all cases but $m = 10$. Repeating the argument we see that, if $n \leqq 2.3.5.7.11.13$, and R_n contains 25 numbers not divisible by 2, 3, 5, 7, 11, or 13, then n must be one of the numbers $n = 2201 + 2310\,m$ ($m = 0, 1, \ldots, 12$). All these turn out to be impossible and, since any R_n may be reduced (mod. $2.3 \ldots 13$), it follows that no R_n can contain more than 24 numbers not divisible by a prime less than or equal to 13. *A fortiori* it follows that $\varrho(97) \leq 24$, and so (on hypothesis X) $\varrho(97) = 24$. Since $\pi(97) = 25$, the difference $\varrho - \pi$ is here unity. Beyond $x = 97$ it would seem that $\varrho(x)$ falls further below $\pi(x)$, at least within any range in which calculation is practicable.

Conclusion.

5. 7. We trust that it will not be supposed that we attach any exaggerated importance to the speculations which we have set out in this last section. We have not forgotten that in pure mathematics, and in the Theory of Numbers in particular, 'it is only proof that counts'. It is quite possible, in the light of the history of the subject, that the whole of our speculations may be ill-founded. Such evidence as there is points, for what it is worth, in the opposite direction. In any case it may be useful that, finding ourselves in possession of an apparently fruitful method, we should develop some of its consequences to the full, even where accurate investigation is beyond our powers.

Postscript.

(1). Prof. Landau has called our attention to the following passage in the *Habilitationsschrift* of Piltz ('Über die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze', Jena, 1884), pp. 46-47: —

'Ferner wiederholen sich gewisse Gruppierungen der Primzahlen mit gewisser Regelmässigkeit, so ist z. B. die durchschnittliche Häufigkeit der Gruppen von je 2 Primzahlen, die in gegebenem Abstand aufeinanderfolgen, für die ungefähre Grösse x der Primzahlen, proportional $\frac{1}{(l\,x)^2}$, wobei allerdings dieser Ausdruck je nach dem gegebenen Abstand mit verschiedenen constanten Faktoren behaftet ist, die Häufigkeit einer Gruppe von 3 Primzahlen proportional $\frac{1}{(l\,x)^3}$ und so fort..... Die nähere Ausführung dieser und andrer Gesetze ... werde ich ein andres Mal folgen lassen.'

All of this is of course in perfect agreement with the results suggested in our concluding section.

(2). We must add a few words concerning the memoirs of Stäckel referred to on p. 34. These have only become accessible to us during the printing of the present memoir, and it is not possible for us even now to give any satisfactory summary of their contents; but Stäckel considers the problem of 'prime-groups' in much detail, and it is clear that he has anticipated some at any rate of the speculations of 5.6. The method of Stäckel, like that of Brun, rests on the use of the sieve of Eratosthenes, followed by a heuristic passage to the limit; but Stäckel's problem is much more general, and he has gone much further than Brun in the determination of the constants in the asymptotic formulae. It seems to be the principal advantage of our transcendental method, considered merely as a machine for the production of heuristic formulae, that these constants are determined naturally in the course of the analysis.

(3). We should also refer to a later memoir of Brun ('Le crible d'Eratosthène et le théorème de Goldbach', *Videnskapsselskapets Skrifter, Mat.-naturv. Klasse*, Kristiania, 1920, No. 3). Brun proves, by elementary methods, (1) that every large even number is the sum of two numbers, each composed of at most 9 prime factors, (2) that the number of prime-pairs ϖ, $\varpi+2$, less than x, cannot exceed a constant multiple of $x(\log x)^{-2}$.

Brun's work enables us to make a substantial improvement in the elementary theorem G. Using the inequalities proved on pp. 32—34 of his memoir, we can show that

$$\varrho(x) < \frac{A\,x}{\log x}.$$

(4). Prof. Landau has pointed out to us an error on p. 9. It is not necessarily true that $C_k = 0$ when χ_k is imprimitive: our argument is only valid when Q is divisible by every prime factor of q.

The inequality (2. 16) is however correct. Suppose first that $q = \varpi^\lambda$ $(\lambda > 0)$. Our argument then holds unless $Q = 1$; in this case χ_k is the principal character and

$$\left|\sum_{m=1}^{q} e_q(m)\, \bar{\chi}_k(m)\right| = 1 \leqq \sqrt{q}.$$

This inequality is then easly generalised to all values of q. If $q=q_1 q_2$, where $(q_1, q_2)=1$, then every χ (mod. q) is the product of a χ_1 (mod. q_1) and a χ_2 (mod. q_2). and it is easily proved that

$$\left|\sum_{m} e_q(m)\, \chi(m)\right| = \left|\chi_1(q_2)\, \chi_2(q_1) \sum_{m_1} e_{q_1}(m_1)\, \chi_1(m_1) \sum_{m_2} e_{q_2}(m_2)\, \chi_2(m_2)\right|$$

$$\leqq \sqrt{q_1}\, \sqrt{q_2} = \sqrt{q}.$$

The conclusion now follows by induction.